ON DISCRETE SPECTRUM IN THE GAPS OF A TWO-DIMENSIONAL PERIODIC ELLIPTIC OPERATOR PERTURBED BY A DECAYING POTENTIAL

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ON DISCRETE SPECTRUM IN THE GAPS OF A TWO-DIMENSIONAL PERIODIC ELLIPTIC OPERATOR PERTURBED BY A DECAYING POTENTIAL

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Abstract. We study the discrete spectrum in the gaps of a two-dimensional periodic elliptic second order operator perturbed by a decaying potential. We are interested in the asymptotics (for large coupling constant) of the number of eigenvalues that have been "born" (or have "died") at the edges of the gap. The high-energy (Weyl) asymptotics and the threshold asymptotics are distinguished. At the right edge of the gap the competition between the Weyl contribution and the threshold contribution to the asymptotics is possible.

§ 0. Introduction

1. Let A be a selfadjoint elliptic periodic second order operator in $L_2(\mathbb{R}^d)$, $d \geq 2$, given by the expression $A = -\operatorname{div} g(\mathbf{x})\nabla + p(\mathbf{x})$, and let V be an operator of multiplication by a function $V(\mathbf{x}) \geq 0$ decaying at infinity. Let the interval (λ_-, λ_+) be a gap in the spectrum of A. We put $A_\pm(\alpha) = A \mp \alpha V(\mathbf{x})$, $\alpha > 0$. By $\mathfrak{N}_+(\alpha, \lambda_+)$ we denote the number of eigenvalues of the operator $A_+(t)$ that have been "born" at the point λ_+ as the coupling constant t has been growing from 0 to α. The function $\mathfrak{N}_-(\alpha, \lambda_-)$ is defined similarly for the operator A_-. We are interested in the asymptotics of these functions as $\alpha \to \infty$ (in the large coupling constant limit). The corresponding asymptotics may be rather diverse, depending both on A and V. They were studied in a number of papers. First of all, we mention [B3, B4, BL] and especially the survey [B2] and the references therein. Another approach to the class of problems under discussion was proposed in [Iv].

The asymptotic behavior of $\mathfrak{N}_\pm(\alpha, \lambda_\pm)$ depends on dimension d and on the character of decay of V. The situation for $d \geq 3$ differs substantially from that for $d = 2$. If $d \geq 3$ and $V \in L_{d/2}(\mathbb{R}^d)$, the function $\mathfrak{N}_+(\alpha, \lambda_+)$ has the Weyl asymptotics

$$\mathfrak{N}_+(\alpha, \lambda_+) \sim (2\pi)^{-d}\omega_d \alpha^{d/2} \int V^{d/2} (\det g)^{-1/2}\, d\mathbf{x}, \quad \alpha \to \infty; \qquad (0.1)$$

here ω_d is the volume of the unit ball in $\mathbb{R}^d$.

If $V \notin L_{d/2}(\mathbb{R}^d)$, then the estimate $\mathfrak{N}_+(\alpha, \lambda_+) = O(\alpha^{d/2})$ is violated, and $\mathfrak{N}_+(\alpha, \lambda_+)$ can have an arbitrary order of growth greater than $d/2$. Essentially, the asymptotics (0.1) has a "high-energy" origin, while the behavior of $\mathfrak{N}_+(\alpha, \lambda_+)$ with $V \notin L_{d/2}(\mathbb{R}^d)$ is determined by the "threshold" effect near the edge of the gap of the unperturbed operator A. (See discussion in [B2, §2].)

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2. If $d = 2$, the situation is much more complicated. Let $V \in L_1(\mathbb{R}^2)$. Already for $A = -\Delta$ in the case of the semi-infinite gap $(-\infty, 0)$, this condition is insufficient for (0.1). Due to the threshold effects, $\mathfrak{N}_+(\alpha, 0)$ may have an arbitrary order of growth greater than $d/2$. Moreover, it may happen that $\mathfrak{N}_+(\alpha, 0) = O(\alpha)$, but the asymptotics is not of Weyl type. In this case the asymptotic coefficient is the sum of the Weyl term and the "threshold" term. A *"special channel"* is responsible for the threshold effect. We mean the problem on the semi-axis that is obtained by restriction of $-\Delta - \alpha V$ to the subspace of functions depending only on $|\mathbf{x}|$. At the same time, the potential V is averaged over the polar angle. These effects were investigated in [BL] in detail. At the level of estimates this channel was discovered before in [S].

In [BLSu], the same effects were studied in the case where A is a periodic elliptic operator of the form $A = -\operatorname{div} g(\mathbf{x})\nabla + p(\mathbf{x})$. Adding an appropriate constant to p allows us to assume that the lower edge of the spectrum is the point $\lambda = 0$. In [BLSu], the *negative* discrete spectrum of the operator $A - \alpha V$, i. e., the case of the semi-bounded gap $(-\infty, 0)$ was studied. The description of the special channel was given in terms of the Floquet-Bloch decomposition for the unperturbed operator A. The answer involves the so-called *tensor of effective masses* at the edge of the spectrum and a *positive periodic solution* φ *of the equation* $A\varphi = 0$. The function φ can be eliminated from the answer under an additional "regularity" condition imposed on V.

The present paper is a continuation of [BLSu], but now we study the case of an internal gap in the spectrum of A. For this, we need to change the technique of investigation substantially.

3. For the study of the functions $\mathfrak{N}_\pm(\alpha, \lambda_\pm)$ in an *internal gap* of A, we impose certain restrictions on the structure of the edges of the gap (see Condition 1.3($\pm$) below). For the lower edge of the spectrum $\lambda = 0$ this condition is fulfilled automatically. The answers are given in terms of the *model operators*, which are simpler than $A_\pm(\alpha)$. The model operators involve the tensors of effective masses at the edges of the gap and the corresponding eigenfunctions. As well as for the semi-infinite gap, it is possible to eliminate the eigenfunctions from the answer under an additional "regularity" condition imposed on V. The main results are formulated in Theorems 2.2($\pm$), 2.5($\pm$). On the right (but not on the left) edge of the gap a competition between the Weyl contribution and the threshold contribution to the asymptotics is possible.

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5. Notation. In what follows, $\mathbb{Q}^2$ is an open unit square in $\mathbb{R}^2$. The symbol $\langle \cdot, \cdot \rangle$ stands for the standard inner product in $\mathbb{C}^m$; $\mathbf{1}$ is the unit (2×2)-matrix. Any integral without indication of the integration domain is over $\mathbb{R}^2$. Further, $\nabla = \operatorname{grad}$, $\nabla^* = -\operatorname{div}$. We denote by H^s, $s \geq 0$, the Sobolev classes.

Many statements and formulas contain the double indices "$\pm$". Unless otherwise explicitly stated, the upper and the lower versions should be read independently.

§ 1. Setting of the problem. Preliminaries

1. Differential operators. The *unperturbed* operator A is formally given by the expression $\mathcal{A}u = \nabla^* g \nabla u + pu$. Here g is a (2×2)-matrix-valued function, p is a real-valued function. We assume that

$$\left.\begin{aligned} &\overline{g} = g > 0, \quad \overline{p} = p, \quad g + g^{-1} \in L_\infty(\mathbb{R}^2), \quad p \in L_\infty(\mathbb{R}^2), \\ &g(\mathbf{x}+\mathbf{n}) = g(\mathbf{x}), \quad p(\mathbf{x}+\mathbf{n}) = p(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \quad \mathbf{n} \in \mathbb{Z}^2. \end{aligned}\right\} \tag{1.1}$$

There is no loss of generality in assuming that the lattice of periods is $\mathbb{Z}^2$. The precise definition of A as a selfadjoint operator in the Hilbert space $L_2(\mathbb{R}^2)$ is given via the closed lower semi-bounded quadratic form

$$a[u,u] = \int (\langle g\nabla u, \nabla u\rangle + p|u|^2)\, d\mathbf{x}, \quad u \in H^1(\mathbb{R}^2). \tag{1.2}$$

Adding an appropriate constant to p allows us to assume that $\inf \operatorname{spec} A = 0$. Under this condition, in $H^1(\mathbb{R}^2)$ the form $a[u,u] + \gamma \int |u|^2\, d\mathbf{x}$, $\gamma > 0$, determines a metric equivalent to the standard one.

A *perturbation* is introduced as the operator of multiplication by a function $V(\mathbf{x})$ such that $V(\mathbf{x}) \geq 0$, $\mathbf{x} \in \mathbb{R}^2$. We impose the following condition on V (cf., e. g., [BL]).

Condition 1.1. For some $\sigma > 1$,

$$\left(\int\limits_{|\mathbf{x}|\leq 1} |V|^\sigma\, d\mathbf{x} \right)^{1/\sigma} + \sum_{k\geq 1} \left(\int\limits_{e^{k-1}\leq|\mathbf{x}|\leq e^k} |V|^\sigma |\mathbf{x}|^{2(\sigma-1)}\, d\mathbf{x} \right)^{1/\sigma} < \infty. \tag{1.3}$$

We mention at once that (1.3) implies that

$$(\!(V)\!)_\sigma := \sum_{\mathbf{n}\in\mathbb{Z}^2} \left(\int\limits_{\mathbb{Q}^2+\mathbf{n}} V^\sigma\, d\mathbf{x} \right)^{1/\sigma} < \infty, \quad \sigma > 1, \tag{1.4}$$

and, moreover, $V \in L_1(\mathbb{R}^2)$. Consider the quadratic form

$$v[u,u] = \int V|u|^2\, d\mathbf{x}.$$

Under condition (1.4) (and, moreover, under condition (1.3)), this form is compact in $H^1(\mathbb{R}^2)$. Consequently, the *form*

$$a_\pm(\alpha)[u,u] := a[u,u] \mp \alpha v[u,u], \quad u \in H^1(\mathbb{R}^2), \quad \alpha > 0,$$

is lower semi-bounded and closed in $L_2(\mathbb{R}^2)$. The form $a_\pm(\alpha)$ generates a selfadjoint operator $A_\pm(\alpha)$ in $L_2(\mathbb{R}^2)$. Formally, the operator $A_\pm(\alpha)$ corresponds to the differential expression $\mathcal{A}_\pm(\alpha)u = \nabla^* g\nabla u + pu \mp \alpha Vu$. The *spectrum of* $A_\pm(\alpha)$ *in the spectral gaps of* A *is discrete.*

First, we recall the result for the semi-infinite gap. Let $\mathfrak{N}_+(\alpha,\lambda; A, V)$, $\alpha > 0$, $\lambda \leq 0$, denote the *number of eigenvalues of the operator* $A_+(\alpha)$, *lying to the left of the point* λ.

For the Weyl asymptotic coefficient we introduce the notation

$$J(V,g) := (4\pi)^{-1} \int V(\det g)^{-1/2}\, d\mathbf{x}. \tag{1.5}$$

Proposition 1.2. *Under condition* (1.4), *we have*

$$\mathfrak{N}_+(\alpha,\lambda;A,V)\le C\alpha(\!(V)\!)_\sigma,\quad C=C(g,p,\sigma,\lambda),\quad \sigma>1,\quad \lambda<0,$$

$$\lim_{\alpha\to\infty}\alpha^{-1}\mathfrak{N}_+(\alpha,\lambda;A,V)=J(V,g),\quad \lambda<0. \tag{1.6}$$

Comments on Proposition 1.2 and necessary references can be found in [BLSu]. For $\lambda = 0$ the Weyl asymptotics (1.6) may fail to occur even under condition (1.3) because of spectral "threshold" effects. These phenomena were studied in the paper [BL] for the operator $-\Delta-\alpha V$ and in [BLSu] in the general case of a periodic operator A. Below we shall impose one more condition on V (see Condition 2.1(q)), which ensures that $\mathfrak{N}_+(\alpha,0;A,V)=O(\alpha^q)$, $\alpha\to\infty$, $q\ge 1$. In the present paper the discrete spectrum of the operators $A_\pm(\alpha)$ in internal gaps of A is studied.

2. The Floquet decomposition. As usual, for the periodic operators we employ the Floquet-Bloch theory. Let $\widetilde{H}^1(\mathbb{Q}^2)$ be the subspace formed by the functions in $H^1(\mathbb{Q}^2)$ whose $\mathbb{Z}^2$-periodic extensions belong to the class $H^1_{\mathrm{loc}}(\mathbb{R}^2)$. Next, we denote by $\widetilde{H}^1_{\boldsymbol{\xi}}(\mathbb{Q}^2)$, $\boldsymbol{\xi}\in\mathbb{R}^2$, the subspace of functions of the form $u(\mathbf{x})=e^{i\langle\mathbf{x},\boldsymbol{\xi}\rangle}v(\mathbf{x})$, $v\in\widetilde{H}^1(\mathbb{Q}^2)$. Consider the following family of quadratic forms in $L_2(\mathbb{Q}^2)$:

$$a_{\boldsymbol{\xi}}[u,u]=\int_{\mathbb{Q}^2}(\langle g\nabla u,\nabla u\rangle+p|u|^2)\,d\mathbf{x},\quad u\in\widetilde{H}^1_{\boldsymbol{\xi}}(\mathbb{Q}^2),\quad \boldsymbol{\xi}\in\mathbb{R}^2. \tag{1.7}$$

The selfadjoint operator in $L_2(\mathbb{Q}^2)$ generated by the form (1.7) is denoted by $A(\boldsymbol{\xi})$. The operator $A(\boldsymbol{\xi})$ corresponds to the expression $\mathcal{A}$ with ($\boldsymbol{\xi}$)-quasi-periodic boundary conditions. Usually it is sufficient to consider $\boldsymbol{\xi}\in\mathbb{T}^2=\mathbb{R}^2/(2\pi\mathbb{Z})^2$. The parameter $\boldsymbol{\xi}$ is called the *quasi-momentum.* All operators $A(\boldsymbol{\xi})$ have discrete spectrum. Let $E_s(\boldsymbol{\xi})$, $s\in\mathbb{N}$, be consecutive eigenvalues (counted with multiplicity) of the operator $A(\boldsymbol{\xi})$; and let $\psi_s(\mathbf{x},\boldsymbol{\xi})$ be the corresponding eigenfunctions normalized in $L_2(\mathbb{Q}^2)$. The functions E_s are continuous and $(2\pi\mathbb{Z})^2$-periodic. The spectrum of A coincides with the union of the intervals (bands) that are the ranges of the functions E_s. The eigenfunctions ψ_s admit representation of the form $\psi_s(\mathbf{x},\boldsymbol{\xi})=e^{i\langle\mathbf{x},\boldsymbol{\xi}\rangle}\varphi_s(\mathbf{x},\boldsymbol{\xi})$, $\varphi_s(\cdot,\boldsymbol{\xi})\in\widetilde{H}^1(\mathbb{Q}^2)$. The functions ψ_s, φ_s are Hölder continuous in $\mathbf{x}$.

We consider the integral operators

$$(\Psi_s u)(\boldsymbol{\xi})=(2\pi)^{-1}\int\overline{\psi_s(\mathbf{x},\boldsymbol{\xi})}u(\mathbf{x})\,d\mathbf{x},\quad s\in\mathbb{N}.$$

The mappings $\Psi_s : L_2(\mathbb{R}^2)\to L_2(\mathbb{T}^2)$ *are partially isometric and surjective.* The operators $\Psi_s^*\Psi_s$, $s\in\mathbb{N}$, are orthoprojections in $L_2(\mathbb{R}^2)$. They are orthogonal to one another and $\sum_{s\in\mathbb{N}}\Psi_s^*\Psi_s=I$. Denoting by $[E_s]$ the operator of multiplication by the function $E_s(\boldsymbol{\xi})$ in $L_2(\mathbb{T}^2)$, we have $A=\sum_{s\in\mathbb{N}}\Psi_s^*[E_s]\Psi_s$.

3. A gap. The spectrum of A may have gaps other than the semi-infinite gap $(-\infty,0)$. Let $\Lambda=(\lambda_-,\lambda_+)$ be a gap. Clearly,

$$\lambda_+=\min_{\boldsymbol{\xi}\in\mathbb{T}^2}E_l(\boldsymbol{\xi}), \tag{1.8+}$$

$$\lambda_-=\max_{\boldsymbol{\xi}\in\mathbb{T}^2}E_{l-1}(\boldsymbol{\xi}), \tag{1.8−}$$

for some $l\in\mathbb{N}$. We impose certain restrictions on the "structure" of the edges of the gap. Let us formulate condition for λ_+.

Condition 1.3(+). a) $\min_{\boldsymbol{\xi}\in\mathbb{T}^2} E_{l+1}(\boldsymbol{\xi}) > \lambda_+$; b) *The minimum* (1.8+) *is attained only at finitely many points* $\boldsymbol{\xi}_j^{(+)} \in \mathbb{T}^2$, $j = 1, \dots, m_+$, *all of which are non-degenerate minimum points for* $E_l(\cdot)$.

Remark 1.4. For the semi-infinite gap $(-\infty, 0)$ Condition 1.3(+) at $\lambda_+ = 0$ is fulfilled automatically with $l = 1$, $m_+ = 1$, $\boldsymbol{\xi}_1^{(+)} = 0$. This fact was used in [BLSu].

By Condition 1.3(+), λ_+ is a simple eigenvalue of the operator $A(\boldsymbol{\xi})$ with $\boldsymbol{\xi} = \boldsymbol{\xi}_j^{(+)}$, $j = 1, \dots, m_+$. Then, for some (sufficiently small) $\delta > 0$ the *eigenvalue* $E_l(\boldsymbol{\xi})$, $|\boldsymbol{\xi} - \boldsymbol{\xi}_j^{(+)}| \le \delta$, *is simple.* This implies the *real analyticity* of $E_l(\boldsymbol{\xi})$ in these neighborhoods of the points $\boldsymbol{\xi}_j^{(+)}$, $j = 1, \dots, m_+$. Then Condition 1.3(+), b) means that

$$E_l(\boldsymbol{\xi}) - \lambda_+ = \mathfrak{b}_j^{(+)}(\boldsymbol{\xi} - \boldsymbol{\xi}_j^{(+)}) + O(|\boldsymbol{\xi} - \boldsymbol{\xi}_j^{(+)}|^3), \quad |\boldsymbol{\xi} - \boldsymbol{\xi}_j^{(+)}| \le \delta, \quad j = 1, \dots, m_+, \tag{1.9+}$$

where $\mathfrak{b}_j^{(+)}$ is a positive definite quadratic form.

Let us formulate condition on λ_-.

Condition 1.3(−). a) $\max_{\boldsymbol{\xi}\in\mathbb{T}^2} E_{l-2}(\boldsymbol{\xi}) < \lambda_-$; b) *The maximum in* (1.8−) *is attained only at finitely many points* $\boldsymbol{\xi}_j^{(-)} \in \mathbb{T}^2$, $j = 1, \dots, m_-$, *all of which are non-degenerate maximum points for* $E_{l-1}(\cdot)$.

Similarly to (1.9+), Condition 1.3(−), b) means that

$$\lambda_- - E_{l-1}(\boldsymbol{\xi}) = \mathfrak{b}_j^{(-)}(\boldsymbol{\xi} - \boldsymbol{\xi}_j^{(-)}) + O(|\boldsymbol{\xi} - \boldsymbol{\xi}_j^{(-)}|^3), \quad |\boldsymbol{\xi} - \boldsymbol{\xi}_j^{(-)}| \le \delta, \quad j = 1, \dots, m_-, \tag{1.9−}$$

where $\mathfrak{b}_j^{(-)}$ is a positive definite quadratic form.

Remark 1.5. We agree that the points $\boldsymbol{\xi}_j^{(\pm)} \in \mathbb{T}^2$ are represented as points of the semi-open cube: $\boldsymbol{\xi}_j^{(\pm)} \in [-\pi, \pi)^2$, $j = 1, \dots, m_\pm$. Accordingly, small neighborhoods of points $\boldsymbol{\xi}_j^{(\pm)} \in \mathbb{T}^2$ are understood as $\mathbb{R}^2$-neighborhoods of points $\boldsymbol{\xi}_j^{(\pm)} \in [-\pi, \pi)^2$.

The form $\mathfrak{b}_j^{(\pm)}(\boldsymbol{\xi} - \boldsymbol{\xi}_j^{(\pm)})$ can be written as

$$\begin{aligned} \mathfrak{b}_j^{(\pm)}(\boldsymbol{\xi} - \boldsymbol{\xi}_j^{(\pm)}) = \langle b_j^{(\pm)}(\boldsymbol{\xi} - \boldsymbol{\xi}_j^{(\pm)}), \boldsymbol{\xi} - \boldsymbol{\xi}_j^{(\pm)} \rangle = |\beta_j^{(\pm)}(\boldsymbol{\xi} - \boldsymbol{\xi}_j^{(\pm)})|^2, \\ \beta_j^{(\pm)} = (b_j^{(\pm)})^{1/2}, \quad j = 1, \dots, m_\pm, \end{aligned} \tag{1.10±}$$

where $b_j^{(\pm)}$ is a constant positive definite matrix. (The matrix $(b_j^{(\pm)})^{-1}$ determines the *tensor of effective masses* for the point $\boldsymbol{\xi}_j^{(\pm)}$.)

We put $E_+ := E_l$, $E_- := E_{l-1}$, $\psi^{(+)} := \psi_l$, $\varphi^{(+)} := \varphi_l$, $\psi^{(-)} := \psi_{l-1}$, $\varphi^{(-)} := \varphi_{l-1}$. The functions $\psi^{(\pm)}$, $\varphi^{(\pm)}$ can be chosen as real-analytic $H^1(\mathbb{Q}^2)$-valued functions of $\boldsymbol{\xi}$ for $|\boldsymbol{\xi} - \boldsymbol{\xi}_j^{(\pm)}| \le \delta$, $j = 1, \dots, m_\pm$. The functions $\psi^{(\pm)}$, $\varphi^{(\pm)}$ are Hölder continuous in $\mathbf{x}$. We introduce the notation

$$\psi_j^{(\pm)}(\mathbf{x}) := \psi^{(\pm)}(\mathbf{x}, \boldsymbol{\xi}_j^{(\pm)}), \quad \varphi_j^{(\pm)}(\mathbf{x}) := \varphi^{(\pm)}(\mathbf{x}, \boldsymbol{\xi}_j^{(\pm)}), \quad j = 1, \dots, m_\pm. \tag{1.11±}$$

4. Let $\lambda \in (\lambda_-, \lambda_+) = \Lambda$. We denote by

$$\mathfrak{N}_\pm(\alpha, \lambda; A, V), \quad \alpha > 0, \quad \lambda \in \Lambda, \tag{1.12}$$

the number of eigenvalues of $A_\pm(t)$ that have crossed λ as the coupling constant t has been growing from 0 to α. The following statement was proved in [B1] (see also [B2, Theorem 3.2]).

Proposition 1.6. *Under condition* (1.4), *we have*

$$\lim_{\alpha\to\infty} \alpha^{-1}\mathfrak{N}_+(\alpha, \lambda; A, V) = J(V, g), \quad \lambda \in \Lambda, \tag{1.13}$$

$$\lim_{\alpha\to\infty} \alpha^{-1}\mathfrak{N}_-(\alpha, \lambda; A, V) = 0, \quad \lambda \in \Lambda.$$

Thus, if the "observation point" λ lies inside the gap, then $\mathfrak{N}_+$ has the Weyl asymptotics (1.13). For $\lambda = \lambda_+$ the asymptotics (1.13) may fail to occur even under condition (1.3). Below we impose an additional condition (Condition 2.1(q)) on V that ensures the finiteness of the following limits:

$$\mathfrak{N}_\pm(\alpha, \lambda_+; A, V) = \lim_{\lambda\to\lambda_+-0} \mathfrak{N}_\pm(\alpha, \lambda; A, V), \tag{1.14+}$$

$$\mathfrak{N}_\pm(\alpha, \lambda_-; A, V) = \lim_{\lambda\to\lambda_-+0} \mathfrak{N}_\pm(\alpha, \lambda; A, V), \tag{1.14−}$$

We are interested in the behavior of functions (1.14) as $\alpha \to \infty$.

5. On compact operators. Let $\mathfrak{H}$ be a separable Hilbert space. The space of continuous linear operators is denoted by $\mathfrak{B}$, and that of compact operators by $\mathfrak{S}_\infty$. Let $T \in \mathfrak{S}_\infty$, and let $s_k(T)$ be the singular numbers of T, i. e., the consecutive eigenvalues (counted with multiplicity) of the operator $(T^*T)^{1/2}$. We denote

$$n(s, T) := \operatorname{card}\{k : s_k(T) > s\}, \quad s > 0.$$

For $T = T^*$, we put $2T_\pm = |T| \pm T$ and $n_\pm(s, T) := n(s, T_\pm)$, $s > 0$. Clearly, $n_+(\cdot, T)$ is the counting function for the sequence $\{\lambda_k^{(+)}(T)\}$ of positive eigenvalues of T. For the sequence $\{\lambda_k^{(-)}(T)\}$ a similar role is played by $n_-(\cdot, T)$ (here $\lambda_k^{(-)}(T) = \lambda_k^{(+)}(-T)$). We have $n(s, T) = n_+(s, T) + n_-(s, T)$, $s > 0$.

We denote by Σ_q, $0 < q < \infty$, the space (ideal) of compact operators distinguished by the condition

$$\| T \|_q^q := \sup_{s>0} s^q n(s, T) < \infty, \quad q > 0.$$

The space Σ_q is complete in the quasi-norm $\|\cdot\|_q$ and non-separable. On the space Σ_q we consider the functionals

$$\begin{aligned} \Delta_q(T) &:= \limsup_{s\to 0} s^q n(s, T), \\ \delta_q(T) &:= \liminf_{s\to 0} s^q n(s, T). \end{aligned} \tag{1.15}$$

The condition $\Delta_q(T) = 0$ determines the separable subspace Σ_q^0 in Σ_q. For $T = T^* \in \Sigma_q$, we put

$$\Delta_q^{(\pm)}(T) := \Delta_q(T_\pm), \quad \delta_q^{(\pm)}(T) := \delta_q(T_\pm). \tag{1.16}$$

Below D_q stands for any of the functionals (1.15), (1.16).

If $T^* = T \in \mathfrak{S}_\infty(\mathfrak{H})$, then the numbers $\lambda_k^{(+)}(T)$ (the numbers $(-\lambda_k^{(-)}(T))$) coincide with the consecutive positive maxima (the negative minima) of the ratio of quadratic forms

$$(Tu, u)_{\mathfrak{H}} / \|u\|_{\mathfrak{H}}^2, \quad u \in \mathfrak{H}. \tag{1.17}$$

Passage from T to the ratio (1.17) facilitates using variational arguments. Therefore, *we shall use the simpler notation* $n_\pm(s, (1.17))$ in place of $n_\pm(s, T)$, $\left|(1.17)\right|_q$ in place of $\left|T\right|_q$, $D_q(1.17)$ in place of $D_q(T)$, etc.

6. An auxiliary problem on the semi-axis. Let $\overline{f} = f \in L_{1,\text{loc}}(\mathbb{R}_+)$. For some $R \geq 1$, consider the ratio of quadratic forms

$$\int_R^\infty f(r)|z(r)|^2 r\, dr \Big/ \int_R^\infty |z'(r)|^2 r\, dr, \quad z(R) = 0, \quad R \geq 1. \tag{1.18}$$

Here z runs through all functions absolutely continuous on $\mathbb{R}_+$ and such that the integral in the denominator is finite. On f we impose the following "implicit" condition: for some $q \geq 1$,

$$\left|(1.18)\right|_q < \infty, \quad q \geq 1. \qquad (1.19)_q$$

This condition is fulfilled (or not fulfilled) simultaneously for all $R \geq 1$. Moreover, under condition $(1.19)_q$ all six functionals $D_q(1.18)$ do not depend on $R \geq 1$.

We can give an elementary *sufficient* condition for $(1.19)_q$ (see [BL], and also [BS], [BLSu]). This condition becomes *necessary* for the non-negative f. Namely, we put $\boldsymbol{\zeta}(f) := \{\zeta_n(f)\}$, $n \in \mathbb{Z}_+$,

$$\zeta_0(f) := \int_0^1 |f(e^t)| e^{2t}\, dt, \quad \zeta_n(f) := \int_{e^{n-1}}^{e^n} t|f(e^t)| e^{2t}\, dt, \quad n \in \mathbb{N};$$

$$\|\boldsymbol{\zeta}(f)\|_{q,\infty}^q := \sup_{s>0} s^q \operatorname{card}\{n : \zeta_n(f) > s\}, \quad q \geq 1,$$

$$\Delta_q(\boldsymbol{\zeta}(f)) := \limsup_{s\to 0} s^q \operatorname{card}\{n : \zeta_n(f) > s\}, \ q \geq 1,$$

$$\delta_q(\boldsymbol{\zeta}(f)) := \liminf_{s\to 0} s^q \operatorname{card}\{n : \zeta_n(f) > s\}, \ q \geq 1.$$

Proposition 1.7. a) *Assume that*

$$\|\boldsymbol{\zeta}(f)\|_{q,\infty} < \infty, \quad q \geq 1. \qquad (1.20)_q$$

Then $(1.19)_q$ *is true, and*

$$\Delta_q(1.18) \leq C(q)\, \Delta_q(\boldsymbol{\zeta}(f)).$$

b) *Assume that* $f(r) \geq 0$, $r \geq R_0$, *for some* $R_0 \geq 1$. *Then* $(1.19)_q$ *implies condition* $(1.20)_q$ *and also the inequalities*

$$\partial_q(1.18) \geq c(q)\, \partial_q(\boldsymbol{\zeta}(f)), \quad \partial = \Delta, \delta.$$

§ 2. Formulation of the main results

1. Our goal is to study the asymptotics of the functions $\mathfrak{N}_\pm(\alpha,\lambda_+;A,V)$, $\mathfrak{N}_\pm(\alpha,\lambda_-;A,V)$ (see (1.14)) as $\alpha\to\infty$. We introduce the following quantities:

$$\Delta_q^{(\pm)}(\lambda_\tau;A,V) := \limsup_{\alpha\to\infty}\alpha^{-q}\mathfrak{N}_\pm(\alpha,\lambda_\tau;A,V),\quad \tau=\pm,\quad q\ge 1, \tag{2.1τ}$$

$$\delta_q^{(\pm)}(\lambda_\tau;A,V) := \liminf_{\alpha\to\infty}\alpha^{-q}\mathfrak{N}_\pm(\alpha,\lambda_\tau;A,V),\quad \tau=\pm,\quad q\ge 1. \tag{2.2τ}$$

For a function $F(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^2$, we put $F_\beta(\mathbf{x})=F(\beta\mathbf{x})$; here β is a positive matrix. Let (r,θ) be the polar coordinates of a point $\mathbf{x}\in\mathbb{R}^2$; we write $F(\mathbf{x})=F(r,\theta)$. We put

$$\langle F\rangle(r) = (2\pi)^{-1}\int_{-\pi}^{\pi} F(r,\theta)\,d\theta.$$

Along with Condition 1.1, we impose the following condition on V.

Condition 2.1(q). *For some $q\ge 1$, $\big|(1.18)\big|_q<\infty$ with $f=\langle V\rangle$.*

Since $V\ge 0$, Condition 2.1(q) is equivalent (see Proposition 1.7) to the following relation: $\|\boldsymbol{\zeta}(\langle V\rangle)\|_{q,\infty}<\infty$. Examples (for any $q\ge 1$) demonstrating the compatibility of Conditions 1.1 and 2.1(q) can be found in [BL] and [BLSu, §8].

For $\varphi\in L_\infty(\mathbb{R}^2)$, we introduce the notation $f_{\beta,\varphi} := \langle(|\varphi|^2V)_\beta\rangle$. Let

$$\Delta_q^{(+)}(V,\beta,\varphi),\quad \delta_q^{(+)}(V,\beta,\varphi),\quad q\ge 1, \tag{2.3}$$

denote the functionals $\Delta_q^{(+)}(1.18)$, $\delta_q^{(+)}(1.18)$ for $f=f_{\beta,\varphi}$. We mention that *Condition* 2.1(q) *with $f=\langle V\rangle$ is equivalent to the same condition with $f=f_{\beta,\varphi}$.*

Also, we note that the functionals (2.3) coincide for potentials asymptotically close as $|\mathbf{x}|\to\infty$ (see Proposition 2.2 from [BLSu]).

2. In [BLSu] it was shown that, if V satisfies Conditions 1.1 and 2.1(q), then

$$\mathfrak{N}_+(\alpha,0;A,V)=O(\alpha^q),\quad \alpha\to\infty, \tag{2.4}$$

and the corresponding asymptotic formulas were established. Earlier the same was established in [BL] in the case where $A=-\Delta$.

Below we formulate two theorems (Theorems 2.2(τ) and 2.4(τ)) about the asymptotics of $\mathfrak{N}_\pm(\alpha,\lambda_\tau;A,V)$, $\tau=\pm$. In Theorems 2.2($\pm$) the answers are formulated in terms of the corresponding *model Schrödinger operators with*, generally speaking, *matrix-valued potentials.* In Theorems 2.4($\pm$) the answers are formulated in terms of the auxiliary problem on the semi-axis, but V is subject to the additional restriction.

The description of the model operators involves the quadratic forms $\mathfrak{b}_j^{(\pm)}$ (see (1.9)) and the corresponding eigenfunctions $\psi_j^{(\pm)}$ (see (1.11)). In the Hilbert space $\mathfrak{H}_\pm=L_2(\mathbb{R}^2;\mathbb{C}^{m_\pm})$, we consider the following diagonal second order elliptic operator with constant coefficients:

$$\mathcal{B}_\pm(\mathbf{D})=\operatorname{diag}(\mathfrak{b}_1^{(\pm)}(\mathbf{D}),\dots,\mathfrak{b}_{m_\pm}^{(\pm)}(\mathbf{D})),\quad \mathbf{D}=-i\nabla. \tag{2.5$\pm$}$$

The expression (2.5±) generates a positive selfadjoint operator $B_\pm$ in $\mathfrak{H}_\pm$. Now, we introduce the matrix-row and the matrix-column

$$\Pi_\pm(\mathbf{x}) := \{\psi_j^{(\pm)}(\mathbf{x})\}_{j=1}^{m_\pm}, \quad \Pi_\pm^*(\mathbf{x}) := \operatorname{col}\{\overline{\psi_j^{(\pm)}(\mathbf{x})}\}_{j=1}^{m_\pm}.$$

We denote $W(\mathbf{x}) = (V(\mathbf{x}))^{1/2}$ and define the non-negative matrix potential $\mathcal{U}_\pm(\mathbf{x}) := (W(\mathbf{x})\Pi_\pm(\mathbf{x}))^* W(\mathbf{x})\Pi_\pm(\mathbf{x})$. The functions $\psi_j^{(\pm)}(\mathbf{x})$ are bounded; therefore, the potential $\mathcal{U}_\pm(\mathbf{x})$ admits a point-wise estimate in terms of $V(\mathbf{x})$. Now, we introduce the model operators

$$B_\pm(\alpha) := B_\pm - \alpha\,\mathcal{U}_\pm(\mathbf{x}), \quad \alpha > 0. \tag{2.6±}$$

By $\mathfrak{N}_+(\alpha,\lambda; B_\pm, \mathcal{U}_\pm)$, $\alpha > 0$, $\lambda \le 0$, we denote the number of eigenvalues of the operator $B_\pm(\alpha)$ lying to the left of the point λ. The estimate (2.4) for $A = -\Delta$ is carried over to the operator (2.6): $\mathfrak{N}_+(\alpha, 0; B_\pm, \mathcal{U}_\pm) = O(\alpha^q)$, $\alpha \to \infty$. Consider the ratio of (finite-dimensional) forms

$$\frac{\langle \mathcal{U}_\pm(\mathbf{x})\mathbf{c}, \mathbf{c}\rangle}{\langle \mathcal{B}_\pm(\boldsymbol{\eta})\mathbf{c}, \mathbf{c}\rangle}, \quad \mathbf{c} \in \mathbb{C}^{m_\pm}; \quad \mathbf{x} \in \mathbb{R}^2, \quad \boldsymbol{\eta} \in \mathbb{R}^2. \tag{2.7±}$$

For $\mu > 0$, by $n^{(\pm)}(\mu; \mathbf{x}, \boldsymbol{\eta})$ we denote the number of eigenvalues of the ratio (2.7±) that are greater than μ.

For $\lambda < 0$, the function $\mathfrak{N}_+(\alpha, \lambda; B_\pm, \mathcal{U}_\pm)$ has the Weyl asymptotics

$$\lim_{\alpha\to\infty} \alpha^{-1}\mathfrak{N}_+(\alpha,\lambda; B_\pm, \mathcal{U}_\pm) = \widetilde{J}(\mathcal{B}_\pm, \mathcal{U}_\pm) := (2\pi)^{-2} \iint n^{(\pm)}(1; \mathbf{x}, \boldsymbol{\eta})\, d\mathbf{x}\, d\boldsymbol{\eta}, \quad \lambda < 0. \tag{2.8±}$$

We introduce the notation

$$\Delta_q(B_\pm, \mathcal{U}_\pm) := \limsup_{\alpha\to\infty} \alpha^{-q}\mathfrak{N}_+(\alpha, 0; B_\pm, \mathcal{U}_\pm), \quad q \ge 1, \tag{2.9±}$$

$$\delta_q(B_\pm, \mathcal{U}_\pm) := \liminf_{\alpha\to\infty} \alpha^{-q}\mathfrak{N}_+(\alpha, 0; B_\pm, \mathcal{U}_\pm), \quad q \ge 1, \tag{2.10±}$$

$$\widetilde{\Delta}_1(B_\pm, \mathcal{U}_\pm) := \Delta_1(B_\pm, \mathcal{U}_\pm) - \widetilde{J}(\mathcal{B}_\pm, \mathcal{U}_\pm), \tag{2.11±}$$

$$\widetilde{\delta}_1(B_\pm, \mathcal{U}_\pm) := \delta_1(B_\pm, \mathcal{U}_\pm) - \widetilde{J}(\mathcal{B}_\pm, \mathcal{U}_\pm). \tag{2.12±}$$

Theorem 2.2(+). *Let the operator A be generated by the form (1.2) under conditions (1.1). Let (λ_-, λ_+) be a gap in the spectrum of A. Suppose that Condition 1.3(+) is satisfied. Suppose that the potential $V \ge 0$ satisfies Conditions 1.1 and 2.1(q). Then the following is true for the quantities (2.1+), (2.2+):*

(a) *If $q = 1$, then*

$$\partial_1^{(+)}(\lambda_+; A, V) = J(V, g) + \widetilde{\partial}_1(B_+, \mathcal{U}_+), \quad \partial = \Delta, \delta, \tag{2.13}$$

$$\Delta_1^{(-)}(\lambda_+; A, V) = 0. \tag{2.14}$$

Here $J(V, g)$ is as in (1.5), and $\widetilde{\Delta}_1(B_+, \mathcal{U}_+)$, $\widetilde{\delta}_1(B_+, \mathcal{U}_+)$ are defined in (2.11+), (2.12+).

For the validity of the Weyl asymptotics

$$\Delta_1^{(+)}(\lambda_+; A, V) = \delta_1^{(+)}(\lambda_+; A, V) = J(V, g) \tag{2.15}$$

it suffices that $\Delta_1^{(+)}(V, \mathbf{1}, 1) = 0$. Here $\Delta_1^{(+)}(V, \mathbf{1}, 1)$ is defined in (2.3).

(b) *If $q > 1$, then (2.14) is fulfilled and*

$$\partial_q^{(+)}(\lambda_+; A, V) = \partial_q(B_+, \mathcal{U}_+), \quad \partial = \Delta, \delta. \tag{2.16}$$

Here $\Delta_q(B_+, \mathcal{U}_+)$, $\delta_q(B_+, \mathcal{U}_+)$ are defined in (2.9+), (2.10+).

Theorem 2.2(−). *Let the operator A be generated by the form* (1.2) *under conditions* (1.1). *Let (λ_-, λ_+) be a gap in the spectrum of A. Suppose that Condition* 1.3(−) *is satisfied. Suppose that the potential $V \geq 0$ satisfies Conditions* 1.1 *and* 2.1(q). *Then the following is true for the quantities* (2.1−), (2.2−):

(a) *If $q = 1$, then*

$$\Delta_1^{(+)}(\lambda_-; A, V) = \delta_1^{(+)}(\lambda_-; A, V) = J(V, g), \tag{2.17}$$

$$\partial_1^{(-)}(\lambda_-; A, V) = \widetilde{\partial}_1(B_-, \mathcal{U}_-), \quad \partial = \Delta, \delta. \tag{2.18}$$

Here $J(V, g)$ is as in (1.5), *and $\widetilde{\Delta}_1(B_-, \mathcal{U}_-)$, $\widetilde{\delta}_1(B_-, \mathcal{U}_-)$ are defined in* (2.11−), (2.12−).

(b) *If $q > 1$, then*

$$\Delta_1^{(+)}(\lambda_-; A, V) \leq J(V, g), \tag{2.19}$$

$$\partial_q^{(-)}(\lambda_-; A, V) = \partial_q(B_-, \mathcal{U}_-), \quad \partial = \Delta, \delta. \tag{2.20}$$

Here $\Delta_q(B_-, \mathcal{U}_-)$, $\delta_q(B_-, \mathcal{U}_-)$ are defined in (2.9−), (2.10−).

3. The model operator (2.6±) involves the forms $\mathfrak{b}_j^{(\pm)}$ or, equivalently, the matrices $\beta_j^{(\pm)}$ (see (1.10)), and also the eigenfunctions $\psi_j^{(\pm)}$. It is impossible to avoid the dependence on $\beta_j^{(\pm)}$ in formulas (2.13), (2.16), (2.18), (2.20). Concerning the (more unpleasant) dependence on the functions $\psi_j^{(\pm)}$, the things are different. It is possible to eliminate these functions from the asymptotic formulas under some supplementary conditions of "regular" behavior of the perturbation V. We proceed to the formulation of Theorem 2.4 that solves this problem.

In addition to Conditions 1.1, 2.1(q), we impose the following condition on V.

Condition 2.3. *There exists a function $\mathcal{S} = \overline{\mathcal{S}}$ satisfying Condition* 1.1 *(with V replaced by S) and such that $V(\mathbf{x}) = \mathcal{S}(\mathbf{x})(1 + o(1))$ as $|\mathbf{x}| \to \infty$. Suppose that the Fourier-image $\Phi\mathcal{S}$ of $\mathcal{S}$ satisfies the following condition: for some $\varkappa > 1$, $\Phi\mathcal{S} \in H^{\varkappa}(\mathbb{R}^2 \setminus B_\varepsilon)$, $\forall \varepsilon > 0$, where $B_\varepsilon = \{\boldsymbol{\xi} \in \mathbb{R}^2 : |\boldsymbol{\xi}| \leq \varepsilon\}$.*

Theorem 2.4(+). *Let the operator A be generated by the form* (1.2) *under conditions* (1.1). *Let (λ_-, λ_+) be a gap in the spectrum of A. Suppose that Condition* 1.3(+) *is satisfied. Suppose that the potential $V \geq 0$ satisfies Conditions* 1.1, 2.1(q), 2.3. *Then the following is true for the quantities* (2.1+), (2.2+):

(a) *If $q = 1$, then* (2.14) *is satisfied and*

$$\partial_1^{(+)}(\lambda_+; A, V) = J(V, g) + \sum_{j=1}^{m_+} \partial_1^{(+)}(V, \beta_j^{(+)}, 1), \quad \partial = \Delta, \delta. \tag{2.21}$$

Here $J(V, g)$ is as in (1.5), *and $\partial_1^{(+)}(V, \beta_j^{(+)}, 1)$, $\partial = \Delta, \delta$, are defined in accordance with* (2.3).

(b) *If $q > 1$, then* (2.14) *is satisfied and*

$$\partial_q^{(+)}(\lambda_+; A, V) = \sum_{j=1}^{m_+} \partial_q^{(+)}(V, \beta_j^{(+)}, 1), \quad \partial = \Delta, \delta. \tag{2.22}$$

Theorem 2.4(−). *Let the operator A be generated by the form* (1.2) *under conditions* (1.1). *Let (λ_-,λ_+) be a gap in the spectrum of A. Suppose that Condition* 1.3(−) *is satisfied. Suppose that the potential $V\ge 0$ satisfies Conditions* 1.1, 2.1(q), 2.3. *Then the following is true for the quantities* (2.1−), (2.2−)*:*

(a) *If $q=1$, then* (2.17) *is satisfied and*

$$\partial_1^{(-)}(\lambda_-;A,V)=\sum_{j=1}^{m_-}\partial_1^{(+)}(V,\beta_j^{(-)},1),\quad \partial=\Delta,\delta. \tag{2.23}$$

(b) *If $q>1$, then* (2.19) *is satisfied and*

$$\partial_q^{(-)}(\lambda_-;A,V)=\sum_{j=1}^{m_-}\partial_q^{(+)}(V,\beta_j^{(-)},1),\quad \partial=\Delta,\delta. \tag{2.24}$$

§ 3. Sketch of the proof

1. Reduction to compact operators. We denote

$$X(\lambda):=W(A-\lambda I)^{-1}W,\quad \lambda\in\Lambda. \tag{3.1}$$

The functions (1.12) are related to the counting functions of the spectrum of the operator $X(\lambda)$:

$$\mathfrak{N}_\pm(\alpha,\lambda;A,V)=n_\pm(t,X(\lambda)),\quad t\alpha=1,\quad \lambda\in\Lambda. \tag{3.2}$$

In (3.2) we cannot pass to the limit as $\lambda\to\lambda_\pm$, because the operators (3.1) do not have limits. Therefore, we need an appropriate regularization.

Proposition 3.1(±). *Suppose that near $\lambda_\pm$ the operator $X(\lambda)$ is represented in the form*

$$X(\lambda)=\Gamma_\pm(\lambda)+Y_\pm(\lambda), \tag{3.3±}$$

where $\Gamma_\pm(\lambda)=\Gamma_\pm(\lambda)^*$, the limit (with respect to the operator norm)

$$\text{(u)-}\lim_{\lambda\to\lambda_\pm}\Gamma_\pm(\lambda)=:\Gamma_\pm\in\Sigma_q,\quad q\ge 1,$$

exists, and (uniformly in λ) $\operatorname{rank}Y_\pm(\lambda)\le r_\pm<\infty$. *Then*

$$\begin{aligned}\partial_q^{(+)}(\lambda_\pm;A,V)&=\partial_q^{(+)}(\Gamma_\pm),\quad \partial=\Delta,\delta,\\ \partial_q^{(-)}(\lambda_\pm;A,V)&=\partial_q^{(-)}(\Gamma_\pm),\quad \partial=\Delta,\delta.\end{aligned} \tag{3.4±}$$

2. For $N>0$, let $\zeta_N(\mathbf{x})$ be the characteristic function of the disc $\{|\mathbf{x}|\le N\}$, and let $\widetilde{\zeta}_N(\mathbf{x}):=1-\zeta_N(\mathbf{x})$. We denote $W_N:=\zeta_N W$, $\widetilde{W}_N:=\widetilde{\zeta}_N W$, $V_N:=\zeta_N V$, $\widetilde{V}_N:=\widetilde{\zeta}_N V$. The operator $X(\lambda)$ is represented in the form

$$X(\lambda)=L_N(\lambda)+K_N(\lambda)+2\operatorname{Re}M_N(\lambda),$$

where $L_N(\lambda):=W_N(A-\lambda I)^{-1}W_N$, $K_N(\lambda):=\widetilde{W}_N(A-\lambda I)^{-1}\widetilde{W}_N$, $M_N(\lambda):=\widetilde{W}_N(A-\lambda I)^{-1}W_N$. We are going to regularize these operators separately and examine the contribution of each of them to the limit quantities (2.1), (2.2).

Below we suppose that *Conditions* 1.1, 2.1(q) *and* 1.3(±) *are satisfied.*

3. The operator $L_N(\lambda)$ is "responsible" for the Weyl contribution to the asymptotics.

Proposition 3.2($\pm$). *We have*

$$L_N(\lambda) = \mathcal{L}_N^{(\pm)}(\lambda) + \widehat{L}_N^{(\pm)}(\lambda),$$

where $\operatorname{rank} \widehat{L}_N^{(\pm)}(\lambda) \le 2m_\pm$, *the limit*

$$(\Sigma_1)\text{-}\lim_{\lambda \to \lambda_\pm} \mathcal{L}_N^{(\pm)}(\lambda) =: \mathcal{L}_N^{(\pm)}(\lambda_\pm) \in \Sigma_1$$

exists, and

$$\Delta_1^{(+)}(\mathcal{L}_N^{(\pm)}(\lambda_\pm)) = \delta_1^{(+)}(\mathcal{L}_N^{(\pm)}(\lambda_\pm)) = J(V_N, g), \qquad (3.5\pm)$$
$$\Delta_1^{(-)}(\mathcal{L}_N^{(\pm)}(\lambda_\pm)) = 0.$$

We have $\mathcal{L}_N^{(\pm)}(\lambda_\pm) = \zeta_N \mathcal{L}_N^{(\pm)}(\lambda_\pm)\zeta_N$.

4. The operator $K_N(\lambda)$ is responsible for the threshold contribution to the asymptotics. We put $\mathcal{E}_j^{(\pm)} := \{\boldsymbol{\xi} : \ |\beta_j^{(\pm)}(\boldsymbol{\xi} - \boldsymbol{\xi}_j^{(\pm)})| \le \delta\}$, $j = 1, \dots, m_\pm$. Choose a number $\delta > 0$ so small that $E_\pm(\boldsymbol{\xi})$ is a simple eigenvalue of the operator $A(\boldsymbol{\xi})$ for $\boldsymbol{\xi} \in \mathcal{E}_j^{(\pm)}$, $j = 1, \dots, m_\pm$, and $\mathcal{E}_j^{(\pm)} \cap \mathcal{E}_k^{(\pm)} = \emptyset$ for $j \ne k$. Let $\chi_j^{(\pm)}$ denote the characteristic function of the ellipse $\mathcal{E}_j^{(\pm)}$.

We introduce the projections

$$\mathcal{X}^{(\pm)} := \sum_{j=1}^{m_\pm} \Psi_\pm^*[\chi_j^{(\pm)}]\Psi_\pm, \quad \widetilde{\mathcal{X}}^{(\pm)} := I - \mathcal{X}^{(\pm)},$$

which commute with A. Here $\Psi_+ := \Psi_l$, $\Psi_- := \Psi_{l-1}$. The operator $K_N(\lambda)$ is represented in the form

$$K_N(\lambda) = Q_N^{(\pm)}(\lambda) + P_N^{(\pm)}(\lambda), \qquad (3.6\pm)$$

where $Q_N^{(\pm)}(\lambda) := \widetilde{W}_N(A - \lambda I)^{-1}\mathcal{X}^{(\pm)}\widetilde{W}_N$, $P_N^{(\pm)}(\lambda) := \widetilde{W}_N(A - \lambda I)^{-1}\widetilde{\mathcal{X}}^{(\pm)}\widetilde{W}_N$.

The operator $P_N^{(\pm)}(\lambda)$ in the limit $N \to \infty$ gives no contribution to the asymptotics.

Proposition 3.3($\pm$). *The limit*

$$(\Sigma_1)\text{-}\lim_{\lambda \to \lambda_\pm} P_N^{(\pm)}(\lambda) =: P_N^{(\pm)}(\lambda_\pm) \in \Sigma_1$$

exists, and

$$\Delta_1^{(+)}(P_N^{(\pm)}(\lambda_\pm)) = \delta_1^{(+)}(P_N^{(\pm)}(\lambda_\pm)) = J(\widetilde{V}_N, g),$$
$$\Delta_1^{(-)}(P_N^{(\pm)}(\lambda_\pm)) = 0.$$

The operator $Q_N^{(\pm)}(\lambda)$ can be written as follows:

$$Q_N^{(\pm)}(\lambda) = \sum_{j,k=1}^{m_\pm} (T_{jN}^{(\pm)}(\lambda)\Psi_\pm) \operatorname{sign}(A - \lambda I)(T_{kN}^{(\pm)}(\lambda)\Psi_\pm)^*, \qquad (3.7\pm)$$

$$T_{jN}^{(\pm)}(\lambda) := \widetilde{W}_N \Psi_\pm^* [\chi_j^{(\pm)} |E_\pm - \lambda|^{-1/2}], \quad j = 1, \ldots, m_\pm.$$

The integral operator $T_{jN}^{(\pm)}(\lambda)$ has the following kernel:

$$(2\pi)^{-1} \widetilde{W}_N(\mathbf{x}) \chi_j^{(\pm)}(\boldsymbol{\xi}) \, |E_\pm(\boldsymbol{\xi}) - \lambda|^{-1/2} \, e^{i\langle \mathbf{x}, \boldsymbol{\xi} \rangle} \varphi^{(\pm)}(\mathbf{x}, \boldsymbol{\xi}).$$

Under regularization, the operator $T_{jN}^{(\pm)}(\lambda)$ can be replaced by the integral operator $\widehat{T}_{jN}^{(\pm)}(\lambda)$ with the following simpler kernel:

$$(2\pi)^{-1} \widetilde{W}_N(\mathbf{x}) \chi_j^{(\pm)}(\boldsymbol{\xi}) \left| \pm \mathfrak{b}_j^{(\pm)}(\boldsymbol{\xi} - \boldsymbol{\xi}_j^{(\pm)}) + \lambda_\pm - \lambda \right|^{-1/2} e^{i\langle \mathbf{x}, \boldsymbol{\xi} \rangle} \varphi_j^{(\pm)}(\mathbf{x}).$$

Recall that $\mathfrak{b}_j^{(\pm)}$ are defined in (1.9), and $\varphi_j^{(\pm)}$ are introduced in (1.11). It is elementary to reduce the operator $\widehat{T}_{jN}^{(\pm)}(\lambda)$ to the model integral operator studied in [BLSu, §3]. The results of [BLSu] imply the following statement.

Proposition 3.4$(\pm)$**.** *The following representations are valid:*

$$\widehat{T}_{jN}^{(\pm)}(\lambda) = \mathcal{T}_{jN}^{(\pm)}(\lambda) + \mathcal{Y}_{jN}^{(\pm)}(\lambda),$$

where $\operatorname{rank} \mathcal{Y}_{jN}^{(\pm)}(\lambda) = 1$ *and the limit*

$$\text{(u)-}\lim_{\lambda \to \lambda_\pm} \mathcal{T}_{jN}^{(\pm)}(\lambda) =: \mathcal{T}_{jN}^{(\pm)}(\lambda_\pm) \in \Sigma_{2q}$$

exists.

As a result, we obtain the following statement for the operator (3.7±).

Proposition 3.5$(\pm)$**.** *We have*

$$Q_N^{(\pm)}(\lambda) = \mathcal{Q}_N^{(\pm)}(\lambda) + \widehat{Q}_N^{(\pm)}(\lambda),$$

where $\operatorname{rank} \widehat{Q}_N^{(\pm)}(\lambda) \le 2m_\pm$, *the limit*

$$\text{(u)-}\lim_{\lambda \to \lambda_\pm} \mathcal{Q}_N^{(\pm)}(\lambda) =: \mathcal{Q}_N^{(\pm)}(\lambda_\pm) \in \Sigma_q$$

exists, and $\mathcal{Q}_N^{(\pm)}(\lambda_\pm) = R_N^{(\pm)} \pmod{\Sigma_q^0}$, *where*

$$R_N^{(\pm)} := \pm \sum_{j,k=1}^{m_\pm} (\mathcal{T}_{jN}^{(\pm)}(\lambda_\pm)\Psi_\pm)(\mathcal{T}_{kN}^{(\pm)}(\lambda_\pm)\Psi_\pm)^*. \tag{3.8±}$$

The quantities $\partial_q^{(+)}(R_N^{(+)}) =: \partial_q^{(+)}(*)$ *and* $\partial_q^{(-)}(R_N^{(-)}) =: \partial_q^{(-)}(*)$, $\partial = \Delta, \delta$, *do not depend on* N.

Relation (3.6±) and Propositions 3.3(±), 3.5(±) imply the following statement.

Proposition 3.6($\pm$). *We have*

$$K_N(\lambda) = \mathcal{K}_N^{(\pm)}(\lambda) + \widehat{K}_N^{(\pm)}(\lambda),$$

where $\operatorname{rank} \widehat{K}_N^{(\pm)}(\lambda) \le 2m_\pm$ *and the limit*

$$\text{(u)-}\lim_{\lambda \to \lambda_\pm} \mathcal{K}_N^{(\pm)}(\lambda) =: \mathcal{K}_N^{(\pm)}(\lambda_\pm) \in \Sigma_q$$

exists. Moreover,

$$\lim_{N\to\infty} \partial_q^{(\pm)}(\mathcal{K}_N^{(\pm)}(\lambda_\pm)) = \partial_q^{(\pm)}(*), \tag{3.9$\pm$}$$

$$\lim_{N\to\infty} \Delta_1^{(\mp)}(\mathcal{K}_N^{(\pm)}(\lambda_\pm)) = 0.$$

We have $\mathcal{K}_N^{(\pm)}(\lambda_\pm) = \widetilde{\zeta}_N \mathcal{K}_N^{(\pm)}(\lambda_\pm)\widetilde{\zeta}_N$.

5. The operator $M_N(\lambda)$ in the limit $N \to \infty$ gives no contribution to the asymptotics.

Proposition 3.7($\pm$). *We have*

$$M_N(\lambda) = \mathcal{M}_N^{(\pm)}(\lambda) + \widehat{M}_N^{(\pm)}(\lambda),$$

where $\operatorname{rank} \widehat{M}_N^{(\pm)}(\lambda) \le 2m_\pm$, *the limit*

$$\text{(u)-}\lim_{\lambda \to \lambda_\pm} \mathcal{M}_N^{(\pm)}(\lambda) =: \mathcal{M}_N^{(\pm)}(\lambda_\pm) \in \Sigma_q$$

exists, and

$$\lim_{N\to\infty} \Delta_q(\mathcal{M}_N^{(\pm)}(\lambda_\pm)) = 0. \tag{3.10$\pm$}$$

6. Now everything is prepared for applying the general method of Subsection 1. The operator $X(\lambda)$ is represented in the form (3.3$\pm$) with the operators

$$\Gamma_N^{(\pm)}(\lambda) := \mathcal{L}_N^{(\pm)}(\lambda) + \mathcal{K}_N^{(\pm)}(\lambda) + 2\mathrm{Re}\,\mathcal{M}_N^{(\pm)}(\lambda),$$

$$Y_N^{(\pm)}(\lambda) := \widehat{L}_N^{(\pm)}(\lambda) + \widehat{K}_N^{(\pm)}(\lambda) + 2\mathrm{Re}\,\widehat{M}_N^{(\pm)}(\lambda)$$

in the role of $\Gamma_\pm(\lambda)$, $Y_\pm(\lambda)$. Herewith, $\operatorname{rank} Y_N^{(\pm)}(\lambda) \le 6m_\pm$ and the limit

$$\text{(u)-}\lim_{\lambda \to \lambda_\pm} \Gamma_N^{(\pm)}(\lambda) =: \Gamma_N^{(\pm)}(\lambda_\pm) = \mathcal{L}_N^{(\pm)}(\lambda_\pm) + \mathcal{K}_N^{(\pm)}(\lambda_\pm) + 2\mathrm{Re}\,\mathcal{M}_N^{(\pm)}(\lambda_\pm) \tag{3.11$\pm$}$$

exists. We write relations (3.4$\pm$) for $\Gamma_N^{(\pm)}(\lambda_\pm)$:

$$\begin{aligned} \partial_q^{(+)}(\lambda_\pm; A, V) &= \partial_q^{(+)}(\Gamma_N^{(\pm)}(\lambda_\pm)), \quad \partial = \Delta, \delta, \\ \partial_q^{(-)}(\lambda_\pm; A, V) &= \partial_q^{(-)}(\Gamma_N^{(\pm)}(\lambda_\pm)), \quad \partial = \Delta, \delta. \end{aligned} \tag{3.12$\pm$}$$

Since the left-hand sides in (3.12±) are independent of N, so are the right-hand sides. We pass to the limit in (3.12±) as $N \to \infty$. Relations (3.5±), (3.9±), (3.10±), (3.11±) and (3.12±) with $q > 1$ imply that

$$\partial_q^{(\pm)}(\lambda_\pm; A, V) = \partial_q^{(\pm)}(*), \quad \partial = \Delta, \delta, \tag{3.13±}$$

$$\Delta_q^{(\mp)}(\lambda_\pm; A, V) = 0. \tag{3.14±}$$

If $q = 1$, the same relations yield the following equalities:

$$\partial_1^{(+)}(\lambda_+; A, V) = J(V, g) + \partial_1^{(+)}(*), \quad \partial = \Delta, \delta, \tag{3.15+}$$

$$\Delta_1^{(+)}(\lambda_-; A, V) = \delta_1^{(+)}(\lambda_-; A, V) = J(V, g), \tag{3.15−}$$

$$\Delta_1^{(-)}(\lambda_+; A, V) = 0, \tag{3.16+}$$

$$\partial_1^{(-)}(\lambda_-; A, V) = \partial_1^{(-)}(*), \quad \partial = \Delta, \delta. \tag{3.16−}$$

Now it is easy to complete the proof of Theorem 2.2, applying a similar regularization for the model operators and comparing the results with formulas (3.13±)–(3.16±).

7. Scheme of the proof of Theorem 2.4. Below $\partial = \Delta, \delta$. The additional Condition 2.3 on V allows us to calculate $\partial_q^{(\pm)}(*)$ (cf. (3.8±)). Under this condition, we have

$$(\mathcal{T}_{jN}^{(\pm)}(\lambda_\pm)\Psi_\pm)(\mathcal{T}_{kN}^{(\pm)}(\lambda_\pm)\Psi_\pm)^* \in \Sigma_q^0, \quad j \neq k.$$

Then $\partial_q^{(\pm)}(*)$ is equal to the sum of the corresponding quantities for the operators $(\mathcal{T}_{jN}^{(\pm)}(\lambda_\pm)\Psi_\pm)(\mathcal{T}_{jN}^{(\pm)}(\lambda_\pm)\Psi_\pm)^*$. In [BLSu, §3], it was shown that these functionals are related to the auxiliary problem on the semi-axis. They coincide with $\partial_q^{(\pm)}(V, \beta_j^{(\pm)}, \varphi_j^{(\pm)})$. Condition 2.3 allows us to eliminate (cf. [BLSu, §7]) the functions $\varphi_j^{(\pm)}$ from the asymptotic coefficients. We have $\partial_q^{(\pm)}(V, \beta_j^{(\pm)}, \varphi_j^{(\pm)}) = \partial_q^{(\pm)}(V, \beta_j^{(\pm)}, 1)$. This leads to the asymptotic formulas (2.21)–(2.24).

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