

**SUPERTRACE DIVERGENCE TERMS
FOR THE WITTEN LAPLACIAN**

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SUPERTRACE DIVERGENCE TERMS FOR THE WITTEN LAPLACIAN

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ABSTRACT. We use invariance theory to compute the divergence term $a_{m+2,m}^{d+\delta}$ in the super trace for the twisted de Rham complex for a closed Riemannian manifold.

1. INTRODUCTION

Let (M, g) be a compact m dimensional Riemannian manifold without boundary. The fundamental solution of the heat equation e^{-tD} for an operator of Laplace type D on M is an infinitely smoothing operator. Let $f \in C^\infty(M)$ be an auxiliary smooth ‘smearing’ function. Work of Seeley [15] shows the smeared heat trace has a complete asymptotic expansion as $t \downarrow 0$ of the form:

$$\mathrm{Tr}_{L^2}(f e^{-tD}) \sim \sum_{n \geq 0} a_{n,m}(f, D) t^{(n-m)/2}.$$

The *heat trace invariants* $a_{n,m}$ vanish if n is odd; if n is even, there are local invariants $a_{n,m}(x, D)$ so that

$$a_{n,m}(f, D) = \int_M f(x) a_{n,m}(x, D) \mathrm{dvol}_g(x).$$

The function f localizes the problem and permits us to recover divergence terms which would otherwise not be detected.

Let ϕ be an auxiliary smooth function called the dilaton. We twist the exterior derivative d and the coderivative δ_g to define

$$d_\phi := e^{-\phi} d e^\phi \quad \text{and} \quad \delta_{\phi,g} := e^\phi \delta_g e^{-\phi}.$$

We denote the associated Laplacian by $\Delta_{\phi,g}^p$ on $C^\infty(\Lambda^p(M))$. It appears in supersymmetric quantum mechanics [2] and in the study of Morse theory [19]. It also is used to study quantum p form fields interacting with a background dilaton [11, 17].

Let $\chi(M) := \sum_p (-1)^p \dim H^p(M; \mathbb{R})$ be the Euler-Poincaré characteristic of M . Arguments of McKean and Singer [12] extend to the twisted setting to show

$$(1.a) \quad \sum_p (-1)^p \mathrm{Tr}_{L^2}(e^{-t\Delta_{\phi,g}^p}) = \chi(M).$$

We define the local supertrace asymptotics by setting:

$$a_{n,m}^{d+\delta}(\phi, g)(x) := \sum_p (-1)^p a_{n,m}(x, \Delta_{\phi,g}^p).$$

We expand the left hand side of equation (1.a) and then equate powers of t to see:

$$(1.b) \quad \int_M a_{n,m}^{d+\delta}(\phi, g)(x) \mathrm{dvol}_g(x) = \begin{cases} \chi(M) & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Let R_{ijkl} be the components of the Riemann curvature tensor relative to a local orthonormal frame for the tangent bundle with the sign convention that $R_{1221} = +1$ on the unit sphere $S^2 \subset \mathbb{R}^3$. We adopt the Einstein convention and sum over repeated indices. If $I = (i_1, \dots, i_m)$ and $J = (j_1, \dots, j_m)$ are m tuples of indices, let

$$\varepsilon_J^I := g(e_{i_1} \wedge \dots \wedge e_{i_m}, e_{j_1} \wedge \dots \wedge e_{j_m})$$

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be the totally anti-symmetric tensor and let

$$\mathcal{R}_{J,s}^{I,t} := R_{i_s i_{s+1} j_{s+1} j_s} \cdots R_{i_{t-1} i_t j_t j_{t-1}};$$

we set $\mathcal{R}_{J,s}^{I,t} = 1$ for $t < s$.

Theorem 1.1. (1) *If $n < m$ or if n is odd, then $a_{n,m}^{d+\delta}(\phi, g)(x) = 0$.*

(2) *If $m = 2\bar{m}$, then $a_{m,m}^{d+\delta}(\phi, g)(x) = \frac{1}{8^{\bar{m}} \pi^{\bar{m}} \bar{m}!} \varepsilon_J^J \mathcal{R}_{J,1}^{I,m}$.*

(3) *If $m = 2\bar{m} + 1$, then $a_{m+1,m}^{d+\delta}(\phi, g)(x) = \frac{1}{\sqrt{\pi} 8^{\bar{m}} \pi^{\bar{m}} \bar{m}!} \varepsilon_J^I \phi_{;i_1 j_1} \mathcal{R}_{J,2}^{I,m}$.*

Assertions (1) and (2) were proved in the untwisted case ($\phi = 0$) by Atiyah, Bott, and Patodi [4], by Gilkey [6], and by Patodi [13]. This provided a heat equation proof of the classical Chern-Gauss-Bonnet [5] theorem. Assertions (1) and (2) in the twisted setting were established in [11] and the divergence term $a_{m+1,m}^{d+\delta}$ was identified in [10]. Our previous paper [10] dealt with the odd dimensional case for manifolds with boundary. The present paper computes $a_{m+2,m}^{d+\delta}$ for closed even dimensional manifolds. This requires significantly different techniques. We note that some information concerning $a_{m+2,m}^{d+\delta}$ was derived earlier in [8] using an entirely different approach.

The main new result of this paper is the following:

Theorem 1.2. *Let M be a closed Riemannian manifold of dimension $m = 2\bar{m}$.*

$$\begin{aligned} a_{m+2,m}^{d+\delta} &= \frac{1}{\pi^{\bar{m}} 8^{\bar{m}} \bar{m}!} \varepsilon_J^I \{ 4\bar{m}(\phi_{;i_1 j_1} \phi_{;i_2} \mathcal{R}_{J,3}^{I,m})_{;j_2} + \frac{1}{12} (\mathcal{R}_{J,1}^{I,m})_{;kk} \\ &\quad + \frac{\bar{m}}{6} (R_{i_1 i_2 k j_1; k} \mathcal{R}_{J,3}^{I,m})_{;j_2} \}. \end{aligned}$$

In Section 2, we recall combinatorial formulas for the invariants $a_{n,m}(x, D)$ for $n = 0, 2, 4, 6$ [9]; formulas for a_8 , and for a_{10} are available [1, 3, 16]. These formulas become very complicated as n increases and it seems hopeless to try to establish Theorem 1.2 via direct computation even for $m = 6$ and $m = 8$. There are also closed formulas available due to Polterovich [14]. However, it does not seem possible to make a direct use of these formulas to derive Theorem 1.2. Instead, we proceed indirectly. In Section 3, we establish some functorial properties of these invariants and use invariance theory to prove the following result:

Lemma 1.3. *If m is even, then there exist universal constants so that*

$$\begin{aligned} a_{m+2,m}^{d+\delta}(\phi, g) &= c_{m+2,m}^1 (\varepsilon_J^I \phi_{;i_1 j_1} \phi_{;i_2} \mathcal{R}_{J,3}^{I,m})_{;j_2} + c_{m+2,m}^2 (\varepsilon_J^I \mathcal{R}_{J,1}^{I,m})_{;kk} \\ &\quad + c_{m+2,m}^3 (\varepsilon_J^I R_{i_1 i_2 k j_1; k} \mathcal{R}_{J,3}^{I,m})_{;j_2}. \end{aligned}$$

We shall complete the proof of Theorem 1.2 in Section 4 by evaluating these normalizing constants. The new features of this investigation are that both H. Weyl's first and second main theorems of invariance theory [18] play a crucial role as does the analysis of the formal cohomology groups of spaces of p form valued invariants [7]. Thus we expect that the techniques presented in this paper will be useful in other similar investigations of this type.

2. LOCAL FORMULAE FOR THE HEAT TRACE INVARIANTS

If D is an operator of Laplace type, then there is a canonical connection ∇ on the underlying vector bundle that, together with the Levi-Civita connection, we use to covariantly differentiate tensors of all types - we denote multiple covariant differentiation by $'$. There is also a canonical endomorphism E so that

$$D = -(\text{Tr}(\nabla^2) + E) \quad \text{i.e.} \quad Du = -(u_{;ii} + Eu).$$

Let Ω_{ij} be the curvature of the connection ∇ . Let $\rho_{ij} = R_{ikkj}$ be the Ricci tensor, and let $\tau = \rho_{jj}$.

The heat trace invariants $a_{n,m}$ for an operator of Laplace type can be expressed in this formalism [9]:

Theorem 2.1.

- (1) $a_{0,m} = (4\pi)^{-m/2} \text{Tr}\{\text{Id}\}$.
- (2) $a_{2,m} = (4\pi)^{-m/2} \frac{1}{6} \text{Tr}\{6E + \tau \text{Id}\}$.
- (3) $a_{4,m} = (4\pi)^{-m/2} \frac{1}{360} \text{Tr}\{60E_{;kk} + 60\tau E + 180E^2 + (12\tau_{;kk} + 5\tau^2 - 2|\rho|^2 + 2|R|^2) \text{Id} + 30\Omega_{ij}\Omega_{ij}\}$.
- (4) $a_{6,m} = (4\pi)^{-m/2} \text{Tr}\{(\frac{18}{7!}\tau_{;iijj} + \frac{17}{7!}\tau_{;k\tau;k} - \frac{2}{7!}\rho_{ij;k}\rho_{ij;k} - \frac{4}{7!}\rho_{jk;n}\rho_{jn;k} + \frac{9}{7!}R_{ijkl;n}R_{ijkl;n} + \frac{28}{7!}\tau\tau_{;nn} - \frac{8}{7!}\rho_{jk}\rho_{jk;nn} + \frac{24}{7!}\rho_{jk}\rho_{jn;kn} + \frac{12}{7!}R_{ijkl}R_{ijkl;nn} + \frac{35}{9\cdot 7!}\tau^3 - \frac{14}{3\cdot 7!}\tau\rho^2 + \frac{14}{3\cdot 7!}\tau R^2 - \frac{208}{9\cdot 7!}\rho_{jk}\rho_{jn}\rho_{kn} - \frac{64}{3\cdot 7!}\rho_{ij}\rho_{kl}R_{ikjl} - \frac{16}{3\cdot 7!}\rho_{jk}R_{jnli}R_{knli} - \frac{44}{9\cdot 7!}R_{ijkn}R_{ijlp}R_{knlp} - \frac{80}{9\cdot 7!}R_{ijkn}R_{ilkp}R_{jlnp}) \text{Id} + \frac{1}{45}\Omega_{ij;k}\Omega_{ij;k} + \frac{1}{180}\Omega_{ij;j}\Omega_{ik;k} + \frac{1}{60}\Omega_{ij;kk}\Omega_{ij} + \frac{1}{60}\Omega_{ij}\Omega_{ij;kk} - \frac{1}{30}\Omega_{ij}\Omega_{jk}\Omega_{ki} - \frac{1}{60}R_{ijkn}\Omega_{ij}\Omega_{kn} - \frac{1}{90}\rho_{jk}\Omega_{jn}\Omega_{kn} + \frac{1}{72}\tau\Omega_{kn}\Omega_{kn} + \frac{1}{60}E_{;iiij} + \frac{1}{6}EE_{;ii} + \frac{1}{12}E_{;i}E_{;i} + \frac{1}{6}E^3 + \frac{1}{12}E\Omega_{ij}\Omega_{ij} + \frac{1}{36}\tau E_{;kk} + \frac{1}{90}\rho_{jk}E_{;jk} + \frac{1}{30}\tau_{;k}E_{;k} + \frac{1}{12}EE\tau + \frac{1}{30}E\tau_{;kk} + \frac{1}{72}E\tau^2 - \frac{1}{180}E|\rho|^2 + \frac{1}{180}E|R|^2\}$.

We refer to [10] for the proof of the following results:

Lemma 2.2. *On the circle, $a_{2,1}^{d+\delta} = \frac{1}{\sqrt{\pi}}\phi_{;11}$.*

Lemma 2.3. *Let $M = (M_1 \times M_2, \phi_1 + \phi_2, g_1 + g_2)$ decouple as a product. Then $a_{n,m}^{d+\delta}(\phi, g)(x_1, x_2) = \sum_{n_1+n_2=n, n_1 \geq m_1, n_2 \geq m_2} a_{n_1, m_1}^{d+\delta}(\phi_1, g_1)(x_1) \cdot a_{n_2, m_2}^{d+\delta}(\phi_2, g_2)(x_2)$.*

3. SPACES OF INVARIANTS

Let \mathcal{Q}_m be the space of $O(m)$ invariant polynomials in the components of the tensors $\{R, \nabla R, \nabla^2 R, \dots, \phi, \nabla\phi, \nabla^2\phi, \dots\}$. Define a grading on \mathcal{Q}_m by setting:

$$\text{weight}(R_{ijkl;\beta}) = |\beta| + 2 \text{ and } \text{weight}(\phi_{;\beta}) = |\beta|.$$

An element $Q \in \mathcal{Q}_m$ is homogeneous of weight n if and only if

$$Q(\phi, c^{-2}g) = c^n Q(\phi, g).$$

Let $\mathcal{Q}_{n,m} \subset \mathcal{Q}_m$ be the set of all $O(m)$ invariant polynomials which are homogeneous of weight n ; we then have a direct sum decomposition:

$$\mathcal{Q}_m = \bigoplus_n \mathcal{Q}_{n,m}.$$

We may use the \mathbb{Z}_2 action $\phi \rightarrow -\phi$ to decompose $\mathcal{Q}_{n,m} = \mathcal{Q}_{n,m}^+ \oplus \mathcal{Q}_{n,m}^-$ where

$$\mathcal{Q}_{n,m}^\pm := \{Q \in \mathcal{Q}_{n,m} : Q(-\phi, g) = \pm Q(\phi, g)\}.$$

The following natural restriction map $r : \mathcal{Q}_{n,m} \rightarrow \mathcal{Q}_{n,m-1}$ will play a crucial role. If (N, g_N, ϕ_N) are structures in dimension $m-1$, then we can define corresponding structures in dimension m by setting

$$(M, \phi_M, g_M) := (N \times S^1, \phi_N, g_N + d\theta^2).$$

If $x \in N$ is the point of evaluation, we take the corresponding point $(x, 1) \in M$ for evaluation; which point on the circle chosen is, of course, irrelevant as S^1 has a rotational symmetry. The restriction map is characterized dually by the formula:

$$r(Q)(\phi_N, g_N)(x) = Q(\phi_N, g_N + d\theta^2)(x, 1).$$

We can also describe the restriction map r in classical terms. H. Weyl's [18] first theorem of invariance theory implies orthogonal invariants are built by contracting indices in pairs, where the indices range from 1 through m . If P is given in terms of such a Weyl spanning set, then $r(P)$ is given in terms of the same Weyl spanning set by restricting the range of summation to be from 1 through $m-1$. Thus necessarily r is surjective. We refer to [10] for the proof of:

Lemma 3.1. *If m is even, then $a_{m+2,m}^{d+\delta} \in \mathcal{Q}_{n,m}^+ \cap \ker(r)$.*

We use H. Weyl's second theorem to see $\mathcal{Q}_{m+2,m}^+ \cap \ker(r)$ is generated by invariants where we contract $2m$ indices using the ε tensor and contract the remaining indices in pairs, we refer to the discussion in [10] for details. A direct calculation shows, after some additional work to eliminate dependencies, that:

$$\begin{aligned} \mathcal{Q}_{m+2,m}^+ \cap \ker(r) = \text{Span}\{ & \varepsilon_J^I \phi_{;i_1 j_1} \phi_{;i_2 j_2} \mathcal{R}_{J,3}^{I,m}, \varepsilon_J^I \phi_{;k} \phi_{;k} \mathcal{R}_{J,1}^{I,m}, \varepsilon_J^I R_{k\ell k} \mathcal{R}_{J,1}^{I,m}, \\ & \varepsilon_J^I \phi_{;i_1} \phi_{;j_1} R_{ki_2 j_2 k} \mathcal{R}_{J,3}^{I,m}, \varepsilon_J^I R_{i_1 i_2 j_2 j_1; k k} \mathcal{R}_{J,3}^{I,m}, \varepsilon_J^I R_{i_1 i_2 j_2 j_1; k} R_{i_3 i_4 j_4 j_3; k} \mathcal{R}_{J,5}^{I,m}, \\ & \varepsilon_J^I R_{ki_1 j_1 k} R_{li_2 j_2 \ell} \mathcal{R}_{J,3}^{I,m}, \varepsilon_J^I R_{kl j_2 j_1} R_{i_1 i_2 k \ell} \mathcal{R}_{J,3}^{I,m}, \varepsilon_J^I R_{ki_1 j_1 \ell} R_{ki_2 j_2 \ell} \mathcal{R}_{J,3}^{I,m}, \\ & \varepsilon_J^I R_{ki_1 j_2 j_1} R_{kj_3 i_3 i_2} R_{li_4 j_5 j_4} R_{lj_6 i_6 i_5} \mathcal{R}_{J,7}^{I,m}\}. \end{aligned}$$

Although this is some gain in simplifying the question, the list of invariants is still quite long. We shall use equation (1.b) to further reduce the number of invariants to be considered and complete the proof of Lemma 1.3.

Let $\mathcal{Q}_{n,m}^{p,+}$ be the space of p form valued invariants in the curvature tensor, the covariant derivatives of the curvature tensor, and the covariant derivatives of ϕ which are even in ϕ . The exterior co-derivative δ_g induces a natural map

$$\delta_g : \mathcal{Q}_{n,m}^{p,+} \rightarrow \mathcal{Q}_{n+1,m}^{p-1,+}.$$

Let $i : N \rightarrow N \times \{1\} \subset N \times S^1$. The analysis of [4] shows that p form valued invariants are constructed by alternating p indices and by contracting the remaining indices in pairs. The restriction map

$$r : \mathcal{Q}_{n,m}^p \rightarrow \mathcal{Q}_{n,m-1}^p$$

is defined by restricting the range of summation of the indices involved; it is characterized by the identity:

$$r(Q)(\phi_N, g_N) = i^* Q(\phi_N, g_N + d\theta^2).$$

One verifies that r is surjective and that

$$r \circ \delta_{g_M} = \delta_{g_N} \circ r.$$

The analysis of the formal cohomology groups of the spaces of invariants of Riemannian manifolds, which was given in [7], then extends immediately to this more general setting to yield:

- Lemma 3.2.** (1) *If $Q \in \mathcal{Q}_{n,m}^+$, if $\int_M Q(\phi, g) = 0$ for all (ϕ, g) , and if $n \neq m$, then there exists $Q^1 \in \mathcal{Q}_{n-1,m}^{1,+}$ so that $\delta_g Q^1 = Q$.*
(2) *If $Q^1 \in \mathcal{Q}_{n,m}^{1,+}$, if $\delta_g Q^1 = 0$, and if $n \neq m-1$, then there exists $Q^2 \in \mathcal{Q}_{n-1,m}^{2,+}$ so that $Q^1 = \delta_g Q^2$.*

The first assertion shows that any scalar invariant which always integrates to zero is canonically a divergence and that any 1 form valued invariant which is co-closed is canonically co-exact. The restriction on the weight is a technical one which plays no role as we shall take $n = m + 2$ in assertion (1) and $n = m + 1$ in assertion (2).

We use this result to show

Lemma 3.3. *If m is even, then there exists a 1 form valued invariant $Q_{m+1,m}^1$ in $\mathcal{Q}_{m+1,m}^{1,+} \cap \ker(r)$ so that $\delta_g Q_{m+1,m}^1 = a_{m+2,m}^{d+\delta}$.*

Proof. By equation (1.b) and Lemma 3.2 (1), there exists $\bar{Q}_{m+1,m}^1 \in \mathcal{Q}_{m+1,m}^{1,+}$ so

$$(3.a) \quad \delta_g \bar{Q}_{m+1,m}^1(\phi, g) = a_{m+2,m}^{d+\delta}(\phi, g).$$

Unfortunately, $r(\bar{Q}_{m+1,m}^1)$ need not be zero and we must correct for this. Since $r(a_{m+2,m}^{d+\delta}) = 0$,

$$0 = r(\delta_g \bar{Q}_{m+1,m}^1) = \delta_g r(\bar{Q}_{m+1,m}^1).$$

Thus by Lemma 3.2 (2), there exists $\bar{Q}_{m,m-1}^2 \in Q_{m,m-1}^2$ so that

$$r(\bar{Q}_{m+1,m}^1) = \delta_g(\bar{Q}_{m,m-1}^2).$$

We observed above that r is surjective. Thus we can find $Q_{m,m}^2 \in Q_{m,m}^{2,+}$ so that:

$$r(Q_{m,m}^2) = \bar{Q}_{m,m-1}^2.$$

We complete the proof by setting $Q_{m+1,m}^1 := \bar{Q}_{m+1,m}^1 - \delta_g(Q_{m,m}^2)$ and computing:

$$\begin{aligned} \delta_g(Q_{m+1,m}^1) &= \delta_g(\bar{Q}_{m+1,m}^1) - \delta_g^2(Q_{m,m}^2) = \delta_g(\bar{Q}_{m+1,m}^1) = a_{m+2,m}^{d+\delta}, \\ r(Q_{m+1,m}^1) &= r(\bar{Q}_{m+1,m}^1) - r(\delta_g(Q_{m,m}^2)) = r(\bar{Q}_{m+1,m}^1) - \delta_g(r(Q_{m,m}^2)) \\ &= r(\bar{Q}_{m+1,m}^1) - \delta_g(\bar{Q}_{m,m-1}^2) = 0. \end{aligned} \quad \square$$

For m even, we define elements of $Q_{m+1,m}^{1,+} \cap \ker(r)$ by setting:

$$\begin{aligned} \Xi_{m+1,m}^{1,\ell} &:= \varepsilon_J^I \phi^\ell \phi_{;i_1 j_1} \phi_{;i_2} \mathcal{R}_{J,3}^{I,m} e^{j_2} & (\ell \text{ even}), \\ \Xi_{m+1,m}^{2,\ell} &:= \varepsilon_J^I \phi^\ell R_{i_1 i_2 j_2 j_1; k} \mathcal{R}_{J,3}^{I,m} e^k & (\ell \text{ even}), \\ \Xi_{m+1,m}^{3,\ell} &:= \varepsilon_J^I \phi^\ell R_{i_1 i_2 k j_1; k} \mathcal{R}_{J,3}^{I,m} e^{j_2} & (\ell \text{ even}), \\ \Xi_{m+1,m}^{4,\ell} &:= \varepsilon_J^I \phi^\ell \phi_{;k} \mathcal{R}_{J,1}^{I,m} e^k, & (\ell \text{ odd}), \\ \Xi_{m+1,m}^{5,\ell} &:= \varepsilon_J^I \phi^\ell \phi_{;i_1} R_{i_2 k k j_2} \mathcal{R}_{J,3}^{I,m} e^{j_1} & (\ell \text{ odd}). \end{aligned}$$

Lemma 3.4. *If $m = 2\bar{m}$ is even, then $\{Q_{m+1,m}^{1,+} \cap \ker(r)\} = \text{Span}\{\Xi_{m+1,m}^{i,\ell}\}_{i,\ell}$.*

Proof. We use H. Weyl's theorem on the invariants of the orthogonal group. Let

$$A = \phi^\ell \phi_{;\alpha_1} \dots \phi_{;\alpha_u} R_{i_1 j_1 k_1 l_1; \beta_1} \dots R_{i_\nu j_\nu k_\nu l_\nu; \beta_\nu} e^n$$

be a typical 1 form valued monomial where $|\alpha_\nu| \geq 1$ and $\ell + u$ is even. Note that:

$$n = \sum_\mu |\alpha_\mu| + \sum_\nu (|\beta_\nu| + 2).$$

We must contract $2m$ indices using the ε tensor and contract the remaining indices in pairs; we refer to [10] where this was discussed in some detail for scalar invariants – the extension to 1 form valued invariants is similar. We may estimate:

$$\begin{aligned} (3.b) \quad 2m &\leq \text{number of indices in } A \\ &= \sum_\mu |\alpha_\mu| + \sum_\nu (|\beta_\nu| + 4) + 1 = n + 2v + 1 \\ (3.c) &= 2n + 1 - \sum_\mu |\alpha_\mu| - \sum_\nu |\beta_\nu| \leq 2n + 1 \end{aligned}$$

We set $n = m + 1$. Since $2m$ and $m + 1 + 2v + 1$ are both even, the inequality in equation (3.c) must be strict and represents an increase either of 1 or of 3.

Suppose first that equation (3.b) is an equality. Then all the $2m$ indices present in A are contracted using the ε tensor. We can commute covariant derivatives at the cost of introducing additional curvature terms. Thus since all indices are to be contracted using the ε tensor, we may assume $|\alpha_\mu| \leq 2$ for all μ . Furthermore, by the first and second Bianchi identity, at most 2 indices can be alternated in $R_{ijkt;\beta}$. Thus $|\beta_\nu| = 0$ for all ν so $\sum_\mu |\alpha_\mu| = 3$. This leads to the invariants $\Xi_{m+1,m}^{1,\ell}$.

Suppose next that equation (3.b) is not an equality. Then there are $2m+2$ indices and one explicit covariant derivative present in A ; $2m$ indices are contracted using the ε tensor and two indices are contracted as a pair. This yields the invariants $\Xi_{m+1,m}^{i,\ell}$ for $i = 2, 3, 4, 5$ and the additional invariants:

$$\begin{aligned} \Theta_{m+1,m}^{1,\ell} &:= \varepsilon_J^I \phi^\ell \phi_{;k} R_{i_1 i_2 j_2 k} \mathcal{R}_{J,3}^{I,m} e^{j_1} & (\ell \text{ odd}), \\ \Theta_{m+1,m}^{2,\ell} &:= \varepsilon_J^I \phi^\ell \phi_{;i_1} R_{i_2 k j_2 j_1} \mathcal{R}_{J,3}^{I,m} e^k & (\ell \text{ odd}), \\ \Theta_{m+1,m}^{3,\ell} &:= \varepsilon_J^I \phi^\ell \phi_{;i_1} R_{i_2 k j_3 j_2} R_{i_3 i_4 j_4 k} \mathcal{R}_{J,5}^{I,m} e^{j_1} & (\ell \text{ odd}), \\ \Theta_{m+1,m}^{4,\ell} &:= \varepsilon_J^I \phi^\ell R_{i_1 i_2 k j_2} R_{i_3 i_4 j_4 j_3; k} \mathcal{R}_{J,5}^{I,m} e^{j_1} & (\ell \text{ even}). \end{aligned}$$

To complete the proof, we must show the invariants $\Theta_{m+1,m}^{i,\ell}$ play no role. Let U and V be collections of $m+1$ indices. Since $\varepsilon_V^U = 0$, we have

$$0 = \varepsilon_V^U \phi^\ell \phi_{;u_1} \mathcal{R}_{V,2}^{U,m+1} e^{v_1}.$$

We set $u_1 = k$ and then set $v_1 = k, v_2 = k, \dots$, and $v_{m+1} = k$ in turn to see:

$$\begin{aligned} 0 &= \varepsilon_J^I \phi^\ell \phi_{;k} \mathcal{R}_{J,1}^{I,m} e^k - m \varepsilon_J^I \phi^\ell \phi_{;k} R_{i_1 i_2 j_2 k} \mathcal{R}_{J,3}^{I,m} e^{j_1} \\ &= \Xi_{m+1,m}^{4,\ell} - m \Theta_{m+1,m}^{1,\ell}. \end{aligned}$$

We set $u_2 = k$ and expand in v to see:

$$\begin{aligned} 0 &= \varepsilon_J^I \phi^\ell \phi_{;i_1} R_{i_2 k j_2 j_1} \mathcal{R}_{J,3}^{I,m} e^k - 2 \varepsilon_J^I \phi^\ell \phi_{;i_1} R_{i_2 k j_2 k} \mathcal{R}_{J,3}^{I,m} e^{j_1} \\ &\quad - (m-2) \varepsilon_J^I \phi^\ell \phi_{;i_1} R_{i_2 k j_3 j_2} R_{i_3 i_4 j_4 k} \mathcal{R}_{J,5}^{I,m} e^{j_1} \\ &= \Theta_{m+1,m}^{2,\ell} + 2 \Xi_{m+1,m}^{5,\ell} - (m-2) \Theta_{m+1,m}^{3,\ell}. \end{aligned}$$

Next, we set $v_1 = k$ and expand in u to see:

$$\begin{aligned} 0 &= \varepsilon_J^I \phi^\ell \phi_{;k} \mathcal{R}_{J,1}^{I,m} e^k + m \varepsilon_J^I \phi^\ell \phi_{;i_1} R_{i_2 k j_2 j_1} \mathcal{R}_{J,3}^{I,m} e^{j_1} \\ &= \Xi_{m+1,m}^{4,\ell} + m \Theta_{m+1,m}^{2,\ell}. \end{aligned}$$

Finally, we set $v_2 = k$ and expand in u to see:

$$\begin{aligned} 0 &= \varepsilon_J^I \phi^\ell \phi_{;k} R_{i_1 i_2 j_2 k} \mathcal{R}_{J,3}^{I,m} e^{j_1} + 2 \varepsilon_J^I \phi^\ell \phi_{;i_1} R_{i_2 k j_2 k} \mathcal{R}_{J,3}^{I,m} e^{j_1} \\ &\quad + (m-2) \varepsilon_J^I \phi^\ell \phi_{;i_1} R_{i_2 i_3 j_2 k} R_{i_4 k j_4 j_3} \mathcal{R}_{J,5}^{I,m} e^{j_1} \\ &= \Theta_{m+1,m}^{1,\ell} - 2 \Xi_{m+1,m}^{5,\ell} + (m-2) \Theta_{m+1,m}^{3,\ell}. \end{aligned}$$

We can show that

$$\{\Theta_{m+1,m}^{1,\ell}, \Theta_{m+1,m}^{2,\ell}, \Theta_{m+1,m}^{3,\ell}\} \subset \text{Span}\{\Xi_{m+1,m}^{i,\ell}\}_{i,\ell}$$

by computing:

$$\begin{aligned} \Theta_{m+1,m}^{1,\ell} &= \frac{1}{m} \Xi_{m+1,m}^{4,\ell} = \frac{1}{m} (2 \Xi_{m+1,m}^{5,\ell} - (m-2) \Theta_{m+1,m}^{3,\ell}), \\ \Theta_{m+1,m}^{2,\ell} &= -\frac{1}{m} \Xi_{m+1,m}^{4,\ell} = -\frac{1}{m} (2 \Xi_{m+1,m}^{5,\ell} - (m-2) \Theta_{m+1,m}^{3,\ell}). \end{aligned}$$

Finally, we put $u_1 = k$ in the identity

$$0 = \varepsilon_V^U \phi^\ell R_{u_2 u_3 v_2; u_1} \mathcal{R}_{V,4}^{U,m+1} e^{v_1}$$

and expand in v to show

$$\begin{aligned} 0 &= \varepsilon_J^I \phi^\ell R_{i_1 i_2 j_2 j_1; k} \mathcal{R}_{J,3}^{I,m} e^k - 2 \varepsilon_J^I \phi^\ell R_{i_1 i_2 j_2 k; k} \mathcal{R}_{J,3}^{I,m} e^{j_1} \\ &\quad - (m-2) \varepsilon_J^I \phi^\ell R_{i_1 i_2 j_3 j_2; k} R_{i_3 i_4 j_4 k} \mathcal{R}_{J,5}^{I,m} e^{j_1} \\ &= \Xi_{m+1,m}^{2,\ell} - 2 \Xi_{m+1,m}^{3,\ell} + (m-2) \Theta_{m+1,m}^{4,\ell}. \end{aligned}$$

This establishes the lemma. \square

We now prove Lemma 1.3. Let $m = 2\bar{m}$ be even. We apply Lemma 3.3 and Lemma 3.4 to see there exist universal constants so

$$a_{m+2,m}^{d+\delta}(\phi, g) = \sum_{i,\ell} c_{m+1,m}^{i,\ell} \delta_g \{\Xi_{m+1,m}^{i,\ell}\},$$

where ℓ is chosen so ϕ appears an even number of times in each expression. Terms which are linear in the 2 jets of ϕ and which are of total weight 2 in ϕ can arise only from $i = 4$ and $i = 5$. Consequently we have

$$a_{m+2,m}^{d+\delta}(\phi, g) = -\sum_{\ell} \phi^\ell \varepsilon_J^I \{c_{m+1,m}^{4,\ell} \phi_{;kk} \mathcal{R}_{J,1}^{I,m} + c_{m+1,m}^{5,\ell} \phi_{;i_1 j_1} R_{i_2 k k j_2} \mathcal{R}_{J,3}^{I,m}\} + \dots$$

where ℓ is odd. Replacing ϕ by $\phi + c$ does not change d_ϕ and $\delta_{g,\phi}$. Thus ϕ^μ does not appear in the formula for $a_{m+2,m}^{d+\delta}$ for $\mu > 0$. Consequently,

$$(3.d) \quad 0 = c_{m+1,m}^{4,\ell} \varepsilon_J^I \phi_{;kk} \mathcal{R}_{J,1}^{I,m} + c_{m+1,m}^{5,\ell} \varepsilon_J^I \phi_{;i_1 j_1} R_{i_2 k k j_2} \mathcal{R}_{J,3}^{I,m}.$$

We consider the expressions:

$$\begin{aligned} A_1 &:= \phi_{;11} R_{1221} R_{3443} \dots R_{m-1,mm,m-1}, \quad \text{and} \\ A_2 &:= \phi_{;12} R_{1332} R_{3443} \dots R_{m-1,mm,m-1}. \end{aligned}$$

We may then expand

$$\begin{aligned} \phi_{;kk} \mathcal{R}_{J,1}^{I,m} &= 4^{\bar{m}} \bar{m}! A_1 + 0 A_2 + \dots, \\ \phi_{;i_1 j_1} R_{i_2 k k j_2} \mathcal{R}_{J,3}^{I,m} &= 4^{\bar{m}-1} (\bar{m}-1)! A_1 - 4 \cdot 4^{\bar{m}-1} (\bar{m}-1)! A_2 + \dots \end{aligned}$$

Consequently equation (3.d) implies $c_{m+1,m}^{4,\ell} = 0$ and $c_{m+1,m}^{5,\ell} = 0$.

We argue similarly to show that if $\ell > 0$, then

$$\begin{aligned} 0 &= c_{m+1,m}^{1,\ell} \varepsilon_J^I \phi_{;j_2} \phi_{;i_1 j_1} \phi_{;i_2} \mathcal{R}_{J,3}^{I,m}, \quad \text{and} \\ 0 &= c_{m+1,m}^{2,\ell} \varepsilon_J^I \phi_{;k} R_{i_1 i_2 j_2 j_1; k} \mathcal{R}_{J,3}^{I,m} + c_{m+1,m}^{3,\ell} \varepsilon_J^I \phi_{;j_2} R_{i_1 i_2 k j_1; k} \mathcal{R}_{J,3}^{I,m}. \end{aligned}$$

This shows $c_{m+1,m}^{\ell} = 0$ for $\ell > 0$. We consider the expressions:

$$\begin{aligned} B_1 &:= \phi_{;1} R_{1221;1} R_{3443} \dots R_{m-1,mm,m-1}, \quad \text{and} \\ B_2 &:= \phi_{;3} R_{1221;3} R_{3443} \dots R_{m-1,mm,m-1} \end{aligned}$$

and expand

$$\begin{aligned} \varepsilon_J^I \phi_{;k} R_{i_1 i_2 j_2 j_1; k} \mathcal{R}_{J,3}^{I,m} &= 4^{\bar{m}} (\bar{m}-1)! B_1 + 4^{\bar{m}} (\bar{m}-1)! B_2 + \dots, \\ \varepsilon_J^I \phi_{;j_2} R_{i_1 i_2 k j_1; k} \mathcal{R}_{J,3}^{I,m} &= 2 \cdot 4^{\bar{m}-1} (\bar{m}-1)! B_1 + 0 B_2 + \dots \end{aligned}$$

to see $c_{m+1,m}^{2,\ell} = 0$ and $c_{m+1,m}^{3,\ell} = 0$ for $\ell > 0$; Lemma 1.3 now follows. \square

4. DETERMINING THE NORMALIZING CONSTANTS

We complete the proof of Theorem 1.2 by evaluating the normalizing constants of Lemma 1.3:

Lemma 4.1. *Let $m = 2\bar{m}$. Then*

$$\begin{aligned} (1) \quad c_{m+2,m}^1 &= \frac{4\bar{m}}{\pi^{\bar{m}} 8^{\bar{m}} \bar{m}!}, \\ (2) \quad c_{m+2,m}^2 &= \frac{1}{12} \frac{1}{\pi^{\bar{m}} 8^{\bar{m}} \bar{m}!}, \\ (3) \quad c_{m+2,m}^3 &= \frac{\bar{m}}{6} \frac{1}{\pi^{\bar{m}} 8^{\bar{m}} \bar{m}!}. \end{aligned}$$

Proof. We shall apply Theorem 1.1, Theorem 2.1, Lemma 2.2, and Lemma 2.3. We use the method of universal examples. Give $M := S^{m-2} \times S^1 \times S^1$ the product metric. Let $\phi = \phi_1(\theta_1) + \phi_2(\theta_2)$. Then:

$$\begin{aligned} a_{m+2,m}^{d+\delta}(\phi, g) &= 2c_{m+2,m}^1 2^{\bar{m}-1} (m-2)! \phi_{;m-1m-1} \phi_{;mm} \\ &= a_{m-2,m-2}^{d+\delta}(0, g_{S^{m-2}}) \cdot a_{2,1}^{d+\delta}(\phi_1, d\theta_1^2) \cdot a_{2,1}^{d+\delta}(\phi_2, d\theta_2^2) \\ &= \frac{1}{8^{\bar{m}-1} \pi^{\bar{m}-1} (\bar{m}-1)!} \cdot 2^{\bar{m}-1} (m-2)! \frac{1}{\pi} \phi_{1;m-1,m-1} \phi_{2;mm}. \end{aligned}$$

We solve this equation for $c_{m+2,m}^1$ to establish assertion (1).

For the remainder of the proof of the Lemma, we set $\phi = 0$ to consider only metric invariants. We express

$$a_{m,m}^{d+\delta} = \mathcal{E}_{m,m} c_{m,m} \quad \text{for} \quad \mathcal{E}_{m,m} = \varepsilon_J^I \mathcal{R}_{J,1}^{I,m} \quad \text{and} \quad c_{m,m} = \frac{1}{8^{\bar{m}} \pi^{\bar{m}} \bar{m}!}.$$

If $m = 2$, then the invariants $(\varepsilon_J^I \mathcal{R}_{J,1}^{I,m})_{;kk}$ and $(\varepsilon_J^I R_{i_1 i_2 k j_1; k} \mathcal{R}_{J,3}^{I,m})_{;j_1}$ are not linearly independent. If (N, g_N) is a Riemann surface, then we may establish assertions (2) and (3) for $m = 2$ by computing:

$$\begin{aligned} a_{2,2}^{d+\delta}(g_N) &= \frac{1}{4\pi} \sum_p (-1)^p \text{Tr}(E^p) \\ a_{4,2}^{d+\delta}(g_N) &= \frac{1}{4\pi} \frac{1}{6} \{ \sum_p (-1)^p \text{Tr}(E^p) \}_{;kk} + O(R^2) \\ &= \frac{1}{6} \{ a_{2,2}^{d+\delta} \}_{;kk} + O(R^2) = \frac{1}{4\pi} \frac{1}{6} R_{ijji;kk} + O(R^2). \end{aligned}$$

Suppose now that $m = 4$. Since $\sum_p (-1)^p \text{Tr}(\dim(\Lambda^p)) = 0$, we compute:

$$\begin{aligned} 0 &= a_{2,4}^{d+\delta}(0, g) = \frac{1}{8\pi^2} \sum_p (-1)^p \text{Tr}(E^p), \\ a_{4,4}^{d+\delta} &= \frac{1}{4^2 \pi^2} \sum_p (-1)^p \text{Tr}\left\{\frac{1}{2} E^p E^p + \frac{1}{12} \text{Tr}(\Omega_{ij}^p \Omega_{ij}^p)\right\}, \\ a_{6,4}^{d+\delta} &= \frac{1}{4^2 \pi^2} \sum_p (-1)^p \text{Tr}\left\{\frac{1}{45} \Omega_{ij;k}^p \Omega_{ij;k}^p + \frac{1}{180} \Omega_{ij;j}^p \Omega_{ik;k}^p + \frac{1}{60} \Omega_{ij;kk}^p \Omega_{ij}^p \right. \\ &\quad \left. + \frac{1}{60} \Omega_{ij}^p \Omega_{ij;kk}^p + \frac{1}{6} E^p E_{;ii}^p + \frac{1}{12} E_{;i}^p E_{;i}^p\right\} + O(R^3). \end{aligned}$$

We study the expressions $C_1 := R_{1221} R_{3443}$ and $C_2 := R_{1221;2} R_{3443;2}$ and suppress other terms. Only the term E^2 can give rise to the expression A_2 and only the term $E_{;i}^p E_{;i}^p$ can give rise to the expression A_3 . We prove assertion (2) if $m = 4$ by computing:

$$\begin{aligned} a_{4,4}^{d+\delta} &= \frac{1}{2 \cdot 4^2 \pi^2} \sum_p (-1)^p \text{Tr}(E^p E^p) + \dots = \frac{32}{8^2 \pi^2 2!} R_{1221} R_{3443} + \dots \\ a_{6,4}^{d+\delta} &= \frac{1}{4^2 \pi^2} \frac{1}{12} \sum_p (-1)^p \text{Tr}(E_{;i}^p E_{;i}^p) + \dots = \frac{1}{4^2 \pi^2} \frac{1}{24} \sum_p (-1)^p \text{Tr}(E^p E^p)_{;ii} + \dots \\ &= \frac{1}{12} (a_{4,4}^{d+\delta})_{;kk} + \dots = \frac{1}{6} \frac{32}{8^2 \pi^2 2!} R_{1221;2} R_{3443;2} + \dots \\ &= 2c_{6,4}^2 \varepsilon_J^I R_{i_1 i_2 j_2 j_1; k} R_{i_3 i_4 j_3 j_4; k} + \dots = 64c_{6,4}^2 R_{1221;2} R_{3443;2} + \dots, \quad \text{so} \\ c_{6,4}^2 &= \frac{1}{12} \frac{1}{8^2 \pi^2 2!}. \end{aligned}$$

If $m > 4$, let $(M, g) := (N^4 \times S^{m-4}, g_N + g_0)$. Assertion (2) follows in general from:

$$\begin{aligned} a_{m+2,m}^{d+\delta}(g) &= \bar{m}(\bar{m}-1)c_{m+2,m}^2 \varepsilon_J^I (R_{i_1 i_2 j_2 j_1; k} R_{i_3 i_4 j_3 j_4; k})(g_N) \mathcal{E}_{m-4, m-4}(g_0) + \dots \\ &= a_{6,4}^{d+\delta}(g_N) a_{m-4, m-4}^{d+\delta}(g_0) + \dots \\ &= 2c_{6,4}^2 c_{m-4, m-4} \varepsilon_J^I (R_{i_1 i_2 j_2 j_1; k} R_{i_3 i_4 j_3 j_4; k})(g_N) \mathcal{E}_{m-4, m-4}(g_0) + \dots \quad \text{so} \\ c_{m+2,m}^2 &= \frac{2}{\bar{m}(\bar{m}-1)} \frac{1}{12} \frac{1}{\pi^2 8^2 2!} \cdot \frac{1}{\pi^{\bar{m}-2} 8^{\bar{m}-2} (\bar{m}-2)!} = \frac{1}{12} \frac{1}{\pi^{\bar{m}} 8^{\bar{m}} \bar{m}!}. \end{aligned}$$

Let $(M, g) = (N^2 \times S^{m-2}, g_N + g_0)$. We derive a relation between the invariants $c_{m+2,m}^2$ and $c_{m+2,m}^3$ and to complete the proof of assertion (3):

$$\begin{aligned} a_{m+2,m}^{d+\delta}(g) &= (2\bar{m}c_{m+2,m}^2 + c_{m+2,m}^3) R_{ijji;kk}(g_N) \cdot \mathcal{E}_{m-2, m-2}(g_0) \\ &= a_{4,2}^{d+\delta}(g_N) a_{m-2, m-2}^{d+\delta}(g_0) = \frac{1}{4\pi} \frac{1}{6} R_{ijji;kk}(g_N) \cdot c_{m-2, m-2} \mathcal{E}_{m-2, m-2}(g_0) \quad \text{so} \\ 2\bar{m}c_{m+2,m}^2 + c_{m+2,m}^3 &= \frac{1}{6} \frac{1}{4\pi} \frac{1}{\pi^{\bar{m}-1} 8^{\bar{m}-1} (\bar{m}-1)!} = \frac{\bar{m}}{3} \frac{1}{\pi^{\bar{m}} 8^{\bar{m}} \bar{m}!}. \quad \square \end{aligned}$$

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