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On the existence of the N-body Efimov effect

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Abstract

In this paper, we study a generalization of the Efimov effect for three-body Schrödinger operators to N-body ones with $N \geq 4$. We prove the infiniteness of the discrete spectrum under the conditions that the bottom of the essential spectrum, E_0, is only attained by the spectra of a unique three-cluster Subhamiltonian and its three associated two-clusters Subhamiltonians, and that at least two of these two-cluster Subhamiltonians have a resonance at E_0. We give a lower bound on the number of eigenvalues of the form $C_0|\log|\lambda - E_0||$, when λ tends to E_0. Here C_0 is a positive constant depending only on the reduced masses in the three-cluster decomposition.

1 Introduction

The Efimov effect describes a phenomenon for three-body Schrödinger operators which can be roughly stated as follows. When the essential spectrum of the three-particle Hamiltonian is the positive real axis, and when at least two of its two-body Subhamiltonians have a resonance at zero, the discrete spectrum of the three-body Schrödinger operator is infinite, even if the interactions are very short-range. This phenomenon is striking if one compares it with the results on the finiteness of eigenvalues of two-body Schrödinger operators or of N-body Schrödinger operators whose bottom of the essential spectrum is only reached by the spectrum of two-cluster Subhamiltonians. See [11, 13, 14, 34]. Since its discovery par V. Efimov in 1970 ([12]), many works are devoted to this subject. See, for example, [2, 3, 4, 6, 9, 10, 21, 22, 25, 29, 30, 31, 32, 36]. In particular, D. Yafaev presented in [36] the first mathematically rigorous proof of the existence of such phenomenon and A. Sobolev established in [29] the asymptotics of the number of eigenvalues near zero in three-body problems.

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For N-body systems with $N \geq 4$, R. D. Amado and F. C. Greenwood ([5]) studied the contribution of resonances at zero of $(N-1)$-particle subsystems to the discrete spectrum of the total system and concluded that there is no Efimov effect in the case $N \geq 4$. The subsequent mathematical researches on this subject are mostly concerned with the case of two-particle resonances and the bottom of essential spectrum of the total Hamiltonian is zero. No evidence on the existence of the Efimov effect has yet been found for $N \geq 4$. In this work, we shall prove that two-cluster resonances of Subhamilonians at the bottom of essential spectrum of an N-body Schrödinger operator, $N \geq 4$, can create an infinite number of discrete eigenvalues.

Let us introduce some notation used in this work. Let P denote the N-body Schrödinger operator obtained after the removal of the center of mass from the total energy operator

$$-\sum_{j=1}^{N} \frac{1}{2m_j}\Delta_{x_j} + \sum_{1\leq i<j\leq N} V_{ij}(x_i - x_j), \quad x_j \in \mathbf{R}^3, 0 < m_j < \infty, \tag{1.1}$$

where x_j and m_j denote the position and mass of the j-th particle, V_{ij} is assumed to be real and relatively compact with respect to $-\Delta$ in $L^2(\mathbf{R}^3)$ and satisfies the decay

$$|V_{ij}(y)| \leq C_{ij} < y >^{-\rho}, \tag{1.2}$$

for $y \in \mathbf{R}^3$. In the most part of this work, we use the assumption $\rho > 2$. But the final result is only valid under the assumption $\rho > 8/3$, due to a technical estimate used in the proof of Proposition 4.2. V_{ij} is assumed to be bounded, but the $L^2_{loc}(\mathbf{R}^3)$-singularities can easily be included. P is regarded as a self-adjoint operator in $L^2(\mathbf{X})$, where $\mathbf{X}$ is the $3(N-1)$ dimensional real vectorial space: $\mathbf{X} = \{(x_1, \cdots, x_N) \in \mathbf{R}^{3N}; \sum_{j=1}^N m_j x_j = 0\}$. Let $\mathcal{A}$ denote the set of all cluster decompositions of the N-particle system. For $a \in \mathcal{A}$, let $\#a$ denote the number of clusters in a and P^a the Subhamiltonian associated with a. For $i, j \in \{1, \cdots, N\}$, we write $(ij) \in a$ if i and j belong to a same cluster in a. If a, b are two cluster decompositions, we write $b \subset a$ if b is a refinement of a. Let a be a k-cluster decomposition $a = (a_1, \cdots, a_k)$. We denote $\mathbf{X}^a = \{x \in \mathbf{X}; \sum_{l\in a_j} m_l x_l = 0, j = 1, \cdots, k\}$ and $\mathbf{X}_a = \{x \in \mathbf{X}; x_i = x_j \text{ if } (ij) \in a_m \text{ for some } m \in \{1, \cdots, k\}\}$. $\mathbf{X}^a$ and $\mathbf{X}_a$ give an orthogonal decomposition for $\mathbf{X}$ relative to the quadratic form $q(x) = \sum_j 2m_j |x_j|^2, x \in \mathbf{X}$. We denote by $\mathbf{X}^*$ (resp., $\mathbf{X}^{a*}, \mathbf{X}^*_a$) the dual space of $\mathbf{X}$ (resp., $\mathbf{X}^a, \mathbf{X}_a$). $\mathbf{X}^*$ is equipped with the dual metric, q^*, of q. In this paper, when coordinates are needed in calculus, we always work with orthonormal basis with respect to q in $\mathbf{X}$ and with respect to q^* in $\mathbf{X}^*$.

Let π^a and π_a denote the orthogonal projection from $\mathbf{X}$ onto $\mathbf{X}^a$ and $\mathbf{X}_a$, respectively. For $x \in \mathbf{X}$, let $x^a = \pi^a x$ and $x_a = \pi_a x$ so that we have the orthogonal decomposition: $x = x^a + x_a$ with $x^a \in \mathbf{X}^a$ and $x_a \in \mathbf{X}_a$. The N-body Schrödinger operator P introduced above can be written in the form $P = P_0 + V(x)$ where P_0 is the Laplace-Beltrami operator on the Euclidean space $(\mathbf{X}, q)$ and $V(x) = \sum_{a\in\mathcal{A}} V_a(x^a)$ with $V_a(x^a) = V_{ij}(x_i - x_j)$ if a is an (N-1)-cluster decomposition and $(ij) \subset a$; $V_a(x^a) = 0$, otherwise. Let $-\Delta^a$ ($-\Delta_a$, resp.) denote the restriction of P_0 on $\mathbf{X}^a$ (on $\mathbf{X}_a$, resp.). For $a \in \mathcal{A}$, denote $P^a = -\Delta^a + \sum_{b\subseteq a} V_b(x^b), P_a = P^a - \Delta_a, I_a(x) = \sum_{b\not\subseteq a} V_b(x^b)$. P^a is the Subhamiltonian associated with the cluster decomposition a and I_a is the sum of all inter-cluster interactions. Let $\mathcal{T} = \cup_{a\in\mathcal{A},\#a\geq 2}\sigma_p(P^a)$ be the set of thresholds of P. The HVZ theorem gives the bottom of the essential spectrum $E_0 \equiv \inf\sigma_{ess}(P)$ of P by the formula

$$E_0 = \min_{a\in\mathcal{A},\#a\geq 2} \inf\sigma(P^a) = \min_{a\in\mathcal{A},\#a=2} \inf\sigma(P^a).$$

The purpose of this paper is to study the infiniteness of the discrete spectrum of N-body Schrödinger operators in the case that E_0 is attained by a unique three-cluster threshold, b, with $b = (b_1, b_2, b_3)$.

$$\inf\sigma(P^b) = E_0 = \inf\sigma_{ess}(P). \tag{1.3}$$

Let $a_k = (b_i \cup b_j, b_k)$, where i, j and k take different values in $\{1,2,3\}$. Then, we have necessarily $\sigma(P^{a_k}) = [E_0, \infty[$. In this paper, we assume that these are the only possible cluster decompositions whose spectra attain the bottom of the essential spectrum of the total Hamiltonian, *i. e.*,

$$\inf\sigma(P^a) = E_0, a \in \{b, a_1, a_2, a_3\}; \quad \inf\sigma(P^a) > E_0, a \notin \{b, a_1, a_2, a_3\}, \#a \geq 2. \tag{1.4}$$

For $N \geq 4$, by the HVZ Theorem, the assumption (1.4) implies that E_0 is in the discrete spectrum of P^b, therefore necessarily, $E_0 < 0$. For simplicity, we assume that the inter-cluster interactions related to b are attractive:

$$V_a(x^a) \leq 0, \text{ for } a \not\subseteq b. \tag{1.5}$$

In this work, unless otherwise stated, the conditions (1.2), (1.4) and (1.5) are always assumed.

For $c, d \in \mathcal{A}$ with $c \subset d$, we define $H^{m,s;c,d}$, $m, s \in \mathbf{R}$, be the weighted Sobolev space on $\mathbf{X}^d$ equipped with the norm

$$||u||_{m,s;c,d} = (\int_{\mathbf{X}^d} |(1+|x_c^d|^2)^{s/2}(1-\triangle_{x^d})^{m/2}u|^2 dx^d)^{1/2}. \tag{1.6}$$

If c and d are just the one-cluster decomposition, the indication about the cluster decompositions will be dropped. For example, for $\#d = 1$, we just write $H^{m,s;c,d}$ as $H^{m,s;c}$. Let $<\cdot,\cdot>$ ($(\cdots,\cdot)_a, <\cdot,\cdot>_a$, resp.) denote the scalar product on $L^2(\mathbf{X})$ (on $L^2(\mathbf{X}^a)$, $L^2(\mathbf{X}_a)$, resp.). $H^{1,-s;c,d}$, $s \geq 0$, to the dual space of $H^{-1,s;c,d}$ with the usual L^2 product as the dual product. Let $\mathcal{L}(m,s;m',s';c,d) = \mathcal{L}(H^{m,s;c,d}, H^{m',s';c,d})$. It will be convenient to regard P as an operator in $\mathcal{L}(1,0;-1,0)$.

Definition. *We say that E_0 is a* resonance *of P^{a_j} if the equation $P^{a_j}u = E_0 u$ has a solution $u \in H^{1,-s}(\mathbf{X}^{a_j}) \setminus L^2(\mathbf{X}^{a_j})$ for any $s > 1/2$.*

Under the assumptions (1.2) and (1.4), the analysis of threshold resonances made in [35] is valid for P^{a_j} and one knows that if E_0 is a resonance of P^{a_j}, then it is simple and the corresponding resonant state u behaves like

$$u(x^{a_j}) = O(e^{-\delta|x^b|}|x_b^{a_j}|^{-1}), \quad |x^{a_j}| \to \infty. \tag{1.7}$$

One can show under the conditions (1.2), (1.4) and (1.5), E_0 is not an eigenvalue of P^{a_j} for $j = 1, 2, 3$.

Theorem 1.1 *Let the conditions (1.2) with $\rho > 8/3$, (1.4) and (1.5) be satisfied. Assume that at least two of the two-cluster Subhamiltonians P^{a_j}, $j = 1, 2, 3$, have a resonance at E_0. Let $N(\lambda)$ denote the number of the eigenvalues of P below $\lambda < E_0$. Then, there exists $C_0 > 0$ depending only on the reduced masses of b_1, b_2, b_3 such that*

$$N(\lambda) \geq C_0 |\log|\lambda - E_0||, \lambda \to E_0. \tag{1.8}$$

Clearly, the positivity of C_0 implies that there exists an infinite number of discrete eigenvalues for the N-body Schrödinger operator satisfying the conditions of Theorem 1.1. Comparing with the known criteria for the finiteness or infiniteness of discrete spectrum of N-body Schrödinger operators ([13, 14]), we can say that these eigenvalues originate from the existence of resonances of two-cluster Subhamiltonians at E_0. This phenomenon is similar to the Efimov effect known in three-body problems.

Let us make some comments on works related to the Efimov effect in N-body problems with $N \geq 4$. For $N = 3$, see [3, 24, 29, 31]. R. D. Amado and F. C. Greenwood claimed in [5] that the zero resonances of $(N-1)$-particle subsystems do not produce an infinite number of eigenvalues in N-body problems when $N \geq 4$. I. M. Sigal announced in [27] three theorems related to the Efimov effect, including the asymptotics of the number of eigenvalues when $N = 3$ and the existence of the Efimov effect in $N \geq 4$ in a special case. No proof about the these two results has appeared. The asymptotics of eigenvalues when $N = 3$ are only obtained by A. Sobolev in 1993 ([29]) and the coupling constant limit by H. Tamura ([31]) by making use of the idea of [29]. In [33], S. A. Vugal'ter and G.M. Zhislin proved that two-particle resonances at zero energy do not produce an infinite number of discrete eigenvalues when $N \geq 4$. The work [18] is also concerned with the case where the essential spectrum of the total Hamiltonian is $\mathbf{R}_+$. In [1], F. Ahia gave another proof of the result of S. A. Vugal'ter and G.M. Zhislin [33]. As remarked by these authors, the question of whether or not the Efimov effect exists for $N \geq 4$ was unsettled.

We emphasize that in spite of the titles, our result is not in controversy with [5]. Different from [5], the bottom of the essential spectrum of the N-body Schrödinger operator in our case is always strictly negative when $N \geq 4$. A mathematical proof of the finiteness of discrete eigenvalues in the situation of [5] is still missing for general $N \geq 4$.

The proof of Theorem 1.1 relies heavily on the ideas and the results given in [29] and [35]. The results of [35] are used to establish the leading term of the resolvent of two-cluster Subhamiltonians. We follow closely A. Sobolev's method to transform the eigenvalue problem to a Toeplitz operator of the Fourier type ([15]). Our main task here is to reduce the additional space dimensions to obtain an integral operator already studied in [29]. Because of the lack of compactness, we only obtain a lower bound on the number of eigenvalues.

The organization of this work is as follows. In Section 2, we reduce the eigenvalue problem by the principle of Birman-Schwinger. In Section 3, the asymptotics of the resolvent of two-cluster Subhamiltonians are derived from the results proved in [35]. In Section 4, we give a lower bound on the number of eigenvalues.

2 A reduction

Let b and $a_j, j = 1,2,3,$ be given in the assumption (1.4). Write $I_b = \sum_{a \not\subseteq b} V_a$ as

$$I_b = -W_1 - W_2 - W_3, \tag{2.1}$$

where

$$W_j = -\sum_{a \not\subseteq b, a \subseteq a_j} V_a, j = 1,2,3.$$

Then $W_j \geq 0$ and one has

$$P_{a_j} = P_b - W_j. \tag{2.2}$$

Denote $R_{a_j}(z) = (P_{a_j} - z)^{-1}$.

For a bounded self-adjoint operator A, we define $n(\lambda, A)$ as

$$n(\mu, A) = \sup\{\dim F; < Au, u >> \mu, u \in F, \|u\| = 1\}.$$

$n(\mu, A)$ is equal to the infinity if μ is in the essential spectrum and if $n(\mu, A)$ is finite, it is equal to the number of the eigenvalues of A bigger than μ.

Proposition 2.1 *Let $\lambda < E_0$. One has*

$$N(\lambda) = n(1, A(\lambda)) \tag{2.3}$$

where

$$A(\lambda) = \begin{pmatrix} 0 & A_{12}(\lambda) & A_{12}(\lambda) \\ A_{21}(\lambda) & 0 & A_{23}(\lambda) \\ A_{31}(\lambda) & A_{32}(\lambda) & 0 \end{pmatrix}$$

with

$$A_{ij}(\lambda) = (1 + W_i^{1/2} R_i(\lambda) W_i^{1/2})^{1/2} W_i^{1/2} R_b(\lambda) W_j^{1/2} (1 + W_j^{1/2} R_j(\lambda) W_j^{1/2})^{1/2}, i \neq j.$$

Proof. This Proposition is deduced by the arguments of [29]. Since $P = P_b + I_b$, and $P_b - \lambda$ is positive and invertible for $\lambda < E_0$, one has

$$\lambda \in \sigma_d(P) \Longleftrightarrow 0 \in \sigma_d(1 + R_b(\lambda)^{1/2} I_b R_b(\lambda)^{1/2})$$

and

$$u \in D(P), < (P-\lambda)u, u >< 0 \Longleftrightarrow < (R_b(\lambda)^{1/2}|I_b|R_b(\lambda)^{1/2} - 1)v, v >> 0, v = (P_b - \lambda)^{1/2}u.$$

It follows that $N(\lambda) = n(1, R_b(\lambda)^{1/2}|I_b|R_b(\lambda)^{1/2})$. Decompose

$$R_b(\lambda)^{1/2}|I_b|R_b(\lambda)^{1/2} = \sum_{j=1}^{3} R_b(\lambda)^{1/2} W_j R_b(\lambda)^{1/2} = B^*B,$$

with $B : L^2 \to (L^2)^3$ defined by

$$B = (W_1^{1/2} R_b(\lambda)^{1/2}, W_2^{1/2} R_b(\lambda)^{1/2}, W_2^{1/2} R_b(\lambda)^{1/2}) \equiv (B_1, B_2, B_3).$$

Then B^*B and BB^* have the same non-zero eigenvalues with the same multiplicity, and the same essential spectrum. It follows that for any $\mu > 0$,

$$n(\mu, B^*B) = n(\mu, BB^*). \tag{2.4}$$

Consequently,

$$N(\lambda) = n(1, BB^*). \tag{2.5}$$

BB^* can be decomposed as $BB^* = D + K$, where

$$D = \begin{pmatrix} B_1B_1^* & 0 & 0 \\ 0 & B_2B_2^* & 0 \\ 0 & 0 & B_3B_3^* \end{pmatrix}, \quad K = \begin{pmatrix} 0 & B_1B_2^* & B_1B_3^* \\ B_2B_1^* & 0 & B_2B_3^* \\ B_3B_1^* & B_3B_2^* & 0 \end{pmatrix}.$$

Since $\sigma(P^{a_j}) = [E_0, \infty[$, $1 - W_j^{1/2}R_b(\lambda)W_j^{1/2}$ is invertible and

$$(1 - W_j^{1/2}R_b(\lambda)W_j^{1/2})^{-1} = 1 + W_j^{1/2}R_{a_j}(\lambda)W_j^{1/2} > 0. \tag{2.6}$$

So $1 - D$ is invertible and positive definite. A direct calculus shows that $n(1, BB^*) = n(1, (1-D)^{-1/2}K(1-D)^{-1/2})$ and that $(1-D)^{-1/2}K(1-D)^{-1/2} = A(\lambda)$. ■

Note that in three-body case, the operator $A(\lambda)$ is compact. This is no longer true if $N \geq 4$, since for $N \geq 4$ and $i \neq j$, $W_i^{1/2}W_j^{1/2}$ gives no decay in x^b variables, therefore $W_i^{1/2}R_{a_j}(\lambda)W_j^{1/2}$ is not compact on $L^2(\mathbf{X})$.

3 Asymptotics of the resolvent for the Subhamiltonians near E_0

Definition. ([17]) *We say that E_0 is a* resonance *of P if the equation $Pu = E_0u$ has a solution $u \in H^{1,-s}(\mathbf{X}) \setminus L^2(\mathbf{X})$ for any $s > 1/2$. We call E_0 a regular point of P if E_0 is neither eigenvalue, nor resonance of P; an exceptional point of the first kind (respectively, the second kind; the third kind) of P if it is a resonance but not an eigenvalue (respectively, an eigenvalue but not a resonance; both an eigenvalue and a resonance) of P.*

Note the $b \subset a_j$ and P^b can be regarded as a two-cluster Subhamiltonian of P^{a_j}. The assumption (1.3) implies that P^b is the unique Subhamiltonian (relatively to the many-particle systems described by P^{a_j}) whose spectrum attains the bottom of the essential spectrum of P^{a_j}. Therefore, the condition (1.3) of [35] is satisfied for P^{a_j}. The following results proved in [35] are important in this work.

Theorem 3.1 *Assume (1.2) with $\rho > 2$ and (1.3). Let $\varphi_0 \in D(P^b)$ be the eigenfunction of P^b associated to E_0*

$$P^b\varphi_0 = E_0\varphi_0, \quad \|\varphi_0\| = 1. \tag{3.1}$$

Let $j = 1,2,3$. *If* $u \in H^{1,-s}(\mathbf{X}^{a_j}), \forall s > 1/2$ *and* $P^{a_j}u = E_0 u$, *then,* u *is a resonant function of* P^{a_j} *if and only if*

$$\int_{\mathbf{X}^{a_j}} W_j(x^{a_j})(x)u(x^{a_j})\varphi_0(x^b)\, dx^{a_j} \neq 0. \tag{3.2}$$

The resonance at E_0, *if it does exist, is simple and the resonant state* u *has the following behavior at the infinity*

$$u(x^{a_j}) = O(e^{-c_0|x^b|}|x_b^{a_j}|^{-1}). \tag{3.3}$$

The property (3.3) follows from the proof of Theorem 1.1 and Proposition 3.3 of [35] and the known behavior of three dimensional 0 resonant state ([31]).

Let $\delta > 0$ be small. Denote $U_\delta(E_0) = \{z \in \mathbf{C}; \Im z > 0, |z - E_0| < \delta\}$ and $U_\delta = U_\delta(0)$. Let $R^m(z) = (P^{a_m} - z)^{-1}$ and

$$R_m(z) = (P_{a_m} - z)^{-1} = (P^{a_m} - \Delta_{x_{a_m}} - z)^{-1}.$$

The asymptotics of the resolvent $R^m(z) = (P^{a_m} - z)^{-1}$, $m = 1,2,3$, can be stated as follows.

Theorem 3.2 *Assume (1.2) with* $\rho > 2$ *and (1.3). Let* $\zeta = z - E_0$ *and* $(\zeta)^{1/2}$ *be the branch such that* $\Im(\zeta)^{1/2} > 0$ *for* $\Im\zeta > 0$. *Set* $l = 0$ *(resp.,* $l = 1,2,3$ *) if* E_0 *is a regular point (resp., an exceptional point of the first, the second, the third kind) of* P^{a_m}. *Then for* $k_l \in \mathbf{Z}$, $k_l \geq -2$, *depending on* ρ *and* l *and* $s > 0$ *depending* k_l *and* ρ, *one has the following asymptotic expansion in* $\mathcal{L}(-1,s;1,-s;b,a_m)$,

$$R^m(z) = \sum_{j=-2}^{k_l} \zeta^{j/2} B_{j,m}^{(l)} + O(|\zeta|^{k_l/2+\epsilon}), \quad ,\zeta \to 0. \tag{3.4}$$

for some $\epsilon > 0$ *depending on* s *and* ρ. *Let* $\rho_0 = [(\rho+1)/2]$ *be the integral part of* $(\rho+1)/2$. *One has the following*

For $l = 0$, $k_0 = \rho_0 - 1$ *and*

$$B_{-2,m}^{(0)} = B_{-1,m}^{(0)} = 0. \tag{3.5}$$

When $l = 1$, $k_1 = \rho_0 - 2$,

$$B_{-2,m}^{(1)} = 0, \quad B_{-1,m}^{(1)} = i < \cdot, u_m > u_m, \tag{3.6}$$

and u_m *is an* E_0 *resonant function of* P^{a_m} *normalized by*

$$\frac{1}{2\sqrt{\pi}} \int_{\mathbf{X}^{a_m}} -W_m(x^{a_m})u_m(x^{a_m})\varphi_0(x^b)\, dx^{a_m} = 1. \tag{3.7}$$

For $l = 2,3$, *let* $\rho > 3$. *Then,* $k_l = \rho_0 - 4$ *and*

$$B_{-2,m}^{(2)} = B_{-2,m}^{(3)} = -\Pi_{E_0,m}, \tag{3.8}$$

$\Pi_{E_0,m}$ *being the orthogonal projection onto the eigenspace of* P^{a_m} *associated with the eigenvalue* E_0.

If $\rho > 3$ *and* $l \in \{0,1,2,3\}$, s *and* ϵ *can be any positive constants satisfying* $(2\rho_0 - 1)/2 < s < \rho - (2\rho_0 - 1)/2$ *and* $0 < \epsilon \le \min\{1/2, (2s' - 2\rho_0 + 1)/4\}$, *for any* $s' < s$. *If* $2 < \rho \le 3$ *and* $l \in \{0,1\}$, *one can take* $1 < s < \rho/2$ *and* $0 < \epsilon < (s-1)$.

The information on ϵ when $\rho > 2$ and $l = 1$ can be derived from the Remark after the proof of Proposition 3.7 in [35]. For $N \ge 4$, $W_m^{1/2}$ does not give the weight $< x_b^{a_m} >^{-s}$ *uniformly* in x^b, needed to give the asymptotics of $W_m^{1/2} R_m(\lambda) W_m^{1/2}$. But the detailed analysis given in [35] enables us to overcome this difficulty.

Theorem 3.3 *Assume (1.2) with* $\rho > 2$, *(1.4) and (1.5). Let* $m \in \{1,2,3\}$.

(a). E_0 *is not an eigenvalue of* P^{a_m}.

(b). Assume that E_0 *is a resonance for* P^{a_m}. *Let* u_m *be the unique resonant state normalized by (3.7). Put*

$$\phi_m = W_m^{1/2} u_m, \quad \nu = E_0 - \lambda. \tag{3.9}$$

Let $\chi \in C_0^\infty(\mathbf{R})$ *be supported in small neighborhood of* 0, $0 \le \chi \le 1$ *and equal to* 1 *near* 0. *Then, one has*

$$W_m^{1/2} R_m(\lambda) W_m^{1/2} = \chi(-\Delta_{x_{a_m}})^2 (-\Delta_{x_{a_m}} + \nu)^{-1/2} ((\cdot, \phi_m)_{a_m} \otimes \phi_m) + r_m(\lambda), \tag{3.10}$$

in $\mathcal{L}(L^2(\mathbf{X})$. *The remainder* $r_m(\lambda)$ *is a family of bounded operators continuous in* $\lambda < E_0$ *satisfying that for any* $\epsilon \in]0, 1/2]$ *with* $\epsilon < (\rho-2)/2$, $(-\Delta_{x_{a_m}} + \nu)^{1/2-\epsilon} r_m(\lambda)$ *is a continuous* $\mathcal{L}(L^2)$*-valued function for* $\lambda \le E_0$.

Proof. (a). Assume that $\eta \in L^2(\mathbf{X}^{a_m})$ is a non trivial solution of $(P^{a_m} - E_0)\eta = 0$, or

$$(P^b - \Delta_{x_b^{a_m}} - E_0)\eta = W_m \eta.$$

Under the assumption (1.4), η is the ground state of P^{a_m}, therefore $\eta > 0$. Let $\psi(x_b^{a_m}) = (\varphi_0, \eta)_b$ and $f(x_b^{a_m}) = (W_m \varphi_0, \eta)_b$. Since φ_0 is the ground state of P^b, we have $\psi \in L^2(\mathbf{X}_b^{a_m})$ with $\dim \mathbf{X}_b^{a_m} = 3$, and $\psi(x_b^{a_m}) > 0$. By (1.5), $f(x_b^{a_m}) \ge 0$. We obtain

$$-\Delta_{x_b^{a_m}} \psi = f \ge 0.$$

Theorem A.3.2 of [28] (with $V_+ = 0$) shows that $\psi \notin L^2$. This contradiction proves that E_0 can not be an eigenvalue of P^{a_m}.

(b). Let Π_0 denote the orthogonal projection induced by the eigenfunction φ_0 of P^b. Let $P_m' = (1 - \Pi_0) P^{a_m} (1 - \Pi_0)$. Then, (3.9) can be easily derived from (3.4) or (4.8) of [35] (with P replaced by P^{a_m} and P^a by P^b) according to the cases whether E_0 is eigenvalue of P'. In the case where E_0 is not an eigenvalue of P_m', (3.4) of [35] says that for z near E_0,

$$(P^{a_m} - z)^{-1} = \epsilon(z) - \epsilon_+(z) \epsilon_{+-}(z)^{-1} \epsilon_-(z). \tag{3.11}$$

where

$$\begin{aligned}\epsilon(z) &= R'_m(z) \equiv (P'_m - z)^{-1}(1-\Pi_0),\\ \epsilon_+(z) &= R'_m(z)W_m(\varphi_0\otimes\cdot) + (\varphi_0\otimes\cdot),\\ \epsilon_-(z) &= (\cdot,\varphi_0)_b + (W_m R'_m(z)\cdot,\varphi_0)_b,\\ \epsilon_{+-}(z) &= (z-E_0) - (-\Delta_{x_b^{a_m}} - (W_m\varphi_0,\varphi_0)_b - (W_m R'_m(z)W_m(\varphi_0\otimes\cdot),\varphi_0)_b).\end{aligned}$$

$z \to R'_m(z) \in \mathcal{L}(L^2(\mathbf{X}^{a_m}))$ is holomorphic near $z = E_0$. Therefore,

$$W_m^{1/2}(P^{a_m} - z)^{-1}W_m^{1/2} = -W_m^{1/2}\epsilon_+(z)\epsilon_{+-}(z)^{-1}\epsilon_-(z)W_m^{1/2} + O(1), \tag{3.12}$$

in $\mathcal{L}(L^2)$. Since φ_0 is exponentially decreasing in $|x^b|$, $W_m^{1/2}\epsilon_+(z)$ is continuous from $L^2(\mathbf{X}_b^{a_m}; < x_b^{a_m} >^{-\rho/2} \ dx_b^{a_m})$ to $L^2(\mathbf{X}^{a_m}; dx^{a_m})$. The analysis given in [35] can be carried over to prove that

$$W_m^{1/2}R^m(z)W_m^{1/2} = W_m^{1/2}(z-E_0)^{-1/2}(i(W_m^{1/2}\cdot,u_m)_{a_m}\otimes u_m) + O(|z-E_0|^{-1/2+\epsilon}), \tag{3.13}$$

in $\mathcal{L}(L^2)$ for any $0 < \epsilon < (\rho-2)/2$. See Theorem 3.2 with $k = 1, s = \rho/2 > 1$. Using (4.8) of [35], we can show that (3.13) remains true when $E_0 \in \sigma_d(P')$.

Now to prove (3.10), it suffices to introduce the decomposition

$$1 = \chi(-\Delta_{x_{a_m}})^2 + (1 - \chi(-\Delta_{x_{a_m}})^2)$$

and to note that the term related to $1 - \chi(-\Delta_{x_{a_m}})^2$ is uniformly bounded in $\mathcal{L}^2(\mathbf{X})$ for λ in a small neighborhood of E_0. ■

Remark that if E_0 is not a resonance of P^{a_m}, (a) of Theorem 3.3 implies that it is a regular point and the proof of (b) of Theorem 3.3 shows that $W_m^{1/2}R_m(\lambda)W_m^{1/2}$ is bounded and continuous in $\lambda \le E_0$. From (3.10), we obtain the following

Corollary 3.4 *Assume that E_0 is a resonance of P^{a_m}. Let $\phi_m = W_m^{1/2}u_m$. Then,*

$$\begin{aligned}&(1 + W_m^{1/2}R_m(\lambda)W_m^{1/2})^{1/2}\\ &= \frac{1}{\|\phi_m\|}\chi(-\Delta_{x_{a_m}})(-\Delta_{x_{a_m}}+\nu)^{-1/4}[(\cdot,\phi_m)_{a_m}\otimes\phi_m] + Q_m(\lambda),\end{aligned} \tag{3.14}$$

in $\mathcal{L}(L^2(\mathbf{X}))$ for λ near E_0. Here $Q_m(\lambda)$ is a family of bounded operators continuous in $\lambda < E_0$. Let $\mathcal{F}_{a_m}$ denote the partial Fourier transform in x_{a_m} variables. Then, $Q_m(\xi_{a_m},\lambda) = \mathcal{F}_{a_m}Q_m(\lambda)\mathcal{F}_{a_m}^*$ is a multiplication in ξ_{a_m}. For each $\delta > 0$, $Q_m(\xi_{a_m},\lambda)$ is bounded in $\mathcal{L}(L^2(\mathbf{X}^{a_m}))$ uniformly in $|\xi_{a_m}|^2 + \nu > \delta$ and

$$Q_m(\xi_{a_m},\lambda)(|\xi_{a_m}|^2+\nu)^{-1/4+\epsilon/2} \to 0$$

when $|\xi_{a_m}|^2 + \nu \to 0$, for any $0 < \epsilon < \min\{1/2, (\rho-2)/2\}$.

Proof. Remark that by Theorem 3.1, $\phi_m \in L^2$ and is non zero, and that since $1 + W_m^{1/2}R_m(\lambda)W_m^{1/2}$ is a positive self-adjoint operator for $\lambda < E_0$, its square root exists. Also,

$$((\cdot,\phi_m)_{a_m}\otimes\phi_m)^{1/2} = \frac{1}{\|\phi_m\|}(\cdot,\phi_m)_{a_m}\otimes\phi_m.$$

Then, (3.14) follows from (3.10) and the well-known inequality ([8])

$$\|A^{1/2} - B^{1/2}\| \leq \|A - B\|^{1/2}$$

for two positive operators A, B. ■

4 Lower bound on the number of eigenvalues

For $\lambda < E_0$, let $\nu = E_0 - \lambda$. Set

$$F(\lambda) = \begin{pmatrix} 0 & F_{12}(\lambda) & F_{13}(\lambda) \\ F_{21}(\lambda) & 0 & F_{23}(\lambda) \\ F_{31}(\lambda) & F_{32}(\lambda) & 0 \end{pmatrix} : (L^2(\mathbf{X}))^3 \to (L^2(\mathbf{X}))^3,$$

with

$$F_{ij}(\lambda) = (1+W_i^{1/2}R_i(\lambda)W_i^{1/2})^{1/2}W_i^{1/2}\Pi_0(-\Delta_b+\nu)^{-1}W_j^{1/2}(1+W_j^{1/2}R_j(\lambda)W_j^{1/2})^{1/2}, i \neq j.$$

$F(\lambda)$ is compact.

If E_0 is a resonance of P^{a_j}, we set

$$G_j(\nu) = \frac{1}{\|\phi_j\|}\chi(-\Delta_{x_{a_j}})(-\Delta_{x_{a_j}} + \nu)^{-1/4}((\cdot, \phi_j)_{a_j} \otimes \phi_j). \tag{4.1}$$

If E_0 is not a resonance of P^{a_j}, we just set $G_j(\nu) = 0$ and $Q_j(\lambda) = (1+W_j^{1/2}R_j(\lambda)W_j^{1/2})^{1/2}$. Then , we have in the both cases

$$(1 + W_j^{1/2}R_j(\lambda)W_j^{1/2})^{1/2} = G_j(\nu) + Q_j(E_0 - \nu),$$

where Q_j verifies the remainder estimate of Corollary 3.4. See the remark after the proof of Theorem 3.3. In the following, we shall only work with indices i and j for which both P^{a_i} and P^{a_j} have a resonance at E_0. If E_0 is not a resonance of one of these two Sub-hamiltonians, the leading term we want to study vanishes.

Let $G(\nu)$ and $Q(\nu)$ denote the diagonal matrices with the entries $G_j(\nu)$ and $Q_j(E_0 - \nu)$, $j = 1, 2, 3$, respectively. Let $H(\nu)$ be the matrix with the entries $H_{ij}(\nu)$, $i, j = 1, 2, 3$, where

$$H_{ij}(\nu) = 0, \text{ if } i = j; \quad H_{ij}(\nu) = W_i^{1/2}\Pi_0(-\Delta_b + \nu)^{-1}W_j^{1/2}, \text{ if } i \neq j. \tag{4.2}$$

Write

$$F(\lambda) = G(\nu)H(\nu)G(\nu) + \{G(\nu)H(\nu)Q(\nu) + Q(\nu)H(\nu)G(\nu) + Q(\nu)H(\nu)Q(\nu)\}. \tag{4.3}$$

We shall first establish the asymptotics of $n(1, G(\nu)H(\nu)G(\nu))$ as $\nu \to 0_+$. Then Theorem 1.1 will be deduced by a perturbation argument based on the following lemma.

Lemma 4.1 *Let $T(\nu) = T_0(\nu) + T_1(\nu)$, where $T_0(\cdot)$ $(T_1(\cdot))$ is a self-adjoint compact operator-valued function in $\nu > 0$ $(\nu \geq 0)$. Assume that there is some function $f(\nu)$ with $f(\nu) \to 0$ when $\nu \to 0_+$ such that $\lim_{\nu\to 0_+} f(\nu)n(\mu, T_0(\nu)) = l(\mu)$, with $l(\mu)$ continuous in μ. Then, the same limit exists for $T(\nu)$: $\lim_{\nu\to 0_+} f(\nu)n(\mu, T(\nu)) = l(\mu)$.*

For the proof of Lemma 4.1, see Lemma 4.9 of [29]).

Proposition 4.2 *Assume (1.2) with $\rho > 8/3$. Then, operator $G(\nu)H(\nu)G(\nu)$ (resp., $G(\nu)H(\nu)Q(\nu) + Q(\nu)H(\nu)G(\nu) + Q(\nu)H(\nu)Q(\nu)$) is compact and continuous in $\nu > 0$ (resp., in $\nu \geq 0$).*

Proof. Since $W_i^{1/2}\Pi_0(-\Delta_b + \nu)^{-1}W_j^{1/2}$ is a compact operator on $L^2(\mathbf{X})$, the result for $\nu > 0$ is clear. It suffices to show that as bounded operator-valued function,

$$G(\nu)H(\nu)Q(\nu) + Q(\nu)H(\nu)G(\nu) + Q(\nu)H(\nu)Q(\nu)$$

can be continuously extended to $\nu = 0$. The limit operator is then necessarily compact. Let us only consider the operator $G(\nu)H(\nu)Q(\nu)$. The other two terms can be treated in the same way. The (i,j)-th entry of this operator is $G_i(\nu)H_{ij}(\nu)Q_j(E_0 - \nu)$ which can be written as

$$\Pi_i[W_i^{1/2}\chi(-\Delta_{x_{a_i}})\Xi_{a_i}(\nu)^{1/4}\Pi_0\Xi_b(\nu)W_j^{1/2}\Xi_{a_j}(\nu)^{1/4-\epsilon/2}]\cdot[\Xi_{a_j}(\nu)^{-1/4+\epsilon/2}Q_j(E_0-\nu)],$$

where

$$\Pi_i = \frac{1}{\|\phi_i\|}(\cdot,\phi_i)_{a_i}\otimes\phi_i, \quad \Xi_a(\nu) = (-\Delta_{x_a} + \nu)^{-1}.$$

Here we assume that E_0 is a resonance of P^{a_i}. Otherwise, $G_i(\nu) = 0$. By Corollary 3.4, $\Xi_{a_j}(\nu)^{-1/4+\epsilon/2}Q_j(E_0 - \nu)$ is continuous up to $\nu = 0$, for any $\epsilon \in]0, (\rho - 2)/2[$, if $2 < \rho \leq 3$, and ϵ can be taken as $1/2$ if $\rho > 3$.

It suffices to show that

$$W_i^{1/2}\Xi_{a_j}(\nu)^{1/4-\epsilon/2}\Pi_0\Xi_b(\nu)\Xi_{a_i}(\nu)^{1/4}W_j^{1/2}$$

is bounded and continuous in ν up to $\nu = 0$. To do this, remark that the $W_{a_i}\Pi_0 = O(<x_{a_j}>^{-\rho/2})$ for $i \neq j$ and that since $\dim \mathbf{X}_{a_j} = 3$,

$$\mathbf{R}_+ \ni \nu \to <x_{a_j}>^{-s}(-\Delta_{x_{a_j}} + \nu)^{-s'} \in \mathcal{L}(L^2(\mathbf{X}_{a_j}))$$

is continuous in $\nu \geq 0$ for $s = s' = 0$ and for any $s > 3/2, s' < 3/2$, since in the later case, the operator is of Hilbert-Schmidt class for $\nu \geq 0$. A complex interpolation gives that for $0 \leq s' < 3/2, s' < s$,

$$\mathbf{R}_+ \ni \nu \to <x_{a_j}>^{-s}(-\Delta_{x_{a_j}} + \nu)^{-s'} \in \mathcal{L}(L^2(\mathbf{X}_{a_j})) \tag{4.4}$$

is uniformly bounded in $\nu \geq 0$. Applying (4.4), we obtain

$$\nu \to W_i^{1/2}\Xi_{a_j}(\nu)^{1/4-\epsilon/2}\Pi_0\Xi_b(\nu)\Xi_{a_i}(\nu)^{1/4}W_j^{1/2} \in \mathcal{L}(L^2(\mathbf{X})) \tag{4.5}$$

is uniformly bounded in $\nu \geq 0$, if

$$\rho > 3 - \epsilon.$$

Under the assumption (1.2) with $\rho > 8/3$, we can choose $\epsilon > 0$ close to $(\rho-2)/2$ so that the above inequality is satisfied.

To see that the above operator can be continuously extended to $\nu = 0$, let $\chi_R(\xi_b)$ be the cut-off $\chi(|\xi_b|^2/R)$, $R > 1$. Let $\chi_1 \in C_0^\infty(\mathbf{R})$ with $\chi_1(t) = 1$ on supp χ. Then, applying (4.5), we obtain

$$W_i^{1/2}\Xi_{a_j}(\nu)^{1/4-\epsilon/2}\Pi_0(1-\chi_R(D_b))\Xi_b(\nu)\Xi_{a_i}(\nu)^{1/4}W_j^{1/2} \to 0, \tag{4.6}$$

$R \to \infty$, uniformly in $\nu \geq 0$. Under the assumption $\rho > 8/3$, one can show that for each fixed $R > 1$,

$$< x_{a_j} >^{-\rho/2} \Xi_{a_j}(\nu)^{3/4-\epsilon/4}\chi_1(-\Delta_{x_{a_j}}/R) \in \mathcal{L}(L^2(\mathbf{X}_{a_j}))$$

and

$$\chi_1(-\Delta_{x_{a_i}}/R)\Xi_{a_i}(\nu)^{3/4-\epsilon/4} < x_{a_i} >^{-\rho/2} \in \mathcal{L}(L^2(\mathbf{X}_{a_i}))$$

are Hilbert-Schmidt operators with sqare integrable kernel depending continuous on $\nu \geq 0$. In particular, they are continuous in $\nu \geq 0$ as bounded operator-valued function. It is clear that

$$\nu \to \Xi_{a_j}(\nu)^{-1/2+\epsilon/4}\chi_R(D_b)\Xi_b(\nu)\Xi_{a_i}(\nu)^{-1/2-\epsilon/4} \in \mathcal{L}(L^2(\mathbf{X}_b)$$

is continuous in $\nu \geq 0$. Consequently, the operator-valued function

$$\nu \to < x_{a_i} >^{-\rho/2} \Xi_{a_j}(\nu)^{1/4-\epsilon/2}\chi_R(D_b)\Xi_b(\nu)\Xi_{a_i}(\nu)^{1/4} < x_{a_j} >^{-\rho/2} \in \mathcal{L}(L^2(\mathbf{X}_b)$$

is continuous up to $\nu = 0$. By (4.6), one derives that

$$W_i^{1/2}\Xi_{a_j}(\nu)^{1/4-\epsilon/2}\Pi_0\Xi_b(\nu)\Xi_{a_i}(\nu)^{1/4}W_j^{1/2}$$

can be continuously extended to $\nu = 0$. This proves Proposition 4.2. ■

Let us now recall some results from [29] which are important in our work. Let $\hat{S}(\sigma) : (L^2(\mathbf{S}^2))^3 \to (L^2(\mathbf{S}^2))^3$ be the integral operator with the kernel

$$\hat{S}(\theta,\omega;\sigma) = \begin{pmatrix} 0 & \hat{S}_{12}(t;\sigma) & \hat{S}_{13}(t;\sigma) \\ \hat{S}_{21}(t;\sigma) & 0 & \hat{S}_{23}(t;\sigma) \\ \hat{S}_{31}(t;\sigma) & \hat{S}_{32}(t;\sigma) & 0 \end{pmatrix}, \quad t = \theta\cdot\omega, \quad \theta,\omega \in \mathbf{S}^2, \tag{4.7}$$

with

$$\hat{S}_{ij}(t;\sigma) = (2\pi)^{-1}\delta_i\delta_j u_{ij} e^{i\tau_{ij}\sigma}\frac{\sinh(\sigma \arccos\ s_{ij}t)}{(1-s_{ij}^2t^2)^{1/2}\sinh(\pi\sigma)}, \quad i \neq j, .$$

Here $\delta_i = 1$ if E_0 is a resonance of P^{a_i}; 0, otherwise. The constants $u_{ij}, s_{ij}, \tau_{ij}$ depend only on the ratio of reduced masses in the clusters b_1, b_2, b_3. In particular, $u_{ij} > 0$ and $|s_{ij}| < 1$. See (4.16) and (4.23). For $\mu > 0$, define

$$\mathcal{U}(\mu) = \frac{1}{4\pi}\int_{\mathbf{R}} n(\mu, \hat{S}(\sigma))\, d\sigma. \tag{4.8}$$

This function was studied in detail in [29] and is very important for the proof of the existence of the Efimov effect. In particular, it is proved in [29] that $\mathcal{U}(\cdot)$ is continuous and that if at least two of the three δ_j's are non zero,

$$\mathcal{U}(1) > 0 \tag{4.9}$$

Theorem 4.3 *For $\mu > 0$, one has*

$$\lim_{\nu \to 0_+} |\log \nu|^{-1} n(\mu, G(\nu)H(\nu)G(\nu)) = \mathcal{U}(\mu). \tag{4.10}$$

Proof. We denote

$$\Pi_j = \frac{1}{\|\phi_j\|}(\cdot, \phi_j)_{a_j} \otimes \phi_j, \quad \Gamma_j(\nu) = \chi(-\Delta_{x_{a_j}})(-\Delta_{x_{a_j}} + \nu)^{-1/4}.$$

Then, $\Pi_j^2 = (\cdot, \phi_j)_{a_j} \otimes \phi_j$ and $G_j(\nu) = \Gamma_j(\nu)\Pi_j(\nu)$. Define

$$\mathcal{E}_{-,j} = (\cdot, \phi_j)_{a_j} : L^2(\mathbf{X}) \to L^2(\mathbf{X}_{a_j}), \text{ and}$$

$$\mathcal{E}_{+,j} = \phi_j \otimes \cdot : L^2(\mathbf{X}_{a_j}) \to L^2(\mathbf{X}),$$

so that $\Pi_j^2 = \mathcal{E}_{+,j}\mathcal{E}_{-,j}$. Let $\mathcal{F}_j : L^2(\mathbf{X}_{a_j}) \to L^2(\mathbf{X}_{a_j}^*)$ be the three-dimensional unitary Fourier transform defined by

$$\mathcal{F}_j u(\xi_{a_j}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{X}_{a_j}} e^{-ix_{a_j}\cdot\xi_{a_j}} u(x_{a_j})\, dx_{a_j}.$$

Let $\gamma_j(\nu)$ denote the multiplication by the function $\chi(|\xi_{a_j}|^2)(|\xi_{a_j}|^2+\nu)^{-1/4}$. Let Π (resp., $\Gamma(\nu)$, $\gamma(\nu)$, $\mathcal{E}_\pm$, $\mathcal{F}$) denote the diagonal matrices with the entries Π_j (resp., $\Gamma_j(\nu)$, $\gamma_j(\nu)$, $\mathcal{E}_{\pm,j}$, $\mathcal{F}_j$, $j = 1, 2, 3$). By an elementary argument, one can see that the non-zero eigenvalues of $G(\nu)H(\nu)G(\nu) = \Pi\Gamma(\nu)H(\nu)\Gamma(\nu)\Pi$ are the same as those of $\Gamma(\nu)H(\nu)\Gamma(\nu)\Pi^2$ with the same multiplicities. The later has the same non-zero eigenvalues as those of the operator

$$\mathcal{E}_-\Gamma(\nu)H(\nu)\Gamma(\nu)\mathcal{E}_+ \in \mathcal{L}(L^2(\mathbf{X}_{a_1}) \times L^2(\mathbf{X}_{a_2}) \times L^2(\mathbf{X}_{a_3})).$$

So, we obtain for $\mu > 0$,

$$n(\mu, G(\nu)H(\nu)G(\nu)) = n(\mu, \mathcal{E}_-\Gamma(\nu)H(\nu)\Gamma(\nu)\mathcal{E}_+). \tag{4.11}$$

Applying Fourier transform $\mathcal{F}$, we obtain

$$n(\mu, G(\nu)H(\nu)G(\nu)) = n(\mu, \gamma(\nu)\mathcal{F}\mathcal{E}_-H(\nu)\mathcal{E}_+\mathcal{F}^*\gamma(\nu)). \tag{4.12}$$

Let

$$\psi_j(x^{a_j}) = W_j(x^{a_j})u_j(x^{a_j})\varphi_0(x^b), \quad j = 1, 2, 3. \tag{4.13}$$

By Theorem 3.1 and the assumption (1.2), one has

$$|x^{a_j}|^\delta \psi_j \in L^1(\mathbf{X}^{a_j}), \text{ for any } 0 < \delta < \rho - 2.$$

The normalization of resonant state gives

$$(\psi_j, 1)_{a_j} = \int_{\mathbf{X}^{a_j}} \psi_j(x^{a_j}) dx^{a_j} = -2\sqrt{\pi}. \tag{4.14}$$

For $i \neq j$, the (i,j) entry of $\mathcal{F}\mathcal{E}_-H(\nu)\mathcal{E}_+\mathcal{F}^*$ is

$$\mathcal{F}_i((-\Delta_b + \nu)^{-1}(\mathcal{F}_j^* \cdot, \psi_j)_b, \psi_i)_{a_i} : L^2(\mathbf{X}_{a_j}^*) \to L^2(\mathbf{X}_{a_i *}).$$

We have the decompositions $x^{a_j} = x^b + x_b^{a_j}$, $\xi_b = \xi_{a_j} + \xi_b^{a_j}$ for $x^{a_j} \in \mathbf{X}^{a_j}$, $\xi_b \in \mathbf{X}_b^*$. Let $\mathcal{F}^i$ denote the partial Fourier transform in $x_b^{a_i}$ variables:

$$\mathcal{F}^i u(\xi_b^{a_i}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{X}_b^{a_i}} e^{-i x_b^{a_i} \cdot \xi_b^{a_i}} u(x_b^{a_i}) dx_b^{a_i}.$$

By the Parseval equality in $x_b^{a_j}$ variables,

$$\begin{aligned} \mathcal{F}_i((-\Delta_b + \nu)^{-1}(\mathcal{F}_j^* \cdot, \psi_j)_b, \psi_i)_{a_i} &= ((|\xi_b|^2 + \nu)^{-1}\mathcal{F}_i[\mathcal{F}^i(\mathcal{F}_j^* \cdot, \psi_j)_b], \mathcal{F}^i\psi_i)_{L^2(\mathbf{X}^b \times \mathbf{X}_b^{a_i *})} \\ &= ((|\xi_b|^2 + \nu)^{-1}\hat{\psi}_j, \hat{\psi}_i)_{L^2(\mathbf{X}_b^{a_i *})}. \end{aligned}$$

Here

$$\hat{\psi}_j(\xi_b^{a_j}) = (2\pi)^{-3/2} \int_{\mathbf{X}^{a_j}} e^{-i\xi_b^{a_j} \cdot x_b^{a_j}} \psi_j(x^{a_j})\, dx^{a_j}).$$

For $i \neq j$, the (i,j)-entry of the operator $\gamma(\nu)\mathcal{F}\mathcal{E}_-H(\nu)\mathcal{E}_+\mathcal{F}^*\gamma(\nu)$ is given by

$$\int_{\mathbf{X}_b^{a_i *}} \gamma_i(\xi_i, \nu)(|\xi_b|^2 + \nu)^{-1}\hat{\psi}_j(\xi_b^{a_j})\overline{\hat{\psi}_i(\xi_b^{a_i})}\gamma_j(\xi_j, \nu)u(\xi_j)\, d\xi_b^{a_i *}, \tag{4.15}$$

for $u \in L^2(\mathbf{X}_j^*)$. To determine the kernel of this operator, we reprensent ξ_{a_i}, and ξ_{a_j} each by three coordinates in a orthonormal basis in $\mathbf{X}_{a_i}^*$ and $\mathbf{X}_{a_i}^*$, respectively. We choose (ξ_{a_i}, ξ_{a_j}), $i \neq j$, as coordinates of $\mathbf{X}_b^*$. This is possible because $\mathbf{X}_b^* = \mathbf{X}_{a_i}^* \oplus \mathbf{X}_{a_j}^*$. We can calculate:

$$\begin{aligned} \xi_b^{a_j} &= c_{ij}^j \xi_{a_i} + b_{ij}^j \xi_{a_j} \\ \xi_b^{a_i} &= c_{ij}^i \xi_{a_i} + b_{ij}^i \xi_{a_j} \\ |\xi_b|^2 &= d_i|\xi_{a_i}|^2 + d_{ij}\xi_{a_i} \cdot \xi_{a_j} + d_j|\xi_{a_j}|^2 \end{aligned} \tag{4.16}$$

Here $c_{ij}^i, b_{ij}^i, c_{ij}^j, b_{ij}^j, d_i > 0, d_j > 0$ and d_{ij} are non-zero constants depending only on the reduced masses of the clusters b_1, b_2, b_3. These constants can be explicitly calculated using, for example, the clustered Jacobi coordinates. Since $|\xi_b|^2$ is positive definite on $\mathbf{X}_b^*$, we have necessarily

$$|d_{ij}|^2 < 4d_i d_j. \tag{4.17}$$

Making the change of variables $\xi_b^{a_i} \to \xi_{a_j}$ in the integral in $\xi_b^{a_i}$, we see that the kernel of the (i,j)-entry of the operator $\gamma(\nu)\mathcal{F}\mathcal{E}_-H(\nu)\mathcal{E}_+\mathcal{F}^*\gamma(\nu)$ is

$$k_{ij} = m_{ij}\gamma_i(\xi_{a_i}, \nu)(|\xi_b|^2 + \nu)^{-1}\hat{\psi}_j(\xi_b^{a_j})\overline{\hat{\psi}_i(\xi_b^{a_i})}\gamma_j(\xi_{a_j}, \nu). \tag{4.18}$$

Here m_{ij} is a positive constant depending only on the masses and $\xi_b, \xi_b^{a_i}, \xi_b^{a_j}$ are expressed in terms of (ξ_{a_i}, ξ_{a_j}) by (4.16). Due to the cut-off $\chi(|\xi_{a_l}|^2)$ in $\gamma_j(\xi_{a_l}, \nu)$ for $l = i, j$, the integral operator defined by k_{ij} can be naturally identified with an operator from $L^2(B(1, a_j))$ to $L^2(B(1, a_j))$. Here

$$B(r, a_j) = \{\xi_{a_j} \in \mathbf{X}^*_{a_j}; |\xi_{a_j}| < r\}, r > 0.$$

Note that $\hat{\psi}_j(0) = -2^{-1/2}\pi^{-1}$ and

$$\hat{\psi}_j(\xi_b^{a_j}) = \hat{\psi}_j(0) + \vartheta(\xi_b^{a_j}), \tag{4.19}$$

where $\vartheta(\xi_b^{a_j})$ is a bounded function behaving like $O(|\xi_b^{a_j}|^\delta)$ as $\xi_b^{a_j} \to 0$. We now remove the cut-off χ and replace $\hat{\psi}_j(\xi_b^{a_j})$ by $\hat{\psi}_j(0)$. Let ς be the characteristic function of $]1, \infty[$. Let

$$K^0(\nu) = \begin{pmatrix} 0 & K^0_{12}(\nu) & K^0_{13}(\nu) \\ K^0_{21}(\nu) & 0 & K^0_{23}(\nu) \\ K^0_{31}(\nu) & K^0_{32}(\nu) & 0 \end{pmatrix}$$

be the operator with the entries

$$K^0_{ij}(\nu) : L^2(B(1, a_j)) \to L^2(B(1, a_i))$$

defined by the integral kernel $k^0_{ij}(\nu)$

$$k^0_{ij}(\nu) = \frac{1}{2\pi^2}\delta_i\delta_j m_{ij}\varsigma(\nu^{-1/2}|\xi_{a_i}|)(|\xi_{a_i}|^2 + \nu)^{-1/4}(|\xi_b|^2 + \nu)^{-1}(|\xi_{a_j}|^2 + \nu)^{-1/4}\varsigma(\nu^{-1/2}|\xi_{a_j}|), \tag{4.20}$$

where $\delta_l = 1$, if E_0 is a resonance of P^{a_l}; 0, otherwise. In (4.20), ξ_b is expressed in terms of the variables ξ_{a_i} and ξ_{a_j}, $i \neq j$. As in Proposition 4.2, it can be shown that $k_{ij} - k^0_{ij}(\nu)$ defines a compact operator continuous in $\nu \geq 0$. So it does not contribute to the leading term of the asymptotics.

The rest of the proof is the same as Theorem 4.7 of [29]. By the dilation $y \to \nu^{1/2}y$, one sees that the operator

$$K^0(\nu) \in \mathcal{L}(L^2(B(1, a_1)) \times L^2(B(1, a_2)) \times L^2(B(1, a_3)))$$

is unitarily equivalent with the operator

$$K^0(1) \in \mathcal{L}(L^2(B(\nu^{-1/2}, a_1)) \times L^2(B(\nu^{-1/2}, a_2)) \times L^2(B(\nu^{-1/2}, a_3))).$$

The integral kernel $k^0_{ij}(1)$ of the (i, j)-th entry of $K^0(1)$ is given by (4.20) with $\nu = 1$:

$$k^0_{ij}(1) = \frac{1}{2\pi^2}\delta_i\delta_j m_{ij}\varsigma(|\xi_{a_i}|)(|\xi_{a_i}|^2 + 1)^{-1/4}(|\xi_b|^2 + 1)^{-1}(|\xi_{a_j}|^2 + 1)^{-1/4}\varsigma(|\xi_{a_j}|).$$

Let $T_0 \in \mathcal{L}(L^2(B(\nu^{-1/2}, a_1)) \times L^2(B(\nu^{-1/2}, a_2)) \times L^2(B(\nu^{-1/2}, a_3)))$ be the integral operator whose integral kernel of the (i, j)-th entry is defined by $t_{ij} = 0$ if $i = j$ and

$$t_{ij} = \frac{1}{2\pi^2}\delta_i\delta_j m_{ij}\varsigma(|\xi_{a_i}|)|\xi_{a_i}|^{-1/2}|\xi_b|^{-2}|\xi_{a_j}|^{-1/2}\varsigma(|\xi_{a_j}|),$$

if $i \neq j$. Replacing $K^0(1)$ by T_0 again gives rise to a remainder which corresponds a compact operator continuous up to $\nu \geq 0$. Let $\mathbf{S}_j$ denote the unit sphere in $\mathbf{X}^*_{a_j}$. Making use of the unitary transformation induced by the change of coordinates $\xi_{a_j} = e^{r_j}\omega_j$, $r_j \in \mathbf{R}, \omega_j \in \mathbf{S}_j$, which maps the set $1 < |\xi_{a_j}| < \nu^{-1/2}$ onto the set $]0, |\log \nu|/2[\times \mathbf{S}_j$, one can check that $T_0(\nu)$ is unitarily equivalent with the operator

$$S_R \in \mathcal{L}(L^2(0,R;L^2(\mathbf{S}_1)) \times L^2(0,R;L^2(\mathbf{S}_2)) \times L^2(0,R;L^2(\mathbf{S}_3))) \tag{4.21}$$

where $R = |\log \nu|/2$ and the entries of S_R are given as follows: $S_{kk} = 0$ and for $i \neq j$, S_{ij} is the operator of integral kernel

$$S_{ij}(r_i - r_j; \omega_i \cdot \omega_j) = \frac{\delta_i \delta_j m_{ij}}{2\pi^2 (d_i e^{(r_i - r_j)} + d_j e^{(r_j - r_i)} + d_{ij}\omega_i \cdot \omega_j)}. \tag{4.22}$$

See the equation (4.16). Set

$$u_{ij} = \frac{m_{ij}}{(d_i d_j)^{1/2}}, \quad s_{ij} = \frac{d_{ij}}{2(d_i d_j)^{1/2}}.$$

Then, $|s_{ij}| < 1$ by (4.17) and

$$S_{ij}(r_i - r_j; \omega_i \cdot \omega_j) = \frac{\delta_i \delta_j u_{ij}}{4\pi^2 (\cosh((r_i - r_j) + \tau_{ij}) + s_{ij}\omega_i \cdot \omega_j)}, \tag{4.23}$$

where

$$\tau_{ij} = \frac{1}{2}\log(d_i/d_j).$$

S_R is a Toeplitz operator of the form $\Pi_R S \Pi_R$ where

$$S \in \mathcal{L}(L^2(\mathbf{R}_+;L^2(\mathbf{S}_1)) \times L^2(\mathbf{R}_+;L^2(\mathbf{S}_2)) \times L^2(\mathbf{R}_+;L^2(\mathbf{S}_3)))$$

is a convolution operator in radial variables with the kernel given by (4.23) and Π_R is the multiplication by the characteristic function of $]0, R[$. This operator is already studied in [29]. Applying Theorem 4.5 of [29], we conclude that

$$\lim_{R \to \infty} R^{-1} n(\mu, S_R) = 2\mathcal{U}(\mu), \tag{4.24}$$

where $\mathcal{U}(\mu)$ is defined by (4.8). Note that $\hat{S}_{ij}(t,\sigma)$ is related to $S_{ij}(r,t)$ by the Fourier transform

$$\hat{S}_{ij}(t,\sigma) = \int_{\mathbf{R}} e^{-i\sigma r} S_{ij}(r,t)\, dr. \tag{4.25}$$

Since $R = \frac{1}{2}|\log \nu|$, we obtain

$$\lim_{\nu \to 0_+} |\log \nu|^{-1} n(\mu, G(\nu)H(\nu)G(\nu)) = \mathcal{U}(\mu).$$

Theorem 4.3 is proved. ■

Proof of Theorem 1.1. It follows from Lemma 4.1, Proposition 4.2 and Theorem 4.3 that

$$\lim_{\nu \to 0_+} |\log \nu|^{-1} n(1, F(E_0 - \nu)) = \mathcal{U}(1) \neq 0.$$

Since $\Pi_0' R_b(\lambda) \Pi_0' \geq 0$ for $\lambda < E_0$, one has $A(\lambda) \geq F(\lambda)$. Therefore,

$$n(1, A(\lambda)) \geq n(1, F(\lambda)). \tag{4.26}$$

Theorem 1.1 follows. ■

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