

**REMARKS ON FLOQUET EIGENVALUES
IN THE SEMI-CLASSICAL REGIME**

B. HELFFER

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Remarks on Floquet eigenvalues in the semi-classical regime

Bernard Helffer

Département de Mathématiques, UMR CNRS 8628,
Université Paris-Sud, Bat. 425
91405 Orsay Cedex, France
e-mail: Bernard.Helffer@math.u-psud.fr

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Abstract

These remarks¹ are based on discussions with T. Suslina and R. Shterenberg at the Mittag-Leffler Institute around the properties of the Floquet eigenvalues of a Schrödinger operator with periodic magnetic and electric field. They are motivated by recent questions about the existence of explicit examples. The aim is to show here how semi-classical analysis can produce various examples on the possible behavior of these eigenvalues.

1 Introduction

If we consider a Schrödinger operator with periodic electric potential and periodic electric magnetic potential, the analysis of perturbations of this operator in the gaps is strongly related to the behavior of the Floquet eigenvalues near their extrema. In many papers² (see for example Raikov [Ra], Birman [Bi1, Bi2], Birman-Suslina [BiSu], Suslina [Sus1]) one is assuming

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that some “generic” assumptions are satisfied, that is for example that the extrema are non degenerate. Usually, one knows that they are satisfied for the first band. Note also the interesting result of genericity obtained by F. Klopp and J. Ralston [KR]. R. Shterenberg has shown in [Sht] that new phenomena can occur in the case of a non zero magnetic potential. Our aim is to sketch how semi-classical analysis can also produce asymptotically various examples. Our basic reference is here [Out], strongly based on [HeSj1], where we implement in addition results by A. Martinez [Mar]. In the case of potential magnetics, we will show how to implement ideas of [HeSj2].

2 Semiclassical analysis

Let us consider

$$S_h := -h^2 \Delta + V \tag{2.1}$$

where V is a C^∞ potential on \mathbb{R}^d , which is periodic :

Assumption 2.1.

$$V(x + T_j e_j) = V(x) , \text{ for } j = 1, \dots, d .$$

Here $T_j > 0$ and e_j is a the unit vector in \mathbb{R}^d , whose coordinates are given by $(e_j)_k = \delta_{jk}$, δ_{jk} being the Kronecker symbol.

We assume moreover, that

Assumption 2.2.

V has a unique minimum per cell, and that this minimum is non degenerate.

After renormalization, we can assume that one minimum is at 0 :

$$V(0) = 0 , \nabla V(0) = 0 , \text{ Hess } V(0) > 0 . \tag{2.2}$$

In this context, one knows in full generality that the spectrum is continuous and is a union of bands possibly overlapping. The ℓ – th band is defined as the range of $\lambda_\ell(h, \theta)$ where for a given $\theta \in \mathbb{R}^d$ $\lambda_\ell(h, \theta)$ is the ℓ -th eigenvalue of the Schrödinger operator defined on the torus \mathbb{T}^d by

$$S_{h,\theta} := \sum_j (hD_{x_j} - h\theta_j)^2 + V . \tag{2.3}$$

We observe that $\theta \mapsto \lambda_\ell(h, \theta)$ is periodic with periods $T_j^* = 2\pi/T_j$.

In continuation of [HeSj1] and [Sim] and under natural generic conditions, A. Outassourt has analyzed the semiclassical behavior of $\lambda_\ell(h, \theta)$ for a fixed ℓ (by fixed we mean h -independent). As an application, he recovers the asymptotic of the length of the first band.

Let us briefly describe his result and the techniques which are used.

The first step is a weak localization called “harmonic approximation”. One simply analyze the spectrum of the Harmonic oscillator :

$$H_h = -h^2 \Delta_y + \frac{1}{2} \langle \text{Hess } V(0)y \mid y \rangle . \quad (2.4)$$

This spectrum is explicitly known, once we know the eigenvalues of $\text{Hess } V(0)$. Using the dilation invariance, we know that they have moreover the form

$$\mu_\ell(h) = h\mu_\ell(1) , \ell \in \mathbb{N}^* . \quad (2.5)$$

We know that $\mu_1(1)$ is always simple (as groundstate energy of a Schrödinger operator).

The first elementary result is that :

Theorem 2.1. .

Under assumptions 2.1 and 2.2, we have

$$\lambda_\ell(h, \theta) = h\mu_\ell(1) + \mathcal{O}(h^{\frac{3}{2}}) , \quad (2.6)$$

uniformly with respect to θ .

This is a very rough result. It is difficult to establish the first reference for this in the case of dimension 1 (see for example Harrell [Ha]). In general dimension, one should mention [Out], [Sim] and Carlsson [Car].

The second rough result, but which is already based on the so-called Agmon estimates measuring the decay of the eigenfunctions, is

Theorem 2.2. .

Under assumption 2.2, there exists $\epsilon > 0$ such that

$$\lambda_\ell(h, \theta) - \lambda_\ell(h, 0) = \mathcal{O}(\exp -\frac{\epsilon}{h}) , \quad (2.7)$$

uniformly with respect to θ .

We are now interested in finer results. In order to simplify later discussions we shall always consider a case when

Assumption 2.3.

$\mu_\ell(1)$ is a simple eigenvalue of the harmonic oscillator.

We recall that this is always satisfied for $\ell = 1$ and notice that this assumption can be relaxed considerably (see [HeSj1], [Mar]).

Before to establish the theorem obtained by Outassourt (see also [Sim] or Carlsson [Car]), let us describe the various assumptions appearing in the statement.

The reference problem

It is useful to introduce a one well reference problem. This will be done by filling all the wells except one. More precisely, we consider

$$V^{(ref)}(x) = V(x) + \sum_{\alpha \in \mathbb{Z}^d \setminus 0} \chi(x - T \cdot \alpha), \quad (2.8)$$

where χ is a positive function with support in a sufficiently small neighborhood of 0 and strictly positive at 0. So the potential $V^{(ref)}(x)$ has a unique minimum at 0 and its infimum limit is strictly positive at ∞ . So for fixed $\ell \in \mathbb{N}^*$, the one-well operator $S_h^{ref} := -h^2\Delta + V^{(ref)}$ has, for h small enough, at least ℓ eigenvalues and this leads to

Definition 2.3.

$\lambda_\ell^{ref}(h)$ is the ℓ -th eigenvalue of the reference one-well operator

$$S_h^{ref} := -h^2\Delta + V^{(ref)}.$$

Agmon distance and decay

The following proposition describes the decay properties of the eigenstates for the reference problem.

Proposition 2.4.

Under assumptions 2.1, 2.2 and 2.3, the corresponding normalized eigenfunction $\phi_{\ell,h}^{ref}$ decays exponentially, that is there exist ν and, for any compact K , constants C_K and $h_0(K)$ such that, for $h \in]0, h_0(K)[$, we have :

$$\| \exp \frac{d^{ref}(x)}{h} \phi_{\ell,h}^{ref} \|_{H^1(K)} \leq C_K h^\nu. \quad (2.9)$$

We can in addition give a weaker control at ∞ of the decay (still exponential) permitting to prove that the contribution at ∞ is irrelevant. Here $d^{ref}(x)$ is the Agmon distance to 0 associated with the metric $V^{ref}(x)dx^2$. We will later note by $d(x, y)$ the Agmon distance between x and y associated with the metric $V(x)dx^2$. Note that $d^{ref}(x) = d(x, 0)$ in a large neighborhood of 0 (where $V = V^{ref}$). We can now define

$$S_j = d(0, T_j e_j) . \quad (2.10)$$

We observe that

$$S_j = d(0, -T_j e_j)$$

and we make the following important assumption that :

Assumption 2.4. .

The nearest neighbours for the Agmon distance are the nearest neighbours in the lattice \mathbb{Z}^d .

$$d(0, \sum \alpha_j T_j e_j) > \sup_j S_j , \forall \alpha \text{ s. t. } \sum |\alpha_j| > 1 . \quad (2.11)$$

Remark 2.5. .

Later on, we will be interested in other interactions where other neighbours play a role. They corresponds to α 's such that

$$d(0, \sum \alpha_j T_j e_j) < 2 \inf_{\beta \neq 0} d(0, \sum \beta_j T_j e_j) . \quad (2.12)$$

This point is immediately related to Formula (3.21) in [Out].

Accurate measure of the tunneling effect .

We now add the new assumptions that

Assumption 2.5. .

Between 0 and $T_j e_j$ there is an unique (or a finite union of) minimal geodesics, and this geodesics is non degenerate (in a natural geometric and generic sense).

The precise definition is given in [HeSj1]. Roughly speaking, the condition is as follows. One considers one minimal geodesics γ between two wells U_1 and U_2 , take a point M_{12} between U_1 and U_2 on this geodesics and an hypersurface S transversal to γ at M_{12} . Then the assumption is that the function defined

locally on S around M_{12} by : $x \mapsto d(x, U_1) + d(x, U_2) - d(U_1, U_2)$, which is positive and C^∞ , has a non-degenerate minimum at M_{12} .

We emphasize that this is important for obtaining equivalents and that it can be relaxed for upperbounds. Then $a_{j,\ell}$ is given by application of the Laplace method to a Laplace integral with a non degenerate real phase. Sometimes it is a sum of such contributions when there are more than one minimal geodesics. One gets the following behavior :

$$a_{j,\ell} \sim h^{\nu_{j,\ell}} \sum_k a_{j,\ell,k} h^k, \quad (2.13)$$

or a sum of such terms, in the case of more than one minimal geodesics. Moreover, in the case when $\ell = 1$, one has

$$\nu_{j,1} = \frac{1}{2}, \quad a_{j,1} < 0. \quad (2.14)$$

In particular, no cancellation can occur in this case when summing the contributions of the various geodesics.

In this context, Outassourt obtains (Theorem 4.3, p. 79-80), under suitably assumptions, an expansion in the form :

Theorem 2.6. .

Under assumptions 2.1, 2.2, 2.3, 2.4 and 2.5, there exists a constant C_ℓ , such that :

$$\lambda_\ell(h, \theta) = \lambda_\ell^{ref}(h) + \sum_{j=1}^d \exp -(S_j/h) a_{j,\ell}(h) \cos T_j \theta_j + r_\ell(h, \theta), \quad (2.15)$$

where

$$|r_\ell(h, \theta)| + |\nabla_\theta r_\ell(h, \theta)| + |\nabla_\theta^2 r_\ell(h, \theta)| \leq C_\ell(\exp -S'/h), \quad (2.16)$$

$\lambda_\ell^{ref}(h)$ is introduced in Definition 2.3, S' is any constant such that

$$S' < \inf_{\sum |\alpha_j| > 1} d(0, \sum \alpha_j T_j e_j), \quad (2.17)$$

and $a_{j,\ell}$ satisfies (2.13)

Remark 2.7. .

Of course, the best result is obtained by taking S' close to $\inf_{\sum |\alpha_j| > 1} d(0, \sum \alpha_j T_j e_j)$. A. Outassourt gets actually instead of (2.16) only the control of $|r_{\ell,h}(\theta)|$. The proof that one can control any derivative is given in [HeSj4] (the control in θ is obtained in a complex neighborhood of the real space). One can also use for this the paper by Carlsson [Car] which permits to analyze the Fourier coefficients of the Floquet eigenvalue.

Observing that (2.14) is satisfied, this gives a very satisfactory answer for the case $\ell = 1$. For $\ell > 1$, the situation is more delicate and not solved in [Out] (outside the case when $d = 1$, which was analyzed earlier by other techniques). It is not excluded that some cancellations could occur. This situation was analyzed by A. Martinez [Mar] in its analysis of the double well problem for excited states. But the terms appearing in its analysis are the same as the terms appearing in the formula of Outassourt.

Remark 2.8. .

Note only a small difference. In the case of the double well, we consider as approximate basis : ϕ_ℓ^{ref} and the function obtained by symmetry with respect to $x_j = \frac{1}{2}T_j$. In the case of Outassourt, one takes the function obtained by a translation by T_j . It is clear that for $\ell = 1$, we get the same function. In general, we have only the same function up to a constant of module 1. This does not change the analysis.

So we can use his result, which is done under the additional assumption that

Assumption 2.6.

$$V(\epsilon_1 x_1, \dots, \epsilon_d x_d) = V(x_1, \dots, x_d), \quad \forall \epsilon \in \{-1, +1\}^d. \quad (2.18)$$

In this case, A. Martinez [Mar] shows

Proposition 2.9. .

Under assumptions 2.1-2.6, there exists reals $\nu'_{j,\ell}$ and $\nu''_{j,\ell}$, and constants C and h_0 , such that :

$$\frac{1}{C} h^{\nu'_{j,\ell}} \leq |a_{j,\ell}| \leq C h^{\nu''_{j,\ell}}, \quad (2.19)$$

for $h \in]0, h_0]$.

3 Some examples of non-degenerate minima for excited states.

We are now in position for treating the question (appearing for example in [Sus1]) of giving simple non trivial explicit examples where a generic of non degeneracy is satisfied. The question is to produce examples of potentials V where $\theta \mapsto \lambda_\ell(h, \theta)$ has a unique non-degenerate minimum (or maximum) when considered as a function on the torus. Of course we recall that such property is known in rather great generality for $\ell = 1$. One possible reference is [KiSi].

Let us treat for simplification the case $d = 2$. Our result is the following :

Theorem 3.1. .

Under the assumptions 2.1-2.6, we can show that there exists $\eta > 0$ and $h_0 > 0$ such that, if

$$|S_1 - S_2| < \eta , \tag{3.1}$$

then, for $h \in]0, h_0]$, the map $\theta \mapsto \lambda_\ell(h, \theta)$ has a unique non-degenerate minimum (or maximum), when considered as a function on the torus.

Sketch of the proof.

The starting point is the formula (2.15) (with (2.16)) with in addition the property (2.19). Without loss of generality, we can assume that :

$$S_1 \geq S_2 . \tag{3.2}$$

After renormalization, translation, we have to analyze if this property is true for the function:

$$(\theta_1, \theta_2) \mapsto \Psi(\theta, h) := (1 - \cos \theta_1) + \alpha(h)(1 - \cos \theta_2) + \varphi(\theta, h) .$$

Here $h \mapsto \alpha(h)$ is given by

$$\alpha(h) = \frac{a_{2,\ell}(h)}{a_{1,\ell}(h)} \exp - \frac{(S_1 - S_2)}{h} ,$$

and satisfies, for h small enough

$$\exp - \frac{2\eta}{h} \leq \alpha(h) \leq 1 .$$

The remainder satisfies, according to (2.16), for some constant \hat{C}

$$|\varphi(\theta, h)| + |\nabla_{\theta}\varphi(\theta, h)| + |\nabla_{\theta}^2\varphi(\theta, h)| \leq \hat{C} \exp -(S' - S_1)/h .$$

The minimum of the function Ψ is below $\hat{C} \exp -(S' - S_1)/h$ (just compute $\Psi(0, h)$) and above $-\hat{C} \exp -(S' - S_1)/h$. So the points of Ψ where Ψ should be exponentially small are near 0 (modulo the translations of the lattice). For $\eta > 0$ small enough, one can apply the implicit function lemma (after a change of variables $\tilde{\theta} = (\theta_1, \sqrt{\alpha(h)}\theta_2)$).

4 The case with periodic magnetic potential

We now consider the case when some magnetic field is included. Our starting point is the family of Schrödinger operators with magnetic fields

$$S_h := \sum_j (hD_{x_j} - tA_j(x))^2 + V(x) , \quad (4.1)$$

where t is a small parameter.

We keep more or less the previous assumptions but have added a periodic potential A and we will concentrate our study on the case of a square lattice in \mathbb{R}^2 . So we assume that :

Assumption 4.1.

$$A(x_1 + 1, x_2) = A(x_1, x_2) , \quad A(x_1, x_2 + 1) = A(x_1, x_2) .$$

We note that the flux over a fundamental cell is 0 :

$$\int_{[0,1] \times [0,1]} B(x) = 0 , \quad (4.2)$$

and that consequently one can use the Floquet theory for analyzing the spectrum and deduce the existence of bands. The harmonic approximation principle is unchanged. If $t = o(1)$ (as $h \rightarrow 0$), the reference Harmonic oscillator is unchanged. If $|t| \leq C\sqrt{h \ln \frac{1}{h}}$, all the techniques used in the case $t = 0$ remain valid as explained in [HeSj2] and [HeSj4].

From now on, we assume that

Assumption 4.2.

$$t = \delta h , \quad \delta \in [-\delta_0, \delta_0] .$$

So all our remarks about the conditions on t for applying [HeSj2] and [HeSj4] are satisfied and there is no need to introduce assumption of analyticity for V .

We will concentrate our analysis near the first level of the harmonic oscillator $h\mu_1$.

We assume that we have the following invariance assumption :

Assumption 4.3.

$$\begin{aligned} V(-x_1, -x_2) &= V(x_1, x_2) , \\ A_1(x_1, x_2) &= -A_1(-x_1, -x_2) , \quad A_2(x_1, x_2) = -A_2(-x_1, -x_2) . \end{aligned} \quad (4.3)$$

We assume that :

Assumption 4.4.

Between $(0, 0)$ and $(1, 0)$ we have two non degenerate minimal geodesics which are consequently symmetric by the map $(x_1, x_2) \mapsto (1 - x_1, -x_2)$.

We assume also that :

Assumption 4.5.

Between $(0, 0)$ and $(0, 1)$ we have a unique minimal geodesics, which is consequently the straight line joining $(0, 0)$ and $(0, 1)$.

Of course these properties are transmitted by symmetry and lattice invariance. The theory developed in [HeSj2], in combination of the results of Outassourt [Out] (see also [He1] and [HeSj4] for the analysis of various Flux effects), gives in this case (to be verified for the details)

Theorem 4.1.

Under assumptions 2.2, 2.4 and 4.3-4.5, the following formula for $\lambda_{1,\delta}(h, \theta)$ holds:

$$\begin{aligned} \lambda_{1,\delta}(h, \theta) &= \\ &\exp -S_1/h \, a_1(\delta, h) \cos(\delta\Phi_\omega + \phi(\delta, h)) \cos \theta_1 \\ &+ \exp -S_2/h \, a_2(\delta, h) \cos \theta_2 \\ &+ r(\delta, \theta, h) . \end{aligned} \quad (4.4)$$

where

$$r(\delta, \theta, h) = \mathcal{O}(\exp -S'/h) \quad (4.5)$$

uniformly with respect to δ and θ , and

$$\phi(\delta, h) = \mathcal{O}(h^2) \quad (4.6)$$

uniformly with respect to δ .

Here the functions $a_1(\delta, h)$ are symbols in h (see (2.13)), with C^∞ coefficients in δ and Φ_ω is the flux of the magnetic field through the surface ω delimited by the two minimal geodesics $\gamma'((0, 0), (1, 0))$ and $\gamma''((0, 0), (1, 0))$ introduced in Assumption 4.4. Note that by symmetry around $(\frac{1}{2}, 0)$ the geodesic $\gamma'((0, 0), (1, 0))$ is sent on $-\gamma''((0, 0), (1, 0))$. We note also that, although the total flux through the cell is zero, the flux Φ_ω can be non-zero. We consequently add the assumption

Assumption 4.6. .

$$\Phi_\omega \neq 0 .$$

By adding a periodic magnetic potential field (with the same symmetries), such that the support of the corresponding magnetic field avoids the minimal geodesics, one can treat the flux Φ_ω as an essentially independent parameter (modulo a very small remainder). The other way which will be enough in our discussion will be to play with the constant δ .

In order to continue in the analysis of the phenomenon, let us assume that $S_1 = S_2$ and that $a_1(\delta, h) \sim a_2(\delta, h)$. This assumption can always be obtained by starting of a potential V (invariant by rotation by $\frac{\pi}{2}$) and by perturbing the potential slightly outside of the wells, in order to create the geometric situation introduced in Assumption 4.4.

From now on, we forget the remainders and **work heuristically**. Probably we shall need a control of some derivatives of the remainder with respect to δ in order to make the arguments rigorous.

After scaling and renormalization, and similarly to the previous section, we have to discuss in function of a parameter $\alpha \in [-1, +1]$, the minima of the function

$$\theta \mapsto \alpha \cos \theta_1 + \cos \theta_2 ,$$

with $\alpha = \cos(\delta\Phi_\omega)$.

When $\alpha > 0$, the minimum is $-1 - \alpha$ and obtained at (π, π) . When $\alpha < 0$, the minimum is $-1 + \alpha$ and obtained at $(0, \pi)$. When $\alpha = 0$, the minimum is degenerate and given by $\theta_2 = \pi$. This is not the picture observed by Shterenberg [Sht].

This suggests to find other examples obtained by adding an interaction between $(0, 0)$ and $(1, 1)$, that is by assuming

Assumption 4.7. .

The points $\beta \in \mathbb{Z}^2$, corresponding to $|\beta_\infty| = 1$ are at an Agmon distance of $(0, 0)$ which is strictly smaller $|\beta_\infty| > 1$.

In this case, one will have to analyze a more sophisticated family of the type

$$\theta \mapsto \alpha \cos \theta_1 + \cos \theta_2 + \epsilon \cos(\theta_1 + \theta_2) .$$

where $\epsilon > 0$ and very small.

When $\alpha = 1$, there is a unique minimum at (π, π) . When $\alpha = -1$ there is a unique minimum at $(0, \pi)$. When $\alpha = \epsilon$, we can see that there are two non degenerate minima near $(\pi/2, \pi)$ and $(-\pi/2, \pi)$.

Remark 4.2. .

Another example, in continuation of the result of the previous subsection, can be probably obtained without introduction of a magnetic field but by playing with the geometry. If instead of $(0, \pm 1)$ and $(\pm 1, 0)$, we have as nearest neighbours of $(0, 0)$ the points $(1, 1)$ and $(-1, -1)$, we will probably get, as rescaled model, :

$$-\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2) = -2 \cos \theta_1 \cos \theta_2 .$$

This function has a unique minimum but has two maxima per cell.

Remark 4.3. .

Let us mention that one can also in the context of the tight binding approximation have rather precise informations on the asymptotic structure of the Floquet eigenvalues. We refer to [Da1, Da2] and references therein for a mathematical analysis justifying rather standard results of the Physics literature.

5 Around other examples of R. Shterenberg: magnetic periodic wells

This section is just a transcription and extension of a discussion with R. Shterenberg. The results are semi-classical but of a completely different type. We are interested in the case of a periodic magnetic potential, whose corresponding periodic magnetic field is not constant. The bottom of the spectrum is then related to the points where the magnetic field vanishes. Here some of

the technics developed in [Mon] and [HeMo1] are relevant. This is rather clear in the analysis of the following simple model in \mathbb{R}^2 :

$$h^2 D_{x_1}^2 + (hD_{x_2} - A_2(x_1))^2, \quad (5.1)$$

where $A_2(x_1)$ is a 2π -periodic. The analysis of the Floquet problem (ground state energy) is reduced to the following family of one dimensional problems:

$$(hD_{x_1} - \kappa_1)^2 + (\kappa_2 - A_2(x_1))^2,$$

on the circle. For analyzing the bottom, one can start by analyzing

$$(hD_{x_1})^2 + (\kappa_2 - A_2(x_1))^2,$$

on a circle. For each κ_2 , the bottom of the spectrum is related to the analysis of the set $\{A_2(x_1) = \kappa_2\}$. So it is clear that for analyzing the bottom of the spectrum, one should first consider

$$\kappa_2 \in A_2(\mathbb{R}) .$$

We try to consider a rather generic situation, with one minimum per period say at 0 and one maximum per period and we assume that A_2' does not vanish outside these points. If $\kappa_2 \neq \inf A_2, \max A_2$, then we have two non degenerate minima in $[-\pi, +\pi[$ and the ground state is of order $\alpha(\kappa_2)h$ with $\alpha_2(\kappa_2) > 0$ as $h \rightarrow 0$.

So one has the feeling that the bottom of the spectrum will be lower for the κ_2 corresponding to the extremas of A_2 . Changing κ_2 into $-\kappa_2$ exchanges the problem near the minimum and the problem near the maximum. Let us treat the problem near the minimum. Let us assume that this minimum corresponds to $A_2(0) = 0$.

It is natural for understanding the semiclassical analysis to replace the problem by the following polynomial approximation:

$$(hD_{x_1})^2 + (\kappa_2 - x_1^2)^2, \quad (5.2)$$

on the line, and to analyze the bottom of the spectrum of this model operator. This problem was already analyzed by Montgomery [Mon], then by Helffer-Mohamed [HeMo1], Helffer-Morame [HeMo2, HeMo3, HeMo4] and Pan-Kwek [PaKw]. We use their results rather freely.

This problem can be treated by homogeneity. We write $x_1 = \rho t$ and get

$$(h\rho^{-1}D_t)^2 + (\kappa_2 - \rho^2 t^2)^2.$$

The natural choice is to choose :

$$\rho = h^{\frac{1}{3}}.$$

We get the following isospectral problem on the line :

$$h^{\frac{4}{3}}(D_t^2 + (t^2 - \beta)^2),$$

with $\beta = h^{-\frac{2}{3}}\kappa_2$.

The question becomes: for which β is the bottom of the spectrum of the operator $D_t^2 + (t^2 - \beta)^2$ minimal.

It is shown in [PaKw] (using also arguments of [Mon], [HeMo1] ...) that there is a unique minimum $\beta_0 > 0$ such that the bottom of the spectrum $\mu(\beta)$ of $(D_t^2 + (t^2 - \beta)^2)$ is minimal with respect to β . So we get that the minimum is obtained for $\kappa_2 = h^{\frac{2}{3}}\beta_0$ for the simplified model and that the bottom of the spectrum is asymptotically $\mu(\beta_0)h^{\frac{4}{3}}$. The error should be of order $\mathcal{O}(h^{\frac{5}{3}})$.

In order to improve the expansion one should have to show (open problem!!) that the minimum of μ is non degenerate. In any case, by analyticity, we know that there exists $\ell \geq 2$ such that :

$$\mu^{(\ell)}(\beta_0) > 0, \mu^{(j)}(\beta_0) = 0, \text{ for } j < \ell. \quad (5.3)$$

The structure for κ_2 very small (more precisely $|h^{-\frac{2}{3}}\kappa_2 - \beta_0|$ small) of the Floquet eigenvalue should be

$$h^{\frac{4}{3}} \left(\mu(\beta_0) + \frac{1}{2}\mu^{(\ell)}(\beta_0)(h^{-\frac{2}{3}}\kappa_2 - \beta_0)^\ell + \mathcal{O} \left(h^{-\frac{2}{3}}\kappa_2 - \beta_0 \right)^{\ell+1} + \mathcal{O}(h^{\frac{1}{3}}) \right). \quad (5.4)$$

Remark 5.1. .

May be the first problem to look at semiclassically is the case when the magnetic field $B(x)$ vanishes exactly at order one on a union of lines invariant by the translations of the lattice. This is indeed the generic problem. If B is periodic, with zero flux on a fundamental cell and non constant, it should have a zero set. The first case is then to analyze is the case when $\{B = 0\}$ is a regular submanifold in \mathbb{R}^2 . One can distinguish

- *the case where each connected component of the zero set of B is bounded (here the techniques developed for the proof of Montgomery- Helffer-Mohamed [Mon, HeMo1] together with Outassourt [Out] will probably apply: rough localization of the first band, upper bound on the size of the band)*
and
- *the case of unbounded components, probably more difficult in general, which contains Shterenberg example as first example.*

Of course, because one can reduce the problem to the torus, the question would be to know if the closed lines of zeros of the magnetic field on the torus are homotopic to a point or of non zero index.

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