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A CLASS OF COUNTEREXAMPLES TO THE FEFFERMAN-PHONG INEQUALITY FOR SYSTEMS

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Abstract. We give here a class of counterexamples to the Fefferman-Phong inequality for systems of pseudodifferential operators, which contains Brummelhuis's one as a particular case. The main ingredient in the proof is the use of "localized operators" associated with the system, and Hörmander's example of a positive-semidefinite matrix whose Weyl quantization is not non-negative. For the considered class the Sharp Gårding inequality cannot be improved.

1. Introduction

The Fefferman-Phong inequality ([5]) states that if $a = a(x,\xi) \geq 0$ is a **scalar** symbol in the class $S^2(\mathbb{R}^n \times \mathbb{R}^n)$, then the relative pseudodifferential operator (ψdo in the sequel) $a(x, D_x)$ is bounded from below in L^2, that is, *there exists a constant* $C > 0$ *such that*

$$(FP) \qquad \mathrm{Re}\,(a(x,D_x)u,u) \geq -C\,\|u\|_0^2, \quad \forall u \in C_0^\infty(\mathbb{R}^n).$$

On the other hand, when $A = A^* = a(x, D_x)$ is a properly supported *classical* ψdo of order m, that is, its (total) symbol $a(x,\xi)$ has an expansion $a(x,\xi) \sim a_m(x,\xi) + a_{m-1}(x,\xi) + \ldots$, where a_{m-j} is positively homogeneous of degree $m-j$ in the fiber variable ξ, one looks for conditions on the *principal symbol* a_m and on the lower order term $a^s_{m-1}(x,\xi) := a_{m-1}(x,\xi) + \frac{i}{2}\sum_{j=1}^n \partial^2_{x_j\xi_j} a_m(x,\xi)$, the *sub-principal symbol* of A, that ensure lower bound estimates for the L^2-quadratic form (Au,u), $u \in C_0^\infty$, associated with A. The reason why one looks at a_m and a^s_{m-1} is that a_m is **invariantly defined** on $T^*\mathbb{R}^n \setminus 0$, and $a^s_{m-1}(x,\xi)$ is **invariantly defined** where a_m vanishes to second order (which is always the case when $a_m \geq 0$).

When $a_m \geq 0$ on $T^*\mathbb{R}^n \setminus 0$, one has the celebrated Sharp Gårding inequality (proved by L.Hörmander) which states: *For every compact* $K \subset \mathbb{R}^n$ *there exists* $C_K > 0$ *such that*

$$(SG) \qquad (Au,u) \geq -C_K\,\|u\|^2_{(m-1)/2}, \quad \forall u \in C_0^\infty(K).$$

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The next step was obtained by A.Melin (see [11]), who proved the following equivalence:

For any given $\varepsilon > 0$ *and any given compact* $K \subset \mathbb{R}^n$ *there exists* $C_{\varepsilon,K} > 0$ *such that*

$$(M) \qquad (Au, u) \geq -\varepsilon \|u\|^2_{(m-1)/2} - C_{\varepsilon,K} \|u\|^2_{(m-2)/2}, \ \forall u \in C_0^\infty(K)$$

(the so-called *Melin inequality*);

$$(1) \qquad \begin{cases} a_m(x,\xi) \geq 0, \ \forall (x,\xi) \in T^*\mathbb{R}^n \setminus 0, \\ \\ a_m(x,\xi) = 0 \implies a^s_{m-1}(x,\xi) + \mathrm{Tr}^+(F(x,\xi)) \geq 0, \end{cases}$$

where $\mathrm{Tr}^+(F(x,\xi))$ is the *positive trace of the fundamental matrix* $F(x,\xi)$ defined by the identity

$$\langle Q_{(x,\xi)} v, w \rangle_{T_{(x,\xi)}T^*\mathbb{R}^n} = \sigma(v, F(x,\xi)w), \ \ v, w \in T_{(x,\xi)}T^*\mathbb{R}^n,$$

$Q_{(x,\xi)}$ being the Hessian of $a_m/2$ at (x,ξ) and with $\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j$ the canonical symplectic form on $T^*\mathbb{R}^n$. Explicitly, $\mathrm{Tr}^+(F(x,\xi)) = \sum_{\mu>0} \mu$ with $i\mu \in \mathrm{Spec}(F(x,\xi))$.
It is important to remark here that the Hessian $Q_{(x,\xi)}$ and the positive trace $\mathrm{Tr}^+(F(x,\xi))$ are **invariantly defined** on the set where a_m vanishes to **second order**. Observe also that in conditions (1) above, no assumption on the geometry of the *characteristic set*

$$\Sigma := \{(x,\xi) \in T^*\mathbb{R}^n \setminus 0; \ a_m(x,\xi) = 0\}$$

is made. In fact, supposing that:

(i) Σ *is a smooth sub-manifold of* $T^*\mathbb{R}^n \setminus 0$,

(ii) σ *has constant rank on the connected components of* Σ, *i.e.* $\Sigma \ni \rho \mapsto \dim\big(T_\rho\Sigma \cap T_\rho\Sigma^\sigma\big)$ *is locally constant, where* $T_\rho\Sigma^\sigma$ *is the symplectic orthogonal of* $T_\rho\Sigma$,

(iii) $a_m(x,\xi)$ *vanishes* **exactly** *to second order on* Σ,

Hörmander proved in [7] the following result: Conditions (1) above is equivalent to

For any given compact $K \subset \mathbb{R}^n$ *there exists* $C_K > 0$ *such that*

$$(H) \qquad (Au, u) \geq -C_K \|u\|^2_{(m-2)/2}, \ \forall u \in C_0^\infty(K).$$

Condition *(iii)* above can also be rewritten as $a_m(x,\xi) \approx |\xi|^m \mathrm{dist}_\Sigma(x, \xi/|\xi|)^2$, where $\mathrm{dist}_\Sigma(x,\xi/|\xi|)$ denotes the distance of $(x,\xi/|\xi|)$ to Σ.

It is important to remark the following fact, that we state already for matrix-valued ψdo's.

Remark 1.1. *If* $A(x,D_x) \in \mathrm{OPS}^m(\mathbb{R}^n;\mathbb{C}^N)$ *(the set of classical, properly supported,* $N \times N$ *matrices of classical* ψdo*'s of order* m*) has symbol* $A(x,\xi) \sim A_m(x,\xi) + A_{m-1}(x,\xi) + \dots$, *where* A_{m-j} *is positively homogeneous of degree* $m-j$ *in* ξ, *and* $A(x,\xi) = A(x,\xi)^*$ *(that is,* $A(x,\xi)$ *is a* **Hermitian matrix of classical symbols***), then* $(A(x,D_x) + A(x,D_x)^*)/2 =:$ $\mathrm{Re}\,(A)(x,D_x) = A^{\mathrm{w}}(x,D_x)$ *(the Weyl-quantization of* $A(x,\xi)$*), with the total* *symbol* $\tilde{A}(x,\xi)$ *of* $\mathrm{Re}\,(A)(x,D_x)$ *computed by the formula (see* [10]*, Vol.III)* $\tilde{A}(x,\xi) = e^{i\langle D_x, D_\xi\rangle/2}A(x,\xi)$*. Hence, at the level of the* L^2*-quadratic forms, one always has* $\mathrm{Re}\,(A(x,D_x)u,u) = (A^{\mathrm{w}}(x,D_x)u,u)$*. We shall hence work with the Weyl-quantization of a classical symbol. In particular, for the subprincipal symbol* $\tilde{A}^s_1$ *of* $\mathrm{Re}\,(A)(x,D_x)$*, one has* $\tilde{A}^s_1(x,\xi) = A_1(x,\xi)$*.*

As regards the vector-valued case, the Fefferman-Phong inequality (FP) is in general false. In fact, in [2] R.Brummelhuis exhibited an example (homogeneous of degree 2) for which (FP) fails. There was already a clue that problems might arise in vector-valued situations. L.Hörmander, in [8], showed that the Weyl-quantization of positive-semidefinite matrices may be **not non-negative**. The purpose of this paper is to introduce (building upon the examples of Hörmander and Brummelhuis) a class of counterexamples to the Fefferman-Phong inequality (FP) that is geometrically characterized (see Section 5 below). Indeed, for such a class, only the Sharp Gårding inequality holds (and no improvement of the kind (WH) below, as suggested by the result of [12], is possible).

We recall that conditions for Melin's inequality (M) to hold in the case of systems were obtained by Brummelhuis [3], Brummelhuis and Nourrigat [4], and by Parenti and Parmeggiani [12] (under conditions on the symplectic type of the characteristic manifold Σ).

As regards Hörmander's inequality (H) for $N \geq 2$, conditions in the case of systems with double characteristics (that is, $\Sigma = \{(x,\xi) \in T^*\mathbb{R}^n \setminus 0;\ \det(a_m(x,\xi)) = 0\}$ is a manifold on which the symplectic form has constant rank, $\det(a_m(x,\xi)) \approx |\xi|^{Nm}\mathrm{dist}_\Sigma(x,\xi/|\xi|)^{2\ell}$, and $\mathrm{dimKer}\; a_m(\rho) = \ell$, for all $\rho \in \Sigma$) were given by Hörmander [7] (case of $\ell = 1$), and by Parenti and Parmeggiani [12] (any ℓ with $1 \leq \ell \leq N$ and Σ symplectic). It is worth noting that, as shown in [12], another inequality that can be obtained by relaxing a bit the hypotheses is the following (Weak-Hörmander): *For any given compact* $K \subset X$ *there exists a constant* $C_K > 0$ *such that*

$$(WH) \qquad (Au,u) \geq -C_K\,\|u\|^2_{(m-3/2)/2}\,, \quad \forall u \in C_0^\infty(K;\mathbb{C}^N).$$

Inequality (WH) is already interesting in its own right.

Hence, inequality (WH) shows that when (FP) (equivalently (H)) fails, there might be room for intermediate inequalities of the following kind.

Definition 1.2. *Let* $A^{\mathrm{w}}(x,D_x) = A^{\mathrm{w}}(x,D_x)^* \in \mathrm{OPS}^m(\mathbb{R}^n;\mathbb{C}^N)$*, with principal symbol* $A_m(x,\xi) \geq 0$ *in the sense of Hermitian matrices in* $\mathbb{C}^N$*. Let* $s \in [0,(m-1)/2)$*. We will say that* $A^{\mathrm{w}}(x,D_x)$ *satisfies inequality* (I_s) *if for any given compact* $K \subset \mathbb{R}^n$ *there exists a constant* $C_{K,s} > 0$ *such that*

$$(I_s) \qquad (A^{\mathrm{w}}(x,D_x)u,u) \geq -C_{K,s}\,\|u\|_s^2\,,\ \forall u \in C_0^\infty(K;\mathbb{C}^N).$$

The reason why we take $s \in [0,(m-1)/2)$ *is that inequality* $(I_{(m-1)/2})$ *is the Sharp Gårding inequality, which already holds true for such an* A^{w}*.*

Remark 1.3. *When considering inequalities such as Fefferman-Phong's inequality for* $A(x, D_x) \in \mathrm{OPS}^2(\mathbb{R}^n; \mathbb{C}^N)$ *(that we also called Hörmander's inequality, because of the presence of homogeneity), one is concerned only with the behavior of* $A_2(x,\xi) + A_1(x,\xi)$*, for the remainder gives rise to a term which is already* $O(\|u\|_0^2)$.

From now on, for definiteness and to relate more clearly the various inequalities to the Fefferman-Phong one, we shall consider $m = 2$.

In Section 2 we will describe the fundamental example by Hörmander of a non-negative matrix whose Weyl-quantization is not non-negative, which is a model for the class we will construct.

In Section 3 we will recall Brummelhuis's counterexample A_B to the Fefferman-Phong inequality. It will be clear in the sequel that the failure of A_B to fulfill inequality (FP) is caused by the fact that the "localized operator" of Brummelhuis's system A_B at the point $(0, x_2^0, 0, 1)$ is unitarily equivalent in L^2 (but also equivalent in $\mathcal{S}$ and $\mathcal{S}'$) to Hörmander's operator A_H, and therefore it cannot be non-negative. Since the non-negativity of the localized operator is a necessary condition for all the inequalities (I_s), $s \in [0, 1/2)$ (see Section 4 below), it follows that inequality (FP) (as well as any of the (I_s), $s \in [0, 1/2)$) cannot hold for A_B.

In Section 4 we will recall what a *localized operator* associated with the system $A(x, D_x)$ at a characteristic point is. As recalled earlier on, its spectral properties give necessary, and in some cases sufficient, conditions for the inequalities we are concerned with to hold.

We will describe the class of pseudodifferential systems that fail to satisfy the Fefferman-Phong inequality in Section 5 and Section 6 below. The class is roughly described as the Weyl-quantization of the quadratic form

$$A_2(x,\xi) = p(x,\xi)^2 \bar{v} \otimes v + 2p(x,\xi)q(x,\xi)\mathrm{Re}(\bar{v} \otimes w) + q(x,\xi)^2 \bar{w} \otimes w,$$

where $v, w \in \mathbb{C}^N$ are orthonormal vectors, and p and q are real and vanish in a "finite-type" fashion on a symplectic (conic) manifold. For this class, it will also be seen at once that the Sharp Gårding inequality cannot be improved, not even to the weaker (WH) inequality above (and in fact, to none of inequalities (I_s) for all $s \in [0, 1/2)$).

We close this introduction by giving some examples (or, better, counterexamples).

Example 1.4. The first one is a 2×2 example in $\mathbb{R}^3$. For $k \in \mathbb{Z}_+$ and $\delta \in [0, 1)$, let

$$A_2(x,\xi) = \begin{bmatrix} \xi_1^2 + x_2^2\xi_3^2 - 2x_2\xi_1\xi_3 & (\xi_1 - x_2\xi_3)(\xi_2 + x_1^{2k+1}\xi_3) \\ (\xi_1 - x_2\xi_3)(\xi_2 + x_1^{2k+1}\xi_3) & \xi_2^2 + x_1^{4k+2}\xi_3^2 + 2x_1^{2k+1}\xi_2\xi_3 \end{bmatrix},$$

$$A_1(x,\xi) = \frac{\delta}{2}\Big(1 + (2k+1)x_1^{2k}\Big)\xi_3 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Then for $A^{\mathrm{w}}(x, D_x)$, $A(x,\xi) = A_2(x,\xi) + A_1(x,\xi)$, the Fefferman-Phong inequality does not hold, and the Sharp Gårding inequality cannot be improved.

Example 1.5. The second one is a 3×3 example in $\mathbb{R}^2$. Let

$$A(x,\xi) = \frac{1}{4}\begin{bmatrix} (\xi_1 + x_1\xi_2)^2 & -i(\xi_1 + x_1\xi_2)^2 & \sqrt{2}\,(\xi_1^2 - x_1^2\xi_2^2) \\ i(\xi_1 + x_1\xi_2)^2 & (\xi_1 + x_1\xi_2)^2 & i\sqrt{2}\,(\xi_1^2 - x_1^2\xi_2^2) \\ \sqrt{2}\,(\xi_1^2 - x_1^2\xi_2^2) & -i\sqrt{2}\,(\xi_1^2 - x_1^2\xi_2^2) & 2(\xi_1 - x_1\xi_2)^2 \end{bmatrix}.$$

Example 1.6. The third one is a 4×4 example in $\mathbb{R}^2$. It is very close to Brummelhuis's example. Let

$$A(x,\xi) = \frac{1}{3}\begin{bmatrix} \xi_1^2 & -ix_1\xi_1\xi_2 & \xi_1(\xi_1 - \xi_2) & \xi_1(\xi_1 + \xi_2) \\ ix_1\xi_1\xi_2 & x_1^2\xi_2^2 & ix_1\xi_2(\xi_1 - \xi_2) & ix_1\xi_2(\xi_1 + \xi_2) \\ \xi_1(\xi_1 - \xi_2) & -ix_1\xi_2(\xi_1 - \xi_2) & (\xi_1 - \xi_2)^2 & \xi_1^2 - \xi_2^2 \\ \xi_1(\xi_1 + \xi_2) & -ix_1\xi_2(\xi_1 + \xi_2) & \xi_1^2 - \xi_2^2 & (\xi_1 + \xi_2)^2. \end{bmatrix}$$

Example 1.7. The last one is a 2×2 example in $\mathbb{R}^2$. It is another kind of generalization of Brummelhuis's example. Let $k \geq 1$ and let

$$A(x,\xi) = \begin{bmatrix} \xi_1^2 & -ix_1^k\xi_1\xi_2 \\ ix_1^k\xi_1\xi_2 & x_1^{2k}\xi_2^2 \end{bmatrix}.$$

Then in all the above examples, the Fefferman-Phong inequality does not hold for $A^{\mathrm{w}}(x, D_x)$, and the Sharp Gårding inequality cannot be improved.

2. Hörmander's non-positivity example

Consider the system

$$A_H(x,\xi) = \begin{bmatrix} x^2 & x\xi \\ x\xi & \xi^2 \end{bmatrix}, \quad (x,\xi) \in \mathbb{R} \times \mathbb{R}. \tag{2}$$

Then $\langle A_H(x,\xi)v, v\rangle \geq 0$ for all $(x,\xi) \in \mathbb{R}^2$ and $v \in \mathbb{C}^2$. In spite of this, Hörmander proved that the the Weyl-quantization A_H^{w} is **not non-negative**. In fact, one has, with $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathcal{S}(\mathbb{R}; \mathbb{C}^2)$ and $D_x = -i\partial_x$,

$$(A_H^{\mathrm{w}}(x, D_x)u, u) = \|xu_1 + D_x u_2\|_0^2 - \mathrm{Im}\,(u_1, u_2).$$

If one chooses, following Hörmander, $u_1 = v''$, $u_2 = i(v - xv')$, for $v \in C_0^\infty(\mathbb{R}; \mathbb{R})$, then the term $\|xu_1 + D_x u_2\|_0^2 = 0$ and the right-hand-side becomes

$$-\mathrm{Im}\,(u_1, u_2) = (v'', v - xv') = -\frac{1}{2}\,\|v'\|_0^2 < 0.$$

It is important to notice that $\det(A_H(x,\xi)) = 0$ on the whole $T^*\mathbb{R}$.

3. Brummelhuis's counterexample to (FP)

Consider the system

$$A_B(x,\xi)=\begin{bmatrix} x_1^2\xi_2^2 & ix_1\xi_1\xi_2 \\ -ix_1\xi_1\xi_2 & \xi_1^2 \end{bmatrix},\ (x,\xi)\in\mathbb{R}^2\times\mathbb{R}^2. \tag{3}$$

One has that the operator $A_B(x,D_x)$, in spite of having non-negative symbol, does not satisfy the Fefferman-Phong inequality (FP). In fact, following Brummelhuis, one considers $\chi,\psi\in C_0^\infty(|x_1|,|x_2|\leq 1)$ such that $\iint \chi(x_1,x_2)\psi(0,x_2)dx_1dx_2=1$. Take then $\mu>0$ a parameter to be picked sufficiently large, and define

$$u(x_1,x_2):=\sqrt{\mu}e^{-i\mu x_2}\chi(\mu x_1,x_2),\qquad v(x_1,x_2):=e^{-i\mu x_2}\psi(x_1,x_2).$$

Then $\|u\|_0^2+\|v\|_0^2=\|\chi\|_0^2+\|\psi\|_0^2$. Writing (recall that $D_{x_j}=i^{-1}\partial_{x_j}$)

$$\mathrm{Re}\left(A_B(x,D_x)\begin{bmatrix}u\\v\end{bmatrix},\begin{bmatrix}u\\v\end{bmatrix}\right)=\|iD_{x_1}v+x_1D_{x_2}u\|_0^2+\mathrm{Re}\,(D_{x_2}u,v),$$

it easily follows that there exists $C>0$ and $\mu_0\geq 1$ such that for all $\mu\geq\mu_0$

$$\mathrm{Re}\left(A_B(x,D_x)\begin{bmatrix}u\\v\end{bmatrix},\begin{bmatrix}u\\v\end{bmatrix}\right)\leq -C\sqrt{\mu}\Big(\|u\|_0^2+\|v\|_0^2\Big).$$

It is interesting to notice that also in this case $\det(A_B(x,\xi))=0$ on the whole $T^*\mathbb{R}^2$.

4. The localized operator

In this section, we shall define the *localized operator* of a pseudodifferential system in a simple case tailored to our purposes (see [1], [3], [4], [12] for more general situations). Suppose we are given $A(x,\xi)=A(x,\xi)^*\sim A_2(x,\xi)+A_1(x,\xi)+\ldots$, an Hermitian $N\times N$ matrix of second-order classical symbols. Suppose that $\det(A_2(x,\xi))=0$ for all $(x,\xi)\in T^*\mathbb{R}^n\setminus 0$, and let Σ be a closed conic symplectic submanifold of $T^*\mathbb{R}^n\setminus 0$ of codimension $2\nu\geq 2$ such that $\mathrm{Ker}\,(A_2(\rho))=\mathbb{C}^N$ for all $\rho\in\Sigma$. Then, for $w,w'\in\mathbb{C}^N$, define

$$\begin{aligned} a^{w',w}(x,\xi) &:= a_2^{w',w}(x,\xi)+a_1^{w',w}(x,\xi)= \\ &= \langle A_2(x,\xi)w,w'\rangle+\langle A_1(x,\xi)w,w'\rangle,\quad (x,\xi)\in T^*\mathbb{R}^n\setminus 0. \end{aligned}$$

Since $a_2^{w,w}\geq 0$ on $T^*\mathbb{R}^n\setminus 0$ $(w=w')$, it follows by polarization that $a_2^{w',w}$ vanishes to second order on Σ for all $w,w'\in\mathbb{C}^N$. Hence we may consider, for it is invariantly defined then, with $\rho_0=(x_0,\xi_0)\in\Sigma$, $|\xi_0|=1$ and $\zeta=(y,\eta)\in T_{\rho_0}T^*\mathbb{R}^n$, the $N\times N$ Hermitian-valued quadratic form $Q_{A_2}(\rho_0;\zeta)=\langle \mathrm{Hess}(A_2/2)(\rho_0)\zeta,\zeta\rangle_{T_{\rho_0}T^*\mathbb{R}^n}$.

Definition 4.1. *(See, e.g.,* [3]*) In this framework, the localized operator* $\mathcal{L}_A^{\mathrm{w}}(\rho_0)$ *of* A^{w} *at* $\rho_0\in\Sigma$ *is the Weyl quantization*

$$\mathcal{L}_A^{\mathrm{w}}(\rho_0;y,D_y):=Q_{A_2}^{\mathrm{w}}(\rho_0;y,D_y)+A_1(\rho_0),$$

thought of as a formally self-adjoint unbounded operator in $L^2(\mathbb{R}^n;\mathbb{C}^N)$.

We then have the following proposition, which provides a necessary condition in order for inequality (I_s) to hold for some $s \in [0, 1/2)$.

Proposition 4.2. *Let* $A(x,\xi) = A(x,\xi)^* \sim A_2(x,\xi) + A_1(x,\xi) + \dots$ *be an Hermitian* $N \times N$ *matrix of classical symbols of order 2. Suppose that at some* $\rho_0 = (x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$, $|\xi_0| = 1$, *one has* $\mathrm{Ker}\,(A_2(\rho_0)) = \mathbb{C}^N$. *If* $A^{\mathrm{w}}(x, D_x) = A^{\mathrm{w}}(x, D_x)^* \in \mathrm{OPS}^2(\mathbb{R}^n; \mathbb{C}^N)$ *satisfies inequality* (I_s) *for some* $s \in [0, 1/2)$, *then*

$$\left(\mathcal{L}_A^{\mathrm{w}}(\rho_0; y, D_y) f, f\right) \geq 0, \quad \forall f \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N). \tag{4}$$

Proof. The proof is classical. One just tests inequality (I_s) as $t \to +\infty$ on a cut-off function of the kind $e^{it^2\langle x, \xi_0\rangle}\psi(t(x - x_0))$, where $t \geq 1$, $\psi \in C_0^\infty(\mathbb{R}^n; \mathbb{C}^N)$. A Taylor-expansion finishes the proof. □

On the other hand, since $T_{\rho_0}T^*\mathbb{R}^n = T_{\rho_0}\Sigma \oplus T_{\rho_0}\Sigma^\sigma$ (symplectic orthogonal direct sum) and $\zeta \in T_{\rho_0}\Sigma \implies Q_{A_2}(\rho_0; \zeta) = 0$, one may reduce matters to considering only $\zeta \in T_{\rho_0}\Sigma^\sigma$. In fact, one may use a homogeneous symplectomorphism χ to locally near ρ_0 flatten Σ into $\{(y', y'', \eta', \eta'') \in \mathbb{R}^\nu \times \mathbb{R}^{n-\nu} \times \mathbb{R}^\nu \times (\mathbb{R}^{n-\nu} \setminus \{0\});\ y' = \eta' = 0,\ \eta'' \neq 0\}$, whose Jacobian matrix hence coordinatizes $T_{\rho_0}\Sigma^\sigma$ as $\mathbb{R}^\nu_t \times (\mathbb{R}^\nu)^*_\tau$ (such a symplectomorphism exists by Thm. 21.2.4 of [10]). Then, by using the invariance of the Weyl-Calculus under linear symplectomorphisms it easy to see that (4) above implies, with

$$L_A^{\mathrm{w}}(\rho_0) := Q^{\mathrm{w}}_{A_2 \circ \chi^{-1}}(\rho_0; t, D_t) + A_1(\rho_0),$$

that

$$(L_A^{\mathrm{w}}(\rho_0) f, f) \geq 0, \quad \forall f \in \mathcal{S}(\mathbb{R}^\nu; \mathbb{C}^N). \tag{5}$$

This latter necessary condition will be used in the proof of Theorem 5.1 below.

5. The class of counterexamples

Let $\nu \geq 1$, and consider *real classical symbols* on $T^*\mathbb{R}^n \setminus 0$, α_j, β_j homogeneous of degree 0, and p_j, q_j homogeneous of degree 1, respectively, $j = 1, \dots, \nu$. Set

$$p(x,\xi) := \sum_{j=1}^{\nu} \alpha_j(x,\xi) p_j(x,\xi), \quad q(x,\xi) := \sum_{j=1}^{\nu} \beta_j(x,\xi) q_j(x,\xi).$$

We suppose that

$$\Sigma_1 = \{(x,\xi) \in T^*\mathbb{R}^n \setminus 0;\ p_1(x,\xi) = \dots = p_\nu(x,\xi) = 0\},$$

and

$$\Sigma_2 = \{(x,\xi) \in T^*\mathbb{R}^n \setminus 0;\ q_1(x,\xi) = \dots = q_\nu(x,\xi) = 0\},$$

are closed conic manifolds of codimension ν with **transversal intersection** $\Sigma = \Sigma_1 \cap \Sigma_2$, that has therefore codimension 2ν. Suppose

$$(H1) \qquad \begin{cases} \{dp_j(\rho)\}_{1\le j\le \nu} \text{ are linearly independent } \forall \rho \in \Sigma_1, \\ \{dq_j(\rho)\}_{1\le j\le \nu} \text{ are linearly independent } \forall \rho \in \Sigma_2, \\ \{p_j, p_k\}\big|_{\Sigma_1} = \{q_j, q_k\}\big|_{\Sigma_2} = 0, \ \forall j,k = 1,\dots,\nu, \end{cases}$$

and

$$(H2) \qquad \text{the matrix } \Big(\{p_j, q_k\}(\rho)\Big)_{1\le j,k\le \nu} \text{ is invertible for all } \rho \in \Sigma.$$

Hence, by $(H1)$, Σ_1 and Σ_2 are involutive manifolds, and by $(H2)$, Σ is symplectic. Moreover, $(H2)$ yields that the differentials dp_j, dq_k, $1 \le j,k \le \nu$ are linearly independent on Σ.

Next, with $\langle \cdot,\cdot \rangle$ the canonical Hermitian product in $\mathbb{C}^N$ and $|\cdot|$ the relative norm, fix orthonormal vectors $v_1, v_2 \in \mathbb{C}^N$, and complete $\{v_1, v_2\}$ into an Hermitian basis of $\mathbb{C}^N$. Let $\{v_j^*\}_{j=1,\dots,N}$ be the relative dual vectors, so that $v_j^*(w) = \langle w, v_j\rangle$, $j = 1,\dots,N$. (Hence, when thinking of v_j as column vectors, the v_j^* are the complex-conjugate row vectors.) Let then, for $(x,\xi) \in T^*\mathbb{R}^n \setminus 0$,

$$(6) \quad L(x,\xi)\colon \mathbb{C} \longrightarrow \mathbb{C}^N,\ L(x,\xi)\colon z \longmapsto z\Big(p(x,\xi)v_1 + q(x,\xi)v_2\Big),\ z \in \mathbb{C},$$

and the related linear form

$$(7) \qquad L(x,\xi)^*\colon \mathbb{C}^N \longrightarrow \mathbb{C},\ L(x,\xi)^*\colon w \longmapsto \langle w, L(x,\xi)\rangle,\ w \in \mathbb{C}^N.$$

Define hence the matrix-valued second-order classical symbol $A_2(x,\xi)$ by

$$(8) \qquad A_2(x,\xi)w = \langle w, L(x,\xi)\rangle L(x,\xi), \quad \forall w \in \mathbb{C}^N.$$

Hence $A_2(x,\xi) = A_2(x,\xi)^* \ge 0$, for

$$\langle A_2(x,\xi)w, w\rangle = |\langle L(x,\xi), w\rangle|^2, \quad \forall w \in \mathbb{C}^N.$$

Moreover, since $(\mathbb{C}^N)^* \otimes \mathbb{C}^N \simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^N, \mathbb{C}^N)$ $(\otimes = \otimes_{\mathbb{C}})$ by the isomorphism that acts on decomposable elements as $v^* \otimes w \longmapsto \Big(u \mapsto v^*(u)w = \langle u, v\rangle w\Big)$, we shall write

$$A_2(x,\xi) = L(x,\xi)^* \otimes L(x,\xi) \equiv L(x,\xi)L(x,\xi)^*,$$

where the last equality is understood when thinking of L as a column vector and L^* as the relative complex-conjugate row vector.

It is clear that

$$(9) \qquad \begin{cases} \det A_2(x,\xi) = 0, \quad \forall (x,\xi) \in T^*\mathbb{R}^n \setminus 0, \\ \mathrm{Ker}\ A_2(\rho) = \mathbb{C}^N, \quad \forall \rho \in \Sigma. \end{cases}$$

One has the following theorem.

Theorem 5.1. *Suppose hypotheses* $(H1)$, $(H2)$ *are satisfied, and consider the classical symbol* $A(x,\xi) = A(x,\xi)^* \sim A_2(x,\xi) + A_1(x,\xi) + \dots$ *with* A_2 *defined in (8). Suppose that there exists* $\rho_0 = (x_0,\xi_0) \in \Sigma$, $|\xi_0| = 1$, *such that*

$$\{p,q\}(\rho_0) = \sum_{j,k=1}^{\nu} \alpha_j(\rho_0)\beta_j(\rho_0)\{p_j,q_k\}(\rho_0) \neq 0,$$

and that there exists $\delta \in [0,1)$ *such that (notice that* $A_1(x,\xi) = A_1(x,\xi)^*$ *for all* $(x,\xi) \in T^*\mathbb{R}^n \setminus 0$*)*

$$\Big\langle A_1(\rho_0)w, w\Big\rangle \leq \delta\,\{p,q\}(\rho_0)\Big\langle \mathrm{Im}\,(v_2^* \otimes v_1)w, w\Big\rangle, \quad \forall w \in \mathbb{C}^N.$$

Then $A^{\mathrm{w}}(x,D_x)$ *does not satisfy the Fefferman-Phong inequality* (FP), *and the Sharp Gårding inequality cannot be improved, that is, none of inequalities* (I_s) *above, with* $s \in [0,1/2)$, *may hold.*

Proof. To prove the theorem, we just compute the localized operator (actually, a unitary equivalent of it) L_A^{w} at ρ_0 and show that it is **not non-negative**. This latter point is based upon Hörmander's example recalled earlier in Section 2. As it is well-known, hypotheses $(H1)$ and $(H2)$ ensure that there exists a conic neighborhood Γ_0 of ρ_0 in $T^*\mathbb{R}^n \setminus 0$, a conic neighborhood $\tilde{\Gamma}$ of $(0, \varepsilon_n = (0,0,\dots,0,1))$ in $\mathbb{R}^{\nu}_{y'} \times \mathbb{R}^{n-\nu}_{y''} \times \mathbb{R}^{\nu}_{\eta'} \times (\mathbb{R}^{n-\nu}_{\eta''} \setminus \{0\}) \simeq T^*\mathbb{R}^{\nu} \times (T^*\mathbb{R}^{n-\nu} \setminus 0)$, and a smooth symplectomorphism, homogeneous of degree 1 in the fibers, $\chi\colon \Gamma_0 \longrightarrow \tilde{\Gamma}$ such that $\chi(\rho_0) = (0,\varepsilon_n)$, and

$$\chi(\Gamma_0 \cap \Sigma_1) = \{(y,\eta) \in \tilde{\Gamma};\ \eta' = 0,\ \eta'' \neq 0\},$$

$$\chi(\Gamma_0 \cap \Sigma_2) = \{(y,\eta) \in \tilde{\Gamma};\ y' = 0,\ \eta'' \neq 0\},$$

$$\chi(\Gamma_0 \cap \Sigma) = \{(y,\eta) \in \tilde{\Gamma};\ y' = \eta' = 0,\ \eta'' \neq 0\}.$$

As a consequence, on $\tilde{\Gamma}$ we have

$$(p_j \circ \chi^{-1})(y,\eta) = \sum_{k=1}^{\nu} a_{jk}(y,\eta)\eta_k, \quad a_{jk} \text{ homogeneous of degree } 0,$$

$$(q_j \circ \chi^{-1})(y,\eta) = |\eta| \sum_{k=1}^{\nu} b_{jk}(y,\eta)y_k, \quad b_{jk} \text{ homogeneous of degree } 0,$$

and for $(x,\xi) \in \Gamma_0 \cap \Sigma$, with $(y,\eta) = \chi(x,\xi)$,

$$\{p_j,q_k\}(x,\xi) = |\eta''| \sum_{r=1}^{\nu} a_{jr}(y,\eta)b_{kr}(y,\eta).$$

Localizing $L(x,\xi)$ at ρ_0 by means of the coordinates (t,τ) gives

$$L_{\rho_0}(t,\tau) := \Big(\sum_{j=1}^{\nu} \tilde{\alpha}_j\tau_j\Big)v_1 + \Big(\sum_{j=1}^{\nu} \tilde{\beta}_j t_j\Big)v_2,$$

where

$$\tilde{\alpha}_j := \sum_{k=1}^{\nu} \alpha_k(\rho_0) a_{kj}(\chi(\rho_0)), \quad \tilde{\beta}_j := \sum_{k=1}^{\nu} \beta_k(\rho_0) b_{kj}(\chi(\rho_0)).$$

Hence, with $\tilde{\alpha} = \begin{bmatrix} \tilde{\alpha}_1 \\ \vdots \\ \tilde{\alpha}_\nu \end{bmatrix}$ and $\tilde{\beta} = \begin{bmatrix} \tilde{\beta}_1 \\ \vdots \\ \tilde{\beta}_\nu \end{bmatrix} \in \mathbb{R}^\nu$, we have

$$\sum_{j=1}^{\nu} \tilde{\alpha}_j \tilde{\beta}_j =: \langle \tilde{\alpha}, \tilde{\beta} \rangle_{(\nu)} = \{p, q\}(\rho_0) \neq 0. \tag{10}$$

Consider now a rotation R of $\mathbb{R}^\nu$ such that $R\tilde{\alpha} = |\tilde{\alpha}|_{(\nu)} e_1$, where $e_1 = {}^t(1, 0, \ldots, 0) \in \mathbb{R}^\nu$ and $|\tilde{\alpha}|_{(\nu)} = \langle \tilde{\alpha}, \tilde{\alpha} \rangle_{(\nu)}^{1/2}$. Then we get a symplectomorphism

$$\varphi\colon \mathbb{R}^\nu \times \mathbb{R}^\nu \to \mathbb{R}^\nu \times \mathbb{R}^\nu,\ \varphi\colon (t, \tau) \longmapsto (|\tilde{\alpha}|_{(\nu)}^{-1/2} Rt, |\tilde{\alpha}|_{(\nu)}^{1/2} R\tau) = (z, \zeta),$$

such that

$$|\tilde{\alpha}|_{(\nu)}^{1/2} \tilde{L}_{\rho_0}(z, \zeta) := (L_{\rho_0} \circ \varphi^{-1})(z, \zeta) = |\tilde{\alpha}|_{(\nu)}^{1/2} \Big(\zeta_1 v_1 + \langle \gamma, z \rangle_{(\nu)} v_2 \Big),$$

where we have put $\gamma := R\tilde{\beta}$. Hence we obtain that *in these coordinates* the localized symbol of A_2 at $\rho_0 \in \Sigma$ is (writing again (t, τ) for (z, ζ))

$$A_{\rho_0}(t, \tau) = |\tilde{\alpha}|_{(\nu)} \tilde{L}_{\rho_0}(t, \tau)^* \otimes \tilde{L}_{\rho_0}(t, \tau) = |\tilde{\alpha}|_{(\nu)} \tilde{L}_{\rho_0}(t, \tau) \tilde{L}_{\rho_0}(t, \tau)^*. \tag{11}$$

We may hence compute

$$\tilde{L}_{\rho_0} \sharp \tilde{L}^*_{\rho_0} = \tilde{L}_{\rho_0} \tilde{L}^*_{\rho_0} - \frac{i}{2} \sum_{j=1}^{\nu} \Big(\partial_{\tau_j} \tilde{L}_{\rho_0} \partial_{t_j} \tilde{L}^*_{\rho_0} - \partial_{t_j} \tilde{L}_{\rho_0} \partial_{\tau_j} \tilde{L}^*_{\rho_0} \Big) = \tag{12}$$

$$= \tilde{L}_{\rho_0} \tilde{L}^*_{\rho_0} - \frac{i}{2} \{ \tilde{L}_{\rho_0}, \tilde{L}^*_{\rho_0} \} = \tilde{L}^*_{\rho_0} \otimes \tilde{L}_{\rho_0} - \frac{i}{2} \gamma_1 \Big(v_2^* \otimes v_1 - v_1^* \otimes v_2 \Big).$$

Remark 5.2. *Notice that*

$$\frac{i}{2} \Big(v_2^* \otimes v_1 - v_1^* \otimes v_2 \Big) = -\mathrm{Im}\,(v_2^* \otimes v_1) = \mathrm{Re}\ i(v_2^* \otimes v_1),$$

is the quadratic form introduced by Hörmander in [6] *(see also* [9]*).*

Upon Weyl-quantization, since to linear symplectomorphisms of $\mathbb{R}^\nu \times \mathbb{R}^\nu$ there correspond (up to factors of modulus 1) unitary transformations in $L^2(\mathbb{R}^\nu)$ that are also automorphisms of $\mathcal{S}(\mathbb{R}^\nu)$ and $\mathcal{S}'(\mathbb{R}^\nu)$, we obtain, using (11) and (12), that the *localized operator* of A_2 at $\rho_0 \in \Sigma$ is *unitarily equivalent*, through a unitary transformation $U \otimes \mathrm{Id}_{\mathbb{C}^N}$ of $L^2(\mathbb{R}^\nu; \mathbb{C}^N) \simeq L^2(\mathbb{R}^\nu) \otimes \mathbb{C}^N$, which is also an automorphism of $\mathcal{S}(\mathbb{R}^\nu; \mathbb{C}^N)$ and of $\mathcal{S}'(\mathbb{R}^\nu; \mathbb{C}^N)$, to

$$A^{\mathrm{w}}_{\rho_0}(t, D_t) = |\tilde{\alpha}|_{(\nu)} \Big[\tilde{L}^{\mathrm{w}}_{\rho_0}(t, D_t) \tilde{L}^{\mathrm{w}}_{\rho_0}(t, D_t)^* + \frac{i}{2} \gamma_1 \Big(v_2^* \otimes v_1 - v_1^* \otimes v_2 \Big) \Big],$$

and hence, with $f=\sum_{j=1}^{N} f_j v_j \in \mathcal{S}(\mathbb{R}^\nu;\mathbb{C}^N)$,

$$\Big(A^{\mathrm{w}}_{\rho_0}(t,D_t)f,f\Big)=|\tilde\alpha|_{(\nu)}\Big(\Big\|\tilde L^{\mathrm{w}}_{\rho_0}(t,D_t)^*f\Big\|_0^2+\gamma_1\mathrm{Im}\,(f_1,f_2)\Big)=$$

$$=|\tilde\alpha|_{(\nu)}\Big(\big\|D_{t_1}f_1+\langle\gamma,t\rangle_{(\nu)}f_2\big\|_0^2+\gamma_1\mathrm{Im}\,(f_1,f_2)\Big).$$

Now set $t=(t_1,t')$. Following Hörmander's example, we find $f_1,f_2\in C_0^\infty(\mathbb{R}^\nu)$ such that

$$D_{t_1}f_1+\langle\gamma,t\rangle_{(\nu)}f_2=0,\quad \textbf{and}\quad \gamma_1\mathrm{Im}\,(f_1,f_2)<0.$$

We choose $f_2\in C_0^\infty(\mathbb{R}^\nu)$ of the form $f_2(t_1,t')=g_1''(t_1)g_2(t')$, for **real valued** $g_1\in C_0^\infty(\mathbb{R})$ and $g_2\in C_0^\infty(\mathbb{R}^{\nu-1})$, so that

$$f_1(t_1,t')=-i\int_{-\infty}^{t_1}\langle\gamma,(s,t')\rangle_{(\nu)}f_2(s,t')ds=$$

$$=-i\Big(\langle\gamma,t\rangle_{(\nu)}g_1'(t_1)g_2(t')-\gamma_1 g_1(t_1)g_2(t')\Big)\in C_0^\infty(\mathbb{R}^\nu).$$

A computation then gives

$$(f_1,f_2)=-\frac{i}{2}\gamma_1\,\|g_1'\|_0^2\,\|g_2\|_0^2.$$

Hence, for $f=f_1v_1+f_2v_2$, f_1,f_2 chosen as above,

$$\Big(A^{\mathrm{w}}_{\rho_0}(t,D_t)f,f\Big)=-|\tilde\alpha|_{(\nu)}\gamma_1\Big(\mathrm{Im}\,(v_2^*\otimes v_1)f,f\Big)=|\tilde\alpha|_{(\nu)}\gamma_1\mathrm{Im}\,(f_1,f_2)=$$

$$=-\frac{1}{2}|\tilde\alpha|_{(\nu)}\gamma_1^2\,\|g_1'\|_0^2\,\|g_2\|_0^2<0$$

provided $\gamma_1\neq 0$, which is true by hypothesis, for we have

$$|\tilde\alpha|_{(\nu)}\gamma_1=\langle|\tilde\alpha|_{(\nu)}e_1,R\tilde\beta\rangle_{(\nu)}=\langle R\tilde\alpha,R\tilde\beta\rangle_{(\nu)}=\{p,q\}(\rho_0)\neq 0.$$

Finally, since $(U\otimes\mathrm{Id}_{\mathbb{C}^N})^*A_1(\rho_0)(U\otimes\mathrm{Id}_{\mathbb{C}^N})=A_1(\rho_0)$, the localized operator of $A^{\mathrm{w}}(x,D_x)$ at ρ_0 is *unitarily equivalent* to $A^{\mathrm{w}}_{\rho_0}(t,D_t)+A_1(\rho_0)$, so that using again f constructed above, one has

$$\Big(\Big[A^{\mathrm{w}}_{\rho_0}(t,D_t)+A_1(\rho_0)\Big]f,f\Big)=\Big(A_1(\rho_0)f,f\Big)-\{p,q\}(\rho_0)\Big(\mathrm{Im}\,(v_2^*\otimes v_1)f,f\Big)$$

$$\leq -(1-\delta)\{p,q\}(\rho_0)\Big(\mathrm{Im}\,(v_2^*\otimes v_1)f,f\Big)<0.$$

Hence the necessary condition (5) is not fulfilled and this concludes the proof of the theorem. □

6. A slight generalization

When $\nu = 1$, we may handle cases with "finite-type" degeneracy (but with no subprincipal term). Let p, q be *real homogeneous symbols* of order 1 in $T^*\mathbb{R}^n \setminus 0$. Let Σ_1 be the simple-characteristic set of p, that is

$$\Sigma_1 = \{(x,\xi) \in T^*\mathbb{R}^n \setminus 0; \quad p(x,\xi) = 0\}$$

with $dp(\rho) \neq 0$ for all $\rho \in \Sigma_1$, and let Σ_2 be another involutive closed conic smooth manifold of $T^*\mathbb{R}^n \setminus 0$ of codimension 1, that intersects **transversally** Σ_1 into a **symplectic** submanifold $\Sigma = \Sigma_1 \cap \Sigma_2$. Suppose that for a fixed $k \geq 2$,

$$(H3) \qquad H_p^j q\big|_{\Sigma_2} = 0, \ \forall j = 0, \dots, k-1 \quad \text{and} \quad H_p^k q(\rho) \neq 0, \ \forall \rho \in \Sigma.$$

Consider then, for L and L^* defined as in (6) and (7) above, the second-order homogeneous symbol

$$A(x,\xi) = A_2(x,\xi) = L(x,\xi)^* \otimes L(x,\xi) =$$

$$= p(x,\xi)^2 v_1^* \otimes v_1 + 2p(x,\xi)q(x,\xi)\mathrm{Re}\,(v_1^* \otimes v_2) + q(x,\xi)^2 v_2^* \otimes v_2.$$

Theorem 6.1. *Suppose hypothesis* $(H3)$ *be satisfied. Take* $\rho_0 = (x_0, \xi_0) \in \Sigma$, $|\xi_0| = 1$. *Then* $A^{\mathrm{w}}(x, D_x)$ *does not satisfy the Fefferman-Phong inequality* (FP), *and the Sharp Gårding inequality cannot be improved, that is, none of inequalities* (I_s) *above, with* $s \in [0, 1/2)$, *may hold.*

Proof. We immediately recall that $A_2^{\mathrm{w}}(x, D_x) = (A_2(x, D_x) + A_2(x, D_x)^*)/2$, so that no subprincipal term is present. Let now $\chi\colon (x,\xi) \longmapsto (y,\eta) = (y_1, y', \eta_1, \eta') \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \times (\mathbb{R}^{n-1} \setminus \{0\})$ be a symplectomorphism that locally flattens Σ_1, Σ_2 and Σ near $\rho_0 \in \Sigma$. We may then suppose $\chi(\rho_0) = (0, 0, 0, \eta_0')$, and

$$(p \circ \chi^{-1})(y,\eta) = \eta_1, \quad (q \circ \chi^{-1})(y,\eta) = \alpha(y,\eta) y_1^k |\eta|,$$

where α is homogeneous of degree 0 in η, and $\alpha(0, y', 0, \eta') \neq 0$. Let F be a properly supported unitary, **scalar** Fourier integral operator associated with the graph of χ^{-1}. Consider $\tilde{\varphi}_\gamma(x) := F\varphi_\gamma(y)$, where $\gamma \geq 1$ and

$$\varphi_\gamma(y_1, y') := e^{i\gamma^{k+1} y' \cdot \eta_0'} \psi_2(\gamma y') \psi_1(\gamma y_1),$$

with $\psi_2 \in C_0^\infty(\mathbb{R}^{n-1}_{y'}; \mathbb{C})$ and $\psi_1 \in C_0^\infty(\mathbb{R}_{y_1}; \mathbb{C}^N)$. Then $\tilde{\varphi}_\gamma \in C_0^\infty(\mathbb{R}^n; \mathbb{C}^N)$. Supposing (I_s) holds for some $s \in [0, 1/2)$, testing (I_s) on $\tilde{\varphi}_\gamma$ and taking the limit as $\gamma \to +\infty$ gives

$$\Big(\tilde{A}^{\mathrm{w}}_{\rho_0}(t, D_t) f, f\Big) \geq 0, \quad \forall f \in \mathcal{S}(\mathbb{R}; \mathbb{C}^N),$$

where this time $\tilde{A}^{\mathrm{w}}_{\rho_0}(t, D_t)$ denotes the Weyl-quantization of the quadratic form

$$\tilde{L}_{\rho_0}(t,\tau)^* \otimes \tilde{L}_{\rho_0}(t,\tau), \quad \text{where} \quad \tilde{L}_{\rho_0}(t,\tau) = \tau v_1 + \alpha(0,0,0,\eta_0') t^k v_2,$$

$(t,\tau)\in\mathbb{R}\times\mathbb{R}$. In other words, we consider

$$\tilde{A}_{\rho_0}(t,\tau):=\sum_{j/k+h=2}\frac{1}{j!h!}(\partial_{y_1}^{j}\partial_{\eta_1}^{h}\tilde{A}_2)(0,0,0,\eta_0')t^j\tau^h,$$

where $\tilde{A}_2(y,\eta)$ is the principal symbol of $F^*A_2(x,D_x)F$. This because no subprincipal term is present in $A_2(x,D_x)$ at Σ (since that is the case for $(p^2)^{\mathrm{w}}(x,D_x)$, $(q^2)^{\mathrm{w}}(x,D_x)$ and $(pq)^{\mathrm{w}}(x,D_x)$). We hence obtain

$$\tilde{A}_{\rho_0}(t,\tau)=\tau^2v_1^*\otimes v_1+2\tilde{\alpha}t^k\tau\mathrm{Re}\,(v_1^*\otimes v_2)+\tilde{\alpha}^2t^{2k}v_2^*\otimes v_2,$$

where we have set $\tilde{\alpha}:=\alpha(0,0,0,\eta_0')$. As before, with $f=\sum_j f_jv_j\in\mathcal{S}(\mathbb{R};\mathbb{C}^N)$, we have

$$\left(\tilde{A}_{\rho_0}^{\mathrm{w}}(t,D_t)f,f\right)=\left\|D_tf_1+\tilde{\alpha}t^kf_2\right\|_0^2+k\tilde{\alpha}\,\mathrm{Im}\,(t^{k-1}f_1,f_2).$$

We now prove that $\tilde{A}_{\rho_0}^{\mathrm{w}}(t,D_t)$ cannot be non-negative in L^2. In fact, we choose $\varphi\in C_0^\infty(\mathbb{R};\mathbb{R})$ and consider

$$f_2(t):=\varphi^{(k+1)}(t),\ \ f_1(t)=-i\tilde{\alpha}\int_{-\infty}^{t}s^k\varphi^{(k+1)}(s)ds.$$

It follows that $f_1,f_2\in C_0^\infty$ and that for $f=f_1v_1+f_2v_2\in C_0^\infty(\mathbb{R};\mathbb{C}^N)$ we have

$$\left(\tilde{A}_{\rho_0}^{\mathrm{w}}(t,D_t)f,f\right)=k\tilde{\alpha}\,\mathrm{Im}\,(t^{k-1}f_1,f_2).$$

A computation yields

$$\begin{aligned}(t^{k-1}f_1,f_2)=-i\tilde{\alpha}\Big[\frac{1}{2}\left\|t^{k-1}\varphi^{(k)}\right\|_0^2+\frac{k(k-1)}{2}\left\|t^{k-2}\varphi^{(k-1)}\right\|_0^2+\\ +k(k-1)^2(k-2)\int t^{k-3}\varphi^{(k-1)}(t)\left(\int_{-\infty}^{t}s^{k-2}\varphi^{(k-1)}(s)ds\right)dt\Big].\end{aligned}$$

Since $t\mapsto\int_{-\infty}^{t}s^{k-3}\varphi^{(k-1)}(s)ds\in C_0^\infty(\mathbb{R};\mathbb{R})$, integrating by parts gives

$$\int t^{k-3}\varphi^{(k-1)}(t)\left(\int_{-\infty}^{t}s^{k-2}\varphi^{(k-1)}(s)ds\right)dt=\frac{1}{2}\left\|\int_{-\infty}^{t}s^{k-3}\varphi^{(k-1)}(s)ds\right\|_0^2.$$

Hence, recalling that $(H_p^kq)(\rho_0)=k!\tilde{\alpha}$, we obtain

$$\begin{aligned}\left(\tilde{A}_{\rho_0}^{\mathrm{w}}(t,D_t)f,f\right)=-\frac{k}{2(k!)^2}(H_p^kq)(\rho_0)^2\Big[\left\|t^{k-1}\varphi^{(k)}\right\|_0^2+k(k-1)\left\|t^{k-2}\varphi^{(k-1)}\right\|_0^2\\ +k(k-1)^2(k-2)\left\|\int_{-\infty}^{t}s^{k-3}\varphi^{(k-1)}(s)ds\right\|_0^2\Big]<0.\end{aligned}$$

Therefore the necessary condition does not hold and this completes the proof of the theorem. ☐

7. The examples of the Introduction

We now show how the examples mentioned in the Introduction fit in the framework we developed. We let $v_1 = e_1$, $v_2 = e_2$ be the canonical vectors of $\mathbb{C}^2$, and consider, for a fixed $k \in \mathbb{Z}_+$, the symbols

$$p(x,\xi) = \xi_1 - x_2\xi_3, \quad q(x,\xi) = \xi_2 + x_1^{2k+1}\xi_3, \quad (x,\xi) \in T^*\mathbb{R}^3.$$

Notice that for $k = 0$, $p+iq$ is the symbol of $-iL_0$, where L_0 is the celebrated Lewy operator (which is non-solvable)

$$L_0 = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + i(x_1 + ix_2)\frac{\partial}{\partial x_3}.$$

Then

$$\Sigma = \{(x,\xi) \in T^*\mathbb{R}^3 \setminus 0;\ \xi_1 = x_2\xi_3,\ \xi_2 = -x_1^{2k+1}\xi_3\},$$

so that $(x,\xi) \in \Sigma \implies \xi_3 \neq 0$, and hence

$$(x,\xi) \in \Sigma \implies \{p,q\}(x,\xi) = \Big((2k+1)x_1^{2k} + 1\Big)\xi_3 \neq 0.$$

In the case of Example 1.4

$$L(x,\xi) = p(x,\xi)e_1 + q(x,\xi)e_2 = \begin{bmatrix} \xi_1 - x_2\xi_3 \\ \xi_2 + x_1^{2k+1}\xi_3 \end{bmatrix},\ (x,\xi) \in T^*\mathbb{R}^3,$$

and $A(x,\xi) = A_2(x,\xi) + A_1(x,\xi)$, where

$$A_2(x,\xi) = L(x,\xi)L(x,\xi)^* = $$
$$= \begin{bmatrix} (\xi_1 - x_2\xi_3)^2 & (\xi_1 - x_2\xi_3)(\xi_2 + x_1^{2k+1}\xi_3) \\ (\xi_1 - x_2\xi_3)(\xi_2 + x_1^{2k+1}\xi_3) & (\xi_2 + x_1^{2k+1}\xi_3)^2 \end{bmatrix},$$

and, for some fixed $\delta \in [0,1)$,

$$A_1(x,\xi) = \delta\{p,q\}(x,\xi)\mathrm{Im}(e_2 \otimes e_1) = \frac{\delta}{2}\Big((2k+1)x_1^{2k} + 1\Big)\xi_3 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

In the case of Example 1.5, $(x,\xi) \in T^*\mathbb{R}^2$, $p(x,\xi) = \xi_1$, $q(x,\xi) = x_1\xi_2$, Σ is defined by $x_1 = \xi_1 = 0$, $\xi_2 \neq 0$,

$$L(x,\xi) = \frac{1}{2}\xi_1 \begin{bmatrix} 1 \\ i \\ \sqrt{2} \end{bmatrix} + \frac{1}{2}x_1\xi_2 \begin{bmatrix} 1 \\ i \\ -\sqrt{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \xi_1 + x_1\xi_2 \\ i(\xi_1 + x_1\xi_2) \\ \sqrt{2}(\xi_1 - x_1\xi_2) \end{bmatrix},$$

and $A(x,\xi) = A_2(x,\xi) = L(x,\xi)L(x,\xi)^*$.

In the case of Example 1.6, $(x,\xi) \in T^*\mathbb{R}^2$, $p(x,\xi) = \xi_1$, $q(x,\xi) = x_1\xi_2$, Σ is again defined by $x_1 = \xi_1 = 0$, $\xi_2 \neq 0$,

$$L(x,\xi) = \frac{1}{\sqrt{3}}\xi_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{\sqrt{3}}x_1\xi_2 \begin{bmatrix} 0 \\ i \\ -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} \xi_1 \\ ix_1\xi_2 \\ \xi_1 - \xi_2 \\ \xi_1 + \xi_2 \end{bmatrix},$$

and $A(x,\xi) = A_2(x,\xi) = L(x,\xi)L(x,\xi)^*$.
In the case of Example 1.7, $(x,\xi) \in T^*\mathbb{R}^2$, $p(x,\xi) = \xi_1$, $q(x,\xi) = x_1^k \xi_2$, $k \geq 1$, Σ is again defined by $x_1 = \xi_1 = 0$, $\xi_2 \neq 0$,

$$L(x,\xi) = \xi_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_1^k \xi_2 \begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} \xi_1 \\ i x_1^k \xi_2 \end{bmatrix},$$

and $A(x,\xi) = A_2(x,\xi) = L(x,\xi)L(x,\xi)^*$.
Theorems 5.1 and 6.1 yield that all the above $A^{\mathrm{w}}(x, D_x)$ do not satisfy the Fefferman-Phong inequality and that the Sharp Gårding inequality cannot be improved in these cases.

We leave it to the reader to write down other possible examples.

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