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REPORT No. 16, 2002/2003, fall

ISSN 1103-467X
ISRN IML-R- -16-02/03- -SE+fall

INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

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January 29, 2003

Abstract

For positive elliptic differential operators Δ, the asymptotic expansion of the heat trace $tr(e^{-t\Delta})$ and its related zeta function $\zeta(s,\Delta) = tr(\Delta^{-s})$ have numerous applications in geometry and physics. This article discusses the general nature of the boundary conditions that must be considered when there is a singular stratum, and presents three examples in which a choice of boundary conditions at the singularity must be made. The first example concerns the signature operator on a manifold with a singular stratum of conic type. The second concerns the "Zaremba problem" for a nonsingular manifold with smooth boundary, posing Dirichlet conditions on part of the boundary and Neumann conditions on the complement; the intersection of these two regions can be viewed as a singular stratum of conic type, and a boundary condition must be imposed along this stratum. The third example is a one-dimensional manifold where the operator at one end has a singularity like that in conic problems, and the choice of boundary conditions affects not just the residues at the poles of the zeta function, but also the very location of the poles.

1 Introduction

For positive elliptic differential operators Δ, the asymptotic expansion of the heat trace $tr(e^{-t\Delta})$ and its related zeta function $\zeta(s,\Delta) = tr(\Delta^{-s})$ have numerous applications in geometry and physics. The first case to be treated was that of smooth manifolds without boundary, by Minakshisundaram and Pleijel (Ref. 1), generalized in (Ref. 2); see also (Refs. 3, 4, 5). The case of operators with differential boundary conditions was treated in (Refs 6, 7, 8, 4), and pseudodifferential boundary conditions were considered in (Refs 9, 10, 11); a much more complete review is found in (Ref. 12). The expansion in the case of manifolds with conic singularities was initiated by Cheeger (Ref 13), and pursued in (Refs 14, 15, 16). Systematic analysis of conic singularities (without particular consideration of the heat trace) was taken up by Kondratiev and his successors (Refs 17, 18, 19). For conic singularities with 0-dimensional singular set, Gil has shown the expansion of the heat trace (Ref 20). The bibliographies in these references serve to supplement this sketchy review.

This article discusses the general nature of the boundary conditions that must be considered when there is a singular stratum of arbitrary dimension, and presents three examples in which a choice of boundary conditions at the singularity must be made. The first example (not yet published), due to Brüning and the author, concerns the signature operator on a manifold with a singular stratum of conic type. The second (Ref 21) concerns the "Zaremba problem" for a nonsingular manifold with smooth boundary, posing Dirichlet conditions on part of the boundary and Neumann conditions on the complement; the intersection of these two regions can be viewed as a singular stratum of conic type, and a boundary condition must be imposed along this stratum. The third example (Ref 22) is a one-dimensional manifold where the operator at one end has a singularity like that in conic problems, and the choice of boundary conditions affects not just the residues at the poles of the zeta function, but also the very location of the poles.

Sections 2 and 3 of this report are based on joint work with J. Brüning. The author is grateful to the Mittag-Leffler Institute for support, to H. Palomir and P. Pisani for bringing the example (Ref 22) to his attention, and to S. A. Nazarov for bringing to his attention (Ref 18) .

2 Boundary Conditions in the Low Eigenspaces

It simplifies the picture to consider a first order Dirac operator which, near a simple conic singularity, can be expressed in the form

$$D = \Gamma[\partial_x + x^{-1}A(x)]1 \tag{1}$$

where x is distance from the vertex of the cone; $A(x)$ is a family of first order self-adjoint elliptic operators on a compact manifold (the cross section of the cone), with appropriate boundary conditions if necessary; and

$$\Gamma = -\Gamma^*, \quad \Gamma\Gamma^* = -I, \quad \Gamma A + A\Gamma = 0.$$

We expand in eigenfunctions $\phi_a$of $A(0)$:

$$u(x) = u_a(x)\phi_a, \qquad ain\ spec(A(0)).2 \tag{2}$$

Then

$$\Gamma^{-1}Du(x) = [\partial_x + x^{-1}a]u_a(x)\phi_a + \widetilde{A}(x)u(x)3 \tag{3}$$

where $\widetilde{A}(x) = x^{-1}[A(x) - A(0)]$ is a smooth family of first order operators. For small x, the term $x^{-1}A(0)$ dominates $\widetilde{A}(x)$, and so the analysis concerns just the eigenfunction sum in (3). (See Ref. 23.) If $[\partial_x + x^{-1}a]u_a(x) = f_a(x)$ is in L^2 then, for $a > -1/2$,

$$u_a(x) = c_a(u)x^{-a} + x^{-a}\int_0^x y^a f_a(y)dy4 \tag{4}$$

while for $a \leq -1/2$

$$u_a(x) = u_a(1)x^{-a} + x^{-a}\int_1^x y^a f_a(y)dy.5 \tag{5}$$

The integral in (4) is $O(\|f_a\|\, x^{1/2})$, and so $c_a(u) = 0$ for $a \geq 1/2$. Similarly, for $a \leq -1/2$, $u_a(x) = O(|x\log x|^{1/2})$. For u in the maximal domain of D we find

$$u(x) =_{|a|<1/2} c_a(u)x^{-a}\phi_a + \widetilde{u}(x), \qquad \widetilde{u}(x) = O(|x\log x|^{1/2}) \quad as \quad x \to 0.$$

The map $u \mapsto_{|a|<1/2} c_a(u)\phi_a$ is continuous from the maximal domain of D to the finite-dimensional space $W =_{|a|<1/2} [A(0) - a]^{-1}(0)$. The closed realizations of D correspond to the subspaces $S \subset W$; the realization D_S has domain defined by $_{|a|<1/2}c_a(u)\phi_a \in S$. The minimal domain corresponds to $S = 0$. The formal adjoint of D is

$$D' = -[\partial_x - x^{-1}A(x)]\Gamma^* = \Gamma[\partial_x - x^{-1}A(x)]$$

and

$$(Du, v) - (u, D'v) =_{|a|<1/2} \langle c_a(u)\Gamma\phi_a, c_{-a}(v)\phi_a\rangle.6 \tag{6}$$

Thus $D_S^* = (D')_{S'}$, where S' is the orthogonal complement of S with respect to the symplectic form on the right hand side of (6).

Imagine a homotopy of A in which an eigenvalue a migrates from $|a| < 1/2$ to $a \geq 1/2$. When a reaches $1/2$ then the constant c_a must be 0; thus, in order to have a continuous homotopy, the boundary conditions should require $c_a(u) = 0$ as $a \to 1/2$. On the other hand, if a migrates to $a \leq -1/2$ then c_a should be unrestricted by the boundary conditions.

The conic operator D in (1) can be compared to the Atiyah-Patodi-Singer operator

$$D_{APS} = \Gamma[\partial_x + A(x)].7 \tag{7}$$

The Atiyah-Patodi-Singer boundary condition for (7) requires that $u(0) =_{a<0} u_a(0)\phi_a$, whereas the condition $u \in L^2$, $Du \in L^2$ for (1) requires that $c_a\phi_a = 0$ for $a \geq 1/2$. Thus the two conditions are essentially the same, except for finitely many small eigenvalues of $A(0)$.

We turn now from singularities of dimension 0 to the case of a conic singular stratum Σ (e.g. the edge of a wedge) with a Dirac operator of the form

$$D = \Gamma[\partial_x + x^{-1}A(x,s) + B(x,s)].8 \tag{8}$$

Near Σ, the manifold is fibred over Σ; x is distance from Σ; s varies in Σ; $A(x,s)$ is a family of first order elliptic operators on the fibres over Σ; and $B(x,s)$ is essentially a family of first order operators along Σ, satisfying appropriate commutation relations with A and Γ. The behavior of D near Σ is controlled by

$$D_0 = \Gamma[\partial_x + x^{-1}A(0,s) + B(0,s)].9 \tag{9}$$

If $A(0,s)$ has eigenvalues with $|a(s)| < 1/2$, then some boundary condition must be imposed along Σ. In favorable cases, there are local conditions involving those low eigenvalues. Since the corresponding eigenspaces vary with s, we are in the homotopy situation envisioned above, and the conditions should conform to the continuity requirements described there.

3 The Signature Operator

The signature operator on a Riemannian manifold M of even dimension m is based on a decomposition of the differential forms into $\pm$ eigenbundles of a certain involution (Ref 4). A manifold with a singular stratum might arise, say, from a group of isometries of a nonsingular manifold with a fixed point set Σ, which becomes a singular stratum for the orbit space of the group action. Near Σ, the manifold takes the form $\mathbb{R}^+ \times N$ with N nonsingular, and fibred over Σ, having fibres of dimension v. The metric on M is essentially

$$dx^2 + g_\Sigma + x^2 g_V(s)$$

where g_Σ is a metric on Σ, and $g_V(s)$ is a family of metrics on the fibres. The signature operator D takes the form (8). The operator $A(0,s)$ acts in the fibres, and has the following eigenvalues:

$$v/2 - j, \quad 0 \le j \le v, \quad from j - dimensional harmonic forms on the fibres;$$

$$\frac{1}{2} \pm \sqrt{\kappa + [(v+1)/2 - j]^2}, \quad 1 \le j < (v+1)/2$$

$$-\frac{1}{2} \pm \sqrt{\lambda + [(v-1)/2 - j]^2}, \quad 0 \le j < (v-1)/2.$$

Here κ (resp λ) is an eigenvalue of the Laplacian on the fibre restricted to the image of d (resp δ). Thus if the $v/2$ dimensional cohomology of the fibre is 0, the eigenvalues divide naturally into two families, one of which remains in $\{a > -1/2\}$ and the other in $\{a < 1/2\}$. These families are stable as s varies in Σ, and also stable with respect to deformations of the metric on the fibres. In the eigenspaces for $\{a > -1/2\}$, an appropriate boundary condition is $u(x) = O(x^{1/2-\varepsilon})$ for all $\varepsilon > 0$; in those for $\{a < 1/2\}$, it is $u(x) = O(x^{-1/2+\delta})$ for some $\delta > 0$. With these boundary conditions, the Laplacian $\Delta = D^*D$ has a heat trace expansion

$$tr(e^{-t\Delta}) \sim t_{n=0}^{-m/2\infty} \left[t^n \int_M a_n + t^{(v+1+n)/2} \int_\Sigma (b_n + b_n' \log t) \right] .10 \qquad (10)$$

Here a_n is the usual locally defined form on M, with the integral suitably regularized; b_n and b_n' are forms defined locally along Σ by spectral data from the operators $A(0,s)$. This expansion is obtained in (Ref 16) for the case where the two families referred to above lie respectively in $\{a \ge 0\}$ and $\{a \le 0\}$, but it is valid in the generality described here.

If the fibre has nontrivial cohomology in dimension $v/2$, the associated harmonic eigenforms define a bundle over Σ corresponding to a stable 0 eigenvalue of A. In this bundle, the operator has the form $\Gamma[\partial_x + B(x,s)]$, essentially an Atiyah-Patodi-Singer type operator on a manifold with boundary Σ. There need not be boundary conditions for this which are local in Σ, and so APS conditions may be required. This case has not yet been sufficiently investigated. Presumably, in addition to the terms in (10), the expansion would contain terms which are global in Σ, such as an eta invariant associated with the operator $B(0,s)$.

4 The Zaremba Problem

As in (Ref 21), consider a manifold M of dimension d, with boundary a disjoint union

$$\partial M = \partial M_D \cup \partial M_N \cup \Sigma$$

where Σ is a smooth embedded submanifold of ∂M. Let Δ be the Laplace operator on M, imposing Dirichlet conditions along ∂M_D and Neumann conditions along ∂M_N. Denote by r the distance from Σ. Near Σ, after an appropriate change of dependent variable, the Laplacian acts essentially as an unbounded operator in $L^2(\mathbb{R}^+ \times [0,\pi] \times \Sigma)$ which takes the form

$$\Delta = -\partial_r^2 + r^{-2}A + \Delta_\Sigma + R.$$

Here Δ_Σ is the Laplacian on Σ, and the remainder R does not affect the outcome qualitatively. The operator A is

$$A = -(\partial_\theta^2 + 1/4) \quad on \ \{f \in H^2(0,\pi) : \ f(0) = 0, \quad f'(\pi) = 0\}$$

thus with Dirichlet condition at $\theta = 0$ and Neumann condition at $\theta = \pi$. The eigenvalues of A are $\{n(n+1)\}_0^\infty$. For this second order operator, the "low eigenvalues" that require some boundary condition are those in the range $-1/4 \leq a < 3/4$; in the present case, we have the eigenvalue 0 in that range. In the corresponding eigenspace, near Σ, the operator reduces to $-\partial_r^2 + \Delta_\Sigma + R$ on $\mathbb{R}^+ \times \Sigma$, and we want to impose some self-adjoint boundary condition at $r = 0$. The Dirichlet condition is often tacitly assumed, but others are of course possible. For the Dirichlet condition along Σ, the analysis in (Ref 16) guarantees an asymptotic expansion

$$tr(e^{-t\Delta}) \sim$$

$$\sim t_{j=0}^{-d/2\infty} t^{j/2} \left[{}_M a_j^M + t^{1/2}({}_{\partial M_D} a_{j+1}^D + {}_{\partial M_N} a_{j+1}^N) + t_\Sigma (a_{j+2}^\Sigma + b_{j+2}^\Sigma \log t). \right] 11 \tag{11}$$

However, the analysis in (Ref 21) shows that in this case the log terms are absent. The first term coming from the singular stratum Σ is $\frac{-1}{16}(4\pi t)^{1-d/2}\mathrm{vol}(\Sigma)$; this particular coefficient is confirmed in (Ref 24). If one used the Neumann condition along Σ, this coefficient should be $\frac{-9}{16}(4\pi t)^{1-d/2}\mathrm{vol}(\Sigma)$, altho that case is not covered by (Ref 21), and the existence of the full expansion has not actually been established.

The Dirichlet and Neumann conditions are essentially scaling invariant, but Robin conditions are not. The next example suggests that Robin conditions on Σ could introduce previously unexpected powers of t in the heat trace expansion.

5 An Interesting Example

In (Ref 22), the authors consider boundary conditions which give rise to powers in the heat expansion other than the usual half integers in (10) and (11). The operator, acting in $L^2(x \geq 0)$, is simply

$$\Delta = -\partial_x^2 + gx^{-2}$$

with a constant $g \geq 0$. This corresponds to a conic singularity where the cross section of the cone is just a single point. For $g < 3/4$, functions u in the maximal domain of Δ satisfy

$$\begin{aligned} u(x) &\sim C_+ x^{a+} + C_- x^{a-} + O(x^{3/2}) \\ a\pm &= \frac{1}{2} \pm \sqrt{g + 1/4}. \end{aligned}$$

Self-adjoint boundary conditions are given by

$$\alpha C_+ + \beta C_- = 012 \qquad (12)$$

with α, β real and $\alpha^2 + \beta^2 = 1$. When either $\alpha = 0$ (Dirichlet) or $\beta = 0$ (Neumann), the domain is invariant under the scaling $x \mapsto cx$, and the powers in the heat expansion are half integers $t^{n/2-1}, n = 0, 1, 2, \ldots$. (The powers begin with $t = -1$, not $t = -1/2$ as for a second order operator on a finite interval; in the present case, the interval is unbounded.) Correspondingly, $\Gamma(s)\zeta(s, \Delta)$ has simple poles at $s = 1 - n/2$. But when $\alpha\beta \neq 0$ (Robin), (Ref 22) shows further poles at

$$s = -N\sqrt{g + 1/4} - 2n, \quad N = 1, 2, \ldots, \quad n = 0, 1, 2, \ldots.$$

The residues depend on the parameter (α, β) in (12), and tend to 0 if the boundary conditions are deformed to Dirichlet or Neumann.

The same phenomenon can be shown if the operator is restricted to a finite interval $[0, 1]$ with ordinary boundary conditions at $x = 1$, and for the natural range $-1/4 \leq g < 3/4$; in this case, the first pole is at $s = 1/2$, but there are still the unusual poles for Robin boundary conditions at the singular point $x = 0$.

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