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A SZEGŐ CONDITION FOR A MULTIDIMENSIONAL SCHRÖDINGER OPERATOR

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ABSTRACT. We consider spectral properties of a Schrödinger operator perturbed by a potential vanishing at infinity and prove that the corresponding spectral measure satisfies a Szegő type condition.

1. INTRODUCTION

In this paper we consider a multidimesional Schrödinger operator in $L^2(\mathbb{R}^d)$ and introduce a version of a Szegő condition (see (2.5)). Szegő's condition is well known for spectral measures of Jacobi matrices (see, for example [3], [8], [9], [10], [11], [13]). Our condition seems to be comparable with the classical Szegő condition only for small energies. The corresponding condition for large energies is much weaker than the expected one and cannot be obtained by the methods of this paper.

One of the motivations of this article is the conjecture of B. Simon from [11].

Conjecture: Let V be a real function on $\mathbb{R}^d$, $d \geq 2$, which obeys

$$\int |x|^{-d+1}|V(x)|^2\,dx < \infty. \tag{1.1}$$

Then $-\Delta + V$ has the a.c. spectrum of infinite multiplicity essentially supported by $[0, \infty)$.

Note that for spherically symmetric potentials this result follows from the paper Deift-Killip [2], where this conjecture is solved for $d = 1$. For $d \geq 2$ it is still open.

The conditions imposed on the potential in our main Theorem 2.1 are much more restrictive than those in (1.1). In fact they are close to the conditions under which absolute continuity of the spectrum can be proven by the methods of the scattering theory. However our work differs from the results obtained in the scattering theory in a critical way: we prove a certain estimate showing that the spectral measure of the Schrödinger operator can not be too small and this estimate turns to be of an independent interest. The

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Szegő condition (2.5) can be interpreted as a version of the Lieb-Thirring inequalities for the a.c. part of the spectrum.

It is known that one can write a so called Lieb-Thiring bound for the eigenvalues of the operator $-\Delta + V$.

When proving the main result we are able to use an analog of Buslaev-Faddeev-Zakharov trace formulae well known for one dimensional Schrödinger operators. The multidimensional case is reduced to a problem for a second order elliptic intergo-differential operator. One of the main difficulties of this appoach is the treatment of the ”potential” type term which appears to be a dissipative integral operator depending on the spectral parameter. The corresponding Fredholm equation for the Jost functions might be not solvable for a discrete subset of the upper half of the complex plane. There is a hope that the corresponding contribution to the trace formulae coming from this subset can be controled by some Lieb-Thirring ineqaulities. Fortunately the positivity of the imaginary parts of the points from this subset appears together with the ”right” sign in the so-called ”first” trace formula. The contribution of these points in the ”second” trace formula is distructive and requires some upper estimates. This explains why in our main theorem we obtain condition involving the first power of the potential V rather than V^2.

2. The main result

Let Ω_1 be the unit ball in $\mathbb{R}^d$ and V be a real valued function on $\mathbb{R}^d \setminus \Omega_1$. We consider the operator $H + V = -\Delta + V$ on $L^2(\mathbb{R}^d \setminus \Omega_1)$ with the Dirichlet boundary conditions on $\partial\Omega_1 = \mathbb{S}^{d-1}$. We can assume without loss of generality that there is $c_1 > 1$ such that

$$V + \frac{\alpha_d}{|x|^2} = 0 \quad \text{for} \quad 1 < |x| < c_1, \tag{2.1}$$

where $\alpha_d = \frac{(d-1)^2}{4} - \frac{d-1}{2}$. Let $E_{H+V}(\delta)$, $\delta \subset \mathbb{R}$, be the spectral projection of the operator $H + V$. We construct a measure μ on the the real line such that for spherically symetric functions f

$$(E_{H+V}(\delta)f, f) = \int_\delta |F(\lambda)|^2 d\mu(\lambda), \quad \delta \subset \mathbb{R}, \tag{2.2}$$

where

$$F(\lambda) = \frac{1}{k}\int_0^{c_1} \sin(k(r-1))f(r)\, r^{(d-1)/2} dr, \quad \operatorname{supp} f \subset \{x : 1 < |x| < c_1\}. \tag{2.3}$$

and $k^2 = \lambda$.

Let $\mathbb{Q} = [0,1)^d$. Then cubes $\mathbb{Q}_n = \mathbb{Q} + n,\ n \in \mathbb{Z}^d$, form a partition of $\mathbb{R}^d$ with which we associate the classes of functions u such that the sequence of (quasi)norms $\{\|u\|_{L^q(\mathbb{Q}_n)}\}_{n=1}^\infty \in \ell^p$, $0 < p, q \le \infty$. These classes are denoted by $\ell^p(\mathbb{Z}^d; L^q(\mathbb{Q}))$. When proving the main result we need the boundedness of the operators in (7.4). For example this can be provided by the following local condition on V from the paper [1]

$$V \in \ell^\infty(\mathbb{Z}^d; L^q(\mathbb{Q})), \quad q > d/2. \tag{2.4}$$

This condition can be weakened by using the characterization of weak Hardy's weights in terms of capacities obtained by Maz'ya (see [7]). Note that under the condition (2.4) the operator $H + V$ can be easily defined in the sense of quadratic forms (see [1]).

Theorem 2.1. *Let V be a real valued function on $\mathbb{R}^d \setminus \Omega_1$ which obeys* (2.4) *and such that*

$$\int_{\mathbb{R}^d\setminus\Omega_1} V_-^{(d+1)/2}(x)\,dx < \infty, \quad \int_{\mathbb{R}^d\setminus\Omega_1} V_+|x|^{-d+1}\,dx < \infty,$$

where $2V_\pm = |V| \pm V$. Then

$$\int_0^\infty \frac{\log(1/\mu'(t))\,dt}{(1+t^{3/2})\sqrt{t}} < \infty. \tag{2.5}$$

If (2.1) *is satisfied then the condition* (2.5) *is equivalent to*

$$\int_0^\infty \frac{\log\big(\frac{d}{d\lambda}(E_{H+V}(\lambda)f, f)\big)\,d\lambda}{(1+\lambda^{3/2})\sqrt{\lambda}} > -\infty, \tag{2.6}$$

for any bounded spherically symmetric function $f \neq 0$ with $\operatorname{supp} f \subset \{x : 1 < |x| < c_1\}$.

Remark 1. Convergence of the integral in (2.5) guaranties that the a.c. spectrum of $-\Delta + V$ is essentially supported by $[0,\infty)$, since $\mu' > 0$ almost everywhere. The condition (2.5) gives a quantative information about the measure μ.

Remark 2. The equivalence of (2.5) and (2.6) follows from the fact that if F is defined as in (2.3) then the function $(1+\lambda^2)^{-1}\log(|F(\lambda)|)$ is in $L^1(\mathbb{R}_+)$ by a classical result from the book [4] (section IIIG2).

3. Reduction to a one-dimensional problem

In this section we assume that $V \in C_0^\infty$ and often use polar coordinates $(r,\theta),\ x = r\theta \in \mathbb{R}^d$, $\theta \in \mathbb{S}^{d-1}$. Denote by $\{Y_j\}_{j=0}^\infty$ the orthonormal in

$L^2(\mathbb{S}^{d-1})$ basis of (real) spherical funcions and let P_j be the orthogonal projection given by

$$P_j u(r,\theta) = Y_j(\theta)\int_{\mathbb{S}^{d-1}} Y_j(\theta')u(r,\theta')\,d\theta'.$$

Clearly $P_0 u$ depends only on r. Denote

$$V_1 = P_0 V P_0, \quad H_1 = P_0 H P_0,$$
$$V_{1,2} = P_0 V(I-P_0), \quad V_{2,1} = V_{1,2}^*,$$
$$V_2 = (I-P_0)V(I-P_0), \quad H_2 = (I-P_0)H(I-P_0).$$

Then the operator $H+V-z$ can be represented as a matrix:

$$H+V-z = \begin{pmatrix} H_1+V_1-z & V_{1,2} \\ V_{2,1} & H_2+V_2-z \end{pmatrix},$$

and the equation

$$(H+V-z)u = P_0 f, \quad \text{Im } z \neq 0,$$

is equivalent to

(3.1) $$(H_1+T_z-z)P_0 u = P_0 f, \qquad (H_2+V_2-z)^{-1}V_{2,1}P_0 u = (P_0-I)u.$$

Here the operator T_z is defined by

$$T_z = V_1 - V_{1,2}(H_2+V_2-z)^{-1}V_{2,1}$$

on $L^2((1,\infty), r^{d-1}\,dr)$.

By using the unitary operator from $L^2((1,\infty),dr)$ to $L^2((1,\infty), r^{d-1}\,dr)$

$$Uu(r) = r^{-(d-1)/2}u$$

we reduce (3.1) to the problem for the following one-dimensional Schrödinger operator in $L^2(1,\infty)$

(3.2) $$A_z u(r) = -\frac{d^2u}{dr^2} + Q_z u, \quad u \in L^2(1,\infty),\ u(1)=0,$$

where

$$Q_z = V_1 + \frac{\alpha_d}{r^2} - V_{1,2}(U^*H_2U + V_2 - z)^{-1}V_{2,1}, \quad \alpha_d = \frac{(d-1)^2}{4} - \frac{d-1}{2}.$$

By considering the potential

$$V - \frac{\alpha_d}{r^2} \quad \text{instead of} \quad V$$

without loss of generality we can assume that

(3.3) $$Q_z = V_1 - V_{1,2}(B+V_2-z)^{-1}V_{2,1},$$

where

(3.4) $$Bu = -\frac{d^2u}{dr^2} - \frac{\Delta_\theta u}{r^2}, \quad u(1,\theta) = 0.$$

Note that the conditions of Theorem 2.1 on V will not be changed.

Then according to (3.1) we obtain

$$P_0(H+V-z)^{-1}P_0 = U(A_z-z)^{-1}U^*. \tag{3.5}$$

We see also that if supp $V \subset \{x \in \mathbb{R}^d : \quad c_1 < |x| < c_2\}$, $c_1 > 1$, then for the operator (3.3) we have

$$Q_z = Q_z\chi = \chi Q_z,$$

where χ is an operator of multiplication by the characteristic function of the interval (c_1, c_2), $c_1 > 0$. It is impotant for us that Q_z is an analytic operator valued function of z with a negative imaginary part in the upper half plane and which has a positive imaginary part in the lower half plane.

4. Green's function.

Let us consider the equation

$$(A_z\psi)(r) := -\frac{d^2}{dr^2}\psi(r) + (Q_z\psi)(r) = z\psi(r), \quad r \geq 1, \ z \in \mathbb{C}, \tag{4.1}$$

with Q_z given by (3.3). For all z except perhaps a discret sequence of points, there exists a unique analytic in k solution $\psi_k(r)$, such that

$$\psi_k(r) = \exp{(ikr)}, \quad k^2 = z, \ \text{Im}\, k > 0, \ \forall r > c_2.$$

Consider the resolvent operator $R(z) = (A_z - z)^{-1}$. If χ_{c_1} is the operator of multiplication by the characteristic function of $(1, c_1)$. Then $R(z)\chi_{c_1}$ is an integral operator whose kernel satisfies the relation:

$$G_z(r,s) = \begin{cases} \frac{\psi_k(s)}{\psi_k(1)}\frac{\sin(k(r-1))}{k}, & \text{for } r < s < c_1, \\ \frac{\psi_k(r)}{\psi_k(1)}\frac{\sin(k(s-1))}{k}, & \text{for } s < \min\{c_1, r\}. \end{cases} \tag{4.2}$$

Indeed, assuming that supp$(f) \subset (1, c_1)$ we can easily check that the function

$$u(r) = \frac{1}{\psi_k(1)}\Big\{\int_r^\infty \frac{\sin(k(r-1))}{k}\psi_k(s)f(s)\,ds + \int_1^r \psi_k(r)\frac{\sin(k(s-1))}{k}f(s)\,ds\Big\}$$

satisfies the equation

$$-\frac{d^2}{dr^2}u(r) + (Q_z u)(r) - zu(r) = f(r), \quad r \geq 1, \ z \in \mathbb{C}, \tag{4.3}$$

and moreover $u(1) = 0$.

5. Wronskian and properties of the M-function.

Let now as above

$$Q_z = V_1 - V_{1,2}(B + V_2 - z)^{-1} V_{2,1}.$$

The function

$$M(k) = \frac{\psi_k'(1)}{\psi_k(1)}$$

is called the Weyl M-function of the operator (4.1). Let us consider the Wronskian

$$W[\overline{\psi_k}, \psi_k](r) = \overline{\psi_k'}(r)\psi_k(r) - \overline{\psi_k}(r)\psi_k'(r). \tag{5.1}$$

Note that $\overline{\psi_k}$ satisfies the equation (4.1) with $Q_{\overline{z}}$ and $\overline{z}$ instead of Q_z and z. Since ψ_k is a solution of the equation (4.1) we find

$$\frac{d}{dr} W[\overline{\psi_k}, \psi_k](r) = (z - \overline{z})\overline{\psi_k}(r)\psi_k(r) + (Q_{\overline{z}}\overline{\psi_k})(r)\psi_k(r) - \overline{\psi_k}(r)(Q_z\psi_k)(r).$$

So we obtain

$$\pm \text{Im}\, \{W[\overline{\psi_k}, \psi_k](c_2) - W[\overline{\psi_k}, \psi_k](c_1)\} \geq 0, \quad \text{for } \pm \text{Im}\, z \geq 0+, \tag{5.2}$$

which means that for all k we have the following inequality

$$\frac{k}{\text{Im}\, M(k)} \leq |\psi_k(1)|^2.$$

Moreover, if we represent the solution ψ_k for real k in the form

$$\psi_k(x) = a(k)e^{ikx} + b(k)e^{-ikx}, \quad x < c_1,$$

then it follows from (5.2) that

$$|a|^2 - |b|^2 \geq 1.$$

Assume that Q_z is the above introduced operator with a smooth compactly supported V. Then for $k^2 = z$

$$M(k) = \psi_k'(1)(\psi_k(1))^{-1} = ik(1 - \rho(k))(1 + \rho(k))^{-1}, \quad \rho(k) := e^{-2ik} b(k) a(k)^{-1}.$$

The latter implies

$$\rho(k) = (ik - M(k))(ik + M(k))^{-1}.$$

Since $|a|^2 - |b|^2 \geq 1$ we obtain that for real k

$$|a(k)|^{-2} \leq 1 - |\rho(k)|^2 =$$

$$\frac{4k \text{Im}\, M}{|ik + M(k)|^2}.$$

Note that since $\text{Im}\, M \geq 0$, then for any $k > 0$ we have

$$|ik + M(k)|^2 = k^2 + |M|^2 + 2k \text{Im}\, M \geq k^2$$

and therefore

$$|a(k)|^{-2} \leq 4k^{-1}\big(\text{Im } M\big), \quad k > 0. \tag{5.3}$$

Note also that

$$\text{Im } M(k) > 0 \quad \text{if Im } k^2 > 0. \tag{5.4}$$

Thus, there are constants $C_0 \in \mathbb{R}$ and $C_1 \geq 0$ and a positive measure μ, such that

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty,$$

where

$$M(k) = C_0 + C_1 z + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big) d\mu(t), \quad k^2 = z. \tag{5.5}$$

Finally, note that $R(z) = P_0(U^*HU + V - z)^{-1}P_0$ and therefore we can write formally that

$$M(k) = \frac{\partial^2}{\partial r \partial s} G_z(r,s)|_{(1,1)} = (P_0(U^*HU + V - z)^{-1}P_0\delta_1', \delta_1'),$$

where δ_1' is the derivative of $\delta(r-1)$. Let χ_{c_1} be the characteristic function of $(1, c_1)$. The representation (4.2) for the resolvent operator gives us the representation for the operator $\chi_{c_1}P_0E_{U^*HU+V}(\delta)P_0\chi_{c_1}$, where $E_{U^*HU+V}(\delta)$ is the spectral measure of $U^*HU + V$:

$$(P_0E_{U^*HU+V}(\delta)P_0f, f) = \int_\delta |F(\lambda)|^2 d\mu(\lambda) \tag{5.6}$$

and where

$$F(\lambda) = \frac{1}{k}\int_0^{c_1} \sin(k(r-1))f(r)\, dr, \quad \text{supp} f \subset (1, c_1), \;\; k^2 = \lambda.$$

Since F is a boundary value of an analytic function, we obtain that $F(\lambda) \neq 0$ for a.e. λ. This means that $E_{H+V}(\delta) \neq 0$ if $\mu' > 0$ a.e. on δ.

6. A TRACE INEQUALITY

In this section we assume that V is not a potential but the operator $\sum_{j=0}^n P_jV\sum_{j=0}^n P_j$, which approximates V for large n. It can be interpreted as an operator of multiplication by a matrix valued function of r. In this case the function V_1 remains the same as before. Since

$$\exp(-ikr)\psi_k(r) = 1 - \frac{1}{2ik}\int_r^\infty (1 - e^{2ik(s-r)})V_1(s)\, ds + o(1/k)$$

we obtain

$$a(k) = \lim_{r\to-\infty} \exp(-ikr)\psi_k(r) = 1 - \frac{1}{2ik}\int V_1\, dr + o(1/k),$$

as $k \to \infty$. Now let $i\beta_m$ and γ_j be zeros and poles of $a(k)$. We shall see in a moment that $\beta_m > 0$. Let $\mathfrak{B}$ be the corresponding Blaschke product

$$\mathfrak{B}(k) = \prod_m \frac{(k - i\beta_m)}{(k + i\beta_m)} \prod_j \frac{(k - \overline{\gamma_j})}{(k - \gamma_j)}.$$

Clearly $|\mathfrak{B}(k)| = 1,\ k \in \mathbb{R}$, and we obtain

$$\int_{-\infty}^{+\infty} \log(a(k)/\mathfrak{B}(k))\, dk = \frac{\pi}{2} \int V_1\, dr + 2\pi\Big(\sum \beta_n - \sum \mathrm{Im}\ \gamma_j\Big). \tag{6.1}$$

In order to prove that $\beta_m > 0$ let us show that $-\beta_m^2$ are the eigenvalues of a certain selfadjoint operator of a Schrödinger type. Namely, let $\hat{H}$ be the operator in $L^2(\mathbb{R}, L^2(\mathbb{S}^{d-1}))$

$$\hat{H}u = -\frac{d^2u}{dr^2} - \chi_1 \frac{\Delta_\theta u}{r^2}, \quad (I - P_0)u(1, \cdot) = 0,$$

where χ_1 is the characteristic function of $(1, \infty)$. Obviously, if $s < c_1 < c_2 < r$, then the kernel of the operator $P_0(\hat{H} + V - z)^{-1}P_0$ equals

$$g(r, s, k) = -\frac{\exp ik(r - s)}{2ika(k)}.$$

On the other hand we can consider the expansion of g near the eigenvalue $-\beta_m^2$. Denote by $\phi_{m,j}(r, \theta)$, $j = 1, 2 \dots n$ the orthonormal system of eigenfunctions corresponding to $-\beta_m^2$. If $\phi_{m,j}^{(0)} = \int_{\mathbb{S}^{d-1}} \phi_{m,j}(r, \theta)\, d\theta$ then

$$g(r, s, k) = \frac{\sum_{j=1}^n \phi_{m,j}^{(0)}(r)\overline{\phi_{m,j}^{(0)}(s)}}{k^2 + \beta_m^2} + g_0(r, s, k), \qquad s < c_1 < c_2 < r,$$

where $g_0(r, s, k) = O(1)$, as $k \to i\beta_m$. This proves that $a(k)$ is a meromorphic funcion and its zeros correspond to the eigenvalues $-\beta_m^2$ of the operator $\hat{H} + V$. Moreover the multiplicities of these zeros are equal to one. The latter arguments were inspired by [5].

Let us prove that by using Lieb-Thirring inequalities [6] we can obtain

$$\sum \beta_n \le C\Big(\int_{\mathbb{R}^d} V_-^{(d+1)/2} dx + \int_{\mathbb{R}^d} V_- |x|^{-d+1} dx\Big). \tag{6.2}$$

Indeed, let $W_- = \sqrt{V_-}$. Then

$$W_-(\hat{H} - z)^{-1}W_- = W_-(B - z)^{-1}W_- + W_-\Theta(z)W_-,$$

where B is defined in (3.4) and $\Theta(z)$ is the operator of rank one with the integral kernel $-e^{2ik(r+s-2)}/2ik$, $k^2 = z$. Therefore

$$||W_-\Theta(z)W_-|| \le \frac{C}{\sqrt{|z|}} \int V_- |x|^{-d+1}\, dx, \quad z < 0. \tag{6.3}$$

Now for any compact operator T and $s > 0$ denote $n_+(s,T) = \operatorname{rank} E_T(s,\infty)$. Then

$$\sum \beta_m = \int_0^\infty n_+(1, W_-(\hat{H}+t)^{-1}W_-)\frac{dt}{2\sqrt{t}} \leq$$

$$\leq \int_0^\infty \big(n_+(1/2, W_-(B+t)^{-1}W_-) + n_+(1/2, W_-\Theta(-t)W_-)\big)\frac{dt}{2\sqrt{t}}.$$

Now the inequality (6.2) follows from

$$\int_0^\infty n_+(1/2, W_-(B+t)^{-1}W_-)\frac{dt}{2\sqrt{t}} \leq C\int_{\mathbb{R}^d} V_-^{(d+1)/2}dx,$$

which is the classical Lieb-Thirring inequality and from

$$\int_0^\infty n_+(1/2, W_-\Theta(-t)W_-)\frac{dt}{2\sqrt{t}} \leq C\int V_-|x|^{-d+1}\,dx,$$

which is implied by (6.3). Consequently, since Im $\gamma_j \geq 0$ the trace formula (6.1) together with (6.2) leads to the inequality

$$\int_{-\infty}^{+\infty} \log|a(k)|\,dk \leq \frac{\pi}{2}\int_{-\infty}^{+\infty} V_1\,dr + +C\Big(\int_{\mathbb{R}^d} V_-^{(d+1)/2}dx + \int_{\mathbb{R}^d} V_-|x|^{-d+1}dx\Big). \tag{6.4}$$

Therefore for any pair of finite numbers $r_2 > r_1 \geq 0$

$$\int_{r_1}^{r_2} \frac{1}{2}\log\frac{k}{4\text{Im}\,M(k)}\,dk \leq \int_{-\infty}^{+\infty} \log|a(k)|\,dk \leq \frac{\pi}{2}\int_{-\infty}^{+\infty} V_1\,dr +C\Big(\int_{\mathbb{R}^d} V_-^{(d+1)/2}dx + \int_{\mathbb{R}^d} V_-|x|^{-d+1}dx\Big). \tag{6.5}$$

7. The end of the proof of Theorem 2.1

Assume that our perturbation V is an arbitrary function satisfying conditions of Theorem 0.1. Then the Weyl function M can be defined for example as $M(k) = \frac{\partial^2}{\partial r\partial s}G_z(r,s)|_{(1,1)}$ where G_z is the integral kernel of the operator $P_0(U^*HU + V - z)^{-1}P_0$. The next proposition allows us to approximate V by compactly supported smooth functions V_n.

Proposition 7.1. *Let V satisfy the conditions of Theorem 2.1. Then there exists a sequence V_n of compactly supported smooth functions converging to V so that*

$$\int (V_n)_-^{(d+1)/2}\,dx < C(V) \quad \text{and} \quad \int (V_n)_+\,|x|^{-d+1}\,dx < C(V) \tag{7.1}$$

and such that the Weyl functions M_n corresponding to V_n converge uniformly when k^2 belongs to any compact subset of the upper half plane:

$$M_n(k) \to M(k).$$

Therefore the sequence of measures μ_n converges weakly to the spectral measure μ.

Proof. Let $W_\pm = \sqrt{V_\pm}$. Since the class C_0^∞ is dense in L^p for any $p > 0$, we can find a pair of sequences W_n^- and $W_n^+ \in C_0^\infty$ satisfying

$$W_n^- \to W_- \ \text{ in } L^{(d+1)}(\mathbb{R}^d); \quad W_n^+ \to W_+ \text{ in } L^2(\mathbb{R}^d, |x|^{-d+1}dx)$$
$$W_n^\pm \to W_\pm \text{ in } \ell^\infty(\mathbb{Z}^d; L^p(\mathbb{Q})), \quad p > d. \tag{7.2}$$

Let us introduce a sequence of functions $\{V_n\}_{n=1}^\infty$ via

$$V_n = (W_n^+)^2 - (W_n^-)^2.$$

Then $V_n \in C_0^\infty$ and the relations (7.1) hold true. Suppose now that $\Gamma_0(z)$ and $\Gamma_n(z)$ are the resolvent operators of $B = U^*(-\Delta)U - \alpha_d/r^2$ and $B_n = B + V_n$ respectively. Denote by δ_1' the derivative of the delta function $\delta(r - 1)$. The expression $\Gamma_0(z)\delta_1'$, $\text{Im}\, z \neq 0$, can be understood as the function

$$\Gamma_0(z)\delta_1' = \exp(ik(r-1)).$$

According to assumptions (7.2) we have that

$$W_n^\pm \Gamma_0(z)\delta_1' \to W_\pm \Gamma_0(z)\delta_1',$$

in $L^2(\mathbb{R}^d)$ (with respect to any weight). Thus in order to prove that the Weyl functions

$$M_n(k) = \frac{\partial^2}{\partial r \partial s} G_{n,z}(r,s)|_{(1,1)} = (\Gamma_n(z)\delta_1', \delta_1')$$
$$= (\Gamma_0(z)\delta_1', \delta_1') - ((W_n^+ - W_n^-)\Gamma_0(z)\delta_1', (W_n^+ + W_n^-)\Gamma_n(\overline{z})\delta_1')$$

converge, it is sufficient to show that

$$(W_n^+ + W_n^-)\Gamma_n(\overline{z})\delta_1' \to (W_+ + W_-)(B + V - \overline{z})^{-1}\delta_1' \tag{7.3}$$

in $L^2(\mathbb{R}^d)$.

Let us denote $W_n = W_n^+ + W_n^-$ and $W_n{}^{(0)} = W_n^+ - W_n^-$. Clearly, if $W_n^\pm \to W_\pm$ in the class (2.4) with $q > d$, as $n \to \infty$, then

$$W_n \Gamma_0(\overline{z}) W_n^{(0)} \to (W_+ + W_-)\Gamma_0(\overline{z})(W_+ - W_-) \tag{7.4}$$

in the operator norm topology.

Then (7.3) follows from the identity

$$W_n \Gamma_n(\overline{z})\delta_1' = (I + W_n \Gamma_0(\overline{z}) W_n^{(0)})^{-1} W_n \Gamma_0(\overline{z})\delta_1'.$$

□

Similarly we can prove the following

Proposition 7.2. *Let V be a compactly supported smooth function. Then the Weyl functions M_l corresponding to $\sum_{j=0}^l P_j V \sum_{j=0}^l P_j$ converge uniformly to M when k^2 belongs to any compact subset K of the upper half plane*

$$M_l(k) \to M(k)$$

and therefore the sequence of measures μ_l converges weakly to the spectral measure μ constructed for V.

Proof. Let us denote $V_l = \sum_{j=0}^l P_j V \sum_{j=0}^l P_j$ let $\Gamma_0(z)$ and let $\Gamma_l(z)$ be the resolvent operators of $B = U^*(-\Delta)U - \alpha_d/r^2$ and $B_l = B + V_l$ respectively. As in Proposition 7.1 the expression $\Gamma_0(z)\delta_1'$, $\mathrm{Im}\, z \neq 0$, is understood as the function $\Gamma_0(z)\delta_1' = \exp(ik(r-1))$. According to our assumptions

$$V_l\Gamma_0(z)\delta_1' = \sum_{j=0}^{l} P_j V \Gamma_0(z)\delta_1' \to V\Gamma_0(z)\delta_1',$$

in $L^2(\mathbb{R}^d)$. Thus in order to prove that the Weyl functions

$$M_l(k) = \frac{\partial^2}{\partial r \partial s} G_{n,z}(r,s)|_{(1,1)} = (\Gamma_l(z)\delta_1', \delta_1')$$
$$= (\Gamma_0(z)\delta_1', \delta_1') - (V_l\Gamma_0(z)\delta_1', \Gamma_l(\overline{z})\delta_1')$$

converge, it is sufficient to show that $\Gamma_l(\overline{z})\delta_1'$ converges to $(B+V-\overline{z})^{-1}\delta_1'$ in $L^2(\mathbb{R}^d)$ uniformly on compact subsets K of the complex plane. The latter follows from the identity

$$\Gamma_l(\overline{z})\delta_1' = (B+V-\overline{z})^{-1}\delta_1' - \Gamma_l(\overline{z})(V_l - V)(B+V-\overline{z})^{-1}\delta_1' =$$
$$= (B+V-\overline{z})^{-1}\delta_1' + \Gamma_l(\overline{z})(I - \sum_{j=0}^{l} P_j)V(B+V-\overline{z})^{-1}\delta_1' +$$
$$+\Gamma_l(\overline{z})\sum_{i=0}^{l} P_i V(I - \sum_{j=0}^{l} P_j)(B+V-\overline{z})^{-1}\delta_1'$$

and from the bound

$$||\Gamma_l(\overline{z})|| \le \frac{1}{\mathrm{Im}\, z} \le C, \quad z \in K.$$

□

Finally according to inequality (6.5) and Propositions 7.1, 7.2 we observe that there exists a sequence of measures μ_l weakly convergent to μ, such that for any fixed $c > 0$

$$\int_0^c \frac{\log(1/\mu_l'(t))\, dt}{(1+t^{3/2})\sqrt{t}} < C(V), \quad \forall l,$$

where $C(V)$ is independent of c. Therefore due to the statement on the upper semicontinuity of an entropy (see [3])

$$\int_0^\infty \frac{\log(1/\mu'(t))\,dt}{(1+t^{3/2})\sqrt{t}} < \infty.$$

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