ABSOLUTE CONTINUITY OF SPECTRA OF TWO-DIMENSIONAL PERIODIC SCHÖDINGER OPERATORS WITH STRONGLY SUBORDINATE MAGNETIC POTENTIALS

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Absolute continuity of spectra of two-dimensional periodic Schrödinger operators with strongly subordinate magnetic potentials

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§ 0. Introduction

1. In the present paper we consider the problem of absolute continuity of the spectrum of the periodic Schrödinger operator with variable metric and electric and magnetic fields. Similar problems were considered in a number of papers starting with the original work of L. Thomas [T]. For a review of the subject, see the books [RSi], [Ku] and the surveys [BSu3], [Su] as well (see also the references therein). Here we consider only the two-dimensional case, where particularly complete results are achievable. The two-dimensional case was investigated in [BSu1,2], [Mo], [BShSu], [Sh1–3], [La]. This paper can be considered as a continuation of [Sh3]. Here we improve the result of [Sh3] in the following sense. Under the same conditions on the metric, the electric potential and the 'weight function' as in [Sh3] we significantly relax the conditions on the vector-valued (magnetic) potential $\mathbf{A}$. In [Sh3] we assumed that

$$\int\limits_{\Omega} |\mathbf{A}|^2 \ln^{\alpha}(1+|\mathbf{A}|)\, d\mathbf{x} < \infty, \quad \alpha > 1. \tag{0.1}$$

Here Ω is an elementary cell of the period lattice. Condition (0.1) was earlier used in [La]. Now we replace condition (0.1) by the condition of 'strong subordination' of $\mathbf{A}$:

$$\int\limits_{\Omega} |\mathbf{A}u|^2 d\mathbf{x} \le \varepsilon \int\limits_{\Omega} |\nabla u|^2 d\mathbf{x} + C(\varepsilon; \mathbf{A}) \int\limits_{\Omega} |u|^2 d\mathbf{x}, \quad \forall \varepsilon \in (0,1), \quad u \in H^1(\Omega). \tag{0.2}$$

The condition (0.2) is significantly wider and more convenient than (0.1). This fact turns out to be more evident when the Schrödinger operator in a periodic waveguide is considered. This problem (previously considered in [SoW], [ShaSo], [ShSu], [Sh4]) is also discussed here. In the present paper we replace the condition of the type (0.1) used in [Sh4] by the condition of the form (0.2). As a result, the condition on the magnetic potential is now expressed in terms of the initial problem and does not involve explicitly the quasi-conformal mapping of the waveguide onto the strip (see § 8).

2. *The main results* are Theorem 1.3 and Theorem 8.5. Theorem 1.3 concerns the absolute continuity of the spectrum of the Schrödinger operator in $L_2(\mathbf{R}^2)$ and Theorem 8.5 concerns

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the absolute continuity of the spectrum of the Schrödinger operator in a simply connected periodic waveguide Π.

As usual, our constructions are based upon the Thomas method first proposed in [T] and then developed in [RSi]. This method involves direct integral decomposition for periodic operators. Next, the operators acting on the fibers of the direct integral are extended to the complex values of the fiber parameter (the quasimomentum). Further considerations reduce the problem to estimating the resolvent of these operators for large imaginary values of the quasimomentum. The estimates required in our case are given in Theorem 2.1, from which we deduce Theorem 1.3.

First, we prove Theorem 2.1 (about the estimates) in the case where the metric g is the flat metric and the weight function η equals to 1. Next, the estimates are carried over to the case of a scalar metric and a nontrivial weight function. Finally, we invoke global isothermal coordinates (cf. [KuL], [Sh2]) and prove Theorem 1.3 in its full generality. Theorem 8.5 is stated here without proof. The proof is actually the same as the proof of Theorem 2.5 from [Sh4] (see also Remark 8.6 below).

Some auxiliary facts are borrowed directly from [Sh3,4].

3. The main definitions and results concerning the Schrödinger operator in $L_2(\mathbf{R}^2)$ are presented in § 1. § 2 contains the necessary material concerning the Thomas approach, together with the statement of the basic Theorem 2.1 about estimates. In § 3 we formulate some auxiliary results from the previous papers. Preparations to the proof of Theorem 2.1 are made in § 4. In § 5 we prove Theorem 2.1 with $g = \mathbf{1}$, $\eta = 1$. In § 6 we include a scalar metric into consideration. Finally, in § 7 we complete the proof of Theorem 2.1 and deduce Theorem 1.3 from Theorem 2.1 in the case of a scalar metric. After that, we establish Theorem 1.3 in its full generality. In conclusive § 8 we state without proof the main result concerning the Schrödinger operator in a simply connected periodic waveguide. Some necessary comments are also given.

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§ 1. Definition of the operator. The main result

1. Notation. Let $\Gamma \subset \mathbf{R}^2$ be a lattice, and let $\mathbf{a}_1, \mathbf{a}_2 \in \mathbf{R}^2$ be a basis of Γ. The set

$$\Omega := \{\mathbf{x} \in \mathbf{R}^2 : \ \mathbf{x} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2, \ 0 \le t_j < 1, \ j = 1, 2\}$$

is an elementary cell of the lattice Γ. We fix an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2$ in $\mathbf{R}^2$ such that $\mathbf{e}_1 = \mathbf{a}_1/|\mathbf{a}_1|$. We use the notation $\nabla = \{\partial/\partial x_1, \partial/\partial x_2\} = \{\partial_1, \partial_2\}$, $\mathbf{D} = \{D_1, D_2\} = -i\nabla$. For a real-valued function f, we put $2f_\pm(\mathbf{x}) := |f(\mathbf{x})| \pm f(\mathbf{x})$. Let $\mathcal{D}$ be an open subset in $\mathbf{R}^2$. The Sobolev classes of the order $s > 0$ with the integrability index p are denoted by $W_p^s(\mathbf{R}^2)$, $W_p^s(\mathcal{D})$; for $p = 2$ we abbreviate this to $H^s(\mathbf{R}^2)$, $H^s(\mathcal{D})$. The subspace formed by the functions $u \in W_p^s(\Omega)$, such that the Γ-periodic extension of u belongs to $W_{p,\,\mathrm{loc}}^s(\mathbf{R}^2)$, is denoted by $\widetilde{W}_p^s(\Omega)$; for $p = 2$ we use the notation $\widetilde{H}^s(\Omega) := \widetilde{W}_2^s(\Omega)$. Similarly, $\widetilde{C}^\infty(\Omega)$ is the class of functions such that their Γ-periodic extensions belong to $C^\infty(\mathbf{R}^2)$.

Next, the symbols $\langle\cdot,\cdot\rangle$, $|\cdot|$ stand for the standard inner product and the standard norm in $\mathbf{C}^2$; by $\mathbf{1}$ we denote the unit (2×2)-matrix. The norm in $L_p(\Omega)$, $1 \le p \le \infty$, is denoted by $\|\cdot\|_p$; for $p = 2$ we omit the index p. All the integrals without indication of the integration domain are over $\mathbf{R}^2$. By C, c we denote various positive constants in estimates. For a matrix h the symbol h^t denotes the transposed matrix. For a measurable subset $\Xi \subset \mathbf{R}^2$, by $\operatorname{meas}\Xi$ we denote the two-dimensional Lebesgue measure of Ξ.

Let $\mathcal{B}$ be a Banach space; then the norm in $\mathcal{B}$ is denoted by $\|\cdot\|_{\mathcal{B}}$. Let $\widetilde{\mathcal{B}}$ be another Banach space. For an operator $\mathcal{T}:\ \mathcal{B}\to\widetilde{\mathcal{B}}$, by $\|\mathcal{T}\|_{\mathcal{B}\to\widetilde{\mathcal{B}}}$ we denote the operator norm. Often we write simply $\|\mathcal{T}\|$ if this does not lead to a confusion.

2. A magnetic potential is determined by a measurable $\mathbf{R}^2$-valued function $\mathbf{A}(\mathbf{x}) = A_1(\mathbf{x})\mathbf{e}_1 + A_2(\mathbf{x})\mathbf{e}_2$ such that

$$\mathbf{A}(\mathbf{x}+\mathbf{a}_j) = \mathbf{A}(\mathbf{x}), \quad j = 1,2, \quad \mathbf{x}\in\mathbf{R}^2. \tag{1.1}$$

We also assume that

$$\int_\Omega |\mathbf{A}|^2|u|^2 d\mathbf{x} \le \varepsilon \int_\Omega |\nabla u|^2 d\mathbf{x} + C(\varepsilon;\Omega,\mathbf{A}) \int_\Omega |u|^2 d\mathbf{x}, \quad \forall\varepsilon\in(0,1),\ u\in\widetilde{H}^1(\Omega). \tag{1.2}$$

A metric is determined by a measurable (2×2)-matrix-valued function $g(\mathbf{x}) = \{g^{jl}(\mathbf{x})\}$ with real-valued entries. We assume that

$$g(\mathbf{x}+\mathbf{a}_j) = g(\mathbf{x}), \quad j = 1,2, \quad \mathbf{x}\in\mathbf{R}^2, \tag{1.3}$$

$$c_0\mathbf{1} \le g(\mathbf{x}) \le c_1\mathbf{1}, \quad 0 < c_0 \le c_1 < \infty. \tag{1.4}$$

We present the metric g in the following form:

$$g(\mathbf{x}) := \omega^2(\mathbf{x}) g_0(\mathbf{x}), \quad \det g_0(\mathbf{x}) = 1,\ \omega(\mathbf{x}) := (\det g(\mathbf{x}))^{1/4}. \tag{1.5}$$

We assume that

$$\omega\in\widetilde{H}^1(\Omega), \tag{1.6}$$

and

$$\int_\Omega |\nabla\omega|^2|u|^2 d\mathbf{x} \le \varepsilon \int_\Omega |\nabla u|^2 d\mathbf{x} + C(\varepsilon;\Omega,\omega) \int_\Omega |u|^2 d\mathbf{x}, \quad \forall\varepsilon\in(0,1),\ u\in\widetilde{H}^1(\Omega). \tag{1.7}$$

The following statement was proved in [S, Lemma 2.1].

Proposition 1.1. *Let F be a measurable function such that*

$$\int_\Omega |F| \ln(1+|F|) d\mathbf{x} < \infty.$$

Then

$$\int\limits_{\Omega} |F||u|^2 d\mathbf{x} \le \varepsilon \int\limits_{\Omega} |\nabla u|^2 d\mathbf{x} + C(\varepsilon;\Omega,F) \int\limits_{\Omega} |u|^2 d\mathbf{x}, \quad \forall \varepsilon \in (0,1),\ u \in H^1(\Omega).$$

So, the estimates (1.2), (1.7) are satisfied if

$$\int\limits_{\Omega} |\mathbf{A}|^2 \ln(1+|\mathbf{A}|) d\mathbf{x} < \infty, \qquad \int\limits_{\Omega} |\nabla\omega|^2 \ln(1+|\nabla\omega|) d\mathbf{x} < \infty.$$

Let $d\nu$ be a real-valued Borel signed measure in $\mathbf{R}^2$ with locally finite variation. Thus,

$$|\nu|(\Xi) := \int\limits_{\Xi} |d\nu| < \infty \tag{1.8}$$

for any Borel bounded set $\Xi \subset \mathbf{R}^2$. We assume that $d\nu$ is Γ-periodic, i.e.,

$$\int\limits_{\Xi + n_1\mathbf{a}_1 + n_2\mathbf{a}_2} d\nu(\mathbf{x}) = \int\limits_{\Xi} d\nu(\mathbf{x}),\ \ \mathbf{n} = \{n_1, n_2\} \in \mathbf{Z}^2. \tag{1.9}$$

Presenting the signed measure $d\nu$ in the form

$$d\nu = d\nu_+ - d\nu_-, \quad 2d\nu_\pm := |d\nu| \pm d\nu,$$

we impose the following conditions on $d\nu$, borrowed from [Sh2].

(i) *The form*

$$m_+[u,u] := \int |\nabla u|^2 d\mathbf{x} + \int |u|^2 d\nu_+, \quad u \in C_0^\infty(\mathbf{R}^2),$$

is closable in $L_2(\mathbf{R}^2)$. (This means that for an arbitrary sequence $\{u_n\}$, $u_n \in C_0^\infty(\mathbf{R}^2)$, which converges to zero in $L_2(\mathbf{R}^2)$ and is a Cauchy sequence with respect to the form m_+, we also have $m_+[u_n,u_n] \to 0$ as $n \to \infty$.) The domain of the closure of the form m_+ is denoted by $\widehat{d}$. The set $\widehat{d}$ $(\subset H^1(\mathbf{R}^2))$, generally speaking, does not coincide with $H^1(\mathbf{R}^2)$; this set depends only on $d\nu_+$.

(ii) *For some* $a < 1$, *we have*

$$\int |u|^2 d\nu_- \le a \left(\int \langle g\nabla u, \nabla u\rangle d\mathbf{x} + \int |u|^2 d\nu_+ \right) + C(a;g,d\nu) \int |u|^2 d\mathbf{x}, \quad u \in C_0^\infty(\mathbf{R}^2),\ a<1. \tag{1.10}$$

Note that the *estimate* (1.10) *remains valid for all functions in* $\widehat{d}$.

(iii) *We have*

$$\int\limits_{\Omega} |u|^2 |d\nu| \leq \varepsilon \left(\int\limits_{\Omega} |\nabla u|^2 d\mathbf{x} + \max_{\mathbf{x}\in\Omega} |u|^2 \right) + C(\varepsilon;\Omega,d\nu) \int\limits_{\Omega} |u|^2 d\mathbf{x}, \qquad \forall \varepsilon \in (0,1),\ u \in \widetilde{C}^\infty(\Omega). \tag{1.11}$$

It is easy to see that condition (i) implies the following condition:

(i$'$) *The form*

$$m_+^\Omega[u,u] := \int\limits_{\Omega} |\nabla u|^2 d\mathbf{x} + \int\limits_{\Omega} |u|^2 d\nu_+, \quad u \in \widetilde{C}^\infty(\Omega),$$

is closable in $L_2(\Omega)$. By $\widehat{d}_\Omega$ we denote the domain of the closure of the form m_+^Ω.

Besides, it is shown in [Sh2, § 2] that (1.10) implies the estimate

$$\int\limits_{\Omega} |u|^2 d\nu_- \leq a \left(\int\limits_{\Omega} \langle g\nabla u, \nabla u\rangle d\mathbf{x} + \int\limits_{\Omega} |u|^2 d\nu_+ \right) + C(a;g,d\nu) \int\limits_{\Omega} |u|^2 d\mathbf{x}, \qquad u \in \widetilde{C}^\infty(\Omega),\ a < 1, \tag{1.12}$$

with the same constants a and $C(a;g,d\nu)$. Obviously, the *estimate* (1.12) *remains valid for all functions in* $\widehat{d}_\Omega$.

Let η be a real-valued measurable function such that

$$\eta(\mathbf{x}+\mathbf{a}_j) = \eta(\mathbf{x}), \quad j=1,2,\ \mathbf{x}\in\mathbf{R}^2, \tag{1.13}$$

$$\eta(\mathbf{x}) > 0, \quad \text{a. a. } \mathbf{x}\in\mathbf{R}^2, \tag{1.14}$$

$$\int\limits_{\Omega} |\eta u|^2 d\mathbf{x} \leq \varepsilon \int\limits_{\Omega} |\nabla u|^2 d\mathbf{x} + C(\varepsilon;\Omega,\eta) \int\limits_{\Omega} |u|^2 d\mathbf{x}, \quad \forall \varepsilon \in (0,1),\ u \in \widetilde{H}^1(\Omega). \tag{1.15}$$

Due to Proposition 1.1, the estimate (1.15) is satisfied provided

$$\int\limits_{\Omega} |\eta|^2 \ln(1+|\eta|) d\mathbf{x} < \infty.$$

Note that, by Lemma 1.5 from [Sh3], the estimates (1.2), (1.7), (1.15) are equivalent to the analogous estimates in the class $H^1(\Omega)$.

3. In the Hilbert space $L_2(\mathbf{R}^2)$ we consider the quadratic form

$$\begin{aligned} &m[u,u] = m(g,\mathbf{A},d\nu,\eta)[u,u] := \\ &\int \langle g(\mathbf{D}-\mathbf{A})\eta^{-1}u, (\mathbf{D}-\mathbf{A})\eta^{-1}u\rangle d\mathbf{x} + \int |\eta^{-1}u|^2\, d\nu, \quad u \in \eta\widehat{d}. \end{aligned} \tag{1.16}$$

First of all, note that by the conditions (1.13)–(1.15), the set $\eta\widehat{d}$ is dense in $L_2(\mathbf{R}^2)$.

Proposition 1.2. *Let* $g(\mathbf{x})$ *be a measurable* (2×2)*-matrix-valued function which satisfies* (1.4). *Suppose that a real-valued function* $\eta(\mathbf{x})$ *satisfies conditions* (1.13)–(1.15), *and a vector-valued function* $\mathbf{A}(\mathbf{x})$ *is subject to conditions* (1.1), (1.2). *Suppose that a real-valued Borel signed measure* $d\nu$ *satisfies* (1.8), (1.9), *as well as* (i), (ii). *Then the form* m *defined by* (1.16) *is lower semibounded and closed in* $L_2(\mathbf{R}^2)$.

Proof is quite similar to the proof of Proposition 1.3 from [Sh3]. •

The closed form m gives rise to a *selfadjoint operator*

$$M = M(g, \mathbf{A}, d\nu, \eta) \tag{1.17}$$

in $L_2(\mathbf{R}^2)$.

4. The main result concerning the periodic Schrödinger operator in $L_2(\mathbf{R}^2)$ is the following theorem.

Theorem 1.3. *Suppose that the magnetic potential* $\mathbf{A}$ *and the metric* g *satisfy conditions* (1.1), (1.2) *and* (1.3)–(1.7) *respectively. Let the weight function* η *be such as in conditions* (1.13)–(1.15). *Suppose that the signed measure* $d\nu$ *satisfies the conditions* (1.8), (1.9), (i)–(iii). *Let the quadratic form* m *be defined by* (1.16). *Let* M *be the selfadjoint operator in* $L_2(\mathbf{R}^2)$ *associated with this form. Then the spectrum of the operator* M *is absolutely continuous.*

Remark 1.4. Using a unitary transformation $u \mapsto u\, e^{i\langle \mathbf{k_A}, \mathbf{x}\rangle}$ in $L_2(\mathbf{R}^2)$ (here $\mathbf{k_A} := (\operatorname{meas}\Omega)^{-1}\int_\Omega \mathbf{A}d\mathbf{x}$), we can, without the loss of generality, subject the magnetic potential $\mathbf{A}$ to the gauge condition

$$\int_\Omega \mathbf{A}d\mathbf{x} = 0. \tag{1.18}$$

§ 2. Direct integral decomposition. The Thomas approach

1. Direct integral. Let $\mathbf{b}_1$, $\mathbf{b}_2$ denote the basis of the dual lattice $\widetilde{\Gamma}$:

$$\langle \mathbf{b}_j, \mathbf{a}_l\rangle = 2\pi\delta_{jl}, \quad j, l = 1, 2.$$

The elementary cell of the dual lattice is denoted by $\widetilde{\Omega}$; we have

$$\widetilde{\Omega} := \{\mathbf{k} = \tau_1\mathbf{b}_1 + \tau_2\mathbf{b}_2 : \quad 0 \le \tau_j < 1, \quad j = 1, 2\}. \tag{2.1}$$

The cell (2.1) is dual to Ω.

The direct integral decomposition for periodic operators is constructed with the help of the *Gelfand transformation* $\mathcal{U}$. Consider the Hilbert space

$$\mathcal{K} := \int_{\widetilde{\Omega}} \oplus L_2(\Omega)\, d\mathbf{k}. \tag{2.2}$$

First, we define the mapping $\mathcal{U}: L_2(\mathbf{R}^2) \to \mathcal{K}$ on the functions f of the Schwartz class $\mathcal{S}(\mathbf{R}^2)$ by the formula

$$(\mathcal{U}f)(\mathbf{k},\mathbf{x}) = (\operatorname{meas}\widetilde{\Omega})^{-1/2} e^{-i\langle \mathbf{k},\mathbf{x}\rangle} \sum_{\mathbf{n}\in \mathbf{Z}^2} e^{-i\langle \mathbf{k}, n_1\mathbf{a}_1+n_2\mathbf{a}_2\rangle} f(\mathbf{x}+n_1\mathbf{a}_1+n_2\mathbf{a}_2).$$

Then $\mathcal{U}$ is extended by continuity to a *unitary* operator acting from $L_2(\mathbf{R}^2)$ onto $\mathcal{K}$.

In $L_2(\Omega)$, for every $\mathbf{k} \in \mathbf{R}^2$ (the parameter $\mathbf{k}$ is called the *quasimomentum*) we consider the quadratic form

$$m(\mathbf{k})[u,u] = m(\mathbf{k}; g, \mathbf{A}, d\nu, \eta)[u,u] := \\ \int_\Omega \langle g(\mathbf{D}-\mathbf{A}+\mathbf{k})\eta^{-1}u, (\mathbf{D}-\mathbf{A}+\overline{\mathbf{k}})\eta^{-1}u\rangle\, d\mathbf{x} + \int_\Omega |\eta^{-1}u|^2\, d\nu, \quad u \in \eta\widehat{d}_\Omega. \tag{2.3}$$

The form (2.3) is closed and lower semibounded (see [Sh3, §2]). Note that the domain of the form (2.3) does not depend on $\mathbf{k}$. The closed form $m(\mathbf{k})$ gives rise to a selfadjoint operator

$$M(\mathbf{k}) := M(\mathbf{k}; g, \mathbf{A}, d\nu, \eta) \tag{2.4}$$

in $L_2(\Omega)$. Standard considerations (see, e.g., [BSu2, § 2], [BShSu, § 2]) show that in the direct integral (2.2) the action of the operator (1.17) reduces to multiplication by the operator-valued function $M(\mathbf{k})$:

$$\mathcal{U}M(g,\mathbf{A},d\nu,\eta)\mathcal{U}^{-1} = \int_{\widetilde{\Omega}} \oplus M(\mathbf{k}; g, \mathbf{A}, d\nu, \eta)\, d\mathbf{k}.$$

2. Complexification. The Thomas method involves extension of the forms $m(\mathbf{k})$ and the operators $M(\mathbf{k})$ to the complex values $\mathbf{k} \in \mathbf{C}^2$ of the quasimomentum. This analytic extension gives rise to *sectorial forms and m-sectorial operators.* These objects were studied systematically in the book [K] by T. Kato.

Formula (2.3) allows us to extend the form $m(\mathbf{k})$ to arbitrary $\mathbf{k} \in \mathbf{C}^2$ analytically. On the domain $\widehat{d}_\Omega$, the *form* $m(\mathbf{k})$ *is closed and sectorial* for $\mathbf{k} \in \mathbf{C}^2$. Such a form gives rise (see [K, Theorems VI.2.1, 2.5, 2.7]) to an m-sectorial operator, still denoted by $M(\mathbf{k}) = M(\mathbf{k}; g, \mathbf{A}, d\nu, \eta)$, $\mathbf{k} \in \mathbf{C}^2$. It is shown in [Sh3, §2] that the *resolvent of the operator* $M(\mathbf{k})$, $\mathbf{k} \in \mathbf{C}^2$, *is compact.*

In what follows, we fix a value $k_2 \in \mathbf{R}$, but assume that $k_1 \in \mathbf{C}$. Then, relative to the parameter $k_1 \in \mathbf{C}$, the operators $M(k_1, k_2)$ constitute a *selfadjoint analytic family of type* (B) *with compact resolvent.* Recall that a family of type (B) arises when we define operators in terms of *sectorial forms with common domain* (see [K, §VII.4]). The selfadjointness of the operator family means that $(M(k_1,k_2))^* = M(\overline{k_1}, k_2)$, $k_1 \in \mathbf{C}$, $k_2 \in \mathbf{R}$.

3. The Thomas approach. As applied to the operator families of type (B), the Thomas approach was described in detail in [BSu3]. These considerations are adaptable to our case as well. Namely, the proof of the absolute continuity of the spectrum of the operator M reduces to the proof of the fact that

$$\|(M(\mu + iy, k_2))^{-1}\| \to 0 \quad \text{as} \quad |y| \to \infty$$

for an appropriate $\mu \in \mathbf{R}$ and all $k_2 \in \mathbf{R}$. In our case, we shall obtain the corresponding estimate for the conformal (scalar) metric g. Namely, let $g_0(\mathbf{x}) = \mathbf{1}$ in (1.5), i.e.,

$$g(\mathbf{x}) = \omega^2(\mathbf{x})\mathbf{1}. \tag{2.5}$$

From (1.3)–(1.5) it follows that

$$\omega(\mathbf{x} + \mathbf{a}_j) = \omega(\mathbf{x}), \quad j = 1, 2, \quad \mathbf{x} \in \mathbf{R}^2, \tag{2.6}$$

and

$$0 < \omega_0 \le \omega(\mathbf{x}) \le \omega_1 < \infty, \quad \mathbf{x} \in \mathbf{R}^2. \tag{2.7}$$

We put

$$k_1 = \mu + iy, \quad \mu = \pi|\mathbf{a}_1|^{-1}, \quad y \in \mathbf{R}, \quad k_2 \in \mathbf{R}. \tag{2.8}$$

In this case, we agree to write $M(y)$ in place of $M(\mathbf{k})$, and similarly for the other operators and forms. The dependence on k_2 is not reflected in the notation.

The following theorem is our main technical result; here the metric is assumed to be conformal.

Theorem 2.1. *Suppose that the signed measure $d\nu$ satisfies conditions* (1.8), (1.11), (i′), (1.12), *and that the magnetic potential $\mathbf{A}$ satisfies conditions* (1.2), (1.18). *Suppose that for the metric g conditions* (1.6), (1.7), (2.5)–(2.7) *are fulfilled, and the weight function η satisfies the conditions* (1.14), (1.15). *Let $M(y) = M(\mathbf{k}; g, \mathbf{A}, d\nu, \eta)$, where $\mathbf{k}$ is defined by* (2.8). *Then there exists a constant*

$$y_0 = y_0(\Omega, a, \omega, \mathbf{A}, d\nu, \eta, k_2)$$

such that the operator $M(y)$ is invertible for $|y| \ge y_0$, and

$$\begin{aligned} &\|(M(y))^{-1}\| \le c(y), \quad |y| \ge y_0, \\ &\qquad c(y) = c(y; \Omega, a, \omega, \mathbf{A}, \eta, k_2) \to 0 \ \text{as}\ |y| \to \infty. \end{aligned} \tag{2.9}$$

If k_2 runs through a bounded subset of $\mathbf{R}$, then y_0 and $c(y)$ can be chosen independent of k_2.

Remark 2.2. In Theorem 2.1 we can omit condition (1.18) on the magnetic potential $\mathbf{A}$. In this case, the statement of Theorem 2.1 remains valid if in (2.8) μ is replaced by $\mu + (k_{\mathbf{A}})_1$.

In the case of scalar metric (2.5), Theorem 1.3 can be deduced from Theorem 2.1 in a standard way (see [BSu3, §2]). Theorem 1.3 in its full generality will be deduced from Theorem 2.1 in § 7.

§ 3. Estimates for the free operator

1. In this section, we obtain the estimates necessary for the proof of Theorem 2.1.

For a function $v \in L_2(\Omega)$, consider its Fourier series

$$v(\mathbf{x}) = (\operatorname{meas}\Omega)^{-1/2} \sum_{\mathbf{n}\in \mathbf{Z}^2} \widehat{v}_{\mathbf{n}} \exp\left(i\langle n_1\mathbf{b}_1 + n_2\mathbf{b}_2, \mathbf{x}\rangle\right). \tag{3.1}$$

Let $\mathcal{F} : L_2(\Omega) \to l_2(\mathbf{Z}^2)$ be the discrete (unitary) Fourier transformation defined by the formula $\mathcal{F}v = \{\widehat{v}_{\mathbf{n}}\}$, $\mathbf{n} \in \mathbf{Z}^2$, in accordance with (3.1).

The operator $M_0(y) := (\mathbf{D} + \mathbf{k})^2$ with $\mathbf{k}$ as in (2.8) is of the form

$$M_0(y) = \mathcal{F}^*\mathbf{h}(y)\mathcal{F}, \tag{3.2}$$

where the *symbol* $\mathbf{h}(y) = \{h_{\mathbf{n}}(y)\}$, $\mathbf{n} \in \mathbf{Z}^2$, is defined by the formula

$$h_{\mathbf{n}}(y) = \mu^2(2n_1 + 1)^2 + (\alpha n_1 + \beta n_2 + k_2)^2 - y^2 + 2iy\mu(2n_1 + 1) \tag{3.3}$$

(cf. [BSu2, § 3]). Here $\alpha = \langle \mathbf{b}_1, \mathbf{e}_2\rangle$, $\beta = \langle \mathbf{b}_2, \mathbf{e}_2\rangle$; observe that $\mathbf{b}_1 = 2\mu\mathbf{e}_1 + \alpha\mathbf{e}_2$, $\mathbf{b}_2 = \beta\mathbf{e}_2$. The symbol (3.3) admits the following factorization:

$$h_{\mathbf{n}}(y) = q^{(+)}_{\mathbf{n}}(y)q^{(-)}_{\mathbf{n}}(y), \quad q^{(\pm)}_{\mathbf{n}}(y) := \mu(2n_1 + 1) \pm i(\alpha n_1 + \beta n_2 + k_2 \pm y). \tag{3.4}$$

Relations (3.2)–(3.4) directly imply that

$$\|(M_0(y))^{-1}\| \le \min\{(2\mu|y|)^{-1}, \ \mu^{-2}\} \le 2\mu^{-1}(\mu + |y|)^{-1}.$$

2. We introduce the operators

$$P_0(y) := \mathcal{F}^*\{|h_{\mathbf{n}}(y)|^{-1/2}\}\mathcal{F}, \ P(y) := \mathcal{F}^*\{(\overline{h_{\mathbf{n}}(y)})^{-1}|h_{\mathbf{n}}(y)|^{1/2}\}\mathcal{F}, \quad \mathbf{n} \in \mathbf{Z}^2; \tag{3.5}$$

$$Q_\pm(y) := \mathcal{F}^*\{q^{(\pm)}_{\mathbf{n}}(y)\}\mathcal{F}, \ |Q_\pm(y)| := \mathcal{F}^*\{|q^{(\pm)}_{\mathbf{n}}(y)|\}\mathcal{F}, \quad \mathbf{n} \in \mathbf{Z}^2. \tag{3.6}$$

Then

$$R_0(y) := (M_0(y))^{-1} = P_0(y)(P(y))^* = (Q_+(y))^{-1}(Q_-(y))^{-1}. \tag{3.7}$$

Note that the operator $Q_\pm(y)$ is invertible and

$$\|(Q_\pm(y))^{-1}\| \le \mu^{-1}.$$

For $R \ge 1$, we put

$$\chi_R(\mathbf{n}; y) := \begin{cases} 1, & \text{if } 2\mu|n_1| \le R|y| \text{ and } |\alpha n_1 + \beta n_2| \le R|y| \\ 0, & \text{otherwise} \end{cases},$$

$$\widetilde{\chi}_R(\mathbf{n}; y) := 1 - \chi_R(\mathbf{n}; y).$$

Next, we put $\boldsymbol{\chi}_R(y) := \{\chi_R(\mathbf{n}; y)\}$, $\mathbf{n} \in \mathbf{Z}^2$;

$$\mathcal{X}_R(y) := \mathcal{F}^* \boldsymbol{\chi}_R(y) \mathcal{F}, \tag{3.8}$$

$$\widetilde{\mathcal{X}}_R(y) := I - \mathcal{X}_R(y). \tag{3.9}$$

In what follows, to simplify the notation, we shall often omit the dependence of operators on the parameters. Without loss of generality, we assume that $|y| \geq 1$. Now, we formulate some auxiliary statements.

Lemma 3.1. *We have*

$$\|P_0^{-1} \widetilde{\mathcal{X}}_R u\|^2 \leq C(\Omega) \|\nabla \widetilde{\mathcal{X}}_R u\|^2, \quad u \in \widetilde{H}^1(\Omega), \ R \geq 1, \ |y| \geq |k_2|; \tag{3.10}$$

$$\|\nabla \widetilde{\mathcal{X}}_R u\| \geq R|y| \|\widetilde{\mathcal{X}}_R u\|, \quad u \in \widetilde{H}^1(\Omega), \ R \geq 1; \tag{3.11}$$

$$\|\nabla \mathcal{X}_R\| \leq \sqrt{2}\, R|y|, \quad R \geq 1; \tag{3.12}$$

$$\|P_0^{-1} u\| \geq C(\Omega) |y|^{1/2} \|u\|, \quad u \in \widetilde{H}^1(\Omega). \tag{3.13}$$

Proof. Estimate (3.13) immediately follows from (3.3), (3.5). To prove (3.10) it suffices to use (3.3), (3.5), (3.9). Estimates (3.11), (3.12) are obvious. •

Proposition 3.2. *Suppose that* $F \in L_4(\Omega)$. *Let* P_0, $|Q_\pm|$ *be the operators* (3.5), (3.6). *Then*

$$\|F |Q_\pm|^{-1/2}\| \leq C(\Omega) \|F\|_4, \tag{3.14}$$

$$\|F P_0\| \leq C(\Omega) \|F\|_4 |y|^{-1/2}. \tag{3.15}$$

Proof. It suffices to refer to [BSu2, (3.16), (3.17), (3.30)]. •

Proposition 3.3. *Suppose that a measurable function* V *satisfies the condition*

$$\int_\Omega |V| |u|^2 d\mathbf{x} \leq \varepsilon \int_\Omega |\nabla u|^2 d\mathbf{x} + C(\varepsilon; \Omega, V) \int_\Omega |u|^2 d\mathbf{x}, \quad \forall \varepsilon \in (0,1), \ u \in \widetilde{H}^1(\Omega). \tag{3.16}$$

Then we have

$$\int_\Omega |V| |P_0 u|^2 d\mathbf{x} \leq c(y) \|u\|^2, \quad u \in L_2(\Omega), \ c(y) = c(y; \Omega, V, k_2) \to 0 \text{ as } |y| \to \infty. \tag{3.17}$$

Proof. It suffices to use (in the very particular case) formulae [Sh1, (4.32), (4.33)] (see also [Sh3, Lemma 1.5]). •

§ 4. Estimates for the magnetic potential

1. Gauge transformation. It is well known that for a vector-valued function $\mathbf{A} \in L_2(\Omega)$ which satisfies condition (1.18), there exists the unique representation of the form

$$\mathbf{A} = \nabla \Phi_{\mathbf{A}} - \mathrm{J}\, \nabla \varphi_{\mathbf{A}}, \quad \mathrm{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.1}$$

Here $\Phi_{\mathbf{A}},\ \varphi_{\mathbf{A}} \in \widetilde{H}^1(\Omega)$ are weak solutions of the equations

$$\Delta \Phi_{\mathbf{A}} = \operatorname{div} \mathbf{A}, \quad \int\limits_{\Omega} \Phi_{\mathbf{A}} d\mathbf{x} = 0;$$

$$\Delta \varphi_{\mathbf{A}} = \operatorname{div} \mathrm{J}\, \mathbf{A}, \quad \int\limits_{\Omega} \varphi_{\mathbf{A}} d\mathbf{x} = 0. \tag{4.2}$$

Let $\Phi \in \widetilde{C}^{\infty}(\Omega)$ be a real-valued function such that

$$\|\nabla \Phi - \nabla \Phi_{\mathbf{A}}\| \leq \varepsilon_{\Phi}. \tag{4.3}$$

A small constant $\varepsilon_{\Phi} \in (0, 1]$ will be fixed later. We put $\mathbf{A}' := \mathbf{A} - \nabla \Phi$. By (1.2), it is clear that $\mathbf{A}'$ satisfies the following condition

$$\int\limits_{\Omega} |\mathbf{A}'|^2 |u|^2 d\mathbf{x} \leq \varepsilon \int\limits_{\Omega} |\nabla u|^2 d\mathbf{x} + C(\varepsilon; \Omega, \mathbf{A}') \int\limits_{\Omega} |u|^2 d\mathbf{x}, \quad \forall \varepsilon \in (0,1), \ u \in \widetilde{H}^1(\Omega). \tag{4.4}$$

We introduce a unitary operator $U_{\Phi}: \ u \mapsto u\, e^{i\Phi(\mathbf{x})}$ in $L_2(\Omega)$. Then we have

$$(U_{\Phi})^{-1} M(\mathbf{k}) U_{\Phi} = M'(\mathbf{k}) := M(\mathbf{k}; g, \mathbf{A}', d\nu, \eta). \tag{4.5}$$

From (4.5) it directly follows that the relation (2.9) is valid for the operators $M(\mathbf{k})$ if and only if it is so for $M'(\mathbf{k})$.

2. Let ϱ be a function such that

$$0 \leq \varrho \in C_0^{\infty}(\mathbf{R}^2), \quad \operatorname{supp} \varrho \subset \Omega, \quad \int \varrho \, d\mathbf{x} = 1.$$

We put

$$\varrho_n(\mathbf{x}) := \sum_{\mathbf{m} \in \mathbf{Z}^2} n^2 \varrho(n(\mathbf{x} + m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2)), \quad n \in \mathbf{N}.$$

Note that ϱ_n satisfy the following conditions

$$0 \leq \varrho_n \in \widetilde{C}^{\infty}(\Omega), \quad \int\limits_{\Omega} \varrho_n d\mathbf{x} = 1, \quad n \in \mathbf{N}.$$

We consider the convolution of $\mathbf{A}$ with the delta-like sequence $\{\varrho_n\}$ on the torus $\mathbf{R}^2/\Gamma$:

$$\mathbf{A}^{(n)}(\mathbf{x}) := \int\limits_{\Omega} \varrho_n(\mathbf{x}-\mathbf{y})\mathbf{A}(\mathbf{y})\,d\mathbf{y}.$$

Obviously, $\mathbf{A}^{(n)} \in \widetilde{C}^{\infty}(\Omega)$.

Let φ_n be a $\widetilde{H}^1(\Omega)$-solution of the equation

$$\Delta\varphi_n = \operatorname{div} \mathrm{J}\,\mathbf{A}^{(n)}, \quad \int\limits_{\Omega} \varphi_n d\mathbf{x} = 0. \tag{4.6}$$

Then $\varphi_n \in \widetilde{C}^{\infty}(\Omega)$. We have $\mathbf{A}^{(n)} \to \mathbf{A}$ in $L_2(\Omega)$ as $n \to \infty$. Thus,

$$\|\varphi_n - \varphi_{\mathbf{A}}\|_{H^1(\Omega)} \le \varepsilon_n \to 0 \text{ as } n \to \infty. \tag{4.7}$$

Later we shall fix $n \in \mathbf{N}$ large enough to ensure that the constant $\varepsilon_n \in (0,1]$ be sufficiently small.

We have

$$\int\limits_{\Omega} |\mathbf{A}^{(n)}|^2|u|^2 d\mathbf{x} \le \varepsilon \int\limits_{\Omega} |\nabla u|^2 d\mathbf{x} + C(\varepsilon;\Omega,\mathbf{A}) \int\limits_{\Omega} |u|^2 d\mathbf{x}, \quad \forall\varepsilon \in (0,1),\ u \in \widetilde{H}^1(\Omega). \tag{4.8}$$

Note that the *estimate* (4.8) *is uniform with respect to n.* Indeed, using the Minkowski's inequality and inequality (1.2), we obtain (here we deal with the functions on the torus $\mathbf{R}^2/\Gamma$)

$$\int\limits_{\Omega} |\mathbf{A}^{(n)}|^2|u|^2 d\mathbf{x} = \int\limits_{\Omega} |u(\mathbf{x})|^2 \left| \int\limits_{\Omega} \mathbf{A}(\mathbf{x}-\mathbf{y})\varrho_n(\mathbf{y})d\mathbf{y} \right|^2 d\mathbf{x} \le$$

$$\left(\int\limits_{\Omega} \varrho_n(\mathbf{y}) \left(\int\limits_{\Omega} |\mathbf{A}(\mathbf{x}-\mathbf{y})|^2 |u(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2} d\mathbf{y} \right)^2 =$$

$$\left(\int\limits_{\Omega} \varrho_n(\mathbf{y}) \left(\int\limits_{\Omega} |\mathbf{A}(\mathbf{z})|^2 |u(\mathbf{z}+\mathbf{y})|^2 d\mathbf{z} \right)^{1/2} d\mathbf{y} \right)^2 \le$$

$$\left(\int\limits_{\Omega} \varrho_n(\mathbf{y}) \left(\varepsilon \int\limits_{\Omega} |\nabla_{\mathbf{z}} u(\mathbf{z}+\mathbf{y})|^2 d\mathbf{z} + C(\varepsilon;\Omega,\mathbf{A}) \int\limits_{\Omega} |u(\mathbf{z}+\mathbf{y})|^2 d\mathbf{z} \right)^{1/2} d\mathbf{y} \right)^2 =$$

$$\left(\int\limits_{\Omega} \varrho_n(\mathbf{y}) d\mathbf{y} \right)^2 \left(\varepsilon \int\limits_{\Omega} |\nabla_{\mathbf{z}} u(\mathbf{z})|^2 d\mathbf{z} + C(\varepsilon;\Omega,\mathbf{A}) \int\limits_{\Omega} |u(\mathbf{z})|^2 d\mathbf{z} \right) =$$

$$\varepsilon \int\limits_{\Omega} |\nabla_{\mathbf{z}} u(\mathbf{z})|^2 d\mathbf{z} + C(\varepsilon;\Omega,\mathbf{A}) \int\limits_{\Omega} |u(\mathbf{z})|^2 d\mathbf{z}, \quad \forall\varepsilon \in (0,1),\ u \in \widetilde{H}^1(\Omega).$$

Next, we have

$$\|e^{\alpha\varphi_{\mathbf{A}}}\| \leq C(\alpha;\Omega,\|\varphi_{\mathbf{A}}\|_{H^1(\Omega)}), \ \alpha \in \mathbf{R}. \tag{4.9}$$

In order to prove (4.9), we use the Sobolev's inequality

$$\|\varphi_{\mathbf{A}}\|_k \leq C(\Omega)k^{1/2}\|\varphi_{\mathbf{A}}\|_{H^1(\Omega)}, \quad k \in \mathbf{N}$$

(see, e.g., [M, (2.3.3/3)]). Then

$$\|e^{\alpha\varphi_{\mathbf{A}}}\|^2 = \int\limits_{\Omega}\left(\sum_{k=0}^{\infty}\frac{(2\alpha)^k}{k!}\varphi_{\mathbf{A}}^k\right)d\mathbf{x} \leq \operatorname{meas}\Omega + \sum_{k=1}^{\infty}\frac{(2|\alpha|)^k}{k!}\|\varphi_{\mathbf{A}}\|_k^k \leq$$
$$\operatorname{meas}\Omega + \sum_{k=1}^{\infty}\left(C(\Omega)|\alpha|\|\varphi_{\mathbf{A}}\|_{H^1(\Omega)}\right)^k\frac{k^{k/2}}{k!} \leq C(\alpha;\Omega,\|\varphi_{\mathbf{A}}\|_{H^1(\Omega)}).$$

Similarly,

$$\|e^{\alpha\varphi_n}\| \leq C(\alpha;\Omega,\|\varphi_n\|_{H^1(\Omega)}) \leq C(\alpha;\Omega,\|\varphi_{\mathbf{A}}\|_{H^1(\Omega)}), \ \alpha \in \mathbf{R}. \tag{4.10}$$

Now we are ready to prove the following lemma.

Lemma 4.1. *Suppose that* $\mathbf{A}$ *satisfies condition* (1.2). *Let* $\varphi_{\mathbf{A}}$ *and* φ_n *be defined by* (4.2) *and* (4.6) *respectively. Then we have*

$$\|e^{\alpha\varphi_{\mathbf{A}}}\|_{H^1(\Omega)} \leq C(\alpha;\Omega,\|\varphi_{\mathbf{A}}\|_{H^1(\Omega)}), \ \alpha \in \mathbf{R}, \tag{4.11}$$

$$\|e^{\alpha\varphi_n}\|_{H^1(\Omega)} \leq C(\alpha;\Omega,\|\varphi_{\mathbf{A}}\|_{H^1(\Omega)}), \ \alpha \in \mathbf{R}. \tag{4.12}$$

Proof. From the definition of φ_n (see (4.6)), it follows that

$$\int\limits_{\Omega}|\nabla(e^{\alpha\varphi_n})|^2 d\mathbf{x} = \alpha^2\int\limits_{\Omega}|\nabla\varphi_n|^2 e^{2\alpha\varphi_n}d\mathbf{x} = \alpha/2\int\limits_{\Omega}\langle\nabla\varphi_n,\nabla(e^{2\alpha\varphi_n})\rangle d\mathbf{x} =$$
$$\alpha/2\int\limits_{\Omega}\langle \mathrm{J}\,\mathbf{A}^{(n)},\nabla(e^{2\alpha\varphi_n})\rangle d\mathbf{x} = \alpha^2\int\limits_{\Omega}\langle \mathrm{J}\,\mathbf{A}^{(n)},\nabla\varphi_n\rangle e^{2\alpha\varphi_n}d\mathbf{x} \leq$$
$$\alpha^2/2\int\limits_{\Omega}|\nabla\varphi_n|^2 e^{2\alpha\varphi_n}d\mathbf{x} + \alpha^2/2\int\limits_{\Omega}|\mathbf{A}^{(n)}|^2 e^{2\alpha\varphi_n}d\mathbf{x}.$$

Combining this with (4.8), we obtain

$$\int\limits_{\Omega}|\nabla(e^{\alpha\varphi_n})|^2 d\mathbf{x} \leq \alpha^2\int\limits_{\Omega}|\mathbf{A}^{(n)}|^2 e^{2\alpha\varphi_n}d\mathbf{x} \leq$$
$$\varepsilon\int\limits_{\Omega}|\nabla(e^{\alpha\varphi_n})|^2 d\mathbf{x} + C(\varepsilon;\alpha,\Omega,\mathbf{A})\int\limits_{\Omega}e^{2\alpha\varphi_n}d\mathbf{x}, \quad \forall\varepsilon \in (0,1).$$

Now, we fix $\varepsilon = 1/2$, and from the last inequality and inequality (4.10) we deduce the estimate (4.12). Taking $n \to \infty$ in (4.12), by (4.7) and Fatou's Lemma, we obtain (4.11). •

3. In [BSu1] the factorization of the Pauli operator was described. This factorization played a crucial role for the investigation of the magnetic Schrödinger operator (see [BSu1,2], [BShSu], [Sh1,3]). Here we shall also use it.

We put $\widetilde{\mathbf{A}}^{(n)} := -\mathrm{J}\,\nabla\varphi_n$, $\widetilde{B}_n := \partial_1 \widetilde{A}_2^{(n)} - \partial_2 \widetilde{A}_1^{(n)} = \Delta\varphi_n$. In $L_2(\mathbf{R}^2)$, we introduce the Pauli operator (more precisely, one of its blocks):

$$\widetilde{M}_n := M(\mathbf{1}, \widetilde{\mathbf{A}}^{(n)}, \widetilde{B}_n d\mathbf{x}, 1) = (\mathbf{D} - \widetilde{\mathbf{A}}^{(n)})^2 + \widetilde{B}_n.$$

Let

$$\widetilde{M}_n(y) := (\mathbf{D} - \widetilde{\mathbf{A}}^{(n)} + \mathbf{k})^2 + \widetilde{B}_n \tag{4.13}$$

be the operator in $L_2(\Omega)$ corresponding to the operator $\widetilde{M}_n$ (here $\mathbf{k}$ is defined by (2.8)). The following multiplicative representation is valid (cf. [BSu1, (3.11)]):

$$\widetilde{M}_n(y) = e^{-\varphi_n} Q_+(y) e^{2\varphi_n} Q_-(y) e^{-\varphi_n}, \quad y \in \mathbf{R}. \tag{4.14}$$

4. In what follows, we shall often deal with the operator

$$T_n(y) := (\widetilde{M}_n(y)^{-1})^* P_0^{-2} = e^{\varphi_n} (Q_+^{-1})^* e^{-2\varphi_n} (Q_-^{-1})^* e^{\varphi_n} P_0^{-2}, \quad y \in \mathbf{R}. \tag{4.15}$$

We denote

$$\widetilde{A}_\pm^{(n)} := [Q_\pm, \varphi_n] = D_1\varphi_n \pm i D_2\varphi_n = \mp \widetilde{A}_1^{(n)} - i\widetilde{A}_2^{(n)}. \tag{4.16}$$

Here $[\cdot, \cdot]$ is a commutator of two operators. We have

$$[Q_\pm^{-1}, f] = -Q_\pm^{-1}[Q_\pm, f]Q_\pm^{-1}, \quad f \in \widetilde{C}^\infty(\Omega). \tag{4.17}$$

The following three statements are proved in [Sh3] (see [Sh3, Lemmas 4.1, 4.2, and Corollary 4.3]).

Lemma 4.2. *We have*

$$\|T_n\| \le C(\Omega, \varphi_n); \tag{4.18}$$

$$\|[\nabla, T_n]\| \le C(\Omega, \varphi_n). \tag{4.19}$$

Lemma 4.3. *Suppose that the signed measure $d\nu$ satisfies the conditions of Theorem* 2.1. *Then we have*

$$\int_\Omega |T_n P_0 \mathcal{X}_R u|^2 |d\nu| \le c(y)\|\mathcal{X}_R u\|^2, \qquad u \in L_2(\Omega),\ c(y) := c(y; \Omega, R, \varphi_n, d\nu, k_2) \to 0 \text{ as } |y| \to \infty. \tag{4.20}$$

Corollary 4.4. *Suppose that a measurable function V satisfies condition* (3.16). *Then we have*

$$\int\limits_{\Omega} |V||T_n P_0 \mathcal{X}_R u|^2 d\mathbf{x} \leq c(y) \|\mathcal{X}_R u\|^2, \tag{4.21}$$
$$u \in L_2(\Omega), \ c(y) := c(y; \Omega, R, \varphi_n, V, k_2) \to 0 \ \textit{as} \ |y| \to \infty.$$

5. The following lemma is crucial for further considerations concerning the magnetic potential. A similar statement was proved in [Sh3, Lemma 4.4]. However, now we have less restrictive conditions on $\mathbf{A}$ than in [Sh3].

Lemma 4.5. *Suppose that $F \in L_2(\Omega)$. Then we have*

$$\|\mathcal{X}_R P_0 T_n^* F P_0\| \leq C(\Omega, R, \mathbf{A}, k_2) \|F\| |y|^{-1}. \tag{4.22}$$

Proof. Using relations (4.16), (4.17), we obtain

$$\begin{gathered}
\widetilde{M}_n^{-1} = e^{\varphi_n} Q_-^{-1} e^{-2\varphi_n} Q_+^{-1} e^{\varphi_n} = Q_-^{-1} Q_+^{-1} + Q_-^{-1} \widetilde{A}_-^{(n)} e^{\varphi_n} Q_-^{-1} e^{-\varphi_n} Q_+^{-1} - \\
Q_-^{-1} e^{-\varphi_n} Q_+^{-1} \widetilde{A}_+^{(n)} e^{\varphi_n} Q_+^{-1} - Q_-^{-1} \widetilde{A}_-^{(n)} e^{\varphi_n} Q_-^{-1} e^{-2\varphi_n} Q_+^{-1} \widetilde{A}_+^{(n)} e^{\varphi_n} Q_+^{-1} = \\
Q_-^{-1} Q_+^{-1} + Q_-^{-1} \widetilde{A}_-^{(n)} Q_-^{-1} Q_+^{-1} + Q_-^{-1} \widetilde{A}_-^{(n)} e^{\varphi_n} Q_-^{-1} \widetilde{A}_-^{(n)} e^{-\varphi_n} Q_-^{-1} Q_+^{-1} - Q_-^{-1} Q_+^{-1} \widetilde{A}_+^{(n)} Q_+^{-1} + \\
Q_-^{-1} Q_+^{-1} \widetilde{A}_+^{(n)} e^{-\varphi_n} Q_+^{-1} \widetilde{A}_+^{(n)} e^{\varphi_n} Q_+^{-1} - Q_-^{-1} \widetilde{A}_-^{(n)} e^{-\varphi_n} Q_-^{-1} Q_+^{-1} \widetilde{A}_+^{(n)} e^{\varphi_n} Q_+^{-1} - \\
Q_-^{-1} \widetilde{A}_-^{(n)} e^{\varphi_n} Q_-^{-1} 2 \widetilde{A}_-^{(n)} e^{-2\varphi_n} Q_-^{-1} Q_+^{-1} \widetilde{A}_+^{(n)} e^{\varphi_n} Q_+^{-1} = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{gathered} \tag{4.23}$$

Representation (4.15) shows that $\mathcal{X}_R P_0 T_n^* F P_0 = \mathcal{X}_R P_0^{-1} \widetilde{M}_n^{-1} F P_0$. We shall estimate the term $\mathcal{X}_R P_0^{-1} I_7 F P_0$. Another six terms can be estimated in the same way or even simpler. Using the identity (3.7), we obtain

$$\begin{gathered}
\|\mathcal{X}_R P_0^{-1} I_7 F P_0\| \leq 2 \|\mathcal{X}_R P_0^{-1} |Q_-|^{-1/2}\| \cdot \||Q_-|^{-1/2} |\widetilde{A}_-^{(n)} e^{\varphi_n}|^{1/2}\| \times \\
\||\widetilde{A}_-^{(n)} e^{\varphi_n}|^{1/2} |Q_-|^{-1/2}\| \cdot \||Q_-|^{-1/2} |\widetilde{A}_-^{(n)} e^{-2\varphi_n}|^{1/2}\| \cdot \||\widetilde{A}_-^{(n)} e^{-2\varphi_n}|^{1/2} P_0\| \times \\
\|P_0 |\widetilde{A}_+^{(n)} e^{\varphi_n}|^{1/2}\| \cdot \||\widetilde{A}_+^{(n)} e^{\varphi_n}|^{1/2} |Q_+|^{-1/2}\| \cdot \||Q_+|^{-1/2} |F|^{1/2}\| \cdot \||F|^{1/2} P_0\|.
\end{gathered} \tag{4.24}$$

Next, by (3.4), (3.6)–(3.8), we have

$$\|\mathcal{X}_R P_0^{-1} |Q_-|^{-1/2}\| = \|\mathcal{X}_R |Q_+|^{1/2}\| \leq C(\Omega, R, k_2) |y|^{1/2}.$$

To estimate the remaining multipliers in (4.24) we use (3.14), (3.15), (4.12) and (4.16). As a result, we obtain

$$\|\mathcal{X}_R P_0^{-1} I_7 F P_0\| \leq C(\Omega, R, \mathbf{A}, k_2) \|F\| |y|^{-1}.$$

Combined with similar estimates for $\mathcal{X}_R P_0^{-1} I_j F P_0, \ j = 1, \dots 6$, this yields (4.22). •

§ 5. Proof of Theorem 2.1 with $g = \mathbf{1}$, $\eta = 1$

Throughout this section we assume that $g = \mathbf{1}$, $\eta = 1$. By (4.5), in order to obtain estimate (2.9) (with $\omega = 1$, $\eta = 1$), it suffices to prove that for some magnetic potential $\mathbf{A}'$ as described in Section 4.1 and every function $0 \neq u \in \widehat{d}_\Omega$, there exists a function $0 \neq v \in \widehat{d}_\Omega$ such that

$$|m'(y)[u,v]| := |m(y; \mathbf{1}, \mathbf{A}', d\nu, 1)[u,v]| \geq (c(y))^{-1} \|u\| \|v\|, \qquad |y| \geq y_0,\ c(y) \to 0 \text{ as } |y| \to \infty. \tag{5.1}$$

Note that the functions Φ and φ_n will be fixed; so, all the constants depending on $\mathbf{A}'$ and φ_n actually depend on $\mathbf{A}$. We present a function $0 \neq u \in \widehat{d}_\Omega$ as $u = \widetilde{\mathcal{X}}_R u + \mathcal{X}_R u$. Obviously, $\mathcal{X}_R u \in \widetilde{C}^\infty(\Omega)$ and therefore $\widetilde{\mathcal{X}}_R u \in \widehat{d}_\Omega$. We put (cf. [Sh2, (4.2)])

$$v := \widetilde{\mathcal{X}}_R u + T_n \mathcal{X}_R u. \tag{5.2}$$

It was explained in [Sh3, §5] that $v \in \widehat{d}_\Omega$ and the following estimate (cf. [Sh3, (5.3)]) is fulfilled:

$$\|P_0^{-1} v\| \leq C(\Omega, \varphi_n) \|P_0^{-1} u\|. \tag{5.3}$$

We have (see (2.3))

$$\begin{aligned} m'(y)[u,v] = &\int_\Omega \langle (\mathbf{D} - \mathbf{A}' + \mathbf{k}) \widetilde{\mathcal{X}}_R u, (\mathbf{D} - \mathbf{A}' + \overline{\mathbf{k}}) \widetilde{\mathcal{X}}_R u \rangle d\mathbf{x} + \\ &\int_\Omega \langle (\mathbf{D} - \mathbf{A}' + \mathbf{k}) \widetilde{\mathcal{X}}_R u, (\mathbf{D} - \mathbf{A}' + \overline{\mathbf{k}}) T_n \mathcal{X}_R u \rangle d\mathbf{x} + \\ &\int_\Omega \langle (\mathbf{D} - \mathbf{A}' + \mathbf{k}) \mathcal{X}_R u, (\mathbf{D} - \mathbf{A}' + \overline{\mathbf{k}}) \widetilde{\mathcal{X}}_R u \rangle d\mathbf{x} + \\ &\int_\Omega \langle (\mathbf{D} - \mathbf{A}' + \mathbf{k}) \mathcal{X}_R u, (\mathbf{D} - \mathbf{A}' + \overline{\mathbf{k}}) T_n \mathcal{X}_R u \rangle d\mathbf{x} + \int_\Omega |\widetilde{\mathcal{X}}_R u|^2 d\nu + \\ &\int_\Omega \widetilde{\mathcal{X}}_R u\, \overline{T_n \mathcal{X}_R u}\, d\nu + \int_\Omega \mathcal{X}_R u\, \overline{\widetilde{\mathcal{X}}_R u}\, d\nu + \int_\Omega \mathcal{X}_R u\, \overline{T_n \mathcal{X}_R u}\, d\nu := J_1 + \ldots + J_8. \end{aligned} \tag{5.4}$$

The term J_1 can be estimated by (2.8), (4.4):

$$\operatorname{Re} J_1 \geq (1 - \varepsilon_1) \|\nabla \widetilde{\mathcal{X}}_R u\|^2 - \left(y^2 + C(\varepsilon_1; \Omega, \mathbf{A}', k_2)\right) \|\widetilde{\mathcal{X}}_R u\|^2, \quad \forall \varepsilon_1 \in (0,1). \tag{5.5}$$

We rewrite J_2 in the form

$$\begin{aligned} J_2 = &\int_\Omega \langle (\mathbf{D} - \widetilde{\mathbf{A}}^{(n)} + \mathbf{k}) \widetilde{\mathcal{X}}_R u, (\mathbf{D} - \widetilde{\mathbf{A}}^{(n)} + \overline{\mathbf{k}}) T_n \mathcal{X}_R u \rangle d\mathbf{x} - \\ &\int_\Omega \langle (\mathbf{D} + \mathbf{k}) \widetilde{\mathcal{X}}_R u, (\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)}) T_n \mathcal{X}_R u \rangle d\mathbf{x} - \int_\Omega \langle (\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)}) \widetilde{\mathcal{X}}_R u, (\mathbf{D} + \overline{\mathbf{k}}) T_n \mathcal{X}_R u \rangle d\mathbf{x} + \\ &\int_\Omega (\mathbf{A}'^2 - (\widetilde{\mathbf{A}}^{(n)})^2) \widetilde{\mathcal{X}}_R u\, \overline{T_n \mathcal{X}_R u}\, d\mathbf{x}. \end{aligned} \tag{5.6}$$

Relations (4.13), (4.15) imply that

$$\int\limits_{\Omega} \langle (\mathbf{D} - \widetilde{\mathbf{A}}^{(n)} + \mathbf{k})\widetilde{\mathcal{X}}_R u, (\mathbf{D} - \widetilde{\mathbf{A}}^{(n)} + \overline{\mathbf{k}})T_n \mathcal{X}_R u \rangle d\mathbf{x} + \int\limits_{\Omega} \widetilde{B}_n \widetilde{\mathcal{X}}_R u \, \overline{T_n \mathcal{X}_R u} \, d\mathbf{x} = 0.$$

Combining this with (5.6), we obtain

$$\begin{aligned} J_2 = -\int\limits_{\Omega} \widetilde{B}_n \widetilde{\mathcal{X}}_R u \, \overline{T_n \mathcal{X}_R u} \, d\mathbf{x} - \\ \int\limits_{\Omega} \langle (\mathbf{D} + \mathbf{k})\widetilde{\mathcal{X}}_R u, (\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)})T_n \mathcal{X}_R u \rangle d\mathbf{x} - \int\limits_{\Omega} \langle (\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)})\widetilde{\mathcal{X}}_R u, (\mathbf{D} + \overline{\mathbf{k}})T_n \mathcal{X}_R u \rangle d\mathbf{x} + \\ \int\limits_{\Omega} (\mathbf{A}'^2 - (\widetilde{\mathbf{A}}^{(n)})^2)\widetilde{\mathcal{X}}_R u \, \overline{T_n \mathcal{X}_R u} \, d\mathbf{x} =: J_{21} + J_{22} + J_{23} + J_{24}. \end{aligned}$$

From the estimate (3.11) it follows that

$$\begin{aligned} & |J_{21} + J_{22} + J_{24}| \leq \\ & \|\widetilde{B}_n \widetilde{\mathcal{X}}_R u\|^2 + \|T_n \mathcal{X}_R u\|^2 + \varepsilon_2' \|\nabla \widetilde{\mathcal{X}}_R u\|^2 + C(\varepsilon_2'; \Omega, k_2) \|(\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)})T_n \mathcal{X}_R u\|^2 + \\ & \| |\mathbf{A}'^2 - (\widetilde{\mathbf{A}}^{(n)})^2|^{1/2} \widetilde{\mathcal{X}}_R u\|^2 + \| |\mathbf{A}'^2 - (\widetilde{\mathbf{A}}^{(n)})^2|^{1/2} T_n \mathcal{X}_R u\|^2, \quad \forall \varepsilon_2' \in (0,1). \end{aligned}$$

The last inequality combined with the smoothness of $\widetilde{\mathbf{A}}^{(n)}$, estimate (4.4) and Corollary 4.4 implies

$$\begin{aligned} |J_{21} + J_{22} + J_{24}| \leq \varepsilon_2 \|\nabla \widetilde{\mathcal{X}}_R u\|^2 + C(\varepsilon_2; \Omega, \mathbf{A}', \varphi_n) \|\widetilde{\mathcal{X}}_R u\|^2 + c(y) \|P_0^{-1} \mathcal{X}_R u\|^2, \\ \forall \varepsilon_2 \in (0,1), \quad c(y) = c(y; \varepsilon_2, \Omega, R, \mathbf{A}', \varphi_n, k_2) \to 0 \text{ as } |y| \to \infty. \end{aligned} \tag{5.7}$$

The term J_{23} is similar to the term $\mathcal{J}_{32}$ from [Sh3, (6.15)]. We have (cf. [Sh3, (6.18)])

$$\begin{aligned} |J_{23}| = \left| \int\limits_{\Omega} \langle (\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)})\widetilde{\mathcal{X}}_R u, T_n(\mathbf{D} + \overline{\mathbf{k}})\mathcal{X}_R u + [\mathbf{D}, T_n]\mathcal{X}_R u \rangle d\mathbf{x} \right| \leq \\ \|\mathcal{X}_R P_0 T_n^* (\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)}) P_0\| \cdot \|(\mathbf{D} + \overline{\mathbf{k}})\mathcal{X}_R\| \cdot \|P_0^{-1} \widetilde{\mathcal{X}}_R u\| \cdot \|P_0^{-1} \mathcal{X}_R u\| + \\ \|(\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)})\widetilde{\mathcal{X}}_R u\| \cdot \|[\mathbf{D}, T_n]\| \cdot \|\mathcal{X}_R u\|. \end{aligned} \tag{5.8}$$

Recall that $(\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)}) = (\nabla \Phi_{\mathbf{A}} - \nabla \Phi) - \mathrm{J}\,(\nabla \varphi_{\mathbf{A}} - \nabla \varphi_n)$. To estimate the first summand in the right-hand side of (5.8) we use (3.10), (3.12), (4.3), (4.7), (4.22); for the second summand we use (3.13), (4.4), (4.19). We obtain

$$\begin{aligned} & |J_{23}| \leq C(\Omega, R, \mathbf{A}, k_2)(\varepsilon_\Phi + \varepsilon_n) \|\nabla \widetilde{\mathcal{X}}_R u\| \|P_0^{-1} \mathcal{X}_R u\| + \\ & \|(\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)})\widetilde{\mathcal{X}}_R u\|^2 + C(\Omega, \varphi_n)|y|^{-1} \|P_0^{-1} \mathcal{X}_R u\|^2 \leq \\ & \varepsilon_3 \|\nabla \widetilde{\mathcal{X}}_R u\|^2 + C(\varepsilon_3; \Omega, \mathbf{A}', \varphi_n) \|\widetilde{\mathcal{X}}_R u\|^2 + \\ & \left(C(\varepsilon_3; \Omega, R, \mathbf{A}, k_2)(\varepsilon_\Phi^2 + \varepsilon_n^2) + C(\Omega, \varphi_n)|y|^{-1} \right) \|P_0^{-1} \mathcal{X}_R u\|^2, \\ & \forall \varepsilon_3 \in (0,1), \ |y| \geq |k_2|. \end{aligned} \tag{5.9}$$

Next, for the term J_3, we have

$$
\begin{aligned}
&J_3 = -\int\limits_{\Omega} \langle \mathbf{A}'\mathcal{X}_R u, (\mathbf{D} + \overline{\mathbf{k}})\widetilde{\mathcal{X}}_R u\rangle d\mathbf{x} - \int\limits_{\Omega} \langle (\mathbf{D} + \mathbf{k})\mathcal{X}_R u, \mathbf{A}'\widetilde{\mathcal{X}}_R u\rangle d\mathbf{x} + \\
&\int\limits_{\Omega} |\mathbf{A}'|\mathcal{X}_R u\, \overline{|\mathbf{A}'|\widetilde{\mathcal{X}}_R u}\, d\mathbf{x} = \\
&-\int\limits_{\Omega} \langle (\mathbf{A}' + \widetilde{\mathbf{A}}^{(n)})\mathcal{X}_R u, (\mathbf{D} + \overline{\mathbf{k}})\widetilde{\mathcal{X}}_R u\rangle d\mathbf{x} - \int\limits_{\Omega} \langle (\mathbf{D} + \mathbf{k})\mathcal{X}_R u, (\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)})\widetilde{\mathcal{X}}_R u\rangle d\mathbf{x} + \\
&\int\limits_{\Omega} |\mathbf{A}'|\mathcal{X}_R u\, \overline{|\mathbf{A}'|\widetilde{\mathcal{X}}_R u}\, d\mathbf{x} =: J_{31} + J_{32} + J_{33}.
\end{aligned}
$$

Here we used the identity $\operatorname{div} \widetilde{\mathbf{A}}^{(n)} = 0$. Applying estimates (3.11), (4.4) and Proposition 3.3, we obtain

$$
\begin{aligned}
&|J_{31} + J_{33}| \le \varepsilon_4 \|\nabla \widetilde{\mathcal{X}}_R u\|^2 + C(\varepsilon_4; \Omega, \mathbf{A}')\|\widetilde{\mathcal{X}}_R u\|^2 + c(y)\|P_0^{-1}\mathcal{X}_R u\|^2, \\
&\qquad \forall \varepsilon_4 \in (0,1), \quad c(y) = c(y; \varepsilon_4, \Omega, \mathbf{A}', \varphi_n, k_2) \to 0 \text{ as } |y| \to \infty.
\end{aligned}
\tag{5.10}
$$

The term J_{32} can be estimated in the same way as J_{23} (cf. $\mathcal{J}_{23}$ from [Sh3 (6.10), (6.13)]). From (3.10), (3.12), (3.15) and (4.3), (4.7) we have

$$
\begin{aligned}
&|J_{32}| \le \|P_0(\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)})P_0\| \cdot \|(\mathbf{D} + \mathbf{k})\mathcal{X}_R\| \cdot \|P_0^{-1}\mathcal{X}_R u\| \cdot \|P_0^{-1}\widetilde{\mathcal{X}}_R u\| \le \\
&C(\Omega)\|P_0|(\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)})|^{1/2}\|^2 \cdot \|(\mathbf{D} + \mathbf{k})\mathcal{X}_R\| \cdot \|P_0^{-1}\mathcal{X}_R u\| \cdot \|\nabla \widetilde{\mathcal{X}}_R u\| \le \\
&C(\Omega, R, k_2)(\varepsilon_\Phi + \varepsilon_n)\|P_0^{-1}\mathcal{X}_R u\| \cdot \|\nabla \widetilde{\mathcal{X}}_R u\| \le \\
&\varepsilon_5 \|\nabla \widetilde{\mathcal{X}}_R u\|^2 + C(\varepsilon_5; \Omega, R, k_2)(\varepsilon_\Phi^2 + \varepsilon_n^2)\|P_0^{-1}\mathcal{X}_R u\|^2, \quad \forall \varepsilon_5 \in (0,1), \ |y| \ge |k_2|.
\end{aligned}
\tag{5.11}
$$

Similarly to (5.6), J_4 can be represented in the following form:

$$
\begin{aligned}
&J_4 = \int\limits_{\Omega} \langle (\mathbf{D} - \widetilde{\mathbf{A}}^{(n)} + \mathbf{k})\mathcal{X}_R u, (\mathbf{D} - \widetilde{\mathbf{A}}^{(n)} + \overline{\mathbf{k}})T_n \mathcal{X}_R u\rangle d\mathbf{x} - \\
&\int\limits_{\Omega} \langle (\mathbf{D} + \mathbf{k})\mathcal{X}_R u, (\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)})T_n \mathcal{X}_R u\rangle d\mathbf{x} - \int\limits_{\Omega} \langle (\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)})\mathcal{X}_R u, (\mathbf{D} + \overline{\mathbf{k}})T_n \mathcal{X}_R u\rangle d\mathbf{x} + \\
&\int\limits_{\Omega} (\mathbf{A}'^2 - (\widetilde{\mathbf{A}}^{(n)})^2)\mathcal{X}_R u\, \overline{T_n \mathcal{X}_R u}\, d\mathbf{x} =: J_{41} + J_{42} + J_{43} + J_{44}.
\end{aligned}
\tag{5.12}
$$

For J_{41}, it follows from (4.13), (4.15) that

$$
J_{41} = \|P_0^{-1}\mathcal{X}_R u\|^2 - \int\limits_{\Omega} \widetilde{B}_n \mathcal{X}_R u\, \overline{T_n \mathcal{X}_R u}\, d\mathbf{x}.
$$

This identity combined with (3.13), (4.18) implies that

$$\operatorname{Re} J_{41} \ge (1 - C(\Omega, \varphi_n)|y|^{-1})\|P_0^{-1}\mathcal{X}_R u\|^2. \tag{5.13}$$

Next, from (3.12), (4.3), (4.7), (4.22) it follows that

$$\begin{gathered}|J_{42}| \le \|\mathcal{X}_R P_0 T_n^* (\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)})^* P_0\| \cdot \|(\mathbf{D} + \mathbf{k})\mathcal{X}_R\| \cdot \|P_0^{-1}\mathcal{X}_R u\|^2 \le \\ C(\Omega, R, \mathbf{A}, k_2)(\varepsilon_\Phi + \varepsilon_n)\|P_0^{-1}\mathcal{X}_R u\|^2.\end{gathered} \tag{5.14}$$

For the term J_{44}, using the smoothness of $\widetilde{\mathbf{A}}^{(n)}$, the estimate (4.4), Proposition 3.3 and Corollary 4.4, we obtain

$$\begin{gathered}|J_{44}| \le \||\mathbf{A}'^2 - (\widetilde{\mathbf{A}}^{(n)})^2|^{1/2} P_0 P_0^{-1}\mathcal{X}_R u\| \cdot \||\mathbf{A}'^2 - (\widetilde{\mathbf{A}}^{(n)})^2|^{1/2} T_n P_0 P_0^{-1}\mathcal{X}_R u\| \le \\ c(y)\|P_0^{-1}\mathcal{X}_R u\|^2, \quad c(y) = c(y; \Omega, R, \mathbf{A}', \varphi_n, k_2) \to 0 \text{ as } |y| \to \infty.\end{gathered} \tag{5.15}$$

Finally, the term J_{43} can be estimated in the same way as J_{23} (see (5.8)) (cf. $\mathcal{J}_{34}$ from [Sh3, (6.15), (6.20)]):

$$\begin{gathered}|J_{43}| \le \|\mathcal{X}_R P_0 T_n^* (\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)}) P_0\| \cdot \|(\mathbf{D} + \overline{\mathbf{k}})\mathcal{X}_R\| \cdot \|P_0^{-1}\mathcal{X}_R u\|^2 + \\ \|(\mathbf{A}' - \widetilde{\mathbf{A}}^{(n)})\mathcal{X}_R u\| \cdot \|[\mathbf{D}, T_n]\| \cdot \|\mathcal{X}_R u\|.\end{gathered}$$

From the last inequality, using (3.12), (4.3), (4.7), (4.22), as well as (3.13), (4.4), (4.19) and Proposition 3.3, we obtain

$$|J_{43}| \le \Big(C(\Omega, R, \mathbf{A}, k_2)(\varepsilon_\Phi + \varepsilon_n) + C(\Omega, \mathbf{A}', \varphi_n, k_2)|y|^{-1/2}\Big) \|P_0^{-1}\mathcal{X}_R u\|^2. \tag{5.16}$$

Substituting (5.13)–(5.16) in (5.12), we arrive at the estimate

$$\begin{gathered}\operatorname{Re} J_4 \ge (1 - C(\Omega, R, \mathbf{A}, k_2)(\varepsilon_\Phi + \varepsilon_n) - c(y))\|P_0^{-1}\mathcal{X}_R u\|^2, \\ c(y) = c(y; \Omega, R, \mathbf{A}', \varphi_n, k_2) \to 0 \text{ as } |y| \to \infty.\end{gathered} \tag{5.17}$$

Terms J_5, J_6, J_7, J_8 were estimated in [Sh3]. The following estimate is proved in [Sh3, (5.14)–(5.17)]:

$$\begin{gathered}J_5 - |J_6 + J_7 + J_8| \ge \\ (1 - a - \varepsilon_6) \int\limits_\Omega |\widetilde{\mathcal{X}}_R u|^2 d\nu_+ - (a + \varepsilon_6)\|\nabla \widetilde{\mathcal{X}}_R u\|^2 - C(\Omega, a, d\nu)\|\widetilde{\mathcal{X}}_R u\|^2 - c(y)\|P_0^{-1}\mathcal{X}_R u\|^2, \\ \forall \varepsilon_6 \in (0, 1), \ c(y) = c(y; \varepsilon_6, \Omega, a, R, \varphi_n, d\nu, k_2) \to 0 \text{ as } |y| \to \infty.\end{gathered} \tag{5.18}$$

Substituting (5.5), (5.7), (5.9)–(5.11), (5.17), (5.18) in (5.4), we obtain

$$\begin{gathered}\operatorname{Re} m'(y)[u, v] \ge (1 - a - \varepsilon) \left(\int\limits_\Omega |\widetilde{\mathcal{X}}_R u|^2 d\nu_+ + \|\nabla \widetilde{\mathcal{X}}_R u\|^2 \right) - \\ \big(y^2 + C(\varepsilon; \Omega, a, \mathbf{A}', \varphi_n, d\nu, k_2)\big) \|\widetilde{\mathcal{X}}_R u\|^2 + \\ (1 - C(\varepsilon; \Omega, R, \mathbf{A}, k_2)(\varepsilon_\Phi + \varepsilon_n) - c(y)) \|P_0^{-1}\mathcal{X}_R u\|^2, \\ \forall \varepsilon \in (0, 1), \quad c(y) = c(y; \varepsilon, \Omega, a, R, \mathbf{A}', \varphi_n, d\nu, k_2) \to 0 \text{ as } |y| \to \infty.\end{gathered} \tag{5.19}$$

We fix $\varepsilon := (1-a)/4$. Next, we fix sufficiently large R such that

$$R^{-2} \le (1-a)/4, \quad R \ge 1.$$

Then, (3.11) and (5.19) imply that

$$\begin{aligned}&\operatorname{Re} m'(y)[u,v] \ge 2^{-1}(1-a)\|\nabla \widetilde{\mathcal{X}}_R u\|^2 - C(\Omega, a, \mathbf{A}', \varphi_n, d\nu, k_2)\|\widetilde{\mathcal{X}}_R u\|^2 + \\ &(1 - C(\Omega, \mathbf{A}, k_2)(\varepsilon_\Phi + \varepsilon_n) - c(y))\,\|P_0^{-1}\mathcal{X}_R u\|^2, \\ &c(y) = c(y; \Omega, a, \mathbf{A}', \varphi_n, d\nu, k_2) \to 0 \text{ as } |y| \to \infty.\end{aligned} \tag{5.20}$$

We fix sufficiently small ε_Φ, ε_n such that

$$C(\Omega, \mathbf{A}, k_2)(\varepsilon_\Phi + \varepsilon_n) \le 1/2.$$

Now, we fix function $\Phi \in \widetilde{C}^\infty(\Omega)$ which satisfies (4.3). It means that $\mathbf{A}' = \mathbf{A} - \nabla\Phi$ is fixed. Next, we fix the function φ_n which satisfies (4.7). From (3.11), (5.20) it follows that

$$\begin{aligned}&\operatorname{Re} m'(y)[u,v] \ge \\ &\left(2^{-1}(1-a) - C(\Omega, a, \mathbf{A}, d\nu, k_2)y^{-2}\right)\|\nabla \widetilde{\mathcal{X}}_R u\|^2 + \left(2^{-1} - c(y)\right)\|P_0^{-1}\mathcal{X}_R u\|^2, \\ &c(y) = c(y; \Omega, a, \mathbf{A}, d\nu, k_2) \to 0 \text{ as } |y| \to \infty.\end{aligned}$$

Thus, for sufficiently large y_0, from the last inequality and (3.10), (5.3) we obtain

$$\operatorname{Re} m(y; \mathbf{1}, \mathbf{A}', d\nu, 1)[u,v] \ge C(\Omega, a)\|P_0^{-1}u\|^2 \ge C(\Omega, a, \mathbf{A})\|P_0^{-1}u\|\|P_0^{-1}v\|, \quad |y| \ge y_0. \tag{5.21}$$

Note that from the first inequality in (5.21) it directly follows that $v \ne 0$. Finally, applying (3.13), from (5.21) we deduce inequality (5.1) with $c(y) := c(\Omega, a, \mathbf{A})|y|^{-1}$. This completes the proof of Theorem 2.1 with $g = \mathbf{1}$, $\eta = 1$.

§ 6. The case of a variable scalar metric

In the present section we drop the restriction $g = \mathbf{1}$, but still assume that $\eta = 1$. The considerations which will allow us to deal with a scalar metric $\omega^2\mathbf{1}$ are essentially the same as in [Sh3, §6].

Along with the form $m(y)$, we consider the form

$$\begin{aligned}m_\omega(y; \omega^2\mathbf{1}, \mathbf{A}, d\nu, 1)[u,u] := m(y; \omega^2\mathbf{1}, \mathbf{A}, d\nu, 1)[\omega^{-1}u, \omega^{-1}u], \\ \operatorname{Dom} m_\omega(y) := \operatorname{Dom} m(y) = \widehat{d}_\Omega.\end{aligned} \tag{6.1}$$

It was shown in [Sh3, §6] that the form (6.1) is closed and sectorial provided that so is the form $m(y)$.

Let $\widetilde{\omega}$ be a smooth real-valued function which approximates ω, i.e.,

$$\begin{aligned}&0 < \omega_0 \le \widetilde{\omega}(\mathbf{x}) \le \omega_1 < \infty, \\ &\widetilde{\omega} \in \widetilde{C}^\infty(\Omega), \quad \|\mathbf{G}_\omega\| \le \varepsilon_\omega \le 1,\end{aligned} \tag{6.2}$$

where (cf. [Sh2, § 5]) $\mathbf{G}_\omega := \widetilde{\omega}^{-1}(\mathbf{D}\widetilde{\omega}) - \omega^{-1}(\mathbf{D}\omega)$. A small constant ε_ω will be fixed later. Following the considerations of [Sh2, § 5], for any functions $u, v \in \widehat{d}_\Omega$ we have

$$\begin{gathered} m'_\omega(y)[u,v] := m_\omega(y; \omega^2\mathbf{1}, \mathbf{A}', d\nu, 1) = \int\limits_\Omega \langle (\mathbf{D} - \mathbf{A}' + \mathbf{k})u, (\mathbf{D} - \mathbf{A}' + \overline{\mathbf{k}})v\rangle d\mathbf{x} + \\ \int\limits_\Omega \langle (\mathbf{D} + \mathbf{k})u, \mathbf{G}_\omega v\rangle d\mathbf{x} + \int\limits_\Omega \langle \mathbf{G}_\omega u, (\mathbf{D} + \overline{\mathbf{k}})v\rangle d\mathbf{x} + \int\limits_\Omega u\overline{v}\, d\widetilde{\nu} =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4, \end{gathered} \tag{6.3}$$

where

$$d\widetilde{\nu} := -2\langle \widetilde{\omega}^{-1}(\mathbf{D}\widetilde{\omega}), \mathbf{G}_\omega\rangle d\mathbf{x} + |\mathbf{G}_\omega|^2 d\mathbf{x} + \widetilde{\omega}^{-1}(\Delta\widetilde{\omega})d\mathbf{x} + \omega^{-2}d\nu. \tag{6.4}$$

We need the following statement which was proved in [Sh3, Lemma 6.1].

Lemma 6.1. *For any* $\widetilde{a} \in (a, 1)$ *we have*

$$\int\limits_\Omega |f|^2 d\widetilde{\nu}_- \le \widetilde{a}\left(\|\nabla f\|^2 + \int\limits_\Omega |f|^2 d\widetilde{\nu}_+ \right) + C(\widetilde{a}; \Omega, \omega, \widetilde{\omega}, d\nu)\|f\|^2, \quad a < \widetilde{a} < 1, \ f \in \widehat{d}_\Omega; \tag{6.5}$$

$$\int\limits_\Omega |f|^2 |d\widetilde{\nu}| \le \varepsilon\left(\|\nabla f\|^2 + \max_{\mathbf{x}\in\Omega} |f|^2 \right) + C(\varepsilon; \Omega, \omega, \widetilde{\omega}, d\nu)\|f\|^2, \quad \forall \varepsilon \in (0,1), \ f \in \widetilde{C}^\infty(\Omega). \tag{6.6}$$

From Lemma 6.1 it follows that the *signed measure* $d\widetilde{\nu}$ *satisfies the estimate of the form* (1.11), *as well as the estimate of the form* (1.12) *with respect to the unit metric.*

In what follows we fix $\widetilde{a} \in (a, 1)$. Let $0 \ne u \in \widehat{d}_\Omega$, and let v *be chosen in accordance with* (5.2). Then, by (6.5), (6.6) the sum $\mathcal{J}_1 + \mathcal{J}_4$ can be estimated in the same way as it was made in § 5 for estimating $m'(y)[u, v]$. So, we have the estimate of the form (5.19) with $\widetilde{a}$ and $d\widetilde{\nu}$ in place of a and $d\nu$. Namely,

$$\begin{gathered} \operatorname{Re}(\mathcal{J}_1 + \mathcal{J}_4) \ge (1 - \widetilde{a} - \varepsilon)\left(\int\limits_\Omega |\widetilde{\mathcal{X}}_R u|^2 d\widetilde{\nu}_+ + \|\nabla\widetilde{\mathcal{X}}_R u\|^2 \right) - \\ \left(y^2 + C(\varepsilon; \Omega, \widetilde{a}, \omega, \widetilde{\omega}, \mathbf{A}', \varphi_n, d\nu, k_2)\right) \|\widetilde{\mathcal{X}}_R u\|^2 + \\ \left(1 - C(\varepsilon; \Omega, R, \mathbf{A}, k_2)(\varepsilon_\Phi + \varepsilon_n) - c(y)\right) \|P_0^{-1}\mathcal{X}_R u\|^2, \\ \forall \varepsilon \in (0,1), \quad c(y) = c(y; \varepsilon, \Omega, \widetilde{a}, R, \omega, \widetilde{\omega}, \mathbf{A}', \varphi_n, d\nu, k_2) \to 0 \text{ as } |y| \to \infty. \end{gathered} \tag{6.7}$$

The terms $\mathcal{J}_2$ and $\mathcal{J}_3$ are the same as in [Sh3, §6]. However, in the present paper we have less restrictive assumptions on magnetic potential than in [Sh3]. Nevertheless, the estimates for $\mathcal{J}_2$, $\mathcal{J}_3$ from [Sh3] remain valid. The proof is formally the same. We only take into account that the crucial estimate (4.22) (which corresponds to the estimate [Sh3, (4.34)]) is now proved under wider conditions on $\mathbf{A}$. So, we have (see [Sh3, (6.11)–(6.14), (6.16), (6.17), (6.19), (6.20)])

$$\begin{gathered} |\mathcal{J}_2 + \mathcal{J}_3| \le \\ \varepsilon\|\nabla\widetilde{\mathcal{X}}_R u\|^2 + C(\varepsilon; \Omega, \omega, \widetilde{\omega}, k_2)\|\widetilde{\mathcal{X}}_R u\|^2 + (C(\varepsilon; \Omega, R, \mathbf{A}, k_2)\varepsilon_\omega + c(y))\|P_0^{-1}\mathcal{X}_R u\|^2, \\ \forall \varepsilon \in (0,1), \quad c(y) = c(y; \varepsilon, \Omega, R, \omega, \widetilde{\omega}, \varphi_n, k_2) \to 0 \text{ as } |y| \to \infty. \end{gathered} \tag{6.8}$$

Substituting (6.7), (6.8) in (6.3), we obtain

$$\begin{aligned} &\operatorname{Re} m'_\omega[u,v] \geq (1-\widetilde{a}-\varepsilon)\left(\int\limits_\Omega |\widetilde{\mathcal{X}}_R u|^2 d\widetilde{\nu}_+ + \|\nabla \widetilde{\mathcal{X}}_R u\|^2\right) - \\ &\left(y^2 + C(\varepsilon;\Omega,\widetilde{a},\omega,\widetilde{\omega},\mathbf{A}',\varphi_n,d\nu,k_2)\right)\|\widetilde{\mathcal{X}}_R u\|^2 + \\ &\left(1 - C(\varepsilon;\Omega,R,\mathbf{A},k_2)(\varepsilon_\Phi+\varepsilon_n+\varepsilon_\omega) - c(y)\right)\|P_0^{-1}\mathcal{X}_R u\|^2, \\ &\forall\varepsilon\in(0,1),\quad c(y) = c(y;\varepsilon,\Omega,\widetilde{a},R,\omega,\widetilde{\omega},\mathbf{A}',\varphi_n,d\nu,k_2) \to 0 \text{ as } |y|\to\infty. \end{aligned} \tag{6.9}$$

We fix $\varepsilon := (1-\widetilde{a})/4$. Next, we fix sufficiently large constant R satisfying

$$R^{-2} \leq (1-\widetilde{a})/4, \quad R \geq 1.$$

Then, from (3.11) and (6.9) we obtain

$$\begin{aligned} &\operatorname{Re} m'_\omega[u,v] \geq \left(2^{-1}(1-\widetilde{a}) - C(\Omega,\widetilde{a},\omega,\widetilde{\omega},\mathbf{A}',\varphi_n,d\nu,k_2)y^{-2}\right)\|\nabla\widetilde{\mathcal{X}}_R u\|^2 + \\ &\left(1 - C(\Omega,\mathbf{A},k_2)(\varepsilon_\Phi+\varepsilon_n+\varepsilon_\omega) - c(y)\right)\|P_0^{-1}\mathcal{X}_R u\|^2, \\ &c(y) = c(y;\Omega,\widetilde{a},\omega,\widetilde{\omega},\mathbf{A}',\varphi_n,d\nu,k_2) \to 0 \text{ as } |y|\to\infty. \end{aligned} \tag{6.10}$$

Now, we fix $\varepsilon_\Phi,\ \varepsilon_n,\ \varepsilon_\omega$ sufficiently small so as to have

$$C(\Omega,\mathbf{A},k_2)(\varepsilon_\Phi+\varepsilon_n+\varepsilon_\omega) \leq 1/2.$$

Then we fix functions Φ, φ_n and $\widetilde{\omega}$ satisfying (4.3), (4.7) and (6.2) respectively. So, by (3.10), (5.3) and (6.10), it follows that, for sufficiently large y_0,

$$\operatorname{Re} m'_\omega[u,v] \geq C(\Omega,a)\|P_0^{-1}u\|^2 \geq C(\Omega,a,\mathbf{A})\|P_0^{-1}u\|\|P_0^{-1}v\|, \quad |y|\geq y_0. \tag{6.11}$$

Now, let $0 \neq \widetilde{u} \in \widehat{d}_\Omega$, and let $u := \omega\widetilde{u} \in \widehat{d}_\Omega$. We define v in terms of u in accordance with (5.2) and put

$$\widetilde{v} := \omega^{-1}v. \tag{6.12}$$

Then relations (6.1) and (6.11) imply the inequality

$$\operatorname{Re} m(y;\omega^2\mathbf{1},\mathbf{A}',d\nu,1)[\widetilde{u},\widetilde{v}] \geq C(\Omega,a,\mathbf{A})\|P_0^{-1}\omega\widetilde{u}\|\|P_0^{-1}\omega\widetilde{v}\|, \quad |y|\geq y_0. \tag{6.13}$$

Combined with (2.7), (3.13), this yields

$$\operatorname{Re} m(y;\omega^2\mathbf{1},\mathbf{A}',d\nu,1)[\widetilde{u},\widetilde{v}] \geq C(\Omega,a,\mathbf{A})\|\omega^{-1}\|_\infty^{-2}|y|\|\widetilde{u}\|\|\widetilde{v}\|, \quad |y|\geq y_0.$$

Applying (4.5) to the last inequality, we complete the proof of Theorem 2.1 with $\eta = 1$.

§ 7. Proof of Theorem 1.3

1. Conclusion of the proof of Theorem 2.1. First of all, we observe that

$$m(y; g, \mathbf{A}, d\nu, \eta)[f, f] = m(y; g, \mathbf{A}, d\nu, 1)[\eta^{-1} f, \eta^{-1} f], \quad f \in \eta \widehat{d}_\Omega. \tag{7.1}$$

Next, put $0 \neq f := \eta \widetilde{u}$, $\widetilde{u} \in \widehat{d}_\Omega$, and $w := \eta \widetilde{v}$, where $\widetilde{v}$ is chosen in accordance with (6.12). Then, by (6.13) and (7.1), for some $\mathbf{A}'$ as described in Section 4.1, we have

$$\operatorname{Re} m(y; \omega^2 \mathbf{1}, \mathbf{A}', d\nu, \eta)[f, w] \geq C(\Omega, a, \mathbf{A}) \| P_0^{-1} \eta^{-1} \omega f \| \| P_0^{-1} \eta^{-1} \omega w \|, \quad |y| \geq y_0. \tag{7.2}$$

From (7.2), taking into account (1.15), (2.7) and Proposition 3.3, we obtain

$$\begin{gathered} \operatorname{Re} m(y; \omega^2 \mathbf{1}, \mathbf{A}', d\nu, \eta)[f, w] \geq (c(y))^{-1} \|f\| \|w\|, \quad |y| \geq y_0, \\ c(y) = c(y; \Omega, a, \omega, \mathbf{A}, \eta, k_2) \to 0 \text{ as } |y| \to \infty. \end{gathered} \tag{7.3}$$

Relation (4.5) and inequality (7.3) directly imply the estimate (2.9). This completes the proof of Theorem 2.1. •

2. Transformation of the operator (1.17) **to the case of a scalar metric.** *In the case of a scalar metric of the form* (2.5), *Theorem* 1.3 *can be deduced from Theorem* 2.1 *in a standard way* (see [BSu3, § 2]). In the general case, to prove Theorem 1.3 we use the change of coordinates in $\mathbf{R}^2$ that reshapes the operator M defined in (1.17) to the operator with a scalar metric, but with a new lattice of periods and new $\mathbf{A}$, $d\nu$ and η. We use the following theorem formulated in [Sh2, § 6] (see also references therein).

Theorem 7.1. *Suppose that a measurable* Γ*-periodic* (2×2)*-matrix-valued function* g_0 *with real-valued entries satisfies condition* (1.4), *and suppose also that* $\det g_0 = 1$. *Then there exists a one-to-one mapping* $\Psi : \mathbf{R}^2 \to \mathbf{R}^2$ *with the following properties.*

1°. *For some* $p > 2$ *we have*

$$\Psi \in W^1_{p,\mathrm{loc}}(\mathbf{R}^2), \ \Psi^{-1} \in W^1_{p,\mathrm{loc}}(\mathbf{R}^2), \quad p > 2. \tag{7.4}$$

Let $\Psi'(\mathbf{x})$ *be the Jacobi matrix of* Ψ. *Then* $0 < |\det \Psi'(\mathbf{x})| < \infty$ *for a. a.* $\mathbf{x} \in \mathbf{R}^2$.

2°. *For some basis* $\mathbf{a}_1^*$, $\mathbf{a}_2^*$ *in* $\mathbf{R}^2$ *we have*

$$\Psi(0) = 0; \quad \Psi(\mathbf{x} + n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2) = \Psi(\mathbf{x}) + n_1 \mathbf{a}_1^* + n_2 \mathbf{a}_2^*, \quad \mathbf{x} \in \mathbf{R}^2, \quad \{n_1, n_2\} \in \mathbf{Z}^2.$$

Thus, the change of coordinates $\boldsymbol{\xi} = \Psi(\mathbf{x})$ *transforms the lattice* Γ *into the lattice*

$$\Gamma_* := \{\boldsymbol{\xi} = n_1 \mathbf{a}_1^* + n_2 \mathbf{a}_2^*, \quad \{n_1, n_2\} \in \mathbf{Z}^2\}. \tag{7.5}$$

3°. *We have*

$$|\det \Psi'(\mathbf{x})|^{-1} \Psi'(\mathbf{x}) g_0(\mathbf{x}) (\Psi'(\mathbf{x}))^t = \mathbf{1}, \quad \boldsymbol{\xi} = \Psi(\mathbf{x}). \tag{7.6}$$

Note that the set $\Psi(\Omega)$ is fundamental with respect to the lattice Γ_*.

Remark 7.2. The relation (7.6) directly implies the following representation

$$\Psi'(\mathbf{x}) = |\det \Psi'(\mathbf{x})|^{1/2} \mathcal{W}(\mathbf{x}) (g_0(\mathbf{x}))^{-1/2}, \tag{7.7}$$

where $\mathcal{W}$ is an orthogonal matrix-valued function.

Lemma 7.3. 1) *Suppose that* $F \in H^1_{\mathrm{loc}}(\mathbf{R}^2)$. *Then* $F \circ \Psi$, $F \circ \Psi^{-1} \in H^1_{\mathrm{loc}}(\mathbf{R}^2)$, *and the standard differential chain rule is valid:*

$$\nabla (F \circ \Psi) = (\Psi')^t \left((\nabla_{\boldsymbol{\xi}} F) \circ \Psi \right).$$

2) *Suppose that* $F \in L_{1,\mathrm{loc}}(\mathbf{R}^2)$. *Then we have*

$$\int\limits_{\Xi} F \, d\boldsymbol{\xi} = \int\limits_{\Psi^{-1}(\Xi)} (F \circ \Psi) \, |\det \Psi'| d\mathbf{x}$$

for an arbitrary Borel bounded set $\Xi \subset \mathbf{R}^2$.

Proof. It suffices to refer to [ABe, Lemma 10 and Theorem 5]. •

In what follows we shall use Lemma 7.3 without any additional comments.

We put

$$\Omega_* := \{ \boldsymbol{\xi} \in \mathbf{R}^2 : \ \boldsymbol{\xi} = t_1 \mathbf{a}_1^* + t_2 \mathbf{a}_2^*, \ 0 \le t_j < 1, \ j = 1, 2 \}.$$

The set Ω_* is an elementary cell of the lattice (7.5). We also introduce the following notation:

$$\psi(\boldsymbol{\xi}) := |\det \Psi'(\mathbf{x})|^{-1/2}, \tag{7.8}$$

$$\mathbf{A}_*(\boldsymbol{\xi}) := \left(\Psi'(\mathbf{x})^t \right)^{-1} \mathbf{A}(\mathbf{x}), \tag{7.9}$$

$$\omega_*(\boldsymbol{\xi}) := \omega(\mathbf{x}), \tag{7.10}$$

$$d\nu_*(\boldsymbol{\xi}) := d\nu(\mathbf{x}), \tag{7.11}$$

$$\eta_*(\boldsymbol{\xi}) := \psi(\boldsymbol{\xi}) \eta(\mathbf{x}), \tag{7.12}$$

where $\boldsymbol{\xi} = \Psi(\mathbf{x})$. From statement 2° of Theorem 7.1 and from (1.1), (1.9), (1.13), (2.6), (7.8)–(7.12) it follows that ψ, $\mathbf{A}_*$, ω_*, $d\nu_*$ and η_* are Γ_*-periodic. By (7.4), (7.8), for the function ψ we have

$$\psi \in L_p(\Omega_*), \quad p > 2, \tag{7.13}$$

$$\psi(\boldsymbol{\xi}) > 0, \quad \text{a. a. } \boldsymbol{\xi} \in \mathbf{R}^2. \tag{7.14}$$

Relation (7.13) and Proposition 1.1 imply the estimate

$$\int\limits_{\Omega_*} |\psi f|^2 d\boldsymbol{\xi} \le \varepsilon \int\limits_{\Omega_*} |\nabla_{\boldsymbol{\xi}} f|^2 d\boldsymbol{\xi} + C(\varepsilon; \Omega_*, \psi) \int\limits_{\Omega_*} |f|^2 d\boldsymbol{\xi}, \quad f \in H^1(\Omega_*), \quad \forall \varepsilon \in (0,1). \tag{7.15}$$

It was shown in [Sh3, Section 7.2] that the coefficients ω_*, $d\nu_*$, η_* satisfy all the conditions of Theorem 1.3 with respect to the lattice Γ_*. In particular, the form

$$m_{*+}[f,f] := \int |\nabla_{\boldsymbol{\xi}} f|^2 d\boldsymbol{\xi} + \int |f|^2 d\nu_{*+}, \quad f \in C_0^\infty(\mathbf{R}^2),$$

is closable in $L_2(\mathbf{R}^2)$ (cf. condition (i)). Let $\widehat{d}_*$ be the domain of the closure of the form m_{*+}. *The transformation* $u \mapsto u \circ \Psi^{-1}$ *is one-to-one from* $\widehat{d}$ *onto* $\widehat{d}_*$ (see [Sh2, Section 6.2]).

Next, for any Γ_*-periodic function $f \in H^1_{\mathrm{loc}}(\mathbf{R}^2)$, by (1.2), (1.4), (1.5), (7.6)–(7.9), (7.15), we have (cf. [Sh3, (7.22)])

$$\begin{aligned}
&\int_{\Omega_*} |\mathbf{A}_* f|^2 d\boldsymbol{\xi} = \int_{\Psi(\Omega)} |\mathbf{A}_* f|^2 d\boldsymbol{\xi} = \int_{\Omega} |\mathcal{W} g_0^{1/2} \mathbf{A}(f\circ\Psi)|^2 d\mathbf{x} \le \\
&\frac{\varepsilon}{2}\int_{\Omega} \langle g_0 \nabla(f\circ\Psi), \nabla(f\circ\Psi)\rangle d\mathbf{x} + C(\varepsilon;\Omega,g_0,\mathbf{A})\int_{\Omega} |f\circ\Psi|^2 d\mathbf{x} = \\
&\qquad\frac{\varepsilon}{2}\int_{\Psi(\Omega)} |\nabla_{\boldsymbol{\xi}} f|^2 d\boldsymbol{\xi} + C(\varepsilon;\Omega,g_0,\mathbf{A})\int_{\Psi(\Omega)} |\psi f|^2 d\boldsymbol{\xi} = \\
&\frac{\varepsilon}{2}\int_{\Omega_*} |\nabla_{\boldsymbol{\xi}} f|^2 d\boldsymbol{\xi} + C(\varepsilon;\Omega,g_0,\mathbf{A})\int_{\Omega_*} |\psi f|^2 d\boldsymbol{\xi} \le \\
&\qquad\varepsilon\int_{\Omega_*} |\nabla_{\boldsymbol{\xi}} f|^2 d\boldsymbol{\xi} + C(\varepsilon;\Omega,\Omega_*,g_0,\mathbf{A},\psi)\int_{\Omega_*} |f|^2 d\boldsymbol{\xi}, \quad \forall \varepsilon \in (0,1).
\end{aligned} \tag{7.16}$$

Thus, $\mathbf{A}_*$, ω_*, $d\nu_*$ and η_* satisfy *all* the assumptions of Theorem 1.3 with respect to the lattice Γ_*. By (7.6), the mapping Ψ sends the quadratic form (1.16) to the form

$$\begin{aligned}
m[u,u] = \int \omega_*^2 |(\mathbf{D}_{\boldsymbol{\xi}} - \mathbf{A}_*)(\eta_*^{-1} v)|^2 d\boldsymbol{\xi} + \int |\eta_*^{-1} v|^2 d\nu_*, \quad u \in \eta\widehat{d}, \\
v(\boldsymbol{\xi}) = \psi(\boldsymbol{\xi}) u(\mathbf{x}), \quad \boldsymbol{\xi} = \Psi(\mathbf{x}).
\end{aligned}$$

At the same time,

$$\int |u(\mathbf{x})|^2 \, d\mathbf{x} = \int |v(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}.$$

This shows that the operator $M = M(g, \mathbf{A}, d\nu, \eta)$ is unitarily equivalent to the operator

$$M_* := M(\omega_*^2 \mathbf{1}, \mathbf{A}_*, d\nu_*, \eta_*).$$

For the case of a scalar metric of the form (2.5), Theorem 1.3 has already been proven. Consequently, the spectrum of the operator M_* is absolutely continuous. Therefore, so is the spectrum of M. This completes the proof of Theorem 1.3.

§ 8. Schrödinger operator in a simply connected periodic waveguide

Theorem 2.1 allows us to relax conditions on the magnetic potential in the case of the Schrödinger operator in a two-dimensional simply connected periodic waveguide as well as in the case of $\mathbf{R}^2$. The widest known conditions on the coefficients which ensure the absolute continuity of the spectrum of the Schrödinger operator in a two-dimensional waveguide were proposed in [Sh4]. Now we are able to relax the conditions on the magnetic potential from [Sh4]. Here we formulate the corresponding theorem (see Theorem 8.5 below). Actually, the proof of Theorem 8.5 repeats the proof of Theorem 2.5 from [Sh4] with natural changes (see Remark 8.6 below). This is why we do not give the detailed proof of Theorem 8.5 here.

1. Notation. Let $\mathcal{D}$ be a domain in $\mathbf{R}^2$. Then we denote $\overline{\mathcal{D}} := \operatorname{clos}\mathcal{D}$. Let $B_R := \{\mathbf{x} \in \mathbf{R}^2 : |\mathbf{x}| < R\}$, $R > 0$. The symbol $C_0(\overline{\mathcal{D}})$ stands for the class of functions $u(\mathbf{x}) \in C(\overline{\mathcal{D}})$ such that $\operatorname{supp} u \subset \overline{\mathcal{D}} \cap B_R$ for some $R < \infty$.

We use the notation $S := \{\mathbf{x} \in \mathbf{R}^2 : 0 < x_2 < 1\}$, $\Omega_R := (-R/2, R/2) \times (0,1)$, $R > 0$. The boundary of the strip S consists of two lines $\Xi_0 := \{\mathbf{x} \in \mathbf{R}^2 : x_2 = 0\}$, $\Xi_1 := \{\mathbf{x} \in \mathbf{R}^2 : x_2 = 1\}$. Let $\beta > 0$, and let $\gamma_\beta := \{n\beta\mathbf{e}_1, \ n \in \mathbf{Z}\}$ be a one-dimensional lattice. The symbol $\widehat{H}^s(\Omega_\beta)$ stands for the subspace formed by the functions $u \in H^s(\Omega_\beta)$, such that the γ_β-periodic extension of u belongs to $H^s(\Omega_R)$ for any $R < \infty$. We denote by $\widehat{C}(\overline{\Omega_\beta})$ the class of functions that are restrictions to $\overline{\Omega_\beta}$ of the γ_β-periodic functions of the class $C(\overline{S})$.

2. Conditions on a waveguide. We put $\mathbf{R}^2_{r+} := \{\mathbf{x} \in \mathbf{R}^2 : x_2 > r\}$, $\mathbf{R}^2_{r-} := \{\mathbf{x} \in \mathbf{R}^2 : x_2 < r\}$, $r \in \mathbf{R}$. We assume that a periodic waveguide $\Pi \subset \mathbf{R}^2$ satisfies the following condition.

Condition 8.1. *The set* $\Pi \subset \mathbf{R}^2$ *is a simply connected domain such that* $\mathbf{R}^2_{r+} \not\subset \Pi$, $\mathbf{R}^2_{r-} \not\subset \Pi$, $\forall r \in \mathbf{R}$. *The domain* Π *is periodic with respect to some lattice* γ_α, $\alpha > 0$*:*

$$\mathbf{x} \in \Pi \Rightarrow \mathbf{x} + n\alpha\mathbf{e}_1 \in \Pi, \quad n \in \mathbf{Z}.$$

Note that from Condition 8.1 it directly follows that the *points* $\pm\infty$ *are different accessible boundary points of the domain* Π *determined by* γ_α*-periodic continuous curves.* The definition of accessible boundary points can be found, for example, in [G, § II.3].

Let Ψ be a conformal mapping from the waveguide Π onto the strip S. As long as the points $\pm\infty$ are accessible, by Koebe's Theorem (see, e.g., [G, § II.3, Theorem 1]), we can choose the mapping Ψ such that

$$\Psi(-\infty) = -\infty, \quad \Psi(+\infty) = +\infty.$$

At the same time, γ_α-periodic continuous curves which determine accessible points $\pm\infty$ in Π transform into continuous curves connecting $\pm\infty$ in S. Then, as it was shown in [Sh4, Section 1.3], the mapping Ψ is fixed up to the shift $\Psi \mapsto \Psi + r\mathbf{e}_1$, $r \in \mathbf{R}$. Besides, for some $\beta > 0$ we have

$$\Psi(\mathbf{x} + n\alpha\mathbf{e}_1) = \Psi(\mathbf{x}) + n\beta\mathbf{e}_1, \quad \mathbf{x} \in \Pi, \ n \in \mathbf{Z}.$$

The boundary of the waveguide Π consists of two disjoint γ_α-periodic closed sets $l_\pm$: $\partial\Pi = l_- \cup l_+$. Namely, put $S_- := \{\mathbf{x} \in \mathbf{R}^2 : \ 0 < x_2 < 1/2\}$, $S_+ := \{\mathbf{x} \in \mathbf{R}^2 : \ 1/2 < x_2 < 1\}$. Then

$$l_- := \partial\Pi \cap \overline{\Psi^{-1}(S_-)}, \quad l_+ := \partial\Pi \cap \overline{\Psi^{-1}(S_+)}.$$

We introduce the notation $\mathcal{O}_R := \Psi^{-1}(\Omega_R)$, $R > 0$. Obviously, $\mathcal{O}_\beta$ is a simply connected fundamental domain in Π with respect to the lattice γ_α. Note that the cell $\mathcal{O}_\beta$, generally speaking, is not bounded.

We denote by $\widehat{H}^1(\mathcal{O}_\beta)$ the class of functions $u \in H^1(\mathcal{O}_\beta)$ such that the γ_α-periodic extension of u belongs to $H^1(\mathcal{O}_R)$ for any $R < \infty$. Along with Condition 8.1 we assume that the following condition is satisfied.

Condition 8.2. *The domain $\mathcal{O}_\beta$ is such that the embedding $\widehat{H}^1(\mathcal{O}_\beta) \subset L_2(\mathcal{O}_\beta)$ is compact.*

Remark 8.3. In Condition 8.2, the domain $\mathcal{O}_\beta$ can be replaced by any (not necessarily connected) open set in Π fundamental with respect to the lattice γ_α. Indeed, the space $\widehat{H}^1(\mathcal{O}_\beta)$ is identical with the space H^1 on the surface of cylinder, formed by factor-set $\Pi\big/\gamma_\alpha$. However, it is more convenient for us to use the fundamental domain $\mathcal{O}_\beta$.

Note that from Condition 8.2 it follows automatically that the domain $\mathcal{O}_\beta$ is of finite measure:

$$\operatorname{meas} \mathcal{O}_\beta < \infty.$$

3. Functional classes. We introduce the following classes of functions in Π: $\mathfrak{C}_0(\Pi) := \{F : \ F = f \circ \Psi, \ f \in C_0(\overline{S})\}$, $\mathfrak{C}(\Pi) := \{F : \ F = f \circ \Psi, \ f \in C(\overline{S})\}$. We denote by $\widehat{\mathfrak{C}}(\mathcal{O}_\beta)$ the class of functions that are restrictions to $\mathcal{O}_\beta$ of the γ_α-periodic functions of class $\mathfrak{C}(\Pi)$. Obviously, $\widehat{\mathfrak{C}}(\mathcal{O}_\beta) = \{F : \ F = f \circ \Psi, \ f \in \widehat{C}(\overline{\Omega_\beta})\}$.

We consider the Schrödinger operator in the waveguide Π with Dirichlet or natural boundary conditions on each of the two parts l_-, l_+ of the boundary $\partial\Pi$. According to these choices we consider the boundary problems of four types denoted by DD, DN, ND, NN. We introduce the following notation, where indices λ and μ mean D or N:

$$C_0^{(\lambda\mu)}(\overline{S}) := \begin{cases} \{f \in C_0(\overline{S}) : \ f|_{\Xi_0} = f|_{\Xi_1} = 0\}, \ \lambda = \mu = D, \\ \{f \in C_0(\overline{S}) : \ f|_{\Xi_0} = 0\}, \ \lambda = D, \ \mu = N, \\ \{f \in C_0(\overline{S}) : \ f|_{\Xi_1} = 0\}, \ \lambda = N, \ \mu = D, \\ C_0(\overline{S}), \ \lambda = \mu = N, \end{cases}$$

$$\mathfrak{C}_0^{(\lambda\mu)}(\Pi) := \{F : \ F = f \circ \Psi, \ f \in C_0^{(\lambda\mu)}(\overline{S})\}, \quad \lambda, \, \mu = D, \, N.$$

4. Conditions on the metric. Let $g(\mathbf{x}) = \{g^{jl}(\mathbf{x})\}$ be a measurable (2×2)-matrix-valued function in Π with real-valued entries. Suppose that

$$g(\mathbf{x} + n\alpha\mathbf{e}_1) = g(\mathbf{x}), \quad \mathbf{x} \in \Pi, \quad n \in \mathbf{Z}, \tag{8.1}$$

$$c_2\mathbf{1} \leq g(\mathbf{x}) \leq c_3\mathbf{1}, \quad \mathbf{x} \in \Pi, \quad 0 < c_2 \leq c_3 < \infty. \tag{8.2}$$

We present the metric $g(\mathbf{x})$ in the following form (cf. (1.5)):

$$g(\mathbf{x}) := \omega^2(\mathbf{x})g_0(\mathbf{x}), \quad \det g_0(\mathbf{x}) = 1, \quad \omega(\mathbf{x}) := (\det g(\mathbf{x}))^{1/4}. \tag{8.3}$$

By (8.1)–(8.3), g_0 and ω are γ_α-periodic and satisfy the conditions

$$c_2'\mathbf{1} \le g_0(\mathbf{x}) \le c_3'\mathbf{1}, \quad \mathbf{x} \in \Pi, \quad 0 < c_2' \le c_3' < \infty,$$

$$0 < \omega_0 \le \omega(\mathbf{x}) \le \omega_1 < \infty, \quad \mathbf{x} \in \Pi.$$

We assume that (cf. (1.6), (1.7))

$$\omega \in \widehat{H}^1(\mathcal{O}_\beta), \tag{8.4}$$

and

$$\int\limits_{\mathcal{O}_\beta} |\nabla\omega|^2 |u|^2 \, d\mathbf{x} \le \varepsilon \int\limits_{\mathcal{O}_\beta} |\nabla u|^2 \, d\mathbf{x} + C(\varepsilon;\omega) \int\limits_{\mathcal{O}_\beta} |u|^2 \, d\mathbf{x}, \quad \forall \varepsilon \in (0,1), \quad u \in \widehat{H}^1(\mathcal{O}_\beta). \tag{8.5}$$

5. Conditions on the weight function. Let $\eta(\mathbf{x})$ be a measurable real-valued function in Π such that (cf. (1.13)–(1.15))

$$\eta(\mathbf{x} + n\alpha\mathbf{e}_1) = \eta(\mathbf{x}), \quad \mathbf{x} \in \Pi, \quad n \in \mathbf{Z}, \tag{8.6}$$

$$\eta(\mathbf{x}) > 0, \quad \text{a. a. } \mathbf{x} \in \Pi, \tag{8.7}$$

$$\int\limits_{\mathcal{O}_\beta} \eta^2 |u|^2 \, d\mathbf{x} \le \varepsilon \int\limits_{\mathcal{O}_\beta} |\nabla u|^2 \, d\mathbf{x} + C(\varepsilon;\eta) \int\limits_{\mathcal{O}_\beta} |u|^2 \, d\mathbf{x}, \quad \forall \varepsilon \in (0,1), \quad u \in \widehat{H}^1(\mathcal{O}_\beta). \tag{8.8}$$

6. Conditions on the magnetic potential. Let $A_1(\mathbf{x})$, $A_2(\mathbf{x})$ be measurable real-valued functions in Π. We put $\mathbf{A}(\mathbf{x}) = A_1(\mathbf{x})\mathbf{e}_1 + A_2(\mathbf{x})\mathbf{e}_2$. We assume that (cf. (1.1), (1.2))

$$\mathbf{A}(\mathbf{x} + n\alpha\mathbf{e}_1) = \mathbf{A}(\mathbf{x}), \quad \mathbf{x} \in \Pi, \quad n \in \mathbf{Z}, \tag{8.9}$$

$$\int\limits_{\mathcal{O}_\beta} |\mathbf{A}|^2 |u|^2 \, d\mathbf{x} \le \varepsilon \int\limits_{\mathcal{O}_\beta} |\nabla u|^2 \, d\mathbf{x} + C(\varepsilon;\mathbf{A}) \int\limits_{\mathcal{O}_\beta} |u|^2 \, d\mathbf{x}, \quad \forall \varepsilon \in (0,1), \quad u \in \widehat{H}^1(\mathcal{O}_\beta). \tag{8.10}$$

7. Conditions on the generalized electric potential. Let $d\nu^\circ$ be a Borel real-valued signed measure in $\overline{S}$ with locally finite variation, i.e.,

$$\int\limits_{\mathcal{B}} |d\nu^\circ| < \infty \tag{8.11}$$

for any Borel bounded set $\mathcal{B} \subset \overline{S}$. We assume that the signed measure $d\nu^\circ$ is γ_β-periodic:

$$\int\limits_{\mathcal{B}+n\beta\mathbf{e}_1} d\nu^\circ = \int\limits_{\mathcal{B}} d\nu^\circ, \quad n \in \mathbf{Z}. \tag{8.12}$$

We introduce two linear continuous functionals ℓ and $\widehat{\ell}$ on the classes $\mathfrak{C}_0(\Pi)$ and $\widehat{\mathfrak{C}}(\mathcal{O}_\beta)$ respectively:

$$\ell[F] := \int\limits_{\overline{S}} F \circ \Psi^{-1}\, d\nu^\circ, \quad F \in \mathfrak{C}_0(\Pi), \tag{8.13}$$

$$\widehat{\ell}[\widehat{F}] := \int\limits_{\widehat{\Omega}_\beta} \widehat{F} \circ \Psi^{-1}\, d\nu^\circ, \quad \widehat{F} \in \widehat{\mathfrak{C}}(\mathcal{O}_\beta). \tag{8.14}$$

Here $\widehat{\Omega}_\beta := [-\beta/2, \beta/2) \times [0,1]$.

Remark 8.4. We use such indirect definition of the functionals ℓ, $\widehat{\ell}$ just because in the case of a waveguide of a general type we cannot directly describe the behavior of the generalized electric potential on the boundary $\partial\Pi$. In [Sh4, Section 2.5] one can find some comments on functionals ℓ, $\widehat{\ell}$ and some examples. In particular, if the boundary of the waveguide Π consists of two disjoint γ_α-periodic Jordan curves, then there is one-to-one correspondence between the set of functionals ℓ and the set of real-valued Borel γ_α-periodic signed measures in $\overline{\Pi}$ with locally finite variation.

As usual, we put $2d\nu^\circ_\pm := |d\nu^\circ| \pm d\nu^\circ$. Similarly to (8.13), (8.14) we define the functionals $|\ell|$, $|\widehat{\ell}|$ in terms of the signed measure $|d\nu^\circ|$; and functionals $\ell_\pm$, $\widehat{\ell}_\pm$ in terms of the signed measure $d\nu^\circ_\pm$. Obviously, $2\ell_\pm = |\ell| \pm \ell$, $2\widehat{\ell}_\pm = |\widehat{\ell}| \pm \widehat{\ell}$.

We assume that ℓ, $\widehat{\ell}$ satisfy the following conditions (cf. (i)–(iii)):

(i°) *The form*

$$\int\limits_\Pi |\nabla u|^2\, d\mathbf{x} + \ell_+[|u|^2], \quad u \in H^1(\Pi) \cap \mathfrak{C}_0(\Pi),$$

is closable in $L_2(\Pi)$.

(ii°) *For some* $a < 1$, *we have*

$$\ell_-[|u|^2] \le a \left(\int\limits_\Pi \langle g\nabla u, \nabla u\rangle\, d\mathbf{x} + \ell_+[|u|^2] \right) + C \int\limits_\Pi |u|^2\, d\mathbf{x},$$
$$u \in H^1(\Pi) \cap \mathfrak{C}_0(\Pi), \quad a < 1, \quad C = C(a; g, \ell).$$

(iii°) *We have*

$$|\widehat{\ell}|[|u|^2] \le \varepsilon \left(\int\limits_{\mathcal{O}_\beta} |\nabla u|^2\, d\mathbf{x} + \sup_{\mathbf{x} \in \mathcal{O}_\beta} |u(\mathbf{x})|^2 \right) + C(\varepsilon; \widehat{\ell}) \int\limits_{\mathcal{O}_\beta} |u|^2\, d\mathbf{x},$$
$$\forall \varepsilon \in (0,1), \quad u \in \widehat{H}^1(\mathcal{O}_\beta) \cap \widehat{\mathfrak{C}}(\mathcal{O}_\beta).$$

8. Definition of the operators. In the Hilbert space $L_2(\Pi)$ we consider the quadratic form

$$m_{\lambda\mu}[v,v] := \int\limits_{\Pi} \langle g(\mathbf{D}-\mathbf{A})\eta^{-1}v, (\mathbf{D}-\mathbf{A})\eta^{-1}v\rangle \, d\mathbf{x} + \ell[|\eta^{-1}v|^2],$$
$$\operatorname{Dom} m_{\lambda\mu} := \{v = \eta u : \; u \in H^1(\Pi) \cap \mathfrak{C}_0^{(\lambda\mu)}(\Pi)\}, \quad \lambda = D, N, \;\; \mu = D, N. \tag{8.15}$$

In [Sh4, Section 2.6], it was shown that the form (8.15) is densely defined, lower semibounded and closable in $L_2(\Pi)$. The closed form $\overline{m_{\lambda\mu}}$ gives rise to a selfadjoint operator $M_{\lambda\mu}$ in $L_2(\Pi)$. The operator M_{DD} corresponds to the Dirichlet boundary problem in Π; the operator M_{DN} (respectively, M_{ND}) corresponds to the Dirichlet condition on l_- (respectively, on l_+). At the same time, the condition on l_+ (respectively, on l_-) is natural. The operator M_{NN} corresponds to the natural boundary condition on $\partial\Pi$.

9. The main result for the Schrödinger operator in a two-dimensional periodic waveguide is the following theorem.

Theorem 8.5. *Suppose that the domain* Π *satisfies Conditions* 8.1, 8.2. *Let the measurable* (2×2)*-matrix-valued function* $g(\mathbf{x})$ *satisfies conditions* (8.1)–(8.5). *Suppose that the real-valued function* $\eta(\mathbf{x})$ *satisfies conditions* (8.6)–(8.8), *and that the vector-valued function* $\mathbf{A}(\mathbf{x})$ *satisfies conditions* (8.9), (8.10). *Let the functionals* ℓ, $\widehat{\ell}$ *be defined by* (8.13), (8.14). *Suppose that conditions* (8.11), (8.12), (i°)–(iii°) *are satisfied. Let the quadratic form* $m_{\lambda\mu}$ *(where* λ *and* μ *mean* D *or* N*) be defined by* (8.15), *and let the form* $\overline{m_{\lambda\mu}}$ *be the closure of the form* $m_{\lambda\mu}$ *in* $L_2(\Pi)$. *Let* $M_{\lambda\mu}$ *be the selfadjoint operator in* $L_2(\Pi)$ *associated with* $\overline{m_{\lambda\mu}}$. *Then the spectrum of each of the four operators* M_{DD}, M_{DN}, M_{ND}, M_{NN} *is absolutely continuous.*

Remark 8.6. 1) In [Sh4], the mapping Ψ was used to reduce the problem in the waveguide Π to the similar problem in the strip S with coefficients (7.9)–(7.12). It turned out that the conditions on the new coefficients in the strip S are similar to those on the initial coefficients in the waveguide Π. The problem in the strip S was investigated in detail in [ShSu]. In [ShSu] the consideration of all the four boundary problems in the strip was reduced to the investigation of the problem with periodic boundary conditions. Next, the periodic operator in $L_2(S)$ was decomposed into the direct integral of operators with periodic boundary conditions acting in $L_2(\Omega_\beta)$ and depending on the quasimomentum $k \in \mathbf{R}$. Then the proof of absolute continuity of the spectrum was reduced to estimating the resolvent of the operator in the fiber for sufficiently large imaginary value of the quasimomentum $k \in \mathbf{C}$. Such estimates were borrowed from [Sh3, Theorem 2.2]. Using more precise Theorem 2.1 from the present paper, we can relax the conditions on the magnetic potential. 2) Formally, condition (8.10) is only slightly better than the one used in [Sh4]. However, it possesses the essential advantage. Namely, condition (8.10) is invariant with respect to the changes of coordinates which transform the waveguide Π onto the strip S. The condition on the magnetic potential used in the previous papers does not possess such property. In [Sh4], this difficulty was overcome either by connecting the condition on the magnetic potential with the definite mapping Ψ (cf. [Sh4, (2.15)]), or by putting some more restrictive conditions on the waveguide (cf. [Sh4, Section 2.4]).

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