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ON THE SPECTRUM AND THE LOWEST EIGENVALUE OF CERTAIN NON-COMMUTATIVE HARMONIC OSCILLATORS

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Abstract. We study here various expansions and approximations of the spectrum of non-commutative harmonic oscillators of the type

$$Q(x,D_x)=A(-\frac{\partial_x^2}{2}+\frac{x^2}{2})+B(x\partial_x+\frac{1}{2}),\ x\in\mathbb{R},$$

with $A,B\in \mathrm{Mat}_2(\mathbb{R})$ constant matrices such that $A={}^tA>0$ (or <0) and $B=-{}^tB\neq 0$, when the Hermitian matrix $A+iB$ is positive (or negative) definite. Special emphasis is put on the lowest eigenvalue. These expansions are written in the limit $\det(A)/\mathrm{pf}(B)^2\to+\infty$, in terms of the matrices A and B.

1. Introduction

In this paper we study various expansions and approximations of the spectrum of non-commutative harmonic oscillators (introduced in [15] and [16]; see also [14] and [11]) of the type

$$Q(x,D_x)=A(-\frac{\partial_x^2}{2}+\frac{x^2}{2})+B(x\partial_x+\frac{1}{2}),\ x\in\mathbb{R},$$

where $A,B\in \mathrm{Mat}_2(\mathbb{R})$ are constant matrices such that $A={}^tA>0$ (or <0) and $B=-{}^tB\neq 0$, and such that the Hermitian matrix $A+iB$ is positive (or negative) definite. In particular, we are interested in better understanding the lowest eigenvalue of such systems. In fact, the structure of the spectrum (along with its multiplicity) was determined in the papers [15] and [16] (see also [14]). However, since the eigenvalues are described by zeros of particular functions defined in general through continued fractions, they are still difficult to be determined explicitly. On the other hand, considering the limit $\det(A)/\mathrm{pf}(B)^2\to+\infty$ allows us to use Rellich's Holomorphic Perturbation Theory (see [18] and [8]) to obtain the eigenvalues (and the relative suitably normalized eigenfunctions) as analytic functions of the parameter $\det(A)/\mathrm{pf}(B)^2$. In particular, we shall be able to select (among the candidate ones found in the papers [15]-[16]) the equation satisfied by the lowest eigenvalue (for $\det(A)/\mathrm{pf}(B)^2$ large; see equation (33) of Theorem 4.11 below). This equation allows a different interpretation of the terms arising through Rellich's method, and an approximation of the lowest eigenvalue

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which is the best one, as compared to Brummelhuis' one (see [1]) and to the one given by the W.K.B. method (which turns out to be the worst one, at least in the simple-minded approach followed in this paper). We remark in passing that, as proved in Corollary 2.4 below and as seen also by Rellich's theory, the lowest eigenvalue of $Q(x, D_x)$ when $A > 0$ and $A + iB > 0$ is strictly smaller than the one of $A(-\partial_x^2 + x^2)/2$.

As proven in [15] (see also [14]), the spectral problem for system $Q(x, D_x)$ is unitarily equivalent to the spectral problem for

$$Q_{(\alpha,\beta)}(x, D_x) = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} (-\frac{\partial_x^2}{2} + \frac{x^2}{2}) + J(x\partial_x + \frac{1}{2}),$$

where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, the parameters α, β belonging to

$$H_+ := \{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+;\ \alpha\beta > 1\},$$

endowed with coordinates

$$\ell := \sqrt{\alpha\beta - 1} \in \mathbb{R}_+, \quad z = \sqrt{\frac{\alpha}{\beta}} \in \mathbb{R}_+.$$

We may hence reduce matters to considering the problem for $Q_{(\alpha,\beta)}(x, D_x)$, with $\alpha, \beta > 0$, in the limit $\sqrt{\alpha\beta} \to +\infty$.

The plan of the paper is the following. In Section 2 we shall set up the perturbation theory used, and prove Corollary 2.4 mentioned above. In Section 3 we shall exploit Rellich's method using the classical Hermite basis, and obtain the expansion of the eigenvalues of $Q(x, D_x)$ in the different instances of a simple eigenvalues and of a higher-multiplicity eigenvalue (with a suitable extra assumption), respectively. In Section 4 we shall compare the different approximations and bounds obtained by the Rellich, Brummelhuis, W.K.B. and Parmeggiani-Wakayama methods, respectively.

We close this introduction by pointing out that the study of the spectrum of matrix-valued oscillators such as $Q(x, D_x)$ is important to the study of lower bounds and solvability of systems of pseudodifferential operators (see, e.g., [1], [2], [6], [12], [13]), and in some semiclassical asymptotics and Morse-type inequality such as the ones of Shubin's paper [20]. Moreover, the study carried out here is to be considered as "complementary" to the (numerical) study done in [9] in the case $\alpha = s\beta$, for increasing $s > 0$.

We hope to be able to apply some of the results of this paper (namely Corollary 3.12 below) to the study of the Lieb-Thirring constant for the system $Q(x, D_x)$.

2. Setting the expansion

In this section we prepare the ground for the Rellich expansion by writing $Q_{(\alpha,\beta)}(x, D_x)$ as a perturbed diagonal harmonic oscillator. We set

$$h(x, D_x) = \frac{-\partial_x^2 + x^2}{2}, \quad \text{and} \quad e(x, D_x) = x\partial_x + \frac{1}{2}, \ x \in \mathbb{R}.$$

Since we may write

$$(1)\quad Q_{(\alpha,\beta)}(x,D_x)=\sqrt{\alpha\beta}\left(\begin{bmatrix}\sqrt{\dfrac{\alpha}{\beta}} & 0\\ 0 & \sqrt{\dfrac{\beta}{\alpha}}\end{bmatrix}h(x,D_x)+\frac{1}{\sqrt{\alpha\beta}}Je(x,D_x)\right),$$

a convenient set of variables in H_+ is therefore given by

$$z=\sqrt{\frac{\alpha}{\beta}},\quad t=\frac{1}{\sqrt{\alpha\beta}}=\frac{1}{\sqrt{\ell^2+1}}.$$

The choice of t in place of ℓ is more convenient for our present purposes. To simplify notation we will also set $A(z):=\begin{bmatrix} z & 0\\ 0 & 1/z\end{bmatrix}$.
We will consider here the case $0<\alpha\le\beta$, that is

$$z\in(0,1]\ \text{ and }\ t\in(0,1),$$

and study the expansion of the eigenvalues, and especially of the lowest eigenvalue, for **fixed** z and for $t\to 0+$ and actually as $t\to 0$ (see the remark below), of the system

$$Q_t(x,D_x)=A(z)h(x,D_x)+tJe(x,D_x)=:Q_0(x,D_x)+tM(x,D_x),\ x\in\mathbb{R}.$$

Remark 2.1. *Because of the particular form of Q_t, we may, and will, consider also $t\in(-1,0)$. We shall hence write the expansion as $t\to 0$. Also, (1) yields that for $t\neq 0$*

$$\lambda\in\operatorname{Spec}(Q_t)\iff\frac{1}{t}\lambda\in\operatorname{Spec}(Q_{(\alpha,\beta)}).$$

Notice finally that the case $0<\beta\le\alpha$, i.e. $z\in[1,+\infty)$, is similar, and gotten from the case $\alpha\le\beta$ by considering the intertwining $Q_t\longmapsto KQ_tK$, where $K=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, and by taking the limit as $t\to 0+$. In fact, $KQ_tK=A(z^{-1})h(x,D_x)-tJe(x,D_x)$ is **isospectral** *to Q_t (since K is an isometry of $L^2(\mathbb{R};\mathbb{C}^2)$).*

For convenience, we now recall the reason why we assume $|t|<1$. Upon denoting by $Q_t(x,\xi)=\begin{bmatrix} z(x^2+\xi^2)/2 & -itx\xi\\ itx\xi & z^{-1}(x^2+\xi^2)/2\end{bmatrix}$ the principal symbol of Q_t, we have for $|t|<1$,

$$\det Q_t(x,\xi)=\left(\frac{x^2-\xi^2}{2}\right)^2+(1-t^2)x^2\xi^2=0\implies x=\xi=0,$$

that is Q_t is a globally elliptic operator (see [19] or [5]) for all $t\in(-1,1)$. Hence

$$Q_t(x,D_x)\colon\mathcal{S}(\mathbb{R};\mathbb{C}^2)\longrightarrow\mathcal{S}(\mathbb{R};\mathbb{C}^2),\ \text{and}\ Q_t(x,D_x)\colon\mathcal{S}'(\mathbb{R};\mathbb{C}^2)\longrightarrow\mathcal{S}'(\mathbb{R};\mathbb{C}^2)$$

are continuous and, as unbounded operators in $L^2(\mathbb{R};\mathbb{C}^2)$, they all have the same domain

$$\mathsf{D}(Q_t)=B^2(\mathbb{R};\mathbb{C}^2)=\{u\in L^2(\mathbb{R};\mathbb{C}^2);\ x^\alpha\partial_x^\beta u\in L^2,\forall\alpha,\beta,0\le\alpha+\beta\le 2\},$$

with norm $\|u\|_{B^2}^2=\sum_{0\le\alpha+\beta\le 2}\|x^\alpha\partial_x^\beta u\|_{L^2}^2$, which is the domain of Q_0, they all are self-adjoint, they all have compact resolvent and have discrete spectrum made of eigenvalues diverging to $+\infty$, with finite multiplicities. Let us now write, for all $t\in(-1,1)$, $\mathrm{Spec}(Q_t)=\{\lambda_k(t)\}_{k\in\mathbb{Z}_+}$, where $\lambda_0(t)\le\lambda_1(t)\le\ldots\le\lambda_k(t)\longrightarrow+\infty$, are the eigenvalues, being repeated according to their multiplicities. Notice that as regards Q_0,

$$\lambda_k(0)=z^{\pm 1}\left(n+\frac{1}{2}\right),\quad \text{for some choice of } \pm \text{ and some } n\in\mathbb{Z}_+.$$

Remark that $\mathcal{S}(\mathbb{R};\mathbb{C}^2)\subset\mathsf{D}(Q_t)=\mathsf{D}(Q_0)\subset\mathsf{D}(M)$, for all $t\in(-1,1)$.

Lemma 2.2. *There exists $C_0>0$, independent of z, such that*

$$\|Mu\|\le C_0\left(z+\frac{1}{z}\right)\Big(\|Q_0u\|+\|u\|\Big),\ \forall u\in B^2(\mathbb{R};\mathbb{C}^2). \tag{2}$$

Hence, for $|t|<1$, $Q_t\longrightarrow Q_0$ as $t\to 0$ in the **strong generalized sense of Kato**. *In particular, for any given $0\le\varepsilon_1<\varepsilon_2$, with $\varepsilon_1,\varepsilon_2\notin\mathrm{Spec}(Q_0)$, there exists $\delta>0$ such that*

$$\dim\Big[\bigoplus_{\lambda\in(\varepsilon_1,\varepsilon_2)}\mathrm{Ker}\Big(Q_t-\lambda I_{L^2}\Big)\Big]=\dim\Big[\bigoplus_{\lambda\in(\varepsilon_1,\varepsilon_2)}\mathrm{Ker}\Big(Q_0-\lambda I_{L^2}\Big)\Big],\ \forall t\in(-\delta,\delta).$$

Recall that by $Q_t\longrightarrow Q_0$ as $t\to 0$ *in the strong generalized convergence of Kato*, we mean that the "gap" between the graphs of the operators Q_t and Q_0 tends to zero as $t\to 0$ (see [8], page 197 for the definition of gap between subspaces of a Banach space and pages 201-202 for the definition of strong generalized convergence).

Proof of the lemma. Since h is globally elliptic, there exists a *parametrix* $p_0(x,D_x)\in\mathrm{OPS}^{-2}_{\mathrm{glob}}$ of h such that $p_0(x,D_x)h(x,D_x)=I_{L^2(\mathbb{R};\mathbb{C})}+r(x,D_x)$, with $r(x,D_x)\in\mathrm{OPS}^{-\infty}_{\mathrm{glob}}$ (see [19] or [5]). Hence $E_0=A(z^{-1})p_0(x,D_x)$ is a parametrix of Q_0, such that $E_0Q_0=I_{L^2(\mathbb{R};\mathbb{C}^2)}+R$, with $R=r(x,D_x)I_{\mathbb{C}^2}\in\mathrm{OPS}^{-\infty}_{\mathrm{glob}}$. From the L^2-boundedness of operators in $\mathrm{OPS}^0_{\mathrm{glob}}$, and the fact that $MR=Je(x,D_x)r(x,D_x)\in\mathrm{OPS}^0_{\mathrm{glob}}$, we have that for any given $u\in B^2(\mathbb{R};\mathbb{C}^2)$, there exists $C_0>0$ (independent of z) such that

$$\|Mu\|=\|M(E_0Q_0-R)u\|\le C_0\left(z+\frac{1}{z}\right)\Big(\|Q_0u\|+\|u\|\Big),$$

which proves (2). We hence have for $|t|<1$

$$\|(Q_t-Q_0)u\|=|t|\|Mu\|\le|t|C_0\left(z+\frac{1}{z}\right)\Big(\|Q_0u\|+\|u\|\Big),\ \forall u\in B^2(\mathbb{R};\mathbb{C}^2), \tag{3}$$

(recall that $\|Q_0 u\| + \|u\| \approx \|u\|_{B^2}$). Theorem 2.24 (page 206) and Theorem 3.16 (page 212-213) of [8] immediately yield the second part of the statement. □

Remark 2.3. *Inequality (3) also yields, by a theorem of Rellich (see* [8]*, Theorem 2.6, page 377-378), that* $\{Q_t\}_{|t|<1}$ *defines a* **holomorphic family of type (A)***, for any given* $t \in I_0 := (-t_0, t_0)$*, where* $t_0 = t_0(z) := \min\Big\{1, \Big[C_0(z^2+1)/z\Big]^{-1}\Big\}$*. It follows that, being all the* Q_t *self-adjoint with compact resolvent for all* $t \in I_0$*, by Theorem 3.9, page 392 of* [8] *(see also* [18]*, page 115),* **all** *the eigenvalues of* Q_t *are given by analytic functions* $\lambda_k \in C^\omega(I_0; \mathbb{R})$*,* $k \in \mathbb{Z}_+$ *(keeping into account repetitions given by multiplicities). Moreover, one may find a sequence of analytic vector-valued functions* $f_k \in C^\omega(I_0; L^2(\mathbb{R}; \mathbb{C}^2))$*,* $k \in \mathbb{Z}_+$*, forming a* **complete** *orthonormal family of the associated eigenfunctions of* Q_t*.*

As a corollary of Lemma 2.2 and Rellich's theory, we have the following a-priori control on the expansion of the lowest eigenvalue of Q_t.

Corollary 2.4. *Let* $\lambda \in C^\omega(I_0; \mathbb{R})$ *be the analytic function that represents the lowest eigenvalue of* Q_t*, which we may suppose, upon shrinking* I_0 *if necessary, to be simple for all* $t \in I_0$ *(for the lowest eigenvalue of* Q_0 *is simple). Then*

$$\partial_t \lambda(0) = 0, \quad \partial_t^2 \lambda(0) < 0.$$

In particular,

$$\lambda(t) < \lambda(0), \quad \text{for } 0 < |t| \text{ sufficiently small.}$$

Proof. We start by considering $u \in C^\omega(I_0; L^2(\mathbb{R}; \mathbb{C}^2))$ with $\|u(t)\| = 1$ for all $t \in I_0$, the relative eigenfunction belonging to λ (i.e. $Q_t u(t) = \lambda(t) u(t)$, $t \in I_0$). We may, and will, suppose u to be real-valued. Let $\lambda_0 = \lambda(0) = \min \operatorname{Spec}\Big(Q_0(x, D_x)\Big)$. Denote for short, $B^0 = L^2(\mathbb{R}; \mathbb{C}^2)$, $B^2 = B^2(\mathbb{R}; \mathbb{C}^2)$.

Lemma 2.5. *There exists* $0 < \delta_* < 1$ *(dependent only on z) such that, with* $I_1 := (-\delta_*, \delta_*) \subset I_0$*, we have*

$$Q_0 u, \ Mu \in C(I_1; B^0).$$

Also, as a consequence, for any fixed $t' \in I_1$*,* $Q_{t'} u \in C(I_1; B^0)$*.*

Proof of the lemma. By estimate (2) we have the existence of $\delta_* \in (0, 1)$, such that

$$\|Q_0 f\| + \|f\| \le 2\Big(\|Q_t f\| + \|f\|\Big), \quad \forall f \in B^2, \ \forall t \in (-\delta_*, \delta_*). \tag{4}$$

Moreover, by elliptic regularity, $u(t) \in B^2$ for all $t \in I_0$. Let now $t, t_0 \in I_1$. Since

$$Q_t\Big(u(t) - u(t_0)\Big) = \lambda(t)u(t) - \lambda(t_0)u(t_0) - (t - t_0)Mu(t_0) \xrightarrow{B^0} 0, \quad \text{as} \quad t \to t_0,$$

we have, by (4),

$$\|Q_0u(t)-Q_0u(t_0)\| = \|Q_0\Big(u(t)-u(t_0)\Big)\|$$
$$\lesssim \|u(t)-u(t_0)\| + \|Q_t\Big(u(t)-u(t_0)\Big)\| \longrightarrow 0,$$

as $t \to t_0$. Again by (2), we therefore have

$$\|Mu(t)-Mu(t_0)\| \lesssim \|Q_0\Big(u(t)-u(t_0)\Big)\| + \|u(t)-u(t_0)\| \longrightarrow 0, \quad \text{as} \quad t \to t_0.$$

The last statement follows from the following inequality (again a consequence of (2)):

$$\|Q_{t'}u(t)-Q_{t'}u(t_0)\| \lesssim \|Q_0\Big(u(t)-u(t_0)\Big)\| + \|u(t)-u(t_0)\|.$$

This conclues the proof of the lemma. □

Lemma 2.6. *We have, for all $t \in I_1$, $\partial_t u(t) \in B^2$, and*

$$\partial_t\big(Q_tu(t)\big) = Q_t\partial_tu(t) + Mu(t), \ \partial_t(Mu,u(t)) = 2(M\partial_tu(t),u(t))$$

(recall that u is real-valued). Moreover, $Q_0\partial_tu \in C(I_1;B^0)$ (and hence also $M\partial_tu \in C(I_1;B^0)$).

Proof of the lemma. Take $t, t_0 \in I_1$. Write

$$\frac{Q_tu(t)-Q_{t_0}u(t_0)}{t-t_0} = Q_0\frac{u(t)-u(t_0)}{t-t_0} + tM\frac{u(t)-u(t_0)}{t-t_0} + Mu(t_0).$$

It follows, for all $f \in \mathcal{S}(\mathbb{R};\mathbb{C}^2)$ (recall that $Q_0 = Q_0^*$ and $M = M^*$ on B^2) as $t \to t_0$,

$$\left(\frac{Q_tu(t)-Q_{t_0}u(t_0)}{t-t_0}, f\right) \longrightarrow (\partial_tu(t_0),Q_0f)+t_0(\partial_tu(t_0),Mf)+(Mu(t_0),f) =$$
$$= \big\langle Q_{t_0}\partial_tu(t),\bar{f}\big\rangle_{\mathcal{S},\mathcal{S}'} + (Mu(t_0),f), \ \forall f \in \mathcal{S}(\mathbb{R};\mathbb{C}^2).$$

On the other hand,

$$\frac{Q_tu(t)-Q_{t_0}u(t_0)}{t-t_0} = \frac{\lambda(t)u(t)-\lambda(t_0)u(t_0)}{t-t_0} \overset{B^0}{\longrightarrow} \partial_t(\lambda u)(t_0)$$

as $t \to t_0$. Then

$$Q_{t_0}\partial_tu(t_0) \overset{\mathcal{S}'}{=} -Mu(t_0) + \partial_t(\lambda u)(t_0) \in B^0, \tag{5}$$

and, by elliptic regularity,

$$\partial_tu(t) \in B^2, \quad \forall t \in I_1. \tag{6}$$

Hence

$$\partial_t(Q_tu(t)) = Q_t\partial_tu(t) + Mu(t), \ \forall t \in I_1. \tag{7}$$

Notice that the same argument gives also

$$Q_0\partial_tu(t), \ M\partial_tu(t) \in B^0, \ \forall t \in I_1.$$

Next we have

$$\frac{(Mu(t),u(t))-(Mu(t_0),u(t_0))}{t-t_0}\longrightarrow(\partial_t u(t_0),Mu(t_0))+(Mu(t_0),\partial_t u(t_0))$$
$$=2(M\partial_t u(t_0),u(t_0)),\quad \text{as}\ \ t\to 0,$$

by (6) and the continuity of $t\mapsto Mu(t)$.
Finally, by formula (5), it follows that

$$Q_t\Big(\partial_t u(t)-\partial_t u(t_0)\Big)=Q_t\partial_t u(t)-Q_{t_0}\partial_t u(t_0)+\Big(Q_{t_0}-Q_t\Big)\partial_t u(t_0)$$
$$=-M\Big(u(t)-u(t_0)\Big)+\partial_t(\lambda u)(t)-\partial_t(\lambda u)(t_0)-(t-t_0)M\partial_t u(t_0)\xrightarrow{B^0}0$$

as $t\to 0$. Hence,

$$\|Q_0\partial_t u(t)-Q_0\partial_t u(t_0)\|\lesssim\|\partial_t u(t)-\partial_t u(t_0)\|+\|Q_t(\partial_t u(t)-\partial_t u(t_0))\|\longrightarrow 0,$$

as $t\to 0$. This proves the lemma. □

We now go back to the proof of the corollary. Write $u(0)=u_0\in\mathrm{Span}\{\varphi_0\otimes e_1\}=\mathrm{Ker}(Q_0-\lambda_0)$, where $\varphi_0=e^{-x^2/2}$ and $e_1=\begin{bmatrix}1\\0\end{bmatrix}$. Then

$$0=\partial_t\Big(\|u(t)\|^2\Big)=2(\partial_t u(t),u(t)),\quad t\in I_1, \tag{8}$$

whence it follows that in particular $\partial_t u(0)\in\mathrm{Ker}(Q_0-\lambda_0)^{\perp}$. Since

$$\lambda(t)=(Q_t u(t),u(t))=(Q_0 u(t),u(t))+t(Mu(t),u(t)),$$

we get

$$\partial_t\lambda(t)=2(Q_t\partial_t u(t),u(t))+(Mu(t),u(t))$$
$$=2\lambda(t)(\partial_t u(t),u(t))+(Mu(t),u(t))\longrightarrow(Mu_0,u_0)=0,\quad\text{as}\ \ t\to 0,$$

for $Mu_0\in\mathrm{Ker}(Q_0-\lambda_0)^{\perp}$ and by (8). Notice that formula (7) and $Mu_0\in\mathrm{Ker}(Q_0-\lambda_0)^{\perp}$ yield $\partial_t u(0)\neq 0$, for otherwise

$$\partial_t(\lambda u)(0)=\partial_t\lambda(0)u(0)=Q_0\partial_t u(0)+Mu(0)=Mu(0),$$

that is $Mu(0)=\partial_t\lambda(0)u(0)$, which is impossible.
Notice, moreover, that

$$\partial_t\lambda(t)=(Mu(t),u(t)),\quad\forall t\in I_1. \tag{9}$$

By Lemma 2.6 we have

$$\partial_t^2\lambda(t)=2(M\partial_t u(t),u(t)), \tag{10}$$

and

$$Q_t\partial_t u(t)=\partial_t\lambda(t)u(t)-Mu(t)+\lambda(t)\partial_t u(t),$$

whence, upon recalling the definition of Q_t,

$$tM\partial_t u(t)=-\Big(Q_0-\lambda(t)\Big)\partial_t u(t)+\Big(\partial_t\lambda(t)-M\Big)u(t).$$

Hence, taking the inner product with $u(t)$ and using (9) yields

$$(M\partial_t u(t), u(t)) = -\frac{1}{t}\Big((Q_0 - \lambda(t))\partial_t u(t), u(t)\Big) \tag{11}$$

$$= -\frac{1}{t}\Big(\partial_t u(t), (Q_0 - \lambda(t))u(t)\Big),$$

for all $t \in I_1$. Let us now write, for $t \in I_1$,

$$\lambda(t) = \lambda_0 + \frac{1}{2}t^2\partial_t^2\lambda(0) + t^2\omega(t), \quad u(t) = u_0 + t\partial_t u(0) + t^2\tilde{u}(t),$$

and remark that $t^2\tilde{u}(t) \in B^0$, for all $t \in I_1$, for $t^2\tilde{u}(t) = u(t) - u_0 - t\partial_t u(0) \in B^0$ for all $t \in J$. Moreover $\tilde{u}(t) \overset{B^0}{\longrightarrow} \partial_t^2 u(0)/2$ as $t \to 0$, by the analiticity of u. Hence

$$(Q_0 - \lambda(t))u(t) = t(Q_0 - \lambda_0)\Big(\partial_t u(0) + t\tilde{u}(t)\Big) - t^2\Big(\frac{1}{2}\partial_t^2\lambda(0) + \omega(t)\Big)u(t),$$

so that

$$\frac{1}{t}\Big((Q_0 - \lambda(t))\partial_t u(t), u(t)\Big) = \tag{12}$$

$$= \Big((Q_0 - \lambda_0)\partial_t u(t), \partial_t u(0) + t\tilde{u}(t)\Big) + o(1) \longrightarrow \Big((Q_0 - \lambda_0)\partial_t u(0), \partial_t u(0)\Big),$$

as $t \to 0$, by Lemma 2.6. Therefore, on the one hand (10) yields

$$(M\partial_t u(t), u(t)) \longrightarrow \frac{1}{2}\partial_t^2\lambda(0), \quad \text{as} \quad t \to 0,$$

and on the other (11) and (12) give

$$(M\partial_t u(t), u(t)) \longrightarrow -\Big((Q_0 - \lambda_0)\partial_t u(0), \partial_t u(0)\Big) \leq$$

$$\leq -\left[\inf_{0 \neq v \in B^2 \cap \mathrm{Ker}(Q_0 - \lambda_0)^\perp} \frac{\Big((Q_0 - \lambda_0)v, v\Big)}{\|v\|^2}\right] \|\partial_t u(0)\|^2 < 0,$$

being $0 \neq \partial_t u(0) \in \mathrm{Ker}(Q_0 - \lambda_0)^\perp$ and $Q_0 - \lambda_0 > 0$ on $B^2 \cap \mathrm{Ker}(Q_0 - \lambda_0)^\perp$. This concludes the proof of the corollary. □

Remark 2.7. *Notice that the above corollary holds in more general situations. For instance, it holds for $|t|$ sufficiently small when $A_t = A_0 + tA_1$, $\mathsf{D}(A_t) = \mathsf{D}(A_0) = B^2$, with $A_0 = A_0^*$ (globally) elliptic, where $A_1 = A_1^*$ satisfies estimate (2), $\lambda_0 = \min \mathrm{Spec}(A_0)$ simple, $A_1 \colon \mathrm{Ker}(A_0 - \lambda_0) \longrightarrow \mathrm{Ker}(A_0 - \lambda_0)^\perp$, $\inf_{0 \neq f \in B^2 \cap \mathrm{Ker}(A_0 - \lambda_0)^\perp} \Big((A_0 - \lambda_0)f, f\Big)/\|f\|^2 > 0$.*

3. The Rellich method

We start this section by recalling a few facts related to the oscillator-representation of $\mathfrak{sl}_2(\mathbb{R})$. Let

$$\psi := \frac{x+\partial_x}{\sqrt{2}}, \quad \psi^\dagger := \frac{x-\partial_x}{\sqrt{2}}.$$

Then $\psi^\dagger$ is the "creation" operator, ψ is the "annihilation" operator and

$$[\psi,\psi^\dagger]=1, \quad h(x,D_x)=\psi\psi^\dagger-\frac{1}{2}, \quad e(x,D_x)=\frac{\psi^2}{2}-\frac{(\psi^\dagger)^2}{2}.$$

Hence, upon defining $\varphi_0=e^{-x^2/2}$ and $\varphi_n=(\psi^\dagger)^n\varphi_0$, $n\in\mathbb{Z}_+$, one gets that $\{\varphi_n\}_{n\in\mathbb{Z}_+}$ is an **orthogonal** basis of $L^2(\mathbb{R};\mathbb{C})$ such that

$$\psi^\dagger\varphi_n=\varphi_{n+1},\ \psi\varphi_0=0,\ \psi\varphi_n=n\varphi_{n-1},\ (\varphi_n,\varphi_{n'})=n!\sqrt{\pi}\,\delta_{nn'},$$

$$h(x,D_x)\varphi_n=(n+\frac{1}{2})\varphi_n=:\mu_n\varphi_n,\ e(x,D_x)\varphi_n=\frac{1}{2}\Big(n(n-1)\varphi_{n-2}-\varphi_{n+2}\Big),$$

for all $n\in\mathbb{Z}_+$. Hence, the basis of $L^2(\mathbb{R};\mathbb{C}^2)$ with respect to which we shall construct the Rellich expansion is $\{\varphi_n\otimes e_j;\ n\in\mathbb{Z}_+, j=1,2\}$, where $\{e_1,e_2\}$ is the canonical basis of $\mathbb{C}^2$. Remark also that for all $t\in\mathbb{R}$, Q_t preserves the parity, for both Q_0 and M do. Hence also the expansion will decompose into one relative to even-eigenfunctions and one relative to odd-eigenfunctions, according to the orthogonal, invariant decomposition of

$$L^2(\mathbb{R};\mathbb{C}^2)=L^2_{\text{even}}(\mathbb{R};\mathbb{C}^2)\oplus L^2_{\text{odd}}(\mathbb{R};\mathbb{C}^2)=:L^2_+(\mathbb{R};\mathbb{C}^2)\oplus L^2_-(\mathbb{R};\mathbb{C}^2).$$

Let us concentrate on the case $z\in(0,1)$, fixed. The case $z=1$ is already explicitly understood in [15], and gives rise to multiplicity 2 eigenvalues. (This fact will also be trasparent in the construction of the expansion.) We will at first consider the case in which the limit-eigenvalue, relative to Q_0, is $\lambda_0(z,n):=z(n+1/2)$, for some **fixed** $n\in\mathbb{Z}_+$. Let us therefore look for

$$u(t,z;\cdot)=\sum_{k=0}^{+\infty}t^k u_k(z;\cdot), \quad \lambda(t)=\sum_{k=0}^{+\infty}t^k\lambda_k(z,n),$$

where it is no restriction to suppose that the u_k's and λ_k are all real. Next, let us consider the eigenvalue equation

$$Q_t(x,D_x)u(t)=\lambda(t)u(t), \quad \text{subject to } \|u(t)\|_{L^2(\mathbb{R};\mathbb{C}^2)}=\|\varphi_n\|_{L^2},$$

which is then written as

$$\sum_{k=0}^{+\infty}t^kA(z)h(x,D_x)u_k+\sum_{k=1}^{+\infty}t^kJe(x,D_x)u_{k-1}=\sum_{k=0}^{+\infty}t^k\sum_{r=0}^{k}\lambda_{k-r}u_r(x),$$

$$\sum_{k=0}^{+\infty}t^k\sum_{r=0}^{k}(u_{k-r},u_r)_{L^2(\mathbb{R};\mathbb{C}^2)}=\|\varphi_n\|^2.$$

We hence get the following recurrence:

$[k=0]$ $$A(z)h(x,D_x)u_0=\lambda_0u_0, \quad \|u_0\|=\|\varphi_n\|$$

$$[k=1] \qquad A(z)h(x,D_x)u_1 + Je(x,D_x)u_0 = \lambda_1 u_0 + \lambda_0 u_1, \quad (u_0,u_1)=0$$

$$[k=2] \qquad \begin{cases} A(z)h(x,D_x)u_2 + Je(x,D_x)u_1 = \lambda_2 u_0 + \lambda_1 u_1 + \lambda_0 u_2 \\ 2(u_0,u_2) + \|u_1\|^2 = 0, \end{cases}$$

$$[\text{step}\, k] \quad \begin{cases} A(z)h(x,D_x)u_k + Je(x,D_x)u_{k-1} = \lambda_k u_0 + \lambda_0 u_k + \displaystyle\sum_{r=1}^{k-1} \lambda_{k-r} u_r \\ \displaystyle\sum_{r=0}^{k} (u_{k-r}, u_r) = 0. \end{cases}$$

Remark 3.1. *It is important to remark that changing* $u_0 \longmapsto s u_0$, $s \in \mathbb{R}$, *changes* u_k *to* $s u_k$, *for all* $k \in \mathbb{Z}_+$. *Hence, the normalization conditions* $\sum_{r=0}^{k} (u_{k-r}, u_r) = 0$, $k \in \mathbb{Z}_+$, *remain unchanged. This is crucial to observe, for it will then be clear in the recurrence for the eigenvalues of* Q_t, *that the expansion of the eigenvalue is independent of* $\|u_0\|$.

It will be useful in the sequel to write the recurrence equations, and the normalization conditions, also in the following form

$$(R) \qquad \begin{cases} (Q_0 - \lambda_0) u_0 = 0, \\ (Q_0 - \lambda_0) u_k = \lambda_k u_0 + \displaystyle\sum_{r=1}^{k-1} \lambda_{k-r} u_r - M u_{k-1}, \; k \geq 1, \end{cases}$$

$$(N) \qquad \|u_0\| = \text{const}, \quad \sum_{r=0}^{k} (u_{k-r}, u_r) = 0, \; k \geq 1,$$

and the terms λ_k, $k \geq 1$, upon taking the scalar product with u_0,

$$(E) \qquad \lambda_k = \Big[(M u_{k-1}, u_0) - \sum_{r=1}^{k-1} \lambda_{k-r} (u_r, u_0) \Big] / \|u_0\|^2.$$

Remark 3.2. *When* λ_0 *is* **simple**, *one has that the above recurrence equations at step* k *are* **all uniquely** *solvable, giving rise to analytic functions* $t \longmapsto \lambda(t)$ *and* $t \longmapsto u(t)$ *(see* [8], [18], [10]*). When* λ_0 *is* **not simple**, *and in our case it will have at most multiplicity 2, things are more complicated, because the recurrence equations may not be solvable at some step* k, *or because of the lack of uniqueness. However, one still has the existence of analytic functions* $\lambda^{(j)}$ *and* $u^{(j)}$, $j = 1,2$, *solutions to* $Q_t u^{(j)}(t) = \lambda^{(j)}(t) u^{(j)}(t)$ *for* t *in a neighborhood of* 0, *with* $\lambda^{(j)}(0) = \lambda_0$, $u^{(j)}(0) = u_0^{(j)}$, $j = 1,2$, *where* $\operatorname{Ker}(Q_0 - \lambda_0) = \operatorname{Span}\{u_0^{(1)}, u_0^{(2)}\}$.

We shall set throughout the sequel $\varphi_{-n} = 0$ for all $n \in \mathbb{Z}_+$.

We end this section by recalling, for convenience, a few elementary facts about the spectrum of $Q_0 = A(z)h(x, D_x)$ in $L^2(\mathbb{R}; \mathbb{C}^2)$. Of course,

$$\mathrm{Spec}(Q_0) = \Big\{\lambda_0(z, r) = z\mu_r;\ r \in \mathbb{Z}_+\Big\} \cup \Big\{\lambda_0(\frac{1}{z}, s) = \frac{\mu_s}{z};\ s \in \mathbb{Z}_+\Big\},$$

where the multiplicity of each eigenvalue is 1, unless $z^2 = \mu_m/\mu_n$ for some positive integers m, n with $m < n$. In such a case, $\lambda_0(z, n) = \lambda_0(1/z, m)$ and the multiplicity is 2.

Definition 3.3. *We denote the set of "resonant" values of $z \in (0, 1)$ by*

$$S = \left\{\sqrt{\frac{\mu_s}{\mu_r}};\ r, s \in \mathbb{Z}_+,\ s < r\right\}.$$

Furthermore, when $z \in S$, we say that (n, m) is **minimal** *when n and m are the* **least** *positive integers for which $z^2 = \mu_m/\mu_n$.*

It is important to observe that once $z \in S$ is fixed as above with (n, m) **minimal**, there are infinitely many eigenvalues of Q_0 that have multiplicity 2. In fact, since for $q, m, m' \in \mathbb{Z}_+$ one has $q\mu_m = \mu_{m'}$ iff $2|q - 1$, i.e. $q = 2\mu_s$ for some $s \in \mathbb{Z}_+$, it follows that

$$z^2 = \frac{\mu_m}{\mu_n} = \frac{\mu_s\mu_m}{\mu_s\mu_n} = \frac{\mu_{m+s+2ms}}{\mu_{n+s+2ns}},\ \forall s \in \mathbb{Z}_+,$$

whence all the eigenvalues of multiplicity 2 are **all uniquely** given by

$$\lambda_0(z, n + s + 2ns) = z\mu_{n+s+2ns} = \frac{1}{z}\mu_{m+s+2ms} = \lambda_0(\frac{1}{z}, m + s + 2ms),$$

for all $s \in \mathbb{Z}_+$. We may therefore state the following fact in the form of a lemma.

Lemma 3.4. *Suppose $z^2 = \mu_m/\mu_n$, for $m < n$ positive integers and (n, m) minimal. Then*

$$\lambda_0(z, r) = \lambda_0(\frac{1}{z}, s) \iff (r, s) \in S_0 := \Big\{\Big(2\mu_n p + n, 2\mu_m p + m\Big);\ p \in \mathbb{Z}_+\Big\}.$$

It hence follows that

$$\min\Big\{\min_{r\neq r'} |\lambda_0(z, r) - \lambda_0(z, r')|, \min_{s\neq s'} |\lambda_0(\frac{1}{z}, s) - \lambda_0(\frac{1}{z}, s')|,$$
$$\min_{(r,s)\notin S_0} |\lambda_0(z, r) - \lambda_0(\frac{1}{z}, s)|\Big\} = \frac{1}{z\mu_n} = \frac{1}{\sqrt{\mu_m\mu_n}}.$$

We will call $(z\mu_n)^{-1} = z/\mu_m = (\mu_m\mu_n)^{-1/2}$, the **spacing** *of (the spectrum of) Q_0.*

Remark 3.5. *Hence $z\mu_n$ is* **simple** *if and only if there is no $m < n$ such that $z\mu_n = z^{-1}\mu_m$. In particular the lowest eigenvalue of Q_0 is* **always** *simple.*

In the next section we will give the formula for the expansion in the case of an eigenvalue $\lambda(t)$ whose "reference eigenvalue" $\lambda(0) = \lambda_0(z, n)$ is **simple**.

3.1. **The case of $\lambda_0(z,n)$ simple.** We want to obtain the expansion of $\lambda(t) = \lambda_0 + t\lambda_1 + t^2\lambda_2 + \dots$, when $\lambda_0 = \lambda(0) = \lambda_0(z,n)$ is simple. We necessarily have to require $z \neq 1$. We hence suppose that $z\mu_n \neq z^{-1}\mu_m$ for all $m \in \mathbb{Z}_+$ with $m < n$.

To get an idea of the formula we should expect, we start by computing a few steps in the expansion.

Step $k = 0$. This is just the eigenvalue equation for Q_0, which yields that

$$u_0 = \varphi_n \otimes e_1.$$

Remark 3.6. *Notice already at this point that supposing $z = 1$ would of course give the same λ_0, and the choice of two functions, orthogonal to one another,*

$$u_0^{(j)} = \varphi_n \otimes e_j, \quad j = 1, 2.$$

Moreover, considering $z^{-1}(n+1/2)$ as "reference eigenvalue" would give rise to a function of the kind $\varphi_n \otimes e_2$.

Step $k = 1$. In this case, we have the equation

$$\Big(A(z)h(x,D_x) - \lambda_0\Big)u_1 = \lambda_1 u_0 - Je(x,D_x)u_0.$$

Since $Ju_0 = \varphi_n \otimes e_2$, we may write

$$\Big(A(z)h(x,D_x) - \lambda_0\Big)u_1 = \lambda_1\varphi_n \otimes e_1 - \frac{1}{2}\Big(n(n-1)\varphi_{n-2} - \varphi_{n+2}\Big) \otimes e_2,$$

whence, with $u_1 = \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}$, the system of equations

$$\begin{cases} z\Big(h(x,D_x) - (n+\frac{1}{2})\Big)u_{11} = \lambda_1\varphi_n \\[2ex] \Big(\frac{1}{z}h(x,D_x) - z(n+\frac{1}{2})\Big)\, u_{12} = -\frac{1}{2}\Big(n(n-1)\varphi_{n-2} - \varphi_{n+2}\Big). \end{cases}$$

The first equation immediately gives

$$\lambda_1 = 0, \;\; u_{11} = c_0^{(1)}\varphi_n, \quad \text{for some} \;\; c_0^{(1)} \in \mathbb{R}.$$

As for the second one, we look for u_{12} of the form $u_{12} = c_{-1}^{(1)}\varphi_{n-2} + c_1^{(1)}\varphi_{n+2}$, for some $c_{-1}^{(1)}, c_1^{(1)} \in \mathbb{R}$ to be determined. Notice that the coordinate of u_{12} with respect to φ_n has to be 0. Then

$$\frac{1}{z}h(x,D_x)u_{12} = \frac{1}{z}\left[c_{-1}^{(1)}\mu_{n-2}\varphi_{n-2} + c_1^{(1)}\mu_{n+2}\varphi_{n+2}\right],$$

and, since $z^{-1}\mu_{n-2} - z\mu_n \neq 0$,

$$c_{-1}^{(1)} = -\frac{n(n-1)}{2\Big(z^{-1}\mu_{n-2} - z\mu_n\Big)}, \qquad c_1^{(1)} = \frac{1}{2\Big(z^{-1}\mu_{n+2} - z\mu_n\Big)}. \tag{13}$$

On the other hand

$$(u_0, u_1) = c_0^{(1)} \|\varphi_n\|^2 = 0 \implies c_0^{(1)} = 0,$$

whence

$$\lambda_1 = 0, \quad u_1 = u_{12} \otimes e_2 := \Big[c_{-1}^{(1)} \varphi_{n-2} + c_1^{(1)} \varphi_{n+2} \Big] \otimes e_2, \quad (u_0, u_1) = 0.$$

We next consider step $k = 2$.

Step $k = 2$**.** In this case, being $\lambda_1 = 0$, we have to consider

$$\Big(A(z)h(x, D_x) - \lambda_0\Big) u_2 = e(x, D_x) \Big[c_{-1}^{(1)} \varphi_{n-2} + c_1^{(1)} \varphi_{n+2} \Big] \otimes e_1 + \lambda_2 \varphi_n \otimes e_1,$$

that is the system, with $u_2 = \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}$,

$$\begin{cases} z\Big(h(x, D_x) - (n + \frac{1}{2})\Big) u_{21} = e(x, D_x) \Big[c_{-1}^{(1)} \varphi_{n-2} + c_1^{(1)} \varphi_{n+2} \Big] + \lambda_2 \varphi_n \\ \Big(z^{-1} h(x, D_x) - z(n + \frac{1}{2})\Big) u_{22} = 0. \end{cases}$$

Being $z^2 \neq 1$ and λ_0 simple, the second equation yields $u_{22} = 0$. The first equation may be written as

$$z\Big(h(x, D_x) - (n + \frac{1}{2})\Big) u_{21} =$$

$$= \frac{1}{2} \Big[c_{-1}^{(1)} (n-2)(n-3) \varphi_{n-4} - c_1^{(1)} \varphi_{n+4} \Big] + \Big(\lambda_2 + \frac{c_1^{(1)}}{2} (n+2)(n+1) - \frac{c_{-1}^{(1)}}{2} \Big) \varphi_n,$$

whence, being $\varphi_n \in \operatorname{Ker}\Big(h(x, D_x) - (n + 1/2)\Big)$, it follows that

$$\lambda_2 = \frac{1}{2} \Big(c_{-1}^{(1)} - (n+2)(n+1) c_1^{(1)} \Big), \quad u_2 = u_{21} \otimes e_1,$$

where $u_{21} = c_{-2}^{(2)} \varphi_{n-4} + c_0^{(2)} \varphi_n + c_2^{(2)} \varphi_{n+4}$, with

$$c_{-2}^{(2)} = -\frac{(n-2)(n-3) c_{-1}^{(1)}}{8z}, \qquad c_2^{(2)} = -\frac{c_1^{(1)}}{8z},$$

and $c_0^{(2)}$ determined by

$$c_0^{(2)} = \frac{(u_0, u_2)}{\|\varphi_n\|^2} = -\frac{\|u_1\|^2}{2\|\varphi_n\|^2}.$$

Notice that choosing $c_0^{(2)} = 0$ would yield $u_1 = u_2 = 0$, so that, as it will be seen below, $\lambda_k = 0$ for all $k \geq 1$.

To properly state the theorem about the expansion, we need the following definition.

Definition 3.7. *Let $n, d \in \mathbb{Z}_+$ and $z \in \mathbb{R}_+$ be fixed. We define the following polynomials in the Hermite functions φ_{n+2j}, $j \in \mathbb{Z}$,*

$$p_{d,n}(z;\varphi) := \begin{cases} \displaystyle\sum_{-d\le j\le d}^{j \text{ even}} c_j^{(d)}(z;n)\varphi_{n+2j}, & \text{when } d \text{ is even}, \\ \displaystyle\sum_{-d\le j\le d}^{j \text{ odd}} c_j^{(d)}(z;n)\varphi_{n+2j}, & \text{when } d \text{ is odd}, \end{cases}$$

where $c_j^{(d)}(z;n) \in \mathbb{C}$. For simplicity, we shall write $c_j^{(d)}$, and it will always be understood that $c_j^{(d)} = 0$ when $j > d$ or $n + 2j < 0$.

We are ready to state the theorem about the expansion.

Theorem 3.8. *Let $z \in (0,1)$. Let $\lambda_0(z,n) = z(n+1/2)$, for some $n \in \mathbb{Z}_+$, with $\lambda_0(z,n)$* **simple**. *One can then find analytic functions $\lambda\colon I_0 \longrightarrow \mathbb{R}$ and $u\colon I_0 \longrightarrow L^2(\mathbb{R};\mathbb{C}^2)$ (with $u(t;\cdot)$ $\mathbb{R}^2$-valued), such that*

$$Q_t u(t) = \lambda(t)u(t), \quad \lambda(t) \longrightarrow \lambda_0(z) \quad \text{as} \quad t \to 0, \quad \|u\| = \|\varphi_n\|, \quad u(0) = \varphi_n \otimes e_1,$$

$$\lambda(t) = \sum_{k\in\mathbb{Z}_+} \lambda_{2k}(z,n)t^{2k}, \quad u(t,z;\cdot) = \sum_{k\in\mathbb{Z}_+} u_k(z;\cdot)t^k,$$

$$u_{2k+1}(z;\cdot) = p_{2k+1,n}(z;\varphi) \otimes e_2, \;\; u_{2k}(z;\cdot) = p_{2k,n}(z;\varphi) \otimes e_1.$$

Hence, in the expansion of $\lambda(t)$ the coefficients $\lambda_{2k+1}(z,n) = 0$, for all $k \in \mathbb{Z}_+$, and the $\lambda_{2k}(z,n)$ and the polynomials $p_{2k,n}$ and $p_{2k+1,n}$ may be explicitly computed by recurrence. More precisely, one has the following formulas:

$$\lambda_{2k}(z,n) = \frac{1}{2}\Big(c_{-1}^{(2k-1)}(z;n) - (n+2)(n+1)c_1^{(2k-1)}(z;n)\Big) + \\ - \sum_{s=1}^{k-1} \lambda_{2(k-s)}(z,n)c_0^{(2s)}(z;n),$$

for $k \ge 1$ (where we set $\sum_{s=1}^{0} = 0$);

$$c_{-1}^{(1)}(z;n) = -\frac{n(n-1)}{2(z^{-1}\mu_{n-2} - z\mu_n)}, \quad c_1^{(1)}(z;n) = \frac{1}{2(z^{-1}\mu_{n+2} - z\mu_n)},$$

with $\mu_k = 0$ for $k < 0$;

$$c_0^{(2)}(z;n) = -\frac{c_{-1}^{(1)}(z;n)^2\|\varphi_{n-2}\|^2 + c_1^{(1)}(z;n)^2\|\varphi_{n+2}\|^2}{2\|\varphi_n\|^2};$$

$$c_0^{(2k)}(z;n) = -\frac{\|u_k\|^2 + 2\displaystyle\sum_{r=1}^{k-1}(u_{2k-r}, u_r)}{2\|\varphi_n\|^2}, \quad \text{for} \quad k \ge 1;$$

and as regards the remaining coefficients of the polynomials $p_{r,n}(z;\varphi)$ one has:

Formulas for the coefficients of the polynomials $p_{2k,n}(\varphi)$*:*

$$c^{(2k)}_{-2k}(z;n) = \begin{cases} -\dfrac{1}{8kz}(n+2-4k)(n+1-4k)c^{(2k-1)}_{1-2k}(z;n), & \textit{if } 4k<n \\ 0, & \textit{otherwise}\ ; \end{cases}$$

$$c^{(2k)}_{2k}(z;n) = -\frac{1}{8kz}c^{(2k-1)}_{2k-1}(z;n);$$

$$c^{(2k)}_{j}(z;n) = \frac{1}{2zj}\Big(\sum_{s=r+1}^{k-1}\lambda_{2(k-s)}(z)c^{(2s)}_{j}(z;n) + \\ +\frac{1}{2}\Big[(n+2j+2)(n+2j+1)c^{(2k-1)}_{j+1}(z;n) - c^{(2k-1)}_{j-1}(z;n)\Big]\Big),$$

when $|j| = 2r+2$, *with* $r = 0,1,\ldots,k-2$.

Formulas for the coefficients of the polynomials $p_{2k+1,n}(\varphi)$*:*

$$c^{(2k+1)}_{-(2k+1)}(z;n) = \begin{cases} -\dfrac{(n-4k)(n-4k-1)}{2\Big(z^{-1}\mu_{n-4k-2} - z\mu_n\Big)}c^{(2k)}_{-2k}(z;n), & \textit{if } 4k+2<n \\ 0, & \textit{otherwise}\ ; \end{cases}$$

$$c^{(2k+1)}_{2k+1}(z;n) = \frac{c^{(2k)}_{2k}(z;n)}{2\Big(z^{-1}\mu_{n+4k+2} - z\mu_n\Big)};$$

$$c^{(2k+1)}_{j}(z;n) = \frac{1}{z^{-1}\mu_{n+2j} - z\mu_n}\Big(\sum_{s=r}^{k-1}\lambda_{2(k-s)}(z)c^{(2s+1)}_{j}(z;n) + \\ +\frac{1}{2}\Big[c^{(2k)}_{j-1}(z;n) - (n+2j+2)(n+2j+1)c^{(2k)}_{j+1}(z;n)\Big]\Big),$$

when $|j| = 2r+1$, *with* $r = 0,1,\ldots,k-1$.

Proof. The first part of the proof consists of proving by induction that the λ_{2k+1} are all zero, and that the functions u_{2k}, u_{2k+1} are indeed of the stated form. The second part deals with computing recursively the coefficients λ_{2k} and the polynomials p_{2k} and p_{2k+1}.

We have already seen that $\lambda_1 = 0$, $u_0 = \varphi_n \otimes e_1 = p_{0,n}(\varphi) \otimes e_1$, and $u_1 = \Big[c^{(1)}_{-1}\varphi_{n-2} + c^{(1)}_{1}\varphi_{n+2}\Big] \otimes e_2 = p_{1,n}(\varphi) \otimes e_2$. So, suppose the induction hypothesis true up to an integer $r-1$.

• Suppose $r = 2k$. We have to prove that $u_{2k} = p_{2k,n}(\varphi)\otimes e_1$. The recurrence above yields for u_{2k} the equation

$$\Big(A(z)h(x,D_x) - \lambda_0\Big)u_{2k} = \lambda_{2k}u_0 + \sum_{s=1}^{2k-1}\lambda_{2k-s}u_s - Je(x,D_x)u_{2k-1}.$$

Hence, by the induction hypothesis, we get

$$\Big(A(z)h(x,D_x)-\lambda_0\Big)u_{2k}=\lambda_{2k}\varphi_n\otimes e_1+ \\ +\sum_{s=1}^{k-1}\lambda_{2(k-s)}p_{2s,n}(\varphi)\otimes e_1-Je(x,D_x)p_{2k-1,n}(\varphi)\otimes e_2,$$

that is

$$\begin{cases} z\Big(h(x,D_x)-(n+\frac{1}{2})\Big)u_{2k,1}=\lambda_{2k}\varphi_n+\sum_{s=1}^{k-1}\lambda_{2(k-s)}p_{2s,n}(\varphi)+ \\ \qquad\qquad\qquad\qquad\qquad\qquad +e(x,D_x)p_{2k-1,n}(\varphi) \\ \Big(z^{-1}h(x,D_x)-z(n+\frac{1}{2})\Big)u_{2k,2}=0. \end{cases}$$

As before, $z^2\neq 1$, $\lambda_0(z,n)$ simple and the second equation yield $u_{2k,2}=0$. Let us now isolate the φ_n coefficient in

$$e(x,D_x)p_{2k-1,n}(\varphi)= \\ =\frac{1}{2}\sum_{|j|\leq 2k-1}^{j\text{ odd}}c_j^{(2k-1)}\Big((n+2j)(n+2j-1)\varphi_{n+2(j-1)}-\varphi_{n+2(j+1)}\Big), \tag{14}$$

which is given by

$$\frac{1}{2}\Big((n+2)(n+1)c_1^{(2k-1)}-c_{-1}^{(2k-1)}\Big).$$

Upon observing that $e(x,D_x)p_{2k-1,n}(\varphi)=\tilde{p}_{2k,n}(\varphi)$ is a polynomial of the form $\sum_{|j|\leq 2k}^{j\text{ even}}\tilde{c}_j^{(2k)}\varphi_{n+2j}$, for suitable coefficients $\tilde{c}_j^{(2k)}$obtained from (14) above, we immediately get

$$u_{2k}=p_{2k,n}(\varphi)\otimes e_1,\ \ \lambda_{2k}=\frac{1}{2}\Big(c_{-1}^{(2k-1)}-(n+2)(n+1)c_1^{(2k-1)}\Big)-\sum_{s=1}^{k-1}\lambda_{2(k-s)}c_0^{(2s)},$$

where we set $c_j^{(r)}=0$ for all $j\in\mathbb{Z}$, whenever $r<0$.
We finally choose the coefficients $c_0^{(2k)}$, as

$$c_0^{(2k)}(z;n)=-\frac{\|u_k\|^2+2\sum_{r=1}^{k-1}(u_{2k-r},u_r)}{2\|\varphi_n\|^2},\quad k\geq 1.$$

The formula for the coefficients of $p_{2k,n}(\varphi)$ is immediately obtained from the system above.

• Suppose now $r=2k+1$. In this case, we have to consider

$$\Big(A(z)h(x,D_x)-\lambda_0\Big)u_{2k+1}=\lambda_{2k+1}u_0+\sum_{s=1}^{2k}\lambda_{2k+1-s}u_s-Je(x,D_x)u_{2k}.$$

Hence, by the induction hypothesis, we get

$$\Big(A(z)h(x,D_x)-\lambda_0\Big)u_{2k+1}=\lambda_{2k+1}\varphi_n\otimes e_1+ \\ +\sum_{s=0}^{k-1}\lambda_{2(k-s)}p_{2s+1,n}(\varphi)\otimes e_2-e(x,D_x)p_{2k,n}(\varphi)\otimes e_2,$$

that is

$$\begin{cases} z\Big(h(x,D_x)-(n+\frac{1}{2})\Big)u_{2k+1,1}=\lambda_{2k+1}\varphi_n \\ \Big(z^{-1}h(x,D_x)-z(n+\frac{1}{2})\Big)u_{2k+1,2}=\sum_{s=0}^{k-1}\lambda_{2(k-s)}p_{2s+1,n}(\varphi)+ \\ \qquad\qquad -e(x,D_x)p_{2k,n}(\varphi). \end{cases}$$

Again, the second equation gives $u_{2k+1,2}=p_{2k+1,n}(\varphi)$, for one has that $e(x,D_x)p_{2k,n}=\tilde{p}_{2k+1,n}(\varphi)$, whereas the first one gives $\lambda_{2k+1}=0$ and $u_{2k+1,1}=c\varphi_n$ for some $c\in\mathbb{R}$. Now,

$$0=\sum_{r=0}^{2k+1}(u_{2k+1-r},u_r)=$$

$$\sum_{r=0}^{k}\Big(p_{2(k-r),n}(\varphi)\otimes e_1,p_{2r+1,n}(\varphi)\otimes e_2\Big)+\Big(p_{2k+1,n}(\varphi)\otimes e_2+c\varphi_n\otimes e_1,\varphi_n\otimes e_1\Big)+ \\ +\sum_{r=1}^{k}\Big(p_{2(k-r)+1,n}(\varphi)\otimes e_2,p_{2r,n}(\varphi)\otimes e_1\Big)=c\|\varphi_n\|^2\implies c=0.$$

It follows from [8] (see also Satz 1, page 360 of [18], and Theorems II and III, pages 356 and 359 of [10]), that such functions $t\longmapsto\lambda(t)$ and $t\longmapsto u(t)$ exist. Equating then the coefficients according to the powers of t in the equation $Q_tu(t)=\lambda(t)u(t)$ gives the above recurrence equations, that we have **uniquely** solved. Hence the coefficients must be given by those we have found above, thereby concluding the proof of the theorem. □

Remark 3.9. *Of course, the formulas for λ_k and u_k can be obtained by the ones in* [18]*, Satz 1 (page 360, formulas* (3a)*-*(3c)*), by specializing matters to the present case.*

Remark 3.10. *Recall that the expansion of a simple eigenvalue $\lambda(t)$ of Q_t holds more generally for all $(z,t)\in\mathbb{R}_+\times I_0\}$ with $z\neq 1$ (value that must be excluded because it gives rise to higher multiplicity eigenvalues).*

For the reader's convenience, we shall give the formula for the expansion in the case the "target" eigenvalue of Q_0 is given by $\frac{1}{z}(m+\frac{1}{2})$ and is *simple*. Notice that in this case there is no integer $n>m$ such that $z\mu_n=z^{-1}\mu_m$ then. One has the following theorem.

Theorem 3.11. *Let $z \in (0,1)$. Let $\lambda_0(z^{-1},m) := z^{-1}(m+1/2)$, for some $m \in \mathbb{Z}_+$, be* **simple**. *One can then find analytic functions $\lambda\colon I_0 \longrightarrow \mathbb{R}$ and $u\colon I_0 \longrightarrow L^2(\mathbb{R};\mathbb{C}^2)$ (with $u(t;\cdot)$ $\mathbb{R}^2$-valued), such that*

$$Q_t u(t) = \lambda(t)u(t), \quad \lambda(t) \longrightarrow \lambda_0(z) \quad as \quad t \to 0, \quad \|\tilde{u}\| = \|\varphi_m\|, \ \tilde{u}(0) = \varphi_m \otimes e_2,$$

$$\lambda(t) = \sum_{k\in\mathbb{Z}_+} \lambda_{2k}(\frac{1}{z},m)t^{2k}, \quad \tilde{u}(t,z;\cdot) = \sum_{k\in\mathbb{Z}_+} \tilde{u}_k(z;\cdot)t^k,$$

$$\tilde{u}_{2k+1}(z;\cdot) = -p_{2k+1,m}(z^{-1};\varphi)\otimes e_1, \quad \tilde{u}_{2k}(z;\cdot) = p_{2k,m}(z^{-1};\varphi)\otimes e_2,$$

where the functions λ_{2k} and the polynomials $p_{r,m}$ are the ones introduced in Theorem 3.8. Hence, for instance, for all $k \geq 1$ one has the formula

$$\lambda_{2k}(\frac{1}{z},m) = \frac{1}{2}\Big(c_{-1}^{(2k-1)}(\frac{1}{z};m) - (m+2)(m+1)c_1^{(2k-1)}(\frac{1}{z};m)\Big) + \\ - \sum_{s=1}^{k-1} \lambda_{2(k-s)}(\frac{1}{z},m)c_0^{(2s)}(\frac{1}{z};m).$$

Proof. The proof follows from Remark 3.10. In fact, one considers as $t \to 0+$

$$KQ_tK = A(\frac{1}{z})h(x,D_x) + (-t)M(x,D_x), \quad \text{where} \ \ K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and applies the construction of Theorem 3.8 to the eigenvalue $\lambda(t)$ that tends to $z^{-1}(m+1/2)$ as $t \to 0$. Hence, it just suffices to replace t by $-t$, z by $1/z$, and e_j by Ke_j, $j=1,2$, in the formulas in Theorem 3.8. □

As a byproduct of the formulas in Theorems 3.8 and 3.11, we have the following corollary on the behavior of finite parts of the spectrum of Q_t as $t \to 0$.

Corollary 3.12. *Fix $N_0 \in \mathbb{Z}_+$, and let $z \notin S$. Suppose that*

$$z^2 > \max_{n=0,\dots,N_0} \left\{ \frac{\mu_n}{\mu_{n+2}}, \frac{\mu_{n-2}}{\mu_n} + \frac{2n(n-1)}{\mu_n(n^2+n+2)} \right\}.$$

Let $\lambda_{\pm,n}(t)$, $n = 0,\dots,N_0$, be the eigenvalues of Q_t such that $\lambda_{\pm,n}(0) = z^{\pm1}\mu_n$, $\pm$-respectively. (We may hence suppose that they are **all simple** *for $n = 0,\dots,N_0$, provided t is sufficiently small.) Then there exists $\delta > 0$ such that*

$$\lambda_{\pm,n}(t) < \lambda_{\pm,n}(0), \quad \forall n = 0,\dots,N_0, \ \forall t \in (-\delta,\delta).$$

Proof. The proof follows immediately from the formulas for $\lambda_2(z^{\pm1},n)$, namely

$$\lambda_2(z,n) = -\frac{z}{4}\left(\frac{n(n-1)}{\mu_{n-2}-z^2\mu_n} + \frac{(n+2)(n+1)}{\mu_{n+2}-z^2\mu_n}\right),$$

and

$$\lambda_2(\frac{1}{z},n) = -\frac{1}{4z}\left(\frac{n(n-1)}{\mu_{n-2}-z^{-2}\mu_n} + \frac{(n+2)(n+1)}{\mu_{n+2}-z^{-2}\mu_n}\right).$$

For the above choice of z one has $\lambda_2(z^{\pm1},n) < 0$, $n = 0,\dots,N_0$. □

3.2. **Expansion for $\lambda_{\min}(t)$ and $\mu_{\min}(\alpha,\beta)$ as $t \to 0$.** We now concentrate on the *lowest* eigenvalue $\lambda_{\min}(t)$ of Q_t. Since $z \in (0,1)$, we have that $\lambda_0 = z/2$ is the lowest eigenvalue of Q_0, and it is isolated. It is therefore interesting to see how the first few terms in the expansion of $\lambda_{\min}(t)$ look like. We have,

$$\lambda_{\min}(t) = \frac{z}{2} - \frac{1}{2\left(\frac{\mu_2}{z} - \mu_0 z\right)} t^2 - \frac{75 - 34z^2 + 7z^4}{32z^3\left(\frac{\mu_6}{z} - z\mu_0\right)\left(\frac{\mu_2}{z} - z\mu_0\right)^3} t^4 + o(t^5),$$

as $t \to 0$, whereas the relative eigenfunction (normalized by $\|u_t\| = \|\varphi_0\|$) "looks like"

$$u_t = \varphi_0 \otimes e_1 + \frac{t}{2\left(\frac{\mu_2}{z} - \mu_0 z\right)} \varphi_2 \otimes e_2 +$$

$$-t^2 \left[\frac{1}{8\left(\frac{\mu_2}{z} - \mu_0 z\right)^2 \|\varphi_0\|^2} \varphi_0 + \frac{1}{16z\left(\frac{\mu_2}{z} - \mu_0 z\right)} \varphi_4 \right] \otimes e_1 + o(t^2), \; t \to 0.$$

In particular, one may write the expansion of the lowest eigenvalue $\mu_{\min}(\alpha,\beta)$ of $Q_{(\alpha,\beta)}(x, D_x)$ as $\sqrt{\alpha\beta} \to +\infty$ when $\alpha < \beta$ and α/β is a constant lying in $(0,1)$ as follows. With the choice of coordinates (z,t) introduced above (hence $z \in (0,1)$ is **constant** and $t \in (0,1)$), one has

$$\alpha = \frac{z}{t}, \quad \beta = \frac{1}{tz}, \quad \mu_{\min}(\alpha,\beta) = \mu_{\min}(\frac{z}{t}, \frac{1}{tz}),$$

and

$$\mu_{\min}(\frac{z}{t}, \frac{1}{tz}) = \frac{1}{t}\left(\frac{z}{2} - \frac{z}{5 - z^2} t^2 - \frac{z(75 - 34z^2 + 7z^4)}{2(13 - z^2)(5 - z^2)^3} t^4 + o(t^5)\right),$$

as $t = 1/\sqrt{\alpha\beta} \to 0+$.

3.3. **Higher multiplicities.** It is interesting to explore higher multiplicities. As shown in [15] and [16] (see also [14]), the multiplicity of an eigenvalue of $Q_{(\alpha,\beta)}(x, D_x)$ relative to $L^2_+(\mathbb{R}; \mathbb{C}^2)$ (resp. $L^2_-(\mathbb{R}; \mathbb{C}^2)$) is at most 2. We are going to consider the case in which the "reference" eigenvalue of Q_0 has multiplicity 2, and is of the kind

$$\lambda_0 = \lambda_0(z,n) = z\left(n + \frac{1}{2}\right) = \frac{1}{z}\left(m + \frac{1}{2}\right) = \lambda_0(\frac{1}{z}, m),$$

for some $n, m \in \mathbb{Z}_+$, that amounts to saying that $z^2 = \mu_m/\mu_n$ (recall that $\mu_n = n + 1/2$ and, since $z \in (0,1)$, we must have $m < n$).
From Remark 2.3, we have two analytic functions in I_0, that we denote $\lambda^{(1)}(t)$ and $\lambda^{(2)}(t)$, for which

$$\lim_{t \to 0} \lambda^{(1)}(t) = \lim_{t \to 0} \lambda^{(2)}(t) = \lambda_0.$$

We also write

$$\lambda^{(1)}(t) = \sum_{k\in\mathbb{Z}_+} \lambda_{2k}(z,n)t^{2k}, \quad \lambda^{(2)}(t) = \sum_{k\in\mathbb{Z}_+} \lambda_{2k}(\frac{1}{z},m)t^{2k}.$$

Let us set

$u_0^{(1)} := \varphi_n \otimes e_1, \quad u_0^{(2)} := \varphi_m \otimes e_2, \quad V_0 := \mathrm{Ker}(Q_0 - \lambda_0) = \mathrm{Span}\{u_0^{(1)}, u_0^{(2)}\},$

and denote by P_0 the orthogonal projection of $L^2(\mathbb{R};\mathbb{C}^2)$ onto V_0.

One may deal with the following problems:

- Study the case where $\lambda^{(1)}(t_0) = \lambda^{(2)}(t_0)$ for some $t_0 \in I_0$;
- Study the case where $\lambda^{(1)}(t) \neq \lambda^{(2)}(t)$ for all $t \neq 0$, sufficiently small;
- Study the case where $\lambda^{(1)}(t) = \lambda^{(2)}(t)$ for all $t \in I_0$.

The first problem is of course more difficult than the other two. We will deal here only with the second one, and actually with a special case of it. The main point is to find out a way for equating the coefficients in the expansions of the functions $\lambda^{(1)}$ and $\lambda^{(2)}$.

Of course, the recurrence procedure introduced in Section 3 has to be adjusted to the higher multiplicity case. As already remarked there, it is well-known that the recurrence changes and it becomes less elementary to get the coefficients of both the expansions of the various branches of the eigenvalues and of the eigenfunctions (in fact, it might happen at some step k that the relative recurrence equation is **not** solvable; this is the main obstruction to the recurrence). At any rate, we know ([8], [10], [18]) that also in this case one can find analytic functions

$$\lambda^{(j)} \colon I_0 \longrightarrow \mathbb{R}, \quad u^{(j)} \colon I_0 \longrightarrow L^2(\mathbb{R};\mathbb{C}^2), \; j = 1, 2,$$

such that (the $u^{(j)}$ may be taken real-valued, for $j = 1, 2$ for Q_t is a *real* operator)

$$Q_t u^{(j)}(t) = \lambda^{(j)}(t)u^{(j)}(t), \; \|u^{(j)}(t)\| = \|u_0^{(j)}\|, \, \forall t \in I_0, \; \lambda^{(j)}(0) = \lambda_0, \, j = 1, 2.$$

Now, suppose $z \in S$ is **not** of the form μ_{n-2-4k_0}/μ_n for some $k_0 \in \mathbb{Z}_+$. Then it is easily seen that

$$M \colon V_0 \longrightarrow V_0^{\perp},$$

whence it follows that if we **assume** that the $u_k^{(j)}$ have one and only one component $v_k^{(j)} = c_{0,j}^{(k)} u_0^{(j)}$ along V_0 in the direction of $u_0^{(j)}$ for all $k \in \mathbb{Z}_+$, then **all** the recurrence equations may be **uniquely** solved, for in this case we always have that the $u_1^{(j)}, \dots, u_{k-1}^{(j)}$ **and** $Mu_{k-1}^{(j)}$ have at most one component along V_0, always in the direction given by $u_0^{(j)}$, which is fixed by the normalization condition, and hence that the only V_0-component of $\lambda_k^{(j)} u_0^{(j)} + \sum_{r=1}^{k-1} \lambda_{k-r}^{(j)} u_r^{(j)} - M u_{k-1}^{(j)}$, which is therefore along $u_0^{(j)}$, can hence be killed by the choice of the relative $\lambda_k^{(j)}$. This is equivalent to solving the recurrence equations (R_j) (the ones relative to $u_0^{(j)}$), along with (N_j) and

(E_j), as if the eigenvalue λ_0 were simple, hence writing things down by using the formulas of Theorem 3.8 (and Theorem 3.11). It is also equivalent to setting, for $j=1,2$,

$$\lambda_1^{(j)} = (Mu_0^{(j)}, u_0^{(j)})/\|u_0^{(j)}\|^2,\ u_1^{(j)} = -R_0 M u_0^{(j)}, \tag{15}$$

$$\lambda_k^{(j)} = \Big[(Mu_{k-1}^{(j)}, u_0^{(j)}) - \sum_{r=1}^{k-1} \lambda_{k-r}^{(j)} (u_{k-r}^{(j)}, u_0^{(j)})\Big]/\|u_0^{(j)}\|^2,\ k \geq 2, \tag{16}$$

$$u_k^{(j)} = -\frac{\sum_{r=1}^{k-1}(u_{k-r}^{(j)}, u_r^{(j)})}{2\|u_0^{(j)}\|^2} u_0^{(j)} + \sum_{r=1}^{k-1} \lambda_{k-r}^{(j)} R_0 u_r^{(j)} - R_0 M u_{k-1}^{(j)},\ k \geq 2, \tag{17}$$

where R_0 is the *pseudoinverse* of $Q_0 - \lambda_0$, which is uniquely defined by

$$(Q_0 - \lambda_0)R_0 = R_0(Q_0 - \lambda_0) = I - P_0,\ \text{and } R_0 u = 0,\ \forall u \in V_0.$$

But equations (15), (16), (17) are exactly equations $(3a)$-$(3c)$ of Satz 1 in [18], and being λ_0 isolated, one has from Rellich's proof that the recurrence converges, and it sums up to analytic functions $t \longmapsto \lambda^{(j)}(t) \in \mathbb{R}$ and $t \longmapsto u^{(j)}(t) \in L^2(\mathbb{R};\mathbb{C}^2)$ that are solutions of $Q_t u^{(j)}(t) = \lambda^{(j)}(t) u^{(j)}(t)$, $j=1,2$, with $\lambda^{(j)}(0) = \lambda_0$ and $u^{(j)}(0) = u_0^{(j)}$, $j = 1,2$, that therefore must coincide with those obtained in [10] and [8] (Satz 2 of [18] does not apply in the present case). We have hence proved the following corollary.

Corollary 3.13. *Suppose $z \in S$, with $z^2 = \mu_m/\mu_n$, such that $m \neq n - 2 - 4k$, for all $k \in \mathbb{Z}_+$. Consider $\lambda_0(z,n) = \lambda_0(1/z, m)$. Then there exist analytic functions $\lambda^{(1)}, \lambda^{(2)} \colon I_0 \longrightarrow \mathbb{R}$ and $u^{(1)}, u^{(2)} \colon I_0 \longrightarrow L^2(\mathbb{R};\mathbb{C}^2)$ such that $\|u^{(1)}\| = \|\varphi_n\|$, $\|u^{(2)}\| = \|\varphi_m\|$ and*

$$u^{(1)}(0) = \varphi_n \otimes e_1, u^{(2)}(0) = \varphi_m \otimes e_2,\ \lambda^{(1)}(0) = \lambda^{(2)}(0) = \lambda_0(z,n) = \lambda_0(\frac{1}{z}, m).$$

Moreover, the coefficients of $\lambda^{(j)}$ and $u^{(j)}$, $j=1,2$, are determined as in Theorems 3.8 and 3.11.

Let us hence take $z^2 = \mu_m/\mu_n$, with (n,m) minimal, $m < n$, and $m \neq n-2-4k$, for all $k \in \mathbb{Z}_+$. One computes

$$\lambda^{(1)}(t) = z\mu_n - \frac{1}{4}\left[\frac{(n+2)(n+1)}{z^{-1}\mu_{n+2} - z\mu_n} + \frac{n(n-1)}{z^{-1}\mu_{n-2} - z\mu_n}\right] t^2 + o(t^2), \quad t \to 0,$$

$$\lambda^{(2)}(t) = \frac{\mu_m}{z} - \frac{1}{4}\left[\frac{(m+2)(m+1)}{z\mu_{m+2} - z^{-1}\mu_m} + \frac{m(m-1)}{z\mu_{m-2} - z^{-1}\mu_m}\right] t^2 + o(t^2), \quad t \to 0.$$

Is is elementary to see that

$$\lambda_2(z,n) \neq \lambda_2(\frac{1}{z}, m).$$

Hence $\lambda^{(1)}(t) \neq \lambda^{(2)}(t)$ as $t \to 0$.

On the other hand, when $z \in S$ all of the other multiple eigenvalues of Q_0 are then of the form

$$\lambda_0(z, n+s+2ns) = \lambda_0(\frac{1}{z}, m+s+2ms),\ \forall s \in \mathbb{Z}_+,$$

and also in this case

$$\lambda_2(z, n+s+2ns) \neq \lambda_2(\frac{1}{z}, m+s+2ms),\ \forall s \in \mathbb{Z}_+.$$

As regards $Q_{(\alpha,\beta)}$ we hence have the following fact.

Proposition 3.14. *Let $\alpha, \beta > 0$ be so chosen that either $z = \sqrt{\alpha/\beta} \notin S$ or $z = \sqrt{\dfrac{2m+1}{2n+1}} \in S$, with $m < n$, (n,m) minimal,* **such that** *$m \neq n-2-4k$ for all $k \in \mathbb{Z}_+$. Then for any given $N \in \mathbb{R}_+$ there exists $C_{N,n,m} > 1$ such that*

$$\Big\{\mu \in \operatorname{Spec}\Big(Q_{(\alpha,\beta)}(x, D_x)\Big);\ \mu < N\Big\}$$

is made of **simple** *eigenvalues for all α, β as above such that $\sqrt{\alpha\beta} \geq C_{N,n,m}$.*

Proof of the proposition. The case $z \notin S$ is immediate. When $z \in S$ is as above, since $\lambda_2(z, n+s+2sn) \neq \lambda_2(z^{-1}, m+s+2ms)$, for all $s \in \mathbb{Z}_+$, also all the related $\lambda^{(1)}(t) \neq \lambda^{(2)}(t)$, for $t > 0$ sufficiently small, depending on n, m and s. By virtue of the spacing condition (see Lemma 3.4), the theorems of [18] and [10] yield the result upon taking the least constant bounding t, that hence depends on N also. □

As a consequence of the above proposition, we may rule out the existence of multiple eigenvalues of multiplicity greater than 1, whose higher multiplicities come, as pointed out in [15] [16] (see also [14]), from the existence of even and of odd eigenfunctions. In the present context, such eigenvalues should be produced in the first approximation already at the level of Q_0. One should therefore choose $z^2 = \mu_m/\mu_n$, with $m < n$, (n,m) minimal and, say, n odd and m even. Then the condition $m \neq n-2-4k$ is fulfilled for all $k \in \mathbb{Z}_+$. By Proposition 3.14 above, it follows that the perturbed eigenvalue is **simple** for all α, β such that $\alpha\beta > 1$ is sufficiently large. Hence, this kind of multiple eigenvalue (of Q_0), when perturbed, splits into two branches, one for each parity. Hence we may say the following.

Corollary 3.15. *For $z \in (0,1)$* **fixed** *as above, for any given $N \in \mathbb{R}_+$ there is $C_{N,n,m} > 1$ sufficiently large such that for all α, β with $\alpha/\beta = z^2$ and $\alpha\beta \geq C_{N,n,m}$ there is no eigenvalue μ of $Q_{(\alpha,\beta)}(x, D_x)$ belonging to $[0, N)$ for which*

$$V_\mu^+ \neq \{0\} \ \ \textbf{and} \ \ V_\mu^- \neq \{0\},$$

where

$$V_\mu^+ = \{u \in L^2(\mathbb{R}; \mathbb{C}^2);\ Q_{(\alpha,\beta)}(x, D_x)u = \mu u \ \ \textbf{and } u \textbf{ is even}\},$$

$$V_\mu^- = \{u \in L^2(\mathbb{R}; \mathbb{C}^2);\ Q_{(\alpha,\beta)}(x, D_x)u = \mu u \ \ \textbf{and } u \textbf{ is odd}\}.$$

When $z^2 = \mu_{n-2}/\mu_n$, with $(n, n-2)$ minimal, let us consider the expansion with respect fo the eigenvalue $\lambda_0 = z\mu_n = \mu_m/z$ of Q_0. In this case the recurrence equations (R) are not solvable at step $k = 1$. The remedy to this was found by Rellich ([18]; see also [10]). Write $\tilde{\varphi}_k := \varphi_k/\|\varphi_k\|$ and recall that $V_0 = \mathrm{Ker}(Q_0 - \lambda_0) = P_0(L^2(\mathbb{R}; \mathbb{C}^2))$. An easy computation shows that in the basis $\{\tilde{\varphi}_n \otimes e_1, \tilde{\varphi}_m \otimes e_2\}$,

$$P_0 M P_0 = \frac{\sqrt{n(n-1)}}{2} K,$$

whence it follows that

$$\mathrm{Spec}(P_0 M P_0) = \left\{ \pm \frac{\sqrt{n(n-1)}}{2} \right\},$$

$$V_0 = \mathrm{Span}\Big\{ \frac{\tilde{\varphi}_n \otimes e_1 + \tilde{\varphi}_m \otimes e_2}{\sqrt{2}}, \frac{\tilde{\varphi}_n \otimes e_1 - \tilde{\varphi}_m \otimes e_2}{\sqrt{2}} \Big\},$$

and, with $u_0^{(j)} = (\tilde{\varphi}_n \otimes e_1 + (-1)^j \tilde{\varphi}_m \otimes e_2)/\sqrt{2}$, $j = 1, 2$

$$(Mu_0^{(j)}, u_0^{(k)}) = (-1)^j \frac{\sqrt{n(n-1)}}{2} \delta_{jk} =: \lambda_1^{(j)} \delta_{jk}, \ \ j, k = 1, 2.$$

Now Satz 2 of [18], page 365, **can be applied**, and, upon recalling that

$$R_0(\varphi_k \otimes e_1) = \frac{1}{z(k-n)} \varphi_k \otimes e_1, , \quad \text{when} \quad k \neq n,$$

and

$$R_0(\varphi_k \otimes e_2) = \frac{z}{k-n-2} \varphi_k \otimes e_2, \quad \text{when} \quad k \neq n-2,$$

we get the following fact.

Proposition 3.16. *Suppose* $z^2 = \mu_{n-2}/\mu_n$ *so that* $z\mu_n = \mu_{n-2}/z$. *In the above notation, one has analytic functions* $\lambda^{(j)} \colon I_0 \longrightarrow \mathbb{R}$, $u^{(j)} \colon I_0 \longrightarrow L^2(\mathbb{R}; \mathbb{C}^2)$ *(real valued),* $j = 1, 2$, *such that* $\|u^{(j)}\| = 1$,

$$\lambda^{(j)}(t) = z\mu_n + (-1)^j \frac{\sqrt{n(n-1)}}{2} t +$$
$$- \frac{1}{32} \left[z(n+2)(n+1) - \frac{(n-2)(n-3)}{z} \right] t^2 + o(t^2),$$

and

$$u^{(j)}(t) = u_0^{(j)} +$$
$$+ \left[\frac{(-1)^{j+1}}{16} \frac{(n+2)(n+1)z + (n-2)(n-3)/z}{\sqrt{n(n-1)}} u_0^{(j)} - R_0 M u_0^{(j)} \right] t + o(t),$$

for $j = 1, 2$. *As regards* $Q_{(\alpha,\beta)}(x, D_x)$, *it follows in particular that also in this case the relative eigenvalue is* **simple** *for large* $\sqrt{\alpha\beta}$.

4. Comparisons

In this section we will compare the approximation of the lowest eigenvalue of $Q_{(\alpha,\beta)}(x, D_x)$ given by Rellich's expansion with the bound of Brummelhuis (see [1]), with the W.K.B. bound obtained by studying the bicharacteristics of the lowest eigenvalue of the principal symbol, and finally with the approximation obtained through the continued fraction introduced in [15].

4.1. **Comparison with the bound of Brummelhuis.** We start by recalling a theorem proved by Brummelhuis in [1] (Theorem 3.5, page 1577), in a form suitable for our purpose.

Theorem 4.1. *Let* $Q(x,\xi) = A\xi^2 + 2Bx\xi + Cx^2$ *be a positive semidefinite* $\mathbb{C}^N$*-Hermitian valued quadratic form on* $\mathbb{R}^2$*, with* $A > 0$*. Then there exists a unique Hermitian matrix* H *solution to*

$$HA^{-1}H + H(iA^{-1}B) + (iA^{-1}B)^*H - (C - BA^{-1}B) = 0$$

for which $\operatorname{Spec}\Big(A^{-1}(iB+H)\Big) \subset \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda \geq 0\}$*. One has, by "completing the squares", the estimate*

$$Q^{\mathrm{w}}(x, D_x) \geq \mu_H := \text{ lowest eigenvalue of } H.$$

The theorem is a consequence of

$$(Q^{\mathrm{w}}u, u) = \|A^{1/2}Du + A^{-1/2}(B - iH)xu\|^2 +$$
$$+((C - (B - iH)^* A^{-1}(B - iH)xu, xu) + (Hu, u),$$

and the theory of Riccati equations (see the reference in [1]).
In the case of $t^{-1}Q_t(x, D_x)$, one then has that

$$H = \frac{1}{t}\frac{\sqrt{1-t^2}}{2}\operatorname{diag}(z, 1/z),$$

and

$$\mu_H(t) = \frac{1}{t}\frac{z\sqrt{1-t^2}}{2} = \frac{z}{2t}\left(1 - \frac{1}{2}t^2 + o(t^2)\right), \quad \text{as} \quad t \to 0+.$$

Since $\dfrac{2}{5} < \dfrac{1}{\mu_2 - \mu_0 z^2} < \dfrac{1}{2}$, we have the following lemma.

Lemma 4.2. *For all* $t > 0$ *such that* $t = 1/\sqrt{\alpha\beta}$ *is sufficiently small, one has*

$$\mu_H(t) < \mu_{\min}(\alpha, \beta). \tag{18}$$

Notice that when $z = 1$ and $t = 1/\alpha < 1$ (i.e. $\alpha = \beta > 1$),

$$\frac{1}{t}\frac{\sqrt{1-t^2}}{2} = \min \operatorname{Spec}\big(Q_{(\alpha,\alpha)}(x, D_x)\big).$$

4.2. **WKB-approximation and the Bohr-Sommerfeld condition.** In this section, we shall study the expansion of the lowest eigenvalue of the system $Q_t(x, D_x)$ obtained by applying the Bohr-Sommerfeld condition to the WKB-construction. We will here slightly change notation, by dropping the explicit dependence on the parameter t. We will then write $Q_t(x, D_x) = Q^{\mathrm{w}}(x, D_x)$, the Weyl-quantization of the matrix-valued quadratic form

$$Q(x, \xi) = A(z)h(x, \xi) + itJx\xi,$$

which is the principal symbol of Q^{w}. The eigenvalues of $Q(x, \xi)$ are

$$\lambda_\pm(x, \xi) = \mu_+ h(x, \xi) \pm \mu_- \sqrt{h(x, \xi)^2 + \frac{t^2x^2\xi^2}{\mu_-^2}},$$

where $\mu_\pm := \dfrac{1}{2}\left(\dfrac{1}{z} \pm z\right) > 0$, since $z \in (0, 1)$. It is straightforward to check that $0 < \lambda_-(x, \xi) < \lambda_+(x, \xi)$ for all $(x, \xi) \neq (0, 0)$, and that for t sufficiently small uniformly in (x, ξ) (and in fact for $1 + t^2/\mu_-^2 < 1/(2z\mu_-)^2$), we have

$$\frac{z}{2}h(x, \xi) \le \left(\mu_+ - \mu_-\sqrt{1 + \frac{t^2}{\mu_-^2}}\right) h(x, \xi) \le \lambda_-(x, \xi) \le zh(x, \xi), \tag{19}$$

for all $(x, \xi) \in \mathbb{R} \times \mathbb{R}$. From now on, we shall suppose t sufficiently small, without any further reference to its smallness. Let us set, for $(x, \xi) \neq (0, 0)$,

$$g(x, \xi) := \sqrt{1 + \frac{t^2x^2\xi^2}{\mu_-^2 h(x, \xi)^2}} - 1 = \frac{\dfrac{t^2x^2\xi^2}{\mu_-^2 h(x, \xi)^2}}{\Big(g(x, \xi) + 2\Big)}, \ \nu(x, \xi) := \sqrt{1 + \frac{g(x, \xi)}{g(x, \xi) + 2}}.$$

For $(x, \xi) \neq (0, 0)$, denote by $\zeta_\pm(x, \xi)$, $\pm$-respectively, the **orthonormal** eigenvectors of $Q(x, \xi)$ belonging to the eigenvalues $\lambda_\pm(x, \xi)$, $\pm$-respectively, defined by

$$\zeta_-(x, \xi) = \frac{1}{\nu(x, \xi)} \begin{bmatrix} i \\ \dfrac{tx\xi}{\mu_- h(x, \xi)\Big(g(x, \xi) + 2\Big)} \end{bmatrix}, \quad \zeta_+(x, \xi) = K\zeta_-(x, \xi),$$

where, recall, $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Notice that $\zeta_\pm$ have the following properties:

$$\zeta_\pm \in C^\infty(\mathbb{R} \times \mathbb{R} \setminus \{(0, 0)\}; \mathbb{C}^2), \quad \zeta_\pm(x, \xi) = \zeta_\pm(\xi, x), \ \forall (x, \xi) \neq (0, 0),$$

and

$$|\zeta_\pm(x, \xi)|_{\mathbb{C}^2} = 1, \quad \langle \zeta_-(x, \xi), \zeta_+(x, \xi) \rangle = 0, \quad \forall (x, \xi) \neq (0, 0),$$

where $|\cdot|_{\mathbb{C}^2}$ and $\langle \cdot, \cdot \rangle$ denote the Hermitian norm and inner-product in $\mathbb{C}^2$, respectively.

Remark 4.3. *It is important to remark that*

$$\zeta_-(x,\xi) \in i\mathbb{R} \times \mathbb{R}, \quad \zeta_+(x,\xi) \in \mathbb{R} \times i\mathbb{R},$$

and, for all $\alpha, \beta \in \mathbb{R}$,

$$\langle \Big(i\alpha A(z) - \beta J\Big)\zeta_-, \zeta_-\rangle, \quad \langle \Big(i\alpha A(z) - \beta J\Big)\zeta_+, \zeta_+\rangle \in i\mathbb{R},$$

$$\langle \Big(i\alpha A(z) - \beta J\Big)\zeta_-, \zeta_+\rangle \in \mathbb{R}.$$

The following lemma will also be useful.

Lemma 4.4. *For all* $(x,\xi) \neq (0,0)$, *in the basis* $\{\zeta_-(x,\xi), \zeta_+(x,\xi)\}$ *of* $\mathbb{C}^2$, *the Hermitian map* $\partial_\xi Q(x,\xi)\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ *is represented by the matrix*

$$\partial_\xi Q(x,\xi) = \begin{bmatrix} \dfrac{\partial \lambda_-}{\partial \xi} & \Big(\lambda_+ - \lambda_-\Big)\langle \partial_\xi \zeta_+, \zeta_-\rangle \\ -\Big(\lambda_+ - \lambda_-\Big)\langle \partial_\xi \zeta_-, \zeta_+\rangle & \dfrac{\partial \lambda_+}{\partial \xi} \end{bmatrix}.$$

By symmetry, the same holds true for the ∂_x *derivative of* Q. *In particular,*

$$\langle \partial_\xi Q(x,\xi)\zeta_-(x,\xi), \zeta_-(x,\xi)\rangle = \frac{\partial \lambda_-}{\partial \xi}(x,\xi),$$

$$\langle \partial_x Q(x,\xi)\zeta_-(x,\xi), \zeta_-(x,\xi)\rangle = \frac{\partial \lambda_-}{\partial x}(x,\xi),$$

for all $(x,\xi) \neq (0,0)$.

Proof. Since $|\zeta_\pm|_{\mathbb{C}^2} = 1$, and $\zeta_- \in i\mathbb{R} \times \mathbb{R}$ and $\zeta_+ \in \mathbb{R} \times i\mathbb{R}$, we immediately get that

$$\partial_\xi \zeta_- \in \mathbb{C}\zeta_+, \quad \partial_\xi \zeta_+ \in \mathbb{C}\zeta_- \quad \text{(at any fixed } (x,\xi) \neq (0,0)).$$

It suffices now to take the ∂_ξ derivative of the equation $(Q - \lambda_\pm)\zeta_\pm = 0$ to get

$$(\partial_\xi Q)\zeta_\pm = \frac{\partial \lambda_\pm}{\partial \xi}\zeta_\pm \pm \Big(\lambda_+ - \lambda_-\Big)\langle \partial_\xi \zeta_\pm, \zeta_\mp\rangle \zeta_\mp.$$

The same argument applies when taking ∂_x. This concludes the proof of the lemma. □

For $E > 0$ let us now define

$$\Lambda_E := \{(x,\xi) \in \mathbb{R} \times \mathbb{R}; \quad \lambda_-(x,\xi) = E\}.$$

Lemma 4.5. *Let* $E > 0$. *The set* Λ_E *is diffeomorphic to the circle. It is externally tangent to the circle* $\{(x,\xi);\ zh(x,\xi) = E\}$ *and internally tangent to the circle* $\{(x,\xi);\ \Big(\mu_+ - \mu_-\sqrt{1 + t^2/\mu_-^2}\Big)h(x,\xi) = E\}$.

Proof. By using polar coordinates we immediately have

$$\Lambda_E = \{(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta);\ \theta \in [0, 2\pi]\},$$

where

$$\rho(\theta) = \left(\frac{2E}{\mu_+ - \mu_- \sqrt{1 + \frac{t^2}{\mu_-^2}\sin^2(2\theta)}} \right)^{1/2}. \tag{20}$$

The final assertions are an immediate consequence of (19) and (20). □

We next set up the WKB approximation. In the first place, we shall do this in $\mathbb{R} \times \mathbb{R}_+$, for the construction may be made global (by taking into account the Maslov index). We shall hence work with the open Lagrangean manifold $\Lambda_E^+ = \Lambda_E \cap (\mathbb{R} \times \mathbb{R}_+)$. From Lemma 4.5 it follows that $\Lambda_E^+ = \{(x, \xi) \in \mathbb{R} \times \mathbb{R}_+;\ \xi = \phi'(x)\}$, where the function $(-\sqrt{2E/z}, \sqrt{2E/z}) \ni x \longmapsto \phi(x)$ is the smooth solution (for t sufficiently small, uniformly in x, ξ, for in this case $\partial_\xi \lambda_- \approx \xi$) to $\lambda_-(x, \phi'(x)) = E$ whose graph lies in $\mathbb{R} \times \mathbb{R}_+$. (Note that ϕ can be extended as a continuous function to the whole interval $[-\sqrt{2E/z}, \sqrt{2E/z}]$.) Consider now the symplectomorphism $\chi\colon (x, \xi) \longmapsto (\gamma^{-1/2}x, \gamma^{1/2}\xi)$, $\gamma > 0$, and $U_\chi\colon f(x) \longmapsto \gamma^{1/4} f(\gamma^{1/2}x)$ the metaplectic operator associated with χ (which is an isometry of $L^2(\mathbb{R}; \mathbb{C}^2)$ and automorphism of both $\mathcal{S}(\mathbb{R}; \mathbb{C}^2)$ and $\mathcal{S}'(\mathbb{R}; \mathbb{C}^2)$). It is well-known that

$$U_\chi^* Q^{\mathrm{w}}(x, D_x) U_\chi = (Q \circ \chi)^{\mathrm{w}}(x, D_x) =: Q_\gamma^{\mathrm{w}}(x, D_x).$$

Our (*forced*) WKB approximation for the spectral problem related to Q^{w} takes hence the form

$$Q_\gamma^{\mathrm{w}}(x, D_x) e^{i\gamma^{-1}\phi_\gamma(x)} v_\gamma(x) = \frac{E}{\gamma} e^{i\gamma^{-1}\phi_\gamma(x)} v_\gamma(x),$$

where

$$\phi_\gamma = \phi_0 + \gamma\phi_1 + \gamma^2\phi_2 + \dots, \qquad v_\gamma = v_0 + \gamma v_1 + \gamma^2 v_2 + \dots.$$

We next compute the first three terms in the approximation (obtained by equating the resulting powers of γ), for we are interested in the first non-trivial correction in the Bohr-Sommerfeld condition (to be written below). We shall **at the end** choose $\gamma = t$. The reason why we follow this procedure (first performing the WKB and then choosing $\gamma = t$ instead of first choosing $\gamma = t$ and then performing the WKB) is that *we want to keep track of the source of the different pieces in the final t-expansion of the lowest eigenvalue.* We therefore have the following equations (of the terms in γ^{-1}, γ^0 and γ^1, respectively):

$$\Big(Q(x, \phi_0'(x)) - E\Big) v_0(x) = 0, \tag{21}$$

$$(22)\qquad \Big(Q(x,\phi_0'(x))-E\Big)v_1(x)=\frac{1}{2}\Big(i\phi_0''(x)A(z)-tJ\Big)v_0(x)+$$
$$+\Big(i\phi_0'(x)A(z)-txJ\Big)i\phi_1'(x)v_0(x)+\Big(i\phi_0'(x)A(z)-txJ\Big)v_0'(x),$$

$$(23)\qquad \Big(Q(x,\phi_0'(x))-E\Big)v_2(x)=\frac{1}{2}\Big(i\phi_0''(x)A(z)-tJ\Big)v_1(x)+$$
$$+\Big(i\phi_0'(x)A(z)-txJ\Big)i\phi_2'(x)v_0(x)+\Big(i\phi_0'(x)A(z)-txJ\Big)v_1'(x)+$$
$$+\frac{1}{2}A(z)\Big[2i\phi_1'(x)v_0'(x)+v_0''(x)+i\phi_1''(x)v_0(x)-\phi_1'(x)^2v_0(x)\Big]+$$
$$+i\phi_1'(x)\Big(i\phi_0'(x)A(z)-txJ\Big)v_1(x).$$

As regards equation (21), we choose $\phi_0=\phi$ and $v_0(x):=\zeta_-(x,\phi'(x))$. Before discussing the other equations, let us remark the following. Upon denoting by H_{λ_-} the Hamiltonian vector-field associated with λ_-, that is $\mathsf{H}_{\lambda_-}=\partial_\xi\lambda_-\partial_x-\partial_x\lambda_-\partial_\xi$, we have $d/dx=\Big(\partial_\xi\lambda_-\big|_{\Lambda_E^+}\Big)^{-1}\mathsf{H}_{\lambda_-}\big|_{\Lambda_E^+}$, so that

$$i\phi''(x)A(z)-tJ=\frac{i}{\partial_\xi\lambda_-\big|_{\Lambda_E^+}}\mathsf{H}_{\lambda_-}(\partial_\xi Q)\big|_{\Lambda_E^+},\ \text{and}\ i\phi'(x)A(z)-txJ=i\partial_\xi Q\big|_{\Lambda_E^+}.$$

To solve (22), we choose ϕ_1 that makes the r.h.s. of (22) orthogonal to v_0. This is possible. In fact, let us take the Hermitean product with v_0. Using Remark 4.3 and Lemma 4.4, we have

$$\frac{1}{2}\langle\Big(i\phi_0''(x)A(z)-tJ\Big)v_0(x),v_0(x)\rangle+\langle\Big(i\phi_0'(x)A(z)-txJ\Big)v_0'(x),v_0(x)\rangle=$$
$$=\frac{1}{2}\frac{d}{dx}\langle i\partial_\xi Q\big|_{\Lambda_E^+}v_0(x),v_0(x)\rangle=\frac{i}{2}\frac{d}{dx}\Big(\partial_\xi\lambda_-\big|_{\Lambda_E^+}\Big).$$

We then obtain that ϕ_1 must satisfy

$$i\phi_1'(x)\langle i\partial_\xi Q\big|_{\Lambda_E^+}v_0(x),v_0(x)\rangle+\frac{1}{2}\frac{d}{dx}\langle i\partial_\xi Q\big|_{\Lambda_E^+}v_0(x),v_0(x)\rangle=0,$$

and hence we may take

$$i\phi_1(x)=\log\frac{1}{\sqrt{\partial_\xi\lambda_-\big|_{\Lambda_E^+}}}=\log\frac{1}{\sqrt{\langle(\partial_\xi Q)\zeta_-,\zeta_-\rangle\big|_{\Lambda_E^+}}}.$$

This term is well-known. We may hence choose $v_1(x)$ orthogonal to $v_0(x)$ as follows. We start by noting that $\phi_1'=(\partial_\xi\lambda_-\big|_{\Lambda_E^+})^{-1}(\mathsf{H}_{\lambda_-}\Phi_1)\big|_{\Lambda_E^+}$, where

$$i\Phi_1(x,\xi):=-\frac{1}{2}\log\Big(\partial_\xi\lambda_-(x,\xi)\Big),\quad (x,\xi)\in\mathbb{R}\times\mathbb{R}_+.$$

Let next, for $(x,\xi)\in\mathbb{R}\times\mathbb{R}_+$,

$$\alpha_1:=\frac{i}{\partial_\xi\lambda_-}\Big(\frac{1}{2}\langle\mathsf{H}_{\lambda_-}(\partial_\xi Q)\zeta_-,\zeta_+\rangle-(i\mathsf{H}_{\lambda_-}\Phi_1)(\lambda_+-\lambda_-)\langle\partial_\xi\zeta_-,\zeta_+\rangle+$$
$$+\partial_\xi\lambda_+\langle\mathsf{H}_{\lambda_-}\zeta_-,\zeta_+\rangle\Big).$$

Since, as it is readily seen,

$$\langle \mathsf{H}_{\lambda_-}(\partial_\xi Q)\zeta_-, \zeta_+\rangle = (\partial_\xi\lambda_- - \partial_\xi\lambda_+)\langle \mathsf{H}_{\lambda_-}\zeta_-, \zeta_+\rangle + \mathsf{H}_{\lambda_-}\Big((\lambda_- - \lambda_+)\langle \partial_\xi\zeta_-, \zeta_+\rangle\Big),$$

we get that for all $(x,\xi) \in \mathbb{R}\times\mathbb{R}_+$,

$$\alpha_1 = \frac{i}{\partial_\xi\lambda_-}\Big[\frac{1}{2}(\partial_\xi\lambda_+ + \partial_\xi\lambda_-)\langle \mathsf{H}_{\lambda_-}\zeta_-, \zeta_+\rangle + \frac{1}{2}\mathsf{H}_{\lambda_-}\Big((\lambda_- - \lambda_+)\langle \partial_\xi\zeta_-, \zeta_+\rangle\Big) + \\ + (\lambda_- - \lambda_+)(i\mathsf{H}_{\lambda_-}\Phi_1)\langle \partial_\xi\zeta_-, \zeta_+\rangle\Big] \in \mathbb{R}.$$

We may finally take

$$v_1 := \left(\frac{\alpha_1}{\lambda_+ - \lambda_-}\zeta_+\right)\Big|_{\Lambda_E^+} \in \mathbb{R}\times i\mathbb{R}.$$

We now deal with (23). Upon taking the Hermitian product of (23) with $v_0(x)$ we get a sum of terms I_1 through I_8, respectively, restricted to Λ_E^+, defined as follows:

$$I_1\big|_{\Lambda_E^+} = \frac{i}{2}(\partial_\xi\lambda_-\big|_{\Lambda_E^+})^{-1}\left(\frac{\alpha_1}{\lambda_+ - \lambda_-}\langle \mathsf{H}_{\lambda_-}(\partial_\xi Q)\zeta_+, \zeta_-\rangle\right)\Big|_{\Lambda_E^+},$$

$$I_2\big|_{\Lambda_E^+} = -\phi_2'\ \partial_\xi\lambda_-\big|_{\Lambda_E^+},$$

$$I_3\big|_{\Lambda_E^+} = \frac{i}{\partial_\xi\lambda_-\big|_{\Lambda_E^+}}\langle (\partial_\xi Q)\mathsf{H}_{\lambda_-}\left(\frac{\alpha_1}{\lambda_+ - \lambda_-}\zeta_+\right), \zeta_-\rangle\Big|_{\Lambda_E^+},$$

$$I_4\big|_{\Lambda_E^+} = -\frac{1}{2}\frac{\mathsf{H}_{\lambda_-}(\partial_\xi\lambda_-)}{(\partial_\xi\lambda_-)^3}\Big|_{\Lambda_E^+}\langle (\partial_\xi^2 Q)\mathsf{H}_{\lambda_-}\zeta_-, \zeta_-\rangle\Big|_{\Lambda_E^+},$$

$$I_5\big|_{\Lambda_E^+} = \frac{(\partial_\xi\lambda_-\big|_{\Lambda_E^+})^{-1}}{2}\Big(\mathsf{H}_{\lambda_-}\left(\frac{1}{\partial_\xi\lambda_-}\langle A(z)\mathsf{H}_{\lambda_-}\zeta_-, \zeta_-\rangle\right) + \\ - \frac{\langle A(z)\mathsf{H}_{\lambda_-}\zeta_-, \mathsf{H}_{\lambda_-}\zeta_-\rangle}{\partial_\xi\lambda_-}\Big)\Big|_{\Lambda_E^+},$$

$$I_6\big|_{\Lambda_E^+} = \frac{1}{2}\left(\frac{1}{\partial_\xi\lambda_-}\mathsf{H}_{\lambda_-}\left(-\frac{1}{2(\partial_\xi\lambda_-)^2}\mathsf{H}_{\lambda_-}(\partial_\xi\lambda_-)\right)\langle A(z)\zeta_-, \zeta_-\rangle\right)\Big|_{\Lambda_E^+},$$

$$I_7\big|_{\Lambda_E^+} = \frac{1}{2}\left(\left(-\frac{1}{2(\partial_\xi\lambda_-)^2}\mathsf{H}_{\lambda_-}(\partial_\xi\lambda_-)\right)^2\langle A(z)\zeta_-, \zeta_-\rangle\right)\Big|_{\Lambda_E^+},$$

$$I_8\big|_{\Lambda_E^+} = -\left(\frac{i}{2(\partial_\xi\lambda_-)^2}\frac{\alpha_1}{\lambda_+ - \lambda_-}\mathsf{H}_{\lambda_-}(\partial_\xi\lambda_-)\langle (\partial_\xi Q)\zeta_+, \zeta_-\rangle\right)\Big|_{\Lambda_E^+}.$$

It is immediate to see that $I_1, \dots, I_8 \in \mathbb{R}$. To solve (23), we therefore have that ϕ_2 must satisfy the differential equation

$$\phi_2'(x) = \frac{1}{(\partial_\xi\lambda_-)\big|_{\Lambda_E^+}}\Big(I_1\big|_{\Lambda_E^+} + \sum_{k=3}^{8} I_k\big|_{\Lambda_E^+}\Big).$$

We are finally in a position to impose the following Bohr-Sommerfeld condition for the (lowest) eigenvalue E. Write $\Lambda_E \cap \{\xi = 0\} = \{x_\pm(E)\} = \{\pm\sqrt{2E/z}\}$, and define the following functions of E :

$$\Psi_0(E) := 2\int_{x_-(E)}^{x_+(E)} \phi'(x)dx = \left|\int_{\Lambda_E} \xi dx\right|,$$

$$\Psi_2(E) := 2 \text{ finite part of } \left(\int_{x_-(E)}^{x_+(E)} \phi_2'(x)dx\right),$$

where we put, following [4],

$$\text{finite part of } \left(\int_{x_-(E)}^{x_+(E)} \phi_2'(x)dx\right) := \lim_{\delta\to 0+}\left(\int_{x_-(E)+\delta}^{x_+(E)-\delta} \phi_2'(x)dx - \hat{\Psi}_2(E,\delta)\right),$$

$\hat{\Psi}_2$ being uniquely defined by the finiteness of the limit.
Remark that the term $\Psi_0(E)$ represents the area of the region of $\mathbb{R}\times\mathbb{R}$, containing the origin, whose boundary is Λ_E. The general Bohr-Sommerfeld condition for the k-th eigenvalue E_k may be written as

$$\mathsf{BS}_\gamma(E_k) := \Psi_0(E_k) + \gamma^2\Psi_2(E_k) = 2\pi(k+\frac{1}{2})\gamma, \quad k \in \mathbb{Z}_+,$$

and in particular the one for the lowest eigenvalue E_0 reads (the Maslov factor being already taken into account in the r.h.s.)

$$\mathsf{BS}_\gamma(E_0) := \Psi_0(E_0) + \gamma^2\Psi_2(E_0) = \pi\gamma.$$

Remark 4.6. *One should add in $\mathsf{BS}_\gamma(E)$ a term $\Psi_1(E)\gamma$, where*

$$\Psi_1(E) = 2\Big[\varphi\Big(x_+(E), \phi'(x_+(E))\Big) - \varphi\Big(x_-(E), \phi'(x_-(E))\Big)\Big],$$

related to the phase function φ associated with $\zeta_\pm\big|_{\Lambda_E}$. Such a term is 0 identically.

Proof of the Remark. Write $\sigma(x,\xi) := \dfrac{x\xi}{\mu_- h(x,\xi)\Big(g(x,\xi)+2\Big)}$, where $x,\xi \neq 0$. Then

$$\zeta_-(x,\xi) = \frac{1}{\sqrt{2}}\left(\frac{t\sigma(x,\xi)+i}{\nu(x,\xi)}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} - \frac{t\sigma(x,\xi)-i}{\nu(x,\xi)}\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}\right) =$$

$$= \frac{1}{\sqrt{2}}\left(e^{i\varphi(x,\xi)}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} - e^{-i\varphi(x,\xi)}\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}\right), \ (x,\xi)\neq(0,0),$$

where

$$\varphi(x,\xi) = \pi/2 - \arctan\Big(t\sigma(x,\xi)\Big), \quad (x,\xi)\neq(0,0).$$

Since $\zeta_+ = K\zeta_-$, both ζ_- and ζ_+ have attached the phases $\pm\varphi$, that therefore appear, restricted to Λ_E^+, in the vectors v_0, v_1 etc. And likewise when

$(x,\xi) \in \Lambda_E^-$. That $\pm\varphi\big|_{\Lambda_E^+}$ and $\pm\varphi\big|_{\Lambda_E^-}$ "paste" correctly when $\xi = 0$ is a trivial consequence of the fact that $\pm\varphi$ are globally (in $\mathbb{R}\times\mathbb{R}\setminus\{(0,0)\}$) defined phases. □

Remark 4.7. *Remark 4.6 agrees with the construction of* [7]. *In fact, one may easily compute that the monodromy operator $Q_-(T)$ relative to Λ_E (T is the period) is the projection $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, whence its eigenvalues are $0,1$ and the phase correction is thereby 0. This is due to the global existence of the above phase φ.*

Since we shall take $t=\gamma$ and find all the correction-terms up to order t^2, we will compute the corrections up to order t^2 in $\Psi_0(E)$, and only the leading term (i.e. order t^0) in $\Psi_2(E)$. We start out with $\Psi_0(E)$. We have

$$\Psi_0(E) = \left|\frac{1}{2}\int_{\Lambda_E} \xi dx - x d\xi\right| = \frac{1}{2}\int_0^{2\pi} \rho(\theta)^2 d\theta =$$

$$= E\int_0^{2\pi} \frac{d\theta}{\mu_+ - \mu_-\sqrt{1+t^2\sin^2(2\theta)/\mu_-^2}},$$

whence it follows, by a Taylor-expansion with respect to t,

$$\Psi_0(E) = \frac{2\pi}{z}E\left(1+\frac{t^2}{4z\mu_-}+o(t^2)\right), \quad \text{as } t\to 0+\,.$$

To examine $\Psi_2(E)$, we perform a Taylor-expansion with respect to t in the terms I_1 through I_8 above. To this purpose, it is useful to remark that

$$\mathsf{H}_{\lambda_-}\big|_{\Lambda_E^+} = \left(\mu_+ - \mu_-\frac{1}{\sqrt{1+t^2\sin^2(2\theta)/\mu_-^2}}\right)\mathsf{H}_h\big|_{\Lambda_E^+} +$$

$$-t^2\frac{\sin(2\theta)}{\mu_-\sqrt{1+t^2\sin^2(2\theta)/\mu_-^2}}\mathsf{H}_{x\xi}\big|_{\Lambda_E^+},$$

and

$$\partial_\xi\lambda_-\big|_{\Lambda_E^+} = \mu_+\xi - \mu_-\frac{2h(x,\xi)\xi + 2t^2x^2\xi/\mu_-^2}{2\sqrt{h(x,\xi)^2+t^2x^2\xi^2/\mu_-^2}}\Big|_{\Lambda_E^+} =$$

$$= \left(\mu_+ - \mu_-\frac{h(x,\xi)+t^2x^2/\mu_-^2}{\sqrt{h(x,\xi)^2+t^2x^2\xi^2/\mu_-^2}}\right)\xi\Big|_{\Lambda_E^+} =$$

$$= \left(\mu_+ - \mu_-\frac{1+2t^2\cos^2\theta/\mu_-^2}{\sqrt{1+t^2\sin^2(2\theta)/\mu_-^2}}\right)\rho(\theta)\sin\theta,\ \ \theta\in(0,\pi).$$

We shall therefore write

$$\Psi_2(E) = -2 \ \text{finite part of} \ \left(\int_0^\pi \phi_2'(x(\theta)) \frac{dx}{d\theta} d\theta \right),$$

and select the t^0-term in the t-expansion of $\phi_2'(x(\theta))dx/d\theta$, as $t \to 0+$ (since Ψ_2 is already a coefficient of a t^2-contribution). Recall that $\mathsf{H}_h = \xi\partial_x - x\partial_\xi$. For convenience, given a function f *depending on* t, x, ξ, we shall say that

$$\mathsf{deg}_t(f) = \begin{cases} 1 & \text{if } t \text{ divides } f \\ 0 & \text{if } t \text{ does not divide } f, \end{cases}$$

and shall write $\mathsf{m}_0(f) = \mathsf{m}_0(f)(x,\xi)$ for the coefficient of t^0 in the t-expansion of the function f. It is easy to see that

$$\mathsf{deg}_t(\partial_\xi \zeta_-(x,\xi)) = 1.$$

Hence by symmetry the same holds for $\partial_x \zeta_-$, and, being $\mathsf{deg}_t(\lambda_-) = 0$, also for $\mathsf{H}_{\lambda_-}\zeta_-$. Since $\mathsf{deg}_t(\lambda_+ - \lambda_-) = 0$, we therefore get $\mathsf{deg}_t(\alpha_1/(\lambda_+ - \lambda_-)) = 1$. As a consequence, being $\mathsf{deg}_t(\partial_\xi \lambda_-) = 0$,

$$\mathsf{deg}_t(I_1) = \mathsf{deg}_t(I_3) = \mathsf{deg}_t(I_4) = \mathsf{deg}_t(I_5) = \mathsf{deg}_t(I_8) = 1,$$

and they will not be considered in the 0th-order term in the t-expansion of Ψ_2. It remains therefore to consider the terms I_6 and I_7. One computes

$$\mathsf{m}_0(I_6) = -\frac{1}{4}\frac{\xi^2 + 2x^2}{\xi^4}, \quad \mathsf{m}_0(I_7) = \frac{1}{8z}\frac{x^2}{\xi^4},$$

so that the term that counts in the intergral of ϕ_2' is given by

$$\frac{1}{z\xi}\left(\frac{1}{8z}\frac{x^2}{\xi^4}(1-4z) - \frac{1}{4\xi^2} \right)\Big|_{\Lambda_E^+}.$$

We must now perform a final expansion in t in the expression for $\Psi_2(E)$, and single out the coefficient of t^0. We shall write $\equiv_{(t)}$ for the equivalence modulo terms divisible by t. Since

$$\frac{dx}{d\theta} \equiv_{(t)} -\rho(\theta)\sin\theta, \quad \rho(\theta) \equiv_{(t)} \sqrt{\frac{2E}{z}},$$

to compute $\hat{\Psi}_2$ (and hence Ψ_2), we have therefore to consider for $\delta > 0$,

$$\begin{aligned} J(\delta) &:= \int_\delta^{\pi-\delta} \frac{1-4z}{8z^2}\frac{\rho(\theta)^2\cos^2\theta}{\rho(\theta)^5\sin^5\theta}\Big(-\rho(\theta)\sin\theta\Big)d\theta - \frac{1}{4z}\int_\delta^{\pi-\delta}\frac{-\rho(\theta)\sin\theta}{\rho(\theta)^3\sin^3\theta}d\theta \\ &\equiv_{(t)} 2\frac{z}{2E}\left(\frac{4z-1}{8z^2}\int_\delta^{\pi/2}\frac{\cos^2\theta}{\sin^4\theta}d\theta + \frac{1}{4z}\int_\delta^{\pi/2}\frac{1}{\sin^2\theta}d\theta \right) = \\ &=: 2\frac{z}{2E}\left(\frac{4z-1}{8z^2}J_1(\delta) + \frac{1}{4z}J_2(\delta) \right). \end{aligned}$$

Upon integrating by parts, we obtain

$$J_1(\delta) = \frac{\cos\delta}{3\sin^3\delta} - \frac{1}{3}J_2(\delta), \quad \text{and} \quad J_2(\delta) = \frac{\cos\delta}{\sin\delta},$$

whence it follows

$$J(\delta) \equiv_{(t)} \frac{\cos\delta}{24z^2\sin^3\delta}\Big(4z-1+(2z+1)\sin^2\delta\Big)\frac{z}{E},$$

and finally that

$$\hat{\Psi}_2(E,\delta) = \frac{\cos\delta}{24z^2\sin^3\delta}\Big(4z-1+(2z+1)\sin^2\delta\Big)\frac{z}{E}.$$

As a consequence, we therefore have

$$\Psi_2(E) = 0,$$

that is, there is no correction term induced by the 2nd-order expansion in $\gamma = t$ of the phase ϕ_γ.
Hence we have, as $t \to 0+$,

$$\mathsf{BS}_t(E) = \pi t \implies \frac{E}{t} = \frac{z}{2}\left(1 - \frac{t^2}{2(1-z^2)} + o(t^2)\right),$$

and finally, upon writing $\mu_{\mathsf{BS}}(t) = E/t^2$,

$$\mu_{\mathsf{BS}}(t) = \frac{z}{2t}\left(1 - \frac{t^2}{2(1-z^2)} + o(t^2)\right), \quad \text{as } t \to 0+.$$

As a consequence,

$$\mu_{\min}(\alpha,\beta) - \mu_H(t) = \frac{z}{2t}\left(t^2\frac{1-z^2}{2(5-z^2)} + o(t^2)\right),$$

$$\mu_H(t) - \mu_{\mathsf{BS}}(t) = \frac{z}{2t}\left(\frac{z^2t^2}{2(1-z^2)} + o(t^2)\right),$$

so that, being $z \in (0,1)$, we have proved the following lemma.

Lemma 4.8. *For all $t > 0$ such that $t = 1/\sqrt{\alpha\beta}$ is sufficiently small, one has*

$$\mu_{\mathsf{BS}}(t) < \mu_H(t) < \mu_{\min}(\alpha,\beta). \tag{24}$$

4.3. **Comparison with the characterization given in** [15]-[16]**.** In this final section, we will study the lowest eigenvalue of $Q_{(\alpha,\beta)}(x,D_x)$ by using the characterization obtained in [15]-[16]. We shall use the notation introduced in the beginning of the paper. Hence the eigenvalues of $Q_{(\alpha,\beta)}(x,D_x)$ are denoted by $\mu(\alpha,\beta)$, the ones of $Q_t(x,D_x)$ by $\lambda(t)$, related by (on abusing notation)

$$\mu(\alpha,\beta) = \mu(\frac{z}{t},\frac{1}{zt}) =: \mu(t) = \frac{1}{t}\lambda(t).$$

The eigenvalues of $(\ell/\sqrt{\alpha\beta})\tilde{Q}_{(\alpha,\beta)}$, which is unitarily equivalent to $Q_{(\alpha,\beta)}$ (see [15] and [14]), are finally denoted by $\ell\zeta(t)/\sqrt{\alpha\beta}$. Hence (being $\sqrt{\alpha\beta} = 1/t$ and $\ell(t) = \sqrt{1-t^2}/t$)

$$\zeta(t) = \frac{\mu(t)\sqrt{\alpha\beta}}{\ell(t)} = \frac{\mu(t)}{\sqrt{1-t^2}} = \frac{\lambda(t)}{t\sqrt{1-t^2}}, \quad t \in (0,1).$$

Denote by $\zeta_{\min}(t)$ the lowest eigenvalue of $\tilde{Q}_{(\alpha,\beta)}$. Since $t\zeta_{\min}(t) \to z/2$ as $t \to 0+$ (and actually as $t \to 0$), we will always take $t > 0$ so small and correspondingly ζ so large that that

$$t\zeta =: y \in I_* := \left[\frac{z}{2}\frac{3z^2+1}{2(1+z^2)}, \frac{z}{2}\frac{3+z^2}{2(1+z^2)}\right].$$

Notice that $I_* = \left[\frac{z}{2}\left(1 - \frac{1}{2(1+z^2)}\right), \frac{z}{2}\left(1 + \frac{1-z^2}{2(1+z^2)}\right)\right]$ is centered in $z/2$, with $\operatorname{diam}(I_*) \approx z/2$. Also, since to the lowest eigenvalue there corresponds an eigenspace spanned by an even eigenfunction, we will consider the even (i.e. $+$) case. Recall that $\mu_N = N + 1/2$, $\mu_\pm = (z^{-1} \pm z)/2 > 0$, for $z \in (0,1)$. We may hence rewrite the functions $\Lambda_{2N}(\zeta)$ and $d_{2N}(\zeta)$ of [15] in the coordinates (z,t) and obtain

$$\Lambda_{2N}(\zeta) = \mu_{2N} - y\mu_+ =: g_{2N}(y),$$

$$d_{2N}(\zeta) = \Big(1 + \frac{2}{\ell(t)^2}\Big)g_{2N}(y)^2 + \frac{2}{\ell(t)^2} y\mu_+ g_{2N}(y) - y^2\mu_-^2 =: p_{2N}(y)$$

$$= \Big(\mu_{2N}^2 + y^2 - 2\mu_{2N}\mu_+ y\Big) + \frac{2}{\ell(t)^2}\mu_{2N}g_{2N}(y) =: p_{2N}^{(\infty)}(y) + \frac{2}{\ell(t)^2}\mu_{2N}g_{2N}(y).$$

Notice that

$$\bigcup_{N\in\mathbb{Z}_+}\Big\{y;\ p_N^{(\infty)}(y) = 0\Big\} = \bigcup_{N\in\mathbb{Z}_+}\Big\{z\mu_N, \frac{1}{z}\mu_N\Big\} = \operatorname{Spec}(Q_0),$$

and our choice of I_* yields that only $p_0^{(\infty)}$ has a zero in I_*, which is exactly $z/2$, and, furthermore,

$$y \in I_* \implies g_N(y), p_N(y) > 0, \quad \forall N \in \mathbb{Z}_+, \tag{25}$$

which shows, once more, that the lowest eigenvalue belongs to Σ_∞^+.
Following [15], we have next to study the solutions $y = t\zeta \in I_*$ of the equation

$$1 - \gamma_0(\zeta) = \cfrac{\gamma_1(\zeta)}{1 - \cfrac{\gamma_2(\zeta)}{1 - \dots}} = f(\zeta), \tag{26}$$

where (transposing the notation of [15] to the present one)

$$\gamma_{N-1}(\zeta) = \frac{2\big(\ell(t)^2+1\big)}{\ell(t)^4}\frac{N(2N-1)g_{2N}(y)g_{2(N-1)}(y)}{p_{2N}(y)p_{2(N-1)}(y)} =: \tilde{\gamma}_{N-1}(y), \quad N \geq 1.$$

These rational functions are actually regular for $y \in I_*$ by virtue of (25). We shall hence consistently define

$$\tilde{f}(y) := \cfrac{\tilde{\gamma}_1(y)}{1 - \cfrac{\tilde{\gamma}_2(y)}{1 - \dots}} = f^{(0)}(y), \quad \text{and, for } k \geq 1,\ \tilde{f}^{(k)}(y) := \cfrac{\tilde{\gamma}_{k+1}(y)}{1 - \cfrac{\tilde{\gamma}_{k+2}(y)}{1 - \dots}},$$

whence

$$\tilde{f}^{(k)}(y) = \frac{\tilde{\gamma}_{k+1}(y)}{1 - \tilde{f}^{(k+1)}(y)}, \ \forall k \in \mathbb{Z}_+. \tag{27}$$

Equation (26) now becomes

$$1 - \tilde{\gamma}_0(y) = \tilde{f}(y), \quad y \in I_*. \tag{28}$$

The next lemma allows us to control the right-hand-side of (28).

Lemma 4.9. *There exists a universal constant $C_* > 0$ (dependent only on z and the diameter of I_*) such that*

$$\max_{y \in I_*} \tilde{\gamma}_N(y) \le C_* \frac{\ell^2 + 1}{\ell^4}, \quad \forall N \ge 1.$$

Proof. Since $g_{2N}(y), p_{2N}^{(\infty)}(y) > 0$ for all $y \in I_*$ and all $N \ge 1$, we immediately have that

$$p_{2N}(y) \ge p_{2N}^{(\infty)}(y), \ \forall y \in I_*, \ \forall N \ge 1.$$

Now,

$$p_{2N}^{(\infty)}(y) = \mu_0^2 + y^2 - 2\mu_0\mu_+ y + 4N^2 + 4N\mu_0 - 4N\mu_+ y =$$

$$= p_0^{(\infty)}(y) + 4N^2 + 4N g_0(y) \ge 4N^2, \ \forall y \in I_*,$$

for $p_0^{(\infty)}(y) \ge 0$ and $g_0(y) > 0$ there. On the other hand, since $g_{2N}(y) \le \mu_{2N}$ on I_*, we get

$$\frac{2N(2N-1)g_{2N}(y)g_{2(N-1)}(y)}{p_{2N}(y)p_{2(N-1)}(y)} \le \frac{2N(2N-1)\mu_{2N}\mu_{2(N-1)}}{4^2N^2(N-1)^2} \le C_*, \quad \forall N \ge 2.$$

This concludes the proof of the lemma. □

Hence, there exists $\ell_* = \ell(t_*) \ge 1$ such that for all $\ell \ge \ell_*$ (that is $0 < t < t_*$; recall that $\ell = t^{-1}\sqrt{1-t^2}$), one has $\max_{y \in I_*} |\tilde{\gamma}_N(y)| \le \frac{1}{4}$, for all $N \ge 2$, so that Worpitzky's Theorem yields the following corollary.

Corollary 4.10. *One has*

$$\max_{y \in I_*} |\tilde{f}^{(k)}(y)| \le \frac{1}{2}, \quad \forall k \ge 1, \ \forall \ell \ge \ell_*, \tag{29}$$

whence it follows, by relation (27), the ℓ-dependent inequality

$$\max_{y \in I_*} \left|\tilde{f}^{(k)}(y)\right| \le 2C_* \frac{\ell^2+1}{\ell^4}, \ \forall k \in \mathbb{Z}_+, \ \forall \ell \ge \ell_*. \tag{30}$$

We now look for $y = y(1/\ell^2) = y_0 + y_2/\ell^2 \in I_*$, as $\ell \to +\infty$ (the expansion in negative powers of ℓ^2 is more convenient at this stage) such that the **equivalent** equation (by virtue of (25))

$$p_2\Big(y(\frac{1}{\ell^2})\Big)p_0\Big(y(\frac{1}{\ell^2})\Big)\left[1-\tilde{\gamma}_0\Big(y(\frac{1}{\ell^2})\Big)-\frac{\tilde{\gamma}_1\Big(y(\frac{1}{\ell^2})\Big)}{1-\tilde{f}^{(1)}\Big(y(\frac{1}{\ell^2})\Big)}\right]=0 \tag{31}$$

is satisfied up to $o(1/\ell^2)$ as $\ell \to +\infty$. On using

$$p_{2N}(y)p_{2N-2}(y) = p_{2N}^{(\infty)}(y)p_{2N-2}^{(\infty)}(y) + \frac{2}{\ell^2}\Big(\mu_{2N}g_{2N}(y)p_{2N-2}^{(\infty)}(y) + \mu_{2N-2}g_{2N-2}(y)p_{2N}^{(\infty)}(y)\Big) + \frac{4}{\ell^4}\mu_{2N}\mu_{2N-2}g_{2N}(y)g_{2N-2}(y),$$

one immediately chooses $y_0 = z/2$, and, as it is easily computed by Taylor-expansion, since $p_0^{(\infty)}(y_0) = 0$, it turns out that y_2 has to satisfy the equation

$$y_2 = -\frac{2\mu_0 g_0(y_0)p_2^{(\infty)}(y_0) - 2g_0(y_0)g_2(y_0)}{p_2^{(\infty)}(y_0)\,\partial_y p_0^{(\infty)}(y_0)},$$

that is $y_2 = \dfrac{z}{2}\dfrac{1-z^2}{2(5-z^2)}$, being $p_2^{(\infty)}(y_0) = 5 - z^2$, $\partial_y p_0^{(\infty)}(y_0) = -\dfrac{1-z^2}{2z}$, $g_0(y_0) = \dfrac{1-z^2}{4}$, and $g_2(y_0) = \dfrac{9-z^2}{4}$. Hence, finally,

$$y(\frac{1}{\ell^2}) = \frac{z}{2}\left(1 + \frac{1-z^2}{2(5-z^2)}\frac{1}{\ell^2}\right),\ \ell \text{ large positive},$$

is the required solution, for one has that the right-hand-side of (31) is $O(1/\ell^4)$ then. Since $\ell(t) = \sqrt{1-t^2}/t$ and $\sqrt{1-t^2}\,\zeta_{\min}(t) = \lambda_{\min}(t)/t$, we at last get (upon expanding $\sqrt{1-t^2}$ and using $\ell(t)^{-2} = t^2(1+o(1))$) the following approximation of $\lambda_{\min}(t)/t = \mu_{\min}(t)$ as $t \to 0+$:

$$\mu_{\mathsf{PW}}(t) = \frac{1}{t}\sqrt{1-t^2}\,y(\frac{1}{\ell(t)^2}) = \tag{32}$$

$$= \frac{1}{t}\Big(1 - \frac{1}{2}t^2 + o(t^2)\Big)\Big(\frac{z}{2} + \frac{z(1-z^2)}{4(5-z^2)}t^2 + o(t^2)\Big) = \frac{1}{t}\Big(\frac{z}{2} - \frac{z}{5-z^2}t^2 + o(t^2)\Big)$$

as $t \to 0+$. Hence $\mu_{\mathsf{PW}}(t)$ is the **best approximation** (to second order) of the lowest eigenvalue of $Q_{(\alpha,\beta)}$ that we have found (as it should of course be). Of course, one may indeed compute (and give a different interpretation of the coefficients arising in the Rellich expansion) by means of equation (31), all the perturbation-coefficients obtained through Rellich's method, and hence an approximation to any order t^{2k} of the lowest eigenvalue. In fact, for instance, to compute the order t^4, one considers $y = y(1/\ell^2) = y_0 + y_2/\ell^2 + y_4\ell^4$, and plug it into equation (31), that can now be written as

$$p_4(y)p_2(y)^2p_0(y)\left[1-\tilde{\gamma}_0(y)-\frac{\tilde{\gamma}_1(y)}{1-\tilde{\gamma}_2(y)\sum_{k\in\mathbb{Z}_+}\left(\tilde{f}^{(2)}(y)\right)^k}\right] =$$

$$= p_4(y)p_2(y)^2p_0(y)\Big[1-\tilde{\gamma}_0(y)-\tilde{\gamma}_1(y)\Big(1+\tilde{\gamma}_2(y)+\tilde{\gamma}_2(y)\tilde{f}^{(2)}(y)+O(\frac{1}{\ell^4})\Big)\Big] = 0.$$

At last, since in $\{\zeta \in \mathbb{R};\ t\zeta \in I_*\}$ there cannot be other eigenvalues of $\tilde{Q}_{(\alpha,\beta)}$ apart from the lowest eigenvalue, one has also proved the following theorem (stated in the present case $0 < \alpha < \beta$).

Theorem 4.11. *For* $\det(A) = \alpha\beta > 1$ *sufficiently large, the lowest eigenvalue* $\mu_{\min}$ *of* $Q_{(\alpha,\beta)}(x, D_x)$ *satisfies equation (26), that is*

$$1 - \gamma_0\Big(\frac{\mu_{\min}\sqrt{\alpha\beta}}{\ell}\Big) = f\Big(\frac{\mu_{\min}\sqrt{\alpha\beta}}{\ell}\Big). \tag{33}$$

We close the paper by remarking that it is of course also interesting to explore eigenvalues of multiplicity greater than or equal to 2 which belong to $\Sigma_0^{\pm}$, or $\Sigma_\infty^{\pm} \cap \Sigma_0^{\mp}$ ($\pm$-respectively). To fix ideas, let us consider the following situation. As remarked in [16], eigenvalues $\mu \in \Sigma_0$ of $Q_{(\alpha,\beta)}(x, D_x)$ are determined by the zeroes of particular polynomials. Consider then the case in which the multiple eigenvalue $\mu_0 := \lambda_0\ell/\sqrt{\alpha\beta}$ is determined by the condition (in the notation of [15] and [16])

$$\Lambda_2(\lambda_0) = 0, \quad \det L_0(\lambda_0) = 0. \tag{34}$$

Using the aforeintroduced notation, condition (34) can be rewritten as

$$g_2(y_0) = 0, \quad p_0(y_0) = 0, \tag{35}$$

whence we get the relation between z and ℓ (and thus t)

$$z(\ell)^2 = \frac{5 - 4\sqrt{1 - \dfrac{1}{2\ell^2}}}{5 + 4\sqrt{1 - \dfrac{1}{2\ell^2}}}, \qquad \ell \geq \frac{1}{\sqrt{2}}. \tag{36}$$

As $\ell \to +\infty$, $z(\ell)$ approaches monotonically the value $1/3$. This shows that in H_+ (with $\alpha \leq \beta$) there is a simple curve $\Gamma := \{(\alpha(\ell), \beta(\ell));\ \ell \geq 1/\sqrt{2}\}$ that starts at $\alpha = \beta$ and is asymptotically tangent to the line $\alpha(\infty) = \beta(\infty)/9$, and such that $Q_{(\alpha(\ell),\beta(\ell))}(x, D_x)$ has a multiple eigenvalue there. The curve Γ intersects the line $\alpha = z^2\beta$ for each fixed $z \in (1/3, 1]$ at a single point that determines ℓ as a function $\ell(z)$. It follows that a lower bound for the constant $C_{N,n,m}$ of Proposition 3.16 is therefore $\sqrt{\ell(z)^2 + 1}$. Of course, the same reasoning can be carried out in all the other cases.

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