STRONGLY REGULAR J-INNER MATRIX FUNCTIONS AND RELATED PROBLEMS

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Strongly Regular J-Inner Matrix Functions and Related Problems

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Abstract. A number of characterizations of the class of strongly regular J-inner matrix valued functions and descriptions of the corresponding reproducing kernel Hilbert spaces and formulas for the reproducing kernels of these spaces are reviewed. Applications to bitangential interpolation problems, bitangential inverse problems for canonical integral and differential systems, J-unitary nodes and Livsic-Brodskii J-nodes are surveyed. Most of the furnished information is adapted from the papers [ArD1]-[ArD11]. However, in the last two sections, some new results are presented.

Table of Contents

1. Introduction

The class $\mathcal{U}_{sR}(J) = \mathcal{U}_{sR}(J, \Omega_+)$ of strongly regular J-inner in Ω_+ mvf's (matrix valued functions) was introduced, investigated and applied to bitangential inverse problems for canonical integral and differential systems in [ArD1]-[ArD8]. A survey

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of some of our results will be given below and some other related results on operator nodes with characteristic matrix functions of the class $\mathcal{U}_{sR}(J)$ will be formulated.

Here and below J is a fixed $m \times m$ signature matrix: $J^* = J^{-1} = J$; Ω_+ is one of the domains $\mathbb{D}_+$ or $\mathbb{C}_+$ in the complex plane $\mathbb{C}$, where

$$\mathbb{D}_+ = \{ \lambda \in \mathbb{C} : |\lambda| < 1\}, \qquad \mathbb{C}_+ = \{ \lambda \in \mathbb{C} : \Im\lambda > 0\}.$$

The matrices

$$(1.1) \quad j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \qquad J_p = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix}, \qquad \mathcal{J}_p = \begin{bmatrix} 0 & -iI_p \\ iI_p & 0 \end{bmatrix}$$

are often considered as a signature matrix J.

The class $\mathcal{U}_{sR}(j_{pq})$ is formed by the resolvent matrices (in the terminology of M.G. Krein) of generalized bitangential interpolation problems in the Schur class $\mathcal{S}^{p\times q}$ of $p \times q$ mvf's that are holomorphic and contractive in Ω_+ in the so called strictly completely indeterminate case. There is an analogous connection between the class $\mathcal{U}_{sR}(J_p)$ and generalized bitangential interpolation problems in the Caratheodory class $\mathcal{C}^{p\times p}$ of $p \times p$-mvf's that are holomorphic with positive real part in Ω_+. These connections were considered in [ArD4] and [ArD6]. They will be discussed in Section 3.

In Section 4, criteria for a mvf in the class $\mathcal{U}(J) = \mathcal{U}(J, \Omega_+)$ of J-inner mvf,s with respect to Ω_+ to belong to $\mathcal{U}_{sR}(J)$ that are taken from [ArD11] will be presented. They are formulated in terms of the Treil-Volberg matrix version of the Muckenhoupt (A_2)-condition on a mvf $\Delta(\zeta)$ that is nonnegative a.e. on the boundary $\partial\Omega_+$ of Ω_+ and is defined by the given mvf $U \in \mathcal{U}(J)$. Earlier in [ArD10] we obtained other criteria that was also formulated in terms of an (A_2)-condition but for other mvf's $\Delta(\zeta)$ and with an extra condition.

To every mvf $U \in \mathcal{U}(J)$ there corresponds a RKHS (reproducing kernel Hilbert space) $\mathcal{H}(U)$, the theory of which was developed and extensively studied by L. de Branges [Br1]–[Br3] and then applied to assorted problems of analysis by L. de Branges himself, partially in collaboration with J. Rovnyak [BrR], and by others, see e.g., [Al], [ADRS], [AlD1]–[AlD3], [An], [Ba], [BaC], [Dy4], [DI], [Re], [Ro], [Sak], [Wi].

In Section 5 the characteristic properties of the space $\mathcal{H}(U)$ that correspond to $U \in \mathcal{U}_{sR}(J)$ are formulated and a description of the space $\mathcal{H}(U)$ for $U \in \mathcal{U}_{sR}(J_p)$ is given. These results were obtained in our papers [ArD4], [ArD7]. In these papers, we also obtained useful formulas for the RK's (reproducing kernels) $K_\omega^U(\lambda)$ for $J = j_{pq}$ and $J = J_p$. These expressions were then applied to obtain formulas for the solutions of bitangential inverse problems for canonical integral and differential systems with matrizants $U_t(\lambda) = U(t, \lambda)$, $0 \le t < d$, that satisfy the extra condition

$$(1.2) \qquad U_t \in \mathcal{U}_{sR}(J), \qquad 0 \le t < d,$$

in [ArD2], [ArD4], [ArD7].

In our investigations, following a stategy that originates with M.G. Krein, we interpret the entire mvf's $U_t(\lambda) \in \mathcal{U}_{sR}(J)$ as the resolvent matrix of a generalized bitangential Krein extension problem for helical mvf in the so called strictly completely indeterminate case. Such GKEP's were investigated in [ArD9]. Some of our results on the bitangential inverse problems will be discussed in Section 6.

We wish to emphasize, that in our formulation of bitangential inverse problems, the frequency characteristic of the canonical system (such as a spectral function, an input impedance matrix, an input scattering matrix or a monodromy matrix) that is usually considered as the given data is augmented by a normalized monotonic continuous chain of pairs of entire inner mvf's, that is associated in a certain sense with the matrizant $U_t(\lambda)$ of the underlying system.

In Section 7, we present formulas for the solution of the bitangential inverse impedance problem that are adapted from [ArD7].

In Section 8, we consider a class of RKHS's $\mathcal{B}(\mathfrak{E})$ that are based on a $p \times 2p$ mvf $\mathfrak{E}(\lambda)$ that is meromorphic in $\mathbb{C}_+$. If $\mathfrak{E}(\lambda)$ is entire, then $\mathcal{B}(\mathfrak{E})$ is a RKHS of entire $p \times 1$ vvf's. The theory of such spaces of entire functions was introduced and extensively developed by L. de Branges [Br2], [Br3] and the references cited therein. These spaces play a useful role in the spectral theory of differential and integral equations and a number of other areas of analysis; see e.g., [Dy1]-[Dy6], [DI], [DK], [DMc], [Dyu], [Gu], citeKW1, [KW2], [Sak], [Wi], [Wo], [Yu]. We characterize those spaces $\mathcal{B}(\mathfrak{E})$ of meromorphic vvf's that are invariant under the backwards shift operator and introduce the notion of left and right strongly regular de Branges spaces in terms of two generalized Fourier transforms. We then present some characterizations of these two classes of strongly regular de Branges spaces that are related to the corresponding characterizations of left and right strongly regular J-inner mvf's.

There is a well known two-sided connection between the theory of mvf's that are J-contractive in Ω_+ and the theory of characteristic mvf's of operator nodes. In particular, it is known that a mvf $U \in U(J)$ that is holomorphic at the point $\lambda = 0$ may be considered as the characteristic mvf of a simple operator node: of a J-unitary node if $\Omega_+ = \mathbb{D}_+$ or of an LB (Livsic-Brodskii) J-node if $\Omega_+ = \mathbb{C}_+$ and $U(0) = I_m$. Moreover, there exist functional models of these nodes, in which the main operators are modelled as backward shift operators in the space $\mathcal{H}(U)$. On the basis of these models and our results on the spaces $\mathcal{H}(U)$ for $U \in \mathcal{U}_{sR}(J)$ the characteristic properties of simple operator nodes with characteristic mvf's $U(\lambda)$ of the class $\mathcal{U}_{sR}(J)$ were obtained by Z.D. Arova in [Ara1]-[Ara4]. Other characterizations of these classes of operator nodes are considered in [Ar] and Section 9 of this survey.

2. Definition of the class $\mathcal{U}_{sR}(J)$

We recall that an $m \times m$ mvf $U(\lambda)$ that is meromorphic in Ω_+ belongs to the class $\mathcal{U}(J) = \mathcal{U}(J, \Omega_+)$ of J-inner mvf's with respect to Ω_+ if it has J-contractive values

$U(\lambda)$ at the points of holomorphy in Ω_+ and J-unitary boundary values $U(\zeta)$ a.e. on $\partial\Omega_+$.

Let $J \neq \pm I_m$ be a signature matrix and let

$$p = \text{rank } (I_m + J), \qquad q = \text{rank } (I_m - J). \tag{2.1}$$

Then there exists a unitary $m \times m$ matrix V such that

$$J = V^* j_{pq} V \qquad (V^*V = VV^* = I_m). \tag{2.2}$$

Moreover,

$$U \in \mathcal{U}(J) \Longleftrightarrow W(\lambda) = VU(\lambda)V^* \text{ belongs to the class } \mathcal{U}(j_{pq}). \tag{2.3}$$

It is well known that the linear fractional transformation

$$T_W[\varepsilon] = (w_{11}\varepsilon + w_{12})\,(w_{21}\varepsilon + w_{22})^{-1} \tag{2.4}$$

that is based on the four-block decomposition

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \tag{2.5}$$

with blocks w_{11} and w_{22} of sizes $p \times p$ and $q \times q$ is well defined on $\mathcal{S}^{p\times q}$ when $W \in \mathcal{U}(j_{pq})$, i.e., $\mathcal{S}^{p\times q} \subset \mathcal{D}(T_W)$, where

$$\mathcal{D}(T_W) = \{\, \varepsilon \,:\, \det(w_{21}\varepsilon + w_{22}) \not\equiv 0 \,\}. \tag{2.6}$$

Moreover,

$$T_W[\mathcal{S}^{p\times q}] \subset \mathcal{S}^{p\times q}, \quad \text{and} \quad T_W[\mathcal{S}_{in}^{p\times q}] \subset \mathcal{S}_{in}^{p\times q}. \tag{2.7}$$

Here we use notations:

$$T_W[E] = \{\, T_W(\varepsilon) \,:\, \varepsilon \in E \,\}, \tag{2.8}$$

$\mathcal{S}_{in}^{p\times q}$ is the class of $p \times q$ mvf's that are inner with respect to Ω_+ if $p \geq q$ and are $*$-inner with respect to Ω_+ if $p < q$. By a theorem of L.A. Simakova [Si], the property (2.7) is characteristic for $W \in \mathcal{U}(j_{pq})$: if W is an $m \times m$ mvf that is meromorphic in Ω_+ with $\det W(\lambda) \not\equiv 0$ such that (2.7) holds, then $\rho(\lambda)W(\lambda) \in \mathcal{U}(j_{pq})$ for some scalar function $\rho(\lambda)$.

Let

$$\overset{o}{S}{}^{p\times q} = \{\, s \in \mathcal{S}^{p\times q} \,:\, \|s\|_\infty < 1 \,\}. \tag{2.9}$$

Definition 2.1. A mvf $W \in \mathcal{U}(j_{pq})$ is said to be *strongly regular* if

$$T_W[\mathcal{S}^{p\times q}] \cap \overset{o}{S}{}^{p\times q} \neq \emptyset. \tag{2.10}$$

Let $J \neq \pm I_m$ and let V be a unitary matrix such that (2.2) holds.

Definition 2.2. A mvf $U \in \mathcal{U}(J)$ is said to be a strongly regular J-inner mvf with respect to Ω_+ if the mvf $W(\lambda) = VU(\lambda)V^*$ is a strongly regular j_{pq}-inner mvf with respect to Ω_+.

The class of strongly regular J-inner mvf,s with respect to Ω_+ will be denoted by the symbol $\mathcal{U}_{sR}(J,\Omega_+)$, or $\mathcal{U}_{sR}(J)$ for short, when the domain is either clear from the context, or not important.

If $J = J_p$, then $q = p$ and

$$J_p = \mathfrak{V}^* j_p \mathfrak{V}, \text{ where } \quad j_p = j_{pp} \quad \text{and} \quad \mathfrak{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} -I_p & I_p \\ I_p & I_p \end{bmatrix}. \tag{2.11}$$

Let V and $\mathfrak{V}$ be unitary matrices that are considered in the relations (2.2) and (2.11) and, for an $m \times m$ mvf $U(\lambda)$, let

$$W(\lambda) = VU(\lambda)V^* \quad \text{and, for } q = p, \text{ let} \quad A(\lambda) = \mathfrak{V}VU(\lambda)V^*\mathfrak{V}^*. \tag{2.12}$$

Then

$$U \in \mathcal{U}(J) \iff W \in \mathcal{U}(j_{pq}) \quad \text{and, if } q = p, \quad U \in \mathcal{U}(J) \iff A \in \mathcal{U}(J_p)$$
$$U \in \mathcal{U}_{sR}(J) \iff W \in \mathcal{U}_{sR}(j_{pq}) \quad \text{and, if } q = p, \quad U \in \mathcal{U}_{sR}(J) \iff A \in \mathcal{U}_{sR}(J_p).$$

Linear fractional transformations based on the four-block decomposition of mvf's $A(\lambda) \in \mathcal{U}(J_p)$

$$A(\lambda) = \begin{bmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) \end{bmatrix} \tag{2.13}$$

with blocks $a_{ij}(\lambda)$ that are $p \times p$ mvf's map $\mathcal{C}^{p\times p} \cap \mathcal{D}(T_A)$ into $\mathcal{C}^{p\times p}$. However, it turns turns out to be more useful to consider the linear fractional transformations based on the mvf's

$$B(\lambda) = \begin{bmatrix} b_{11}(\lambda) & b_{12}(\lambda) \\ b_{21}(\lambda) & b_{22}(\lambda) \end{bmatrix} = A(\lambda)\mathfrak{V}, \tag{2.14}$$

since the set

$$\mathcal{C}(A) = T_B[\mathcal{S}^{p\times p} \cap \mathcal{D}(T_B)] \tag{2.15}$$

is a larger subset of $\mathcal{C}^{p\times p}$ than $T_A[\mathcal{C}^{p\times p} \cap \mathcal{D}(T_A)]$. We remark that

$$\mathcal{C}^{p\times p} = T_{\mathfrak{V}}[\mathcal{S}^{p\times p} \cap \mathcal{D}(T_{\mathfrak{V}})] \tag{2.16}$$

and define

$$\overset{o}{\mathcal{C}}{}^{p\times p} = T_{\mathfrak{V}}[\overset{o}{\mathcal{S}}{}^{p\times p}]. \tag{2.17}$$

Then

$$\overset{o}{\mathcal{C}}{}^{p\times p} = \{\, c \in \mathcal{C}^{p\times p} \,:\, c \in H_\infty^{p\times p} \text{ and } (\Re c(\zeta))^{-1} \in L_\infty^{p\times p} \,\}. \tag{2.18}$$

A mvf $A \in \mathcal{U}(J_p)$ belongs to the class $\mathcal{U}_{sR}(J_p)$ if and only if

$$\mathcal{C}(A) \cap \overset{o}{\mathcal{C}}{}^{p\times p} \neq \emptyset. \tag{2.19}$$

In the sequel we shall present a number of different characterizations of the class $\mathcal{U}_{sR}(J)$ and, somewhat later in the development, shall rename the class $\mathcal{U}_{sR}(J)$ to the class $\mathcal{U}_{rsR}(J)$ of right strongly regular J-inner mvf's and shall introduce a second class $\mathcal{U}_{lsR}(J)$ of left strongly regular J-inner mvf's.

3. Connection with generalized bitangential Nevanlinna-Pick problems

The generalized Schur Interpolation Problem GSIP $(b_1, b_2; s^o)$ based on a given set of mvf's $b_1 \in \mathcal{S}_{in}^{p\times p}$, $b_2 \in \mathcal{S}_{in}^{q\times q}$ and $s^o \in \mathcal{S}^{p\times q}$ is to describe the set

$$S(b_1, b_2; s^o) = \{\, s \in \mathcal{S}^{p\times q} \,:\, b_1^{-1}(s - s^o)b_2^{-1} \in H_\infty^{p\times q}\,\}. \tag{3.1}$$

This problem is said to be *strictly completely indeterminate* if

$$\mathcal{S}(b_1, b_2; s^o) \cap \overset{o}{\mathcal{S}}{}^{p\times q} \neq \emptyset. \tag{3.2}$$

There is a two-sided connection between such problems and the class $\mathcal{U}_{sR}(j_{pq})$. To explain this, recall that the Potapov-Ginzburg transform $\mathcal{S} = PG(W)$ of a mvf $W \in \mathcal{U}(j_{pq})$ is defined by the formula

$$\mathcal{S} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} (w_{11}^{\#})^{-1} & w_{12}w_{22}^{-1} \\ -w_{22}^{-1}w_{21} & w_{22}^{-1} \end{bmatrix}, \tag{3.3}$$

where $f^{\#}(\lambda) = f(\lambda^{\sim})^*$,

$$\lambda^{\sim} = \begin{cases} 1/\overline{\lambda}, & \text{if} \quad \Omega_+ = \mathbb{D}_+, \\ \overline{\lambda}, & \text{if} \quad \Omega_+ = \mathbb{C}_+, \end{cases} \tag{3.4}$$

and the pseudocontinuation of a mvf $U \in \mathcal{U}(J)$ into the domain $\Omega_- = \text{Ext}\,\Omega_+$ is defined by the J-symmetry principle:

$$U(\lambda) = J(U^{\#}(\lambda))^{-1}J. \tag{3.5}$$

The diagonal blocks blocks of $\mathcal{S}(\lambda)$ satisfy the conditions $s_{11} \in \mathcal{S}^{p\times p}, \det s_{11} \not\equiv 0$ and $s_{22} \in \mathcal{S}^{q\times q}, \det s_{22} \not\equiv 0$ and hence admit inner-outer and outer-inner factorizations:

$$s_{11} = b_1\varphi_1, \qquad b_1 \in \mathcal{S}_{in}^{p\times p}, \quad \varphi_1 \in \mathcal{S}_{out}^{p\times p}, \tag{3.6}$$

$$s_{22} = \varphi_2 b_2, \qquad b_2 \in \mathcal{S}_{in}^{q\times q}, \quad \varphi_q \in \mathcal{S}_{out}^{q\times q}. \tag{3.7}$$

The pair $\{b_1, b_2\}$ of inner mvf's considered in (3.6), (3.7) is called an *associated pair* of the j_{pq}-inner mvf W and we denote such a pair by writing $\{b_1, b_2\} \in \text{ap}\,(W)$. From this definition and general results on GSIP's we obtained the following two theorems.

Theorem 3.1. [ArD4]. *Let the GSIP $(b_1, b_2; s^o)$ be strictly completely indeterminate. Then there exists a mvf $W \in \mathcal{U}(j_{pq})$ such that*

$$\mathcal{S}(b_1, b_2; s^o) = T_W[\mathcal{S}^{p\times q}]. \tag{3.8}$$

Moreover, every such mvf $W(\lambda)$ belongs to the class $\mathcal{U}_{sR}(j_{pq})$ and, in the set of these mvf's, there exists a $W \in \mathcal{U}_{sR}(j_{pq})$ such that

$$\{b_1, b_2\} \in ap\,(W). \tag{3.9}$$

A mvf $W \in \mathcal{U}_{sR}(j_{pq})$ for which (3.8) and (3.9) hold is defined up to a constant j_{pq}-unitary right multiplier. If $b_1(\lambda)$ and $b_2(\lambda)$ are both entire inner mvf's, then

$W(\lambda)$ is also an entire mvf and may be uniquely specified by the conditions (3.8), (3.9) and the normalization condition $W(0) = I_m$.

Theorem 3.2. [ArD4]. *Let $W \in \mathcal{U}_{sR}(j_{pq})$. Let*

$$\{b_1, b_2\} \in ap\ (W) \quad and \quad s^o \in T_W[\mathcal{S}^{p\times q}]. \tag{3.10}$$

Then the GSIP $(b_1, b_2; s^o)$ is strictly completely indeterminate and the formula (3.8) *holds. If $W(\lambda)$ is entire, then $b_1(\lambda)$ and $b_2(\lambda)$ are both entire.*

There exist analogous results on GCIP's (Generalized Caratheodory Interpolation Problems) and the class $\mathcal{U}_{sR}(J_p)$. To formulate them for a mvf $A(\lambda) \in \mathcal{U}(J_p)$ we consider the corresponding mvf $B(\lambda)$, defined by formula (2.14).

The mvf's ${b_{21}^{\#}}^{-1}$ and b_{22}^{-1} have inner-outer and outer-inner factorizations in the Smirnov class [1] $\mathcal{N}_+^{p\times p}$ in Ω_+:

$$\begin{aligned} {b_{21}^{\#}}^{-1} &= b_3\varphi_3, \quad \text{where} \quad b_3 \in \mathcal{S}_{in}^{p\times p}, \quad \varphi_3 \in \mathcal{N}_{out}^{p\times p}, && (3.11)\\ b_{22}^{-1} &= \varphi_4 b_4, \quad \text{where} \quad b_4 \in \mathcal{S}_{in}^{p\times p}, \quad \varphi_4 \in \mathcal{N}_{out}^{p\times p}. && (3.12) \end{aligned}$$

The pair $\{b_3, b_4\}$ is called an *associated pair of the second kind* of the mvf $A(\lambda)$ and we write $\{b_3, b_4\} \in \mathrm{ap}_{\mathrm{II}}(A)$. (If $W(\lambda) = \mathfrak{V}A(\lambda)\mathfrak{V}$ and $\{b_1, b_2\} \in \mathrm{ap}\ (W)$, then $\{b_1, b_2\}$ is called an *associated pair of the first kind* of $A(\lambda)$ and we write $\{b_1, b_2\} \in \mathrm{ap}_{\mathrm{I}}(A)$.)

Let $b_3 \in \mathcal{S}_{in}^{p\times p}$, $b_4 \in \mathcal{S}_{in}^{p\times p}$ and $c^o \in \mathcal{C}^{p\times p}$ be given mvf's. Then GCIP $(b_3, b_4; c^o)$ is to describe the set

$$\mathcal{C}(b_3, b_4; c^o) = \{\, c \in \mathcal{C}^{p\times p} \,:\, b_3^{-1}(c - c^o)b_4^{-1} \in \mathcal{N}_+^{p\times p}\}. \tag{3.13}$$

This problem is said to be strictly completely indeterminate, if

$$\mathcal{C}(b_3, b_4; c^o) \cap \overset{o}{\mathcal{C}}{}^{p\times p} \neq \emptyset. \tag{3.14}$$

Theorem 3.3. [ArD6]. *Let the GCIP $(b_3, b_4; c^o)$ be strictly completely indeterminate. Then there exists a mvf $A(\lambda) \in \mathcal{U}(J_p)$ such that*

$$\mathcal{C}(b_3, b_4; c^o) = \mathcal{C}(A). \tag{3.15}$$

Moreover every such mvf $A(\lambda)$ belongs to the class $\mathcal{U}_{sR}(J_p)$ and, in the set of these mvf's, there exists an $A \in \mathcal{U}_{sR}(J_p)$ such that

$$\{b_3, b_4\} \in ap_{\mathrm{II}}(A). \tag{3.16}$$

A mvf $A \in \mathcal{U}_{sR}(J_p)$ for which (3.15) and (3.16) hold is defined up to a constant J_p-unitary right multiplier. If $b_3(\lambda)$ and $b_4(\lambda)$ are both entire inner mvf's, then $A(\lambda)$ is also an entire mvf and may be uniquely specified by the conditions (3.15), (3.16) and the normalization condition $A(0) = I_m$.

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[1] $\mathcal{N}_+^{p\times p} = \{g/h : g \in \mathcal{S}^{p\times p}$ and $h \in \mathcal{S}_{out}^{1\times 1}\}$; $\mathcal{N}_{out}^{p\times p} = \{g/h : g \in \mathcal{S}_{out}^{p\times p}$ and $h \in \mathcal{S}_{out}^{1\times 1}\}$.

Theorem 3.4. [ArD6]. *Let* $A \in \mathcal{U}_{sR}(J_p)$. *Let*

$$\{b_3, b_4\} \in ap_{\rm II}(A) \quad and \quad c^o \in \mathcal{C}(A). \tag{3.17}$$

Then the GCIP $(b_3, b_4; c^o)$ *is strictly completely indeterminate and formula* (3.15) *holds. If* $A(\lambda)$ *is entire, then* $b_3(\lambda)$ *and* $b_4(\lambda)$ *are both entire.*

A GCIP $(b_3, b_4; c^o)$ that is based on entire mvf's $b_3(\lambda)$ and $b_4(\lambda)$, is equivalent to a generalized bitangential Krein extension problem for the so called helical mvf's $g(t)$ that correspond to mvf's $c(\lambda) \in \mathcal{C}^{p\times p}$ by the formula

$$c(\lambda) = \lambda^2 \int_0^\infty e^{i\lambda t} g(t)\, dt, \quad g(0) \le 0. \tag{3.18}$$

This GKEP was considered in [ArD9].

4. Criteria for $U \in \mathcal{U}_{sR}(J)$ in terms of Muckenhoupt conditions

In [ArD10] we obtained a necessary and sufficient condition on $U \in \mathcal{U}(J)$ under which $U \in \mathcal{U}_{sR}(J)$. In this formulation there are two conditions, one of which is the Treil-Volberg matrix version of the Muckenhoupt (A_2)-condition

$$\sup_I \|(\Delta_I)^{1/2}((\Delta^{-1})_I)^{1/2}\| < \infty, \tag{4.1}$$

where

$$\Delta_I = \frac{1}{|I|} \int_I \Delta(\zeta)\, |d\zeta|, \tag{4.2}$$

$\Delta(\zeta)$ is a mvf that is defined in terms of $U(\lambda)$ (and J) and is nonnegative a.e. on $\partial\Omega_+$, I is a subarc of $\mathbb{T} = \{\zeta : |\zeta| = 1\}$ if $\Omega_+ = \mathbb{D}_+$ and I is a finite subinterval of $\mathbb{R}$, if $\Omega_+ = \mathbb{C}_+$. In both settings, $|I|$ denotes the length of I.

Remark 4.1. The condition (4.1) is equivalent to the determinant condition

$$\sup_I((\det \Delta_I) \cdot (\det((\Delta^{-1})_I))) < \infty; \tag{4.3}$$

see [ArD11].

In [ArD11] we obtained new and simpler criteria for a mvf $U \in \mathcal{U}(J)$ to be strongly regular that is based on a different choice of the mvf $\Delta(\zeta)$ than was considered in [ArD10] and is summarized below. In fact, in both [ArD10] and [ArD11], we considered two notions of strong regularity: left and right, and introduced the classes $\mathcal{U}_{lsR}(J)$ and $\mathcal{U}_{rsR}(J)$ of left and right strongly regular J-inner mvf's. The latter coincides with the class $\mathcal{U}_{sR}(J)$ that was considered earlier. These two classes are connected by the relation

$$U(\lambda) \in \mathcal{U}_{lsR}(J) \Longleftrightarrow U^\sim(\lambda) \in \mathcal{U}_{rsR}(J),$$

where

$$U^\sim(\lambda) = JU^\#(-\lambda)J.$$

Let

$$P = \frac{1}{2}(I_m + J), \quad Q = \frac{1}{2}(I_m - J) \tag{4.4}$$

and, for a mvf $U \in \mathcal{U}(J)$ with $J \neq \pm I_m$, let

(4.5)

$$G_r(\zeta) = P + U(\zeta)^* Q U(\zeta) \quad \text{and} \quad G_l(\zeta) = P + U(\zeta) Q U(\zeta)^* \quad \text{a.e.} \quad \zeta \in \partial\Omega_+.$$

Let

$$W(\lambda) = VU(\lambda)V^* = \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix}, \tag{4.6}$$

for $U \in \mathcal{U}(J)$, where $J \neq \pm I_m$ and V satisfies (2.2). Let

$$s_{21}(\lambda) = -w_{22}^{-1}(\lambda) w_{21}(\lambda) \text{ and } s_{12}(\lambda) = w_{12}(\lambda) w_{22}(\lambda)^{-1}. \tag{4.7}$$

Then

$$V^*(G_r(\zeta))^{-1}V = \begin{bmatrix} I_p & s_{21}(\zeta)^* \\ s_{21}(\zeta) & I_q \end{bmatrix} =: \Delta_r(\zeta), \tag{4.8}$$

$$V^*(G_l(\zeta))^{-1}V = \begin{bmatrix} I_p & -s_{12}(\zeta) \\ -s_{12}(\zeta)^* & I_q \end{bmatrix} =: \Delta_l(\zeta) \tag{4.9}$$

and we have the following conclusions:

Theorem 4.2. [ArD11]. *Let $U \in \mathcal{U}(J)$. Then:*

(a)

$$\begin{aligned} U \in \mathcal{U}_{rsR}(J) &\iff \Delta(\zeta) = \Delta_r(\zeta) \textit{ satisfies the } (A_2)\textit{-condition } (4.1) \\ &\iff \Delta(\zeta) = G_r(\zeta) \textit{ satisfies the } (A_2)\textit{-condition } (4.1). \end{aligned}$$

(b)

$$\begin{aligned} U \in \mathcal{U}_{lsR}(J) &\iff \Delta(\zeta) = \Delta_l(\zeta) \textit{ satisfies the } (A_2)\textit{-condition } (4.1) \\ &\iff \Delta(\zeta) = G_l(\zeta) \textit{ satisfies the } (A_2)\textit{-condition } (4.1). \end{aligned}$$

5. The de Branges spaces $\mathcal{H}(U)$ for $U \in \mathcal{U}_{sR}(J)$

The RK $K_\omega^U(\lambda)$ of the RKHS $\mathcal{H}(U)$ based on a mvf $U \in \mathcal{U}(J)$ is defined by the formula

$$K_\omega^U(\lambda) = (\rho_\omega(\lambda))^{-1}(J - U(\lambda)JU(\omega)^*), \qquad \omega \neq \lambda^\sim, \tag{5.1}$$

where

$$\rho_\omega(\lambda) = \begin{cases} 1 - \lambda\overline{\omega}, & \text{if} \quad \Omega_+ = \mathbb{D}_+, \\ -2\pi i(\lambda - \overline{\omega}), & \text{if} \quad \Omega_+ = \mathbb{C}_+. \end{cases} \tag{5.2}$$

A mvf $U \in \mathcal{U}(J)$ and the vvf's $g \in \mathcal{H}(U)$ have pseudocontinuations into the extended complex plane $\mathbb{C} \cup \{\infty\}$ and we consider $U(\lambda)$ and $g(\lambda)$ on the sets of analyticity $\mathfrak{h}_U$ and $\mathfrak{h}_g$ of these extended functions in $\mathbb{C} \cup \{\infty\}$. The RK K_ω^U may be considered on $\mathfrak{h}_U \times \mathfrak{h}_U$. The vvf's $g(\lambda) \in \mathcal{H}(U)$ have nontangential boundary values

$g(\zeta)$ at a.e. point $\zeta \in \partial\Omega_+$, when λ tends to ζ nontangentially (with respect to $\partial\Omega_+$) from $\mathbb{C} \setminus \partial\Omega_+$. We write $\mathcal{H}(U) \subset L_2^m$ if $g(\zeta) \in L_2^m(\partial\Omega_+)$ for every $g \in \mathcal{H}(U)$.

Theorem 5.1. [ArD1]. *Let $U \in \mathcal{U}(J)$, where $J \neq \pm I_m$. Then $U \in \mathcal{U}_{sR}(J)$ if and only if one of the following equivalent properties holds:*

1) $\mathcal{H}(U) \subset L_2^m$.
2) $\mathcal{H}(U)$ *is a (closed) subspace of* L_2^m.
3)

$$\gamma_1 \|g\|_{L_2^m} \leq \|g\|_{\mathcal{H}(U)} \leq \gamma_2 \|g\|_{L_2^m} \tag{5.3}$$

for every $g \in \mathcal{H}(U)$, where γ_1 and γ_2 are finite constants such that $0 < \gamma_1 \leq \gamma_2$.

Remark 5.2. The preceding theorem implies that $\mathcal{U}(J) \cap L_\infty^{m\times m} \subset \mathcal{U}_{sR}(J)$. Moreover, this inclusion is proper. Examples that illustrate this are furnished in Subection 7.6 of [ArD10].

Let $H_2^m(\Omega_\pm)$ be the Hardy spaces of $m \times 1$ vvf's $g(\lambda)$ that are holomorphic in $\Omega_\pm$ and have finite H_2-norm. (If $\Omega_- = \mathbb{D}_- = \text{Ext}\, \mathbb{D}_+$ and $g \in H_2^m(\Omega_-)$ then $g(\infty) = 0$.)

Upon identifying the vvf's $g(\lambda)$ with their boundary values $g(\zeta)$ we have:

$$K_2^m \oplus H_2^m = L_2^m, \quad \text{where} \quad H_2^m = H_2^m(\Omega_+) \quad \text{and} \quad K_2^m = H_2^m(\Omega_-). \tag{5.4}$$

Let

$$\mathcal{H}_\pm(U) = \mathcal{H}(U) \cap H_2^m(\Omega_\pm) \tag{5.5}$$

for $U \in \mathcal{U}(J)$.

Theorem 5.3. [ArD1]. *Let $U \in \mathcal{U}_{sR}(J)$, where $J \neq \pm I_m$. Then*

$$\mathcal{H}(U) = \mathcal{H}_-(U) \dot{+} \mathcal{H}_+(U). \tag{5.6}$$

Let M_c be the operator in $L_2^p = L_2^p(\mathbb{R})$ of multiplication by a mvf $c \in L_\infty^{p\times p}$ and, for $b \in \mathcal{S}_{in}^{p\times p}$, let

$$\mathcal{H}(b) = H_2^p \ominus bH_2^p, \qquad \mathcal{H}_*(b) = K_2^p \ominus b^{-1}K_2^p. \tag{5.7}$$

These two spaces are RKHS's with respect to the standard inner product in L_2^p with RK's

$$k_\omega^b(\lambda) = \frac{I_p - b(\lambda)b(\omega)^*}{\rho_\omega(\lambda)} \text{ and } l_\omega^b(\lambda) = \frac{b(\lambda)^{-1}b(\omega)^{-*} - I_p}{\rho_\omega(\lambda)},$$

respectively. The following description of the space $\mathcal{H}(A)$ for $A \in \mathcal{U}_{sR}(J, \mathbb{C}_+)$ is given in [ArD7] in terms of the three mvf's

$$\{b_3, b_4\} \in \text{ap}_{\text{II}}(A) \tag{5.8}$$

and

$$c \in \mathcal{C}(A) \cap H_\infty^{p\times p}. \tag{5.9}$$

(5.10) $\Phi_{11} = \Pi_{\mathcal{H}(b_3)} M_c\big|_{H_2^p}, \quad \Phi_{12} = \Pi_{\mathcal{H}(b_3)} M_c\big|_{\mathcal{H}_*(b_4)}, \quad \Phi_{22} = \Pi_{K_2^p} M_c\big|_{\mathcal{H}_*(b_4)},$

where Π_L is the orthoprojection onto L.

Theorem 5.4. [ArD7]. *Let $A \in \mathcal{U}_{sR}(J, \mathbb{C}_+)$. Then the operator L_A that is defined by the rule*

$$L_A : \begin{bmatrix} g \\ h \end{bmatrix} \longrightarrow \begin{bmatrix} -\Phi_{11}^* g + \Phi_{22} h \\ g + h \end{bmatrix} \tag{5.11}$$

is a bounded linear map from $\mathcal{H}(b_3) \oplus \mathcal{H}_*(b_4)$ onto $\mathcal{H}(A)$:

$$\mathcal{H}(A) = \left\{ \begin{bmatrix} -\Phi_{11}^* g + +\Phi_{22} h \\ g + h \end{bmatrix} : g \in \mathcal{H}(b_3) \ \text{ and } \ h \in \mathcal{H}_*(b_4) \right\}. \tag{5.12}$$

Moreover, if

$$f = L_A \begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} -\Phi_{11}^* g + +\Phi_{22} h \\ g + h \end{bmatrix}$$

for some choice of $g \in \mathcal{H}(b_3)$ and $h \in \mathcal{H}_*(b_4)$, and

$$\Delta_A = 2\Re \begin{bmatrix} \Phi_{11}|_{\mathcal{H}(b_3)} & \Phi_{12} \\ 0 & \Pi_{\mathcal{H}(b_4)} \Phi_{22} \end{bmatrix},$$

then

$$\begin{aligned} \|f\|^2_{\mathcal{H}(A)} &= \langle \Delta_A \begin{bmatrix} g \\ h \end{bmatrix}, \begin{bmatrix} g \\ h \end{bmatrix} \rangle_{L_2^m} \\ &= \langle (c + c^*)(g + h), (g + h) \rangle_{L_2^p}. \end{aligned} \tag{5.13}$$

In [ArD7] we also obtained a formula for the RK K_ω^A in terms of the solution of a linear system of equations with coefficients $2\Re\Phi_{11}$, Φ_{12}, Φ_{12}^* and $2\Re\Phi_{22}$ and a right hand side that is defined by the mvf's c, b_3 and b_4 and a dual pair $\dot{b}_3, \dot{b}_4$ of inner $p \times p$ inner mvf's that are defined essentially uniquely by the relations

$$\Phi_{11}^* \mathcal{H}(b_3) = \mathcal{H}(\dot{b}_3) \text{ and } \Phi_{22}^* \mathcal{H}_*(b_4) = \mathcal{H}_*(\dot{b}_4).$$

In applications to inverse problems and operator nodes, the case $0 \in \mathfrak{h}_A$ and formulas for $K_0^A(\lambda)$ are of particular interest; see Section 7 for a sample. In this case, the formulas referred to above yield the following result:

Theorem 5.5. [ArD7]. *Let $A \in \mathcal{U}_{sR}(J, \mathbb{C}_+)$, let $0 \in \mathfrak{h}_A$ and let b_5, b_6 be any pair of inner $p \times p$ mvf's such that*

$$(\dot{b}_3)^{-1} b_5 \in \mathcal{S}_{in}^{p\times p} \text{ and } b_6 (\dot{b}_4)^{-1} \in \mathcal{S}_{in}^{p\times p}.$$

Then

$$K_0^A = L_A \begin{bmatrix} \widehat{u}_{11} & \widehat{u}_{12} \\ \widehat{u}_{21} & \widehat{u}_{22} \end{bmatrix}, \tag{5.14}$$

where the $\widehat{u}_{ij} = \widehat{u}_{ij}(\lambda)$ *are* $p \times p$ *mvf's that are obtained as the solutions of the system of equations*

$$\Delta_A \begin{bmatrix} \widehat{u}_{11} & \widehat{u}_{12} \\ \widehat{u}_{21} & \widehat{u}_{22} \end{bmatrix} = \begin{bmatrix} -\Phi_{11} k_0^{b_5} & k_0^{b_3} \\ \Phi_{22}^* l_0^{b_6} & l_0^{b_4} \end{bmatrix} \tag{5.15}$$

and the operators in formulas (5.14) and (5.15) act on the indicated matrix arrays column by column. In particular, the columns of $\widehat{u}_{11}(\lambda)$ *and* $\widehat{u}_{12}(\lambda)$ *belong to* $\mathcal{H}(b_3)$ *and the columns of* $\widehat{u}_{21}(\lambda)$ *and* $\widehat{u}_{22}(\lambda)$ *belong to* $\mathcal{H}_*(b_4)$*.*

Earlier analogous results on the description of the RKHS $\mathcal{H}(W)$ and a formula for the RK $K_\omega^W(\lambda)$ were obtained for $W \in \mathcal{U}_{sR}(j_{pq})$ in [ArD4].

6. Canonical systems with matrizants of the class $\mathcal{U}_{sR}(J)$

In our papers [ArD2], [ArD4], [ArD7], we obtained formulas for the solutions of bitangential inverse problems for canonical integral systems of the form

$$u(t,\lambda) = u(0,\lambda) + i\lambda \int_0^t u(s,\lambda)\, d\mathrm{M}(s) J, \quad 0 \le t < d, \tag{6.1}$$

and the corresponding canonical differential systems of the form

$$\frac{du}{dt} = izu(t,\lambda)H(t)J, \qquad 0 \le t < d \tag{6.2}$$

with $J = j_{pq}$ and $J = J_p$ from the formulas for $K_0^W(\lambda)$ and $K_0^A(\lambda)$.

In our analysis, we assume that the mvf $\mathrm{M}(t)$ that appears in the system (6.1) is a continuous nondecreasing $m \times m$ mvf on the interval $[0,d)$ with $M(0) = 0$ and that $u(t,\lambda)$ is a continuous $k \times m$ mvf on $[0,d)$. In the system (6.2) $H(t)$ is assumed to be locally summable $m \times m$ mvf that is nonnegative a.e. on $[0,d)$ and $u(t,\lambda)$ is a $k \times m$ mvf that is absolutely continuous on every closed subinterval of $[0,d)$. The system (6.2) is equivalent to the system (6.1) with

$$\mathrm{M}(t) = \int_0^t H(s)\, ds, \quad 0 \le t < d. \tag{6.3}$$

The matrizant $U_t(\lambda) = U(t,\lambda)$ of the system (6.1) (or (6.2)) is the solution of this system with initial condition $U(0,\lambda) = I_m$. It belongs to the class $\mathcal{E} \cap \mathcal{U}(J)$ of entire J-inner mvf's and $U_t(0) = I_m$ for every $t \in [0,d)$. Moreover

$$(U_{t_1})^{-1} U_{t_2} \in \mathcal{E} \cap \mathcal{U}(J), \quad \text{if} \quad 0 \le t_1 \le t_2 < d. \tag{6.4}$$

Our next objective is to introduce the notions of input scattering matrices and input impedance matrices for systems (6.1) and (6.2 with arbitrary signature matrices $J \neq \pm I_m$. To this end, we set

$$W_t(\lambda) = VU_t(\lambda)V^* \quad \text{and} \quad A_t(\lambda) = \mathfrak{V} W_t(\lambda) \mathfrak{V}^*, \quad 0 \le t < d, \tag{6.5}$$

where V and $\mathfrak{V}$ are defined by (2.2) and (2.11), and let

$$\mathcal{S}_{scat} = \bigcap_{0\le t<d} T_{W_t}[\mathcal{S}^{p\times q}] \quad \text{and} \quad \mathcal{C}_{imp} = \bigcap_{0\le t<d} \mathcal{C}(A_t). \tag{6.6}$$

The set $\mathcal{S}_{scat}$ of *input scattering matrices* is not empty because

$$T_{W_{t_2}} \subset T_{W_{t_1}} \subset \mathcal{S}^{p\times q} \quad \text{for} \quad t_1 \le t_2 < d$$

and $\mathcal{S}^{p\times q}$ is sequentially compact. We shall call a mvf $s \in \mathcal{S}^{p\times q}$ an input scattering matrix of the considered system. The sets $\mathcal{C}(A_t)$ are monotone non decreasing nonempty subsets of $\mathcal{C}^{p\times p}$. However, the set $\mathcal{C}_{imp}$ of *input impedance matrices* may be empty unless additional restrictions are imposed because $\mathcal{C}^{p\times p}$ is not sequentially compact. We shall call a mvf $c \in \mathcal{C}_{imp}$ an input impedance matrix of the considered system.

The mvf's $b_j^t(\lambda)$ in the chains of associated pairs

$$\{b_1^t, b_2^t\} \in \text{ap}\,(W_t), \qquad 0 \le t < d \tag{6.7}$$

and

$$\{b_3^t, b_4^t\} \in \text{ap}_{\text{II}}\,(A_t), \qquad 0 \le t < d \tag{6.8}$$

of inner mvf's are entire and they may be normalized by the condition $b_j^t(0) = I$, $1 \le j \le 4$. Moreover, these two chains are monotonic in the following sense:

$$(b_1^{t_1})^{-1} b_1^{t_2} \in \mathcal{E} \cap \mathcal{S}_{in}^{p\times p}, \qquad b_2^{t_2}(b_2^{t_1})^{-1} \in \mathcal{E} \cap \mathcal{S}_{in}^{q\times q}, \tag{6.9}$$

$$(b_3^{t_1})^{-1} b_3^{t_2} \in \mathcal{E} \cap \mathcal{S}_{in}^{p\times p}, \qquad b_4^{t_2}(b_4^{t_1})^{-1} \in \mathcal{E} \cap \mathcal{S}_{in}^{p\times p}, \tag{6.10}$$

if $0 \le t_1 \le t_2 < d$. This property follows from (6.4).

In [ArD1]-[ArD8] we investigated canonical systems of the form (6.1) with property (1.2). We proved that for such systems the chains of normalized associated pairs are continuous in the sense that the mvf's $b_j^t(\lambda)$ are continuous with respect to t on the interval $[0, d)$ for every fixed $\lambda \in \mathbb{C}$. In our considerations, the given data for a bitangential inverse problem is not just a frequency characteristic of the system such as an input scattering matrix $s(\lambda)$, or an input impedance matrix $c(\lambda)$, or a spectral function or a monodromy matrix, but also a normalized monotonic continuous chain of pairs of entire mvf's $\{b_1^t, b_2^t\}$, $0 \le t < d$ or $\{b_3^t, b_4^t\}$, $0 \le t < d$, that relate to the system via (6.7) or (6.8), respectively.

The next theorem, which guarantees the existence and uniqueness of a solution to the bitangential inverse impedance problem with given data $(c, b_3^t, b_4^t, 0 \le t < d)$ is established in [ArD6]. The proof of this theorem is based on Theorem 3.3. This theorem is applied to obtain similar conclusions for the bitangential inverse spectral problem in [ArD8]. Analogous results were obtained earlier for the bitangential inverse monodromy problem in [ArD2] and for the bitangential inverse input scattering problem in [ArD4].

Theorem 6.1. *Let $c \in \mathcal{C}^{p\times p}$, let $\{b_3^t(\lambda), b_4^t(\lambda)\}, 0 \le t < d$, be a normalized monotonic continuous chain of entire inner $p \times p$ mvf's and assume that*

$$\mathcal{C}(b_3^t, b_4^t; c) \cap \mathring{\mathcal{C}}^{p\times p} \neq \emptyset \text{ for every } t \in [0, d) \ . \tag{6.11}$$

Then there exists exactly one canonical integral system (6.1) with matrizant $A_t(\lambda) = A(t, \lambda)$, $0 \le t < d$, that satisfies the following conditions:

$$c \in \mathcal{C}_{imp} \ . \tag{1}$$

$$\{b_3^t, b_4^t\} \in ap_{II}(A_t) \text{ for every } t \in [0, d) \ . \tag{2}$$

$$A_t \in \mathcal{U}_{sR}(J_p) \text{ for every } t \in [0, d) \ . \tag{3}$$

In particular, the condition (6.11) is satisfied if $c \in \mathring{\mathcal{C}}^{p\times p}$.

7. Formulas for the solution $M(t)$ of the bitangential inverse impedance problem and for the corresponding matrizant.

Under assumption (6.11), there exists a mvf

$$c^t \in \mathcal{C}(A_t) \cap H_\infty^{p\times p} \tag{7.1}$$

for every $t \in [0, d)$ and hence, upon writing Φ_{ij}^t for the operators Φ_{ij} defined in formula (5.10) with $b_3^t(\lambda)$ in place of $b_3(\lambda)$, $b_4^t(\lambda)$ in place of $b_4(\lambda)$ and $c^t(\lambda)$ in place of $c(\lambda)$, we obtain

(7.2)
$$\Phi_{11}^t = \Pi_{\mathcal{H}(b_3^t)} M_{c^t}\Big|_{H_2^p} \ , \ \Phi_{22}^t = \Pi_- M_{c^t}\Big|_{\mathcal{H}_*(b_4^t)} \ \text{ and } \ \Phi_{12}^t = \Pi_{\mathcal{H}(b_3^t)} M_{c^t}\Big|_{\mathcal{H}_*(b_4^t)} \ .$$

Correspondingly, let

$$Y_1^t = \Pi_{\mathcal{H}(b_3^t)}\Big\{M_{c^t} + (M_{c^t})^*\Big\}\Big|_{\mathcal{H}(b_3^t)} = 2\Re\Big(\Phi_{11}^t\Big|_{\mathcal{H}(b_3^t)}\Big) \ , \tag{7.3}$$

$$Y_2^t = \Pi_{\mathcal{H}_*(b_4^t)}\Big\{M_{c^t} + (M_{c^t})^*\Big\}\Big|_{\mathcal{H}_*(b_4^t)} = 2\Re\Big(\Pi_{\mathcal{H}_*(b_4^t)}\Phi_{22}^t\Big) \ , \tag{7.4}$$

$$\tau_3(t) = \overline{\lim}_{\nu\uparrow\infty} \frac{\log \|b_3^t(-i\nu)\|}{\nu} \text{ and } \tau_4(t) = \overline{\lim}_{\nu\uparrow\infty} \frac{\log \|b_4^t(-i\nu)\|}{\nu} \ . \tag{7.5}$$

Then, the entire inner $p \times p$ mvf $b_j^t(\lambda)$ is of exponential type $\tau_j(t)$, $\mathcal{H}(b_3^t) \subset \mathcal{H}(e_{\tau_3(t)}I_p)$, $\mathcal{H}_*(b_4^t) \subset \mathcal{H}(e_{\tau_4(t)}I_p)$ and, as the type of $\dot{b}_j^t$ is equal to the type of b_j^t,

$$\Phi_{11}^*\mathcal{H}(b_3^t) \subseteq \mathcal{H}(e_{\tau_3(t)}I_p) \text{ and } \Phi_{22}\ \mathcal{H}_*(b_4^t) \subseteq \mathcal{H}_*(e_{\tau_4(t)}I_p) \ .$$

Therefore, we can choose

$$b_5(\lambda) = b_5^t(\lambda) = e^{i\lambda\tau_3(t)}I_p \text{ and } b_6(\lambda) = b_6^t(\lambda) = e^{i\lambda\tau_4(t)}I_p$$

in Theorem 5.5.

In order to keep the notation relatively simple, an operator T that acts in the space of $p\times 1$ vvf's will be applied to $p\times p$ mvf's with columns $f_1,\dots,f_p$ column by column: $T[f_1\cdots f_p]=[Tf_1\cdots Tf_p]$.

Theorem 7.1. *In the setting of the last theorem, let $\tau_3(t)$ and $\tau_4(t)$ be defined by formula (7.5), and recalling that $(R_0f)(\lambda)=\{f(\lambda)-f(0)\}/\lambda$, let*

$$\text{(7.6)}\quad \widehat{y}_{11}^t(\lambda) = -\frac{1}{i}\left(\Phi_{11}^t(R_0e_{\tau_3(t)}I_p)\right)(\lambda)\ ,\ \widehat{y}_{12}^t(\lambda)=\frac{1}{i}(R_0b_3^t)(\lambda)$$

$$\text{(7.7)}\quad \widehat{y}_{21}^t(\lambda) = \frac{-1}{i}\left((\Phi_{22}^t)^*(R_0e_{-\tau_4(t)}I_p)\right)(\lambda)\ ,\ \widehat{y}_{22}^t(\lambda)=-\frac{1}{i}(R_0(b_4^t)^{-1})(\lambda)\ .$$

Then the RK $K_\omega^t(\lambda)$ of the RKHS $\mathcal{H}(A_t)$ evaluated at $\omega=0$ is given by the formula

$$\text{(7.8)}\qquad K_0^t(\lambda)=\frac{1}{2\pi}\begin{bmatrix}\widehat{x}_{11}^t(\lambda)+\widehat{x}_{21}^t(\lambda) & \widehat{x}_{12}^t(\lambda)+\widehat{x}_{22}^t(\lambda)\\ \widehat{u}_{11}^t(\lambda)+\widehat{u}_{21}^t(\lambda) & \widehat{u}_{12}^t(\lambda)+\widehat{u}_{22}^t(\lambda)\end{bmatrix}\ ,$$

where:

(1) *The $\widehat{u}_{ij}^t(\lambda)$ are $p\times p$ mvf's such that the columns of $\widehat{u}_{1j}^t(\lambda)$ and $\widehat{u}_{12}^t(\lambda)$ belong to $\mathcal{H}(b_3^t)$ and the columns of $\widehat{u}_{21}^t(\lambda)$ and $\widehat{u}_{22}^t(\lambda)$ belong to $\mathcal{H}_*(b_4^t)$. The $\widehat{u}_{ij}^t(\lambda)$ may be defined as the solutions of the systems of equations:*

$$\text{(7.9)}\qquad Y_1^t\widehat{u}_{1j}^t+\Phi_{12}^t\widehat{u}_{2j}^t = \widehat{y}_{1j}^t(\lambda)$$

$$\text{(7.10)}\qquad (\Phi_{12}^t)^*\widehat{u}_{1j}^t+Y_2^t\widehat{u}_{2j}^t = \widehat{y}_{2j}^t(\lambda)\ ,\ j=1,2\ .$$

(2) *The mvf's $\widehat{x}_{ij}^t(\lambda)$ are defined by the formulas*

$$\text{(7.11)}\qquad \begin{aligned}\widehat{x}_{1j}^t(\lambda) &= -(\Phi_{11}^t)^*\widehat{u}_{1j}^t\ ,\\ \widehat{x}_{2j}^t(\lambda) &= \Phi_{22}^t\widehat{u}_{2j}^t\ ,\ j=1,2\ .\end{aligned}$$

Remark 7.2. In the one-sided cases, when either $b_4^t(\lambda)=I_p$ or $b_3^t(\lambda)=I_p$, the formulas for recovering $M(t)$ are simpler. For example, if $b_4^t(\lambda)=I_p$, then $\tau_4(t)=0$ and $\mathcal{H}_*(b_4^t)=\{0\}$ and hence $\widehat{u}_{2j}^t=0,\widehat{x}_{2j}^t=0$, and the equation for equations $\widehat{u}_{1j}^t$ and the formula for $\widehat{x}_{1j}^t$ simplify to

$$\text{(7.12)}\qquad Y_1^t\widehat{u}_{1j}^t=\widehat{y}_{1j}^t(\lambda)\ ,\ j=1,2,$$

and

$$\text{(7.13)}\qquad \widehat{x}_{1j}^t=-(\Phi_{11}^t)^*\widehat{u}_{1j}^t\ ,\ j=1,2,$$

Theorem 7.3. *Let $\{c(\lambda);b_3^t(\lambda),b_4^t(\lambda),\ 0\le t<d\}$ be given, where $c\in\mathcal{C}^{p\times p}$ and $\{b_3^t(\lambda),b_4^t(\lambda)\}$, $0\le t<d$, is a normalized monotonic continuous chain of pairs of entire inner $p\times p$ mvf's and let assumption (6.11) be in force. Then the unique solution $M(t)$ of the inverse input impedance problem considered in Theorem 6.1 is given by the formula*

(7.14)

$$M(t)=2\pi K_0^t(0)=\int_0^{\tau_3(t)}\begin{bmatrix}x_{11}^t(a) & x_{12}^t(a)\\ u_{11}^t(a) & u_{12}^t(a)\end{bmatrix}da+\int_{-\tau_4(t)}^0\begin{bmatrix}x_{21}^t(a) & x_{22}^t(a)\\ u_{21}^t(a) & u_{22}^t(a)\end{bmatrix}da$$

and the corresponding matrizant may be defined by the formula

$$A_t(\lambda) = I_m + 2\pi i K_0^t(\lambda) J_p \ , \tag{7.15}$$

where $K_0^t(\lambda)$ *is specified by formula (7.8) and* $x_{ij}^t(a)$ *and* $u_{ij}^t(a)$ *designate the inverse Fourier transforms of* $\widehat{x}_{ij}^t(\lambda)$ *and* $\widehat{u}_{ij}^t(\lambda)$, *respectively.*

Remark 7.4. Upon writing L_t for L_{A_t}, Δ_t for Δ_{A_t}, and $K_\omega^t(\lambda)$ for $K_\omega^{A_t}(\lambda)$, the formula for the RK $K_0^t(\lambda)u = K_0^{A_t}(\lambda)u$ can be expressed in the form

$$K_0^t(\lambda)u = L_t\Delta_t^{-1}G_t u, \text{ where } G_t = \begin{bmatrix} -\Phi_{11}^t k_0^{b_5^t} & k_0^{b_3^t} \\ (\Phi_{22}^t)^* l_0^{b_6^t} & l_0^{b_4^t} \end{bmatrix},$$

for every $u \in \mathbb{C}^m$ and hence

$$\begin{aligned} u^* M(t) u = 2\pi \langle K_0^t u, K_0^t u\rangle_{\mathcal{H}(A_t)} &= 2\pi\langle L_t\Delta_t^{-1}G_t u, L_t\Delta_t^{-1}G_t u\rangle_{\mathcal{H}(A_t)} \\ &= 2\pi\langle \Delta_t^{-1}G_t u, G_t u\rangle_{st}. \end{aligned}$$

8. The de Branges spaces $\mathcal{B}(\mathfrak{E})$

We turn next to a class of RKHS's $\mathcal{B}(\mathfrak{E})$ that were introduced and exploited to study spectral problems for integral and differential systems by L. de Branges [Br1], [Br2]; for subsequent developments and applications, see also the references cited in the first section. In the spectral theory of integral and differential systems, the spaces $\mathcal{B}(\mathfrak{E})$ are spaces of entire vvf's. However, in other problems of analysis it is useful to consider spaces $\mathcal{B}(\mathfrak{E})$ of vvf's that are meromorphic in Ω_+. In this more general setting, the space $\mathcal{B}(\mathfrak{E})$ may be defined in terms of a $p \times 2p$ mvf

$$\mathfrak{E}(\lambda) = [E_-(\lambda) \quad E_+(\lambda)], \tag{8.1}$$

where $\{E_-(\lambda), E_+(\lambda)\}$ is a pair of $p \times p$ mvf's that are meromorphic in $\mathbb{C}_+$ and are such that

$$\det E_+(\lambda) \not\equiv 0 \quad \text{and} \quad \chi = E_+^{-1}E_- \text{ belongs to the class } \mathcal{S}_{in}^{p\times p}. \tag{8.2}$$

We shall call a pair of mvf's that meet this property a de Branges pair and shall refer to the corresponding $p \times 2p$ mvf $\mathfrak{E}(\lambda)$ as a de Branges function. With each such de Branges function, there is an associated RKHS $\mathcal{B}(\mathfrak{E})$ of $p \times 1$ vvf's with RK

$$K_\omega^{\mathfrak{E}}(\lambda) = -\frac{\mathfrak{E}(\lambda) j_p \mathfrak{E}(\omega)^*}{\rho_\omega(\lambda)} = \frac{E_+(\lambda)E_+(\omega)^* - E_-(\lambda)E_-(\omega)^*}{\rho_\omega(\lambda)}. \tag{8.3}$$

The formula

$$f \longrightarrow E_+^{-1} f$$

defines a unitary map from $\mathcal{B}(\mathfrak{E})$ onto $\mathcal{H}(\chi)$.

For the rest of this section we shall restrict attention to the case $\Omega_+ = \mathbb{C}_+$. In this setting, if $E_+(\lambda)$ tends to a limit $E_+(\mu)$ as λ tends non tangentially to

$\mu \in \mathbb{R}$ at a.e. point $\mu \in \mathbb{R}$ and $\det E_+(\mu) \neq 0$ a.e., then $f(\lambda)$ has nontangential limits $f(\mu)$ at a.e point $\mu \in \mathbb{R}$ for every $f \in \mathcal{B}(\mathfrak{E})$ and

$$\|f\|^2_{\mathcal{B}(\mathfrak{E})} = \|E_+^{-1}f\|^2_{st} = \int_{-\infty}^{\infty} f(\mu)^* \Delta_{\mathfrak{E}}(\mu) f(\mu) d\mu,$$

where

$$\Delta_{\mathfrak{E}}(\mu) = E_+^{-1}(\mu)^* E_+^{-1}(\mu). \tag{8.4}$$

Moreover,

$$\begin{aligned} f \in \mathcal{B}(\mathfrak{E}) \quad &\Longleftrightarrow \quad E_+^{-1} f \in \mathcal{H}(\chi) \\ &\Longleftrightarrow \quad E_-^{-1} f \in \mathcal{H}_*(\chi) \\ &\Longleftrightarrow \quad E_+^{-1} f \in H_2^p \text{ and } E_-^{-1} f \in K_2^p \end{aligned}$$

We turn next to de Branges spaces $\mathcal{B}(\mathfrak{E})$ based on mvf's $\mathfrak{E}(\lambda)$ that meet the following additional constraints:

$$0 \in \mathfrak{h}_{\mathfrak{E}}, E_+(0) = I_p, E_-(0) = I_p \text{ and } -i\chi'(0) > 0. \tag{8.5}$$

The significance of this extra assumption rests on the fact that

$$-i\chi'(0) > 0 \Longleftrightarrow K_0^{\mathfrak{E}}(0) > 0 \Longleftrightarrow \{f(0) : f \in \mathcal{B}(\mathfrak{E})\} = \mathbb{C}^p; \tag{8.6}$$

see [ArD7] for the justification. It is convenient to set

$$G_+(\lambda) = \frac{E_+(\lambda) - I_p}{\lambda} \quad \text{and} \quad G_-(\lambda) = \frac{E_-(\lambda) - I_p}{\lambda} \tag{8.7}$$

when these constraints are in effect.

Theorem 8.1. *Let $\{E_+, E_-\}$ be a de Branges pair that satisfies the extra constraints (8.5). Then the following conditions are equivalent:*

1. *The RKHS $\mathcal{B}(\mathfrak{E})$ is invariant under the action of the backwards shift operator*

$$R_0 : f \longrightarrow \frac{f(\lambda) - f(0)}{\lambda}. \tag{8.8}$$

2. *$G_+u \in \mathcal{B}(\mathfrak{E})$ and $G_-u \in \mathcal{B}(\mathfrak{E})$ for every $u \in \mathbb{C}^p$.*
3.

$$E_+ \in \Pi^{p\times p},\ E_+^{-1}G_+ \in H_2^{p\times p}, E_- \in \Pi^{p\times p} \text{ and } E_-^{-1}G_- \in K_2^{p\times p}. \tag{8.9}$$

4. *There exists an $m \times m$ mvf $A \in \mathcal{U}(J_p, \mathbb{C}_+)$ such that*

$$0 \in \mathfrak{h}_A, A(0) = I_m \text{ and } \mathfrak{E}(\lambda) = \sqrt{2}[0 \quad I_p]A(\lambda)\mathfrak{V}. \tag{8.10}$$

Proof. Suppose first that (1) is in force and that $g \in \mathcal{B}(\mathfrak{E})$. Then the vvf's $g(\lambda)$ and $h(\lambda) = (E_+^{-1}g)(\lambda)$ are both holomorphic at the point $\lambda = 0$ and the identity

$$\frac{h(\lambda) - h(0)}{\lambda} = E_+^{-1}(\lambda)(R_0 g)(\lambda) + \frac{E_+^{-1}(\lambda) - I_p}{\lambda} g(0)$$

implies that the second term on the right belongs to $\mathcal{H}(\chi)$. Moreover, in view of the constraint (8.5) and the observation (8.6), it follows that

$$\frac{E_+^{-1}(\lambda) - I_p}{\lambda} u \in \mathcal{H}(\chi)$$

for every choice of $u \in \mathbb{C}^p$.

Next, since $\chi^{-1}\mathcal{H}(\chi) = \mathcal{H}_*(\chi)$ and $\lambda^{-1}(\chi^{-1}(\lambda) - I_p)u \in \mathcal{H}_*(\chi)$, the identity

$$\chi^{-1}(\lambda) \left\{ \frac{E_+^{-1}(\lambda) - I_p}{\lambda} \right\} u = \frac{E_-^{-1}(\lambda) - I_p}{\lambda} u - \frac{\chi^{-1}(\lambda) - I_p}{\lambda} u$$

leads immediately to the conclusion that

$$\frac{E_+^{-1}(\lambda) - I_p}{\lambda} u \in \mathcal{H}(\chi) \Longleftrightarrow \frac{E_-^{-1}(\lambda) - I_p}{\lambda} u \in \mathcal{H}_*(\chi).$$

Therefore, since

$$\frac{E_+^{-1}(\lambda) - I_p}{\lambda} u \in \mathcal{H}(\chi) \Longleftrightarrow G_+ u \in \mathcal{B}(\mathfrak{E})$$

and

$$\frac{E_-^{-1}(\lambda) - I_p}{\lambda} u \in \mathcal{H}_*(\chi) \Longleftrightarrow G_- u \in \mathcal{B}(\mathfrak{E})$$

for every $u \in \mathbb{C}^p$ it is readily seen that (1) implies (2) and (2) implies (3).

Next, with the help of the evaluation

$$\langle \frac{h(\mu) - h(0)}{\mu}, f(\mu) \rangle_{st} = \langle h(\mu), \frac{f(\mu) - \chi(\mu) f(0)}{\mu} \rangle_{st},$$

which is valid for every choice of h and f in $\mathcal{H}(\chi)$, it may be shown that the adjoint of the backwards shift operator R_0 acting in $\mathcal{B}(\mathfrak{E})$ is given by the formula

$$\begin{aligned} (R_0^* g)(\lambda) &= (R_0 g)(\lambda) \\ &- i \left\{ G_-(\lambda) \frac{1}{2\pi} \int_{-\infty}^{\infty} G_-(\mu)^* \Delta_{\mathfrak{E}}(\mu) g(\mu) d\mu - G_+(\lambda) \frac{1}{2\pi} \int_{-\infty}^{\infty} G_+(\mu)^* \Delta_{\mathfrak{E}}(\mu) g(\mu) d\mu \right\}. \end{aligned}$$

Let

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \tag{8.11}$$

where

$$C_1 : g \in \mathcal{B}(\mathfrak{E}) \longrightarrow \frac{\sqrt{\pi}}{2\pi i} \int_{-\infty}^{\infty} (G_+(\mu) + G_-(\mu))^* \Delta_{\mathfrak{E}}(\mu) g(\mu) d\mu \tag{8.12}$$

$$\begin{aligned} C_2 : g \in \mathcal{B}(\mathfrak{E}) \longrightarrow & \frac{\sqrt{\pi}}{2\pi i} \int_{-\infty}^{\infty} (G_+(\mu) - G_-(\mu))^* \Delta_{\mathfrak{E}}(\mu) g(\mu) d\mu \\ & = \sqrt{\pi} g(0). \end{aligned} \tag{8.13}$$

Then

$$R_0 - R_0^* = i C^* J_p C \tag{8.14}$$

and hence the $m \times m$ mvf

(8.15) $$A_{\mathfrak{E}}(\lambda) = I_m + i\lambda C(I - \lambda R_0)^{-1}C^* J_p$$

satisfies the identity

(8.16) $$J_p - A_{\mathfrak{E}}(\lambda)J_p A_{\mathfrak{E}}(\omega)^* = -i(\lambda - \overline{\omega})C(I - \lambda R_0)^{-1}(I - \overline{\omega}R_0^*)^{-1}C^*.$$

Thus, $A_{\mathfrak{E}}(\lambda)$ belongs to the class $\mathcal{U}(J_p, \mathbb{C}_+)$ and meets the constraints (8.10). This completes the proof that (3) implies (4).

It remains only to prove that (4) implies (1), but that is selfevident. □

Remark 8.2. The mvf $A(\lambda) = A_{\mathfrak{E}}(\lambda)$ that was constructed in the proof of the last theorem satisfies the condition

(8.17) $$T_A[I_p] = i\alpha + \frac{1}{i\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\lambda - \mu} - \frac{\mu}{1 + \mu^2}\right)\Delta_{\mathfrak{E}}(\mu)d\mu$$

for some Hermitian matrix $\alpha \in \mathbb{C}^{p\times p}$. It is in fact the only mvf in the class $\mathcal{U}(J_p, \mathbb{C}_+)$ that meets both (8.10) and this last condition.

We shall say that a de Branges function $\mathfrak{E}(\lambda)$ is *regular* if it meets both of the conditions (8.5) and (8.9). Correspondingly, we shall say that the de Branges space $\mathcal{B}(\mathfrak{E})$ is *regular* if $\mathfrak{E}(\lambda)$ meets the constraint (8.5) and $\mathcal{B}(\mathfrak{E})$ is invariant under the backwards shift operator R_0. In view of the last theorem, the $p \times 2p$ mvf $\mathfrak{E}(\lambda)$ is regular if and only if the space $\mathcal{B}(\mathfrak{E})$ is regular. The class of regular de Branges spaces $\mathcal{B}(\mathfrak{E})$ of entire vvf's play a significant role in the spectral theory of integral and differential systems.

We now introduce a pair of generalized Fourier transforms on regular de Branges spaces $\mathcal{B}(\mathfrak{E})$ by the formulas

$$\begin{aligned}\mathcal{F}_r(\lambda] &= \frac{1}{\sqrt{2\pi}}C(I - \lambda R_0)^{-1}g \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{2\pi i}\int_{-\infty}^{\infty}(G_+(\mu) + G_-(\mu))^*\,\Delta_{\mathfrak{E}}(\mu)\frac{\lambda g(\lambda) - \mu g(\mu)}{\lambda - \mu}d\mu \\ g(\lambda)\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_l(\lambda) &= \frac{1}{\sqrt{2\pi}}C(I + \lambda R_0^*)^{-1}g \\ &= A_{\mathfrak{E}}^{\sim}(\lambda)\mathcal{F}_r(-\lambda).\end{aligned}$$

The identities (8.16) and

(8.18) $$J_p - A_{\mathfrak{E}}^{\sim}(\lambda)J_p A_{\mathfrak{E}}^{\sim}(\omega)^* = -i(\lambda - \overline{\omega})C(I - \lambda R_0)^{-1}(I - \overline{\omega}R_0^*)^{-1}C^*.$$

imply that $\mathcal{F}_r$ is a unitary map from $\mathcal{B}(\mathfrak{E})$ onto $\mathcal{H}(A_{\mathfrak{E}})$ and that $\mathcal{F}_l$ is a unitary map from $\mathcal{B}(\mathfrak{E})$ onto $\mathcal{H}(A_{\mathfrak{E}}^{\sim})$. Consequently, $\mathcal{F}_r g \in \Pi^m$ and $\mathcal{F}_l g \in \Pi^m$ for every $g \in \mathcal{B}(\mathfrak{E})$. Therefore, both transforms have boundary values for a.e. $\mu \in \mathbb{R}$.

We shall say that a regular de Branges function $\mathfrak{E}(\lambda)$ is *right strongly regular* if $\mathcal{F}_r g \in L_2^m$ for every $g \in \mathcal{B}(\mathfrak{E})$ and that it is *left strongly regular* if $\mathcal{F}_l g \in L_2^m(\mathbb{R})$

for every $g \in \mathcal{B}(\mathfrak{E})$. Correspondingly, we shall say that the space $\mathcal{B}(\mathfrak{E})$ is left or right strongly regular according as $\mathfrak{E}(\lambda)$ is left or right strongly regular.

Theorem 8.3. *Let $\mathfrak{E}(\lambda)$ be a regular de Branges function. Then the following statements are equivalent:*

1. *$\mathfrak{E}(\lambda)$ is right strongly regular.*
2. *$\{\mathcal{F}_r g : g \in \mathcal{B}(\mathfrak{E})$ is a closed subspace of $L_2^m(\mathbb{R})$.*
3. *There exist a pair of positive constants γ_1 and γ_2 such that*

$$\gamma_1 \|\mathcal{F}_r g\|_{st} \le \|g\|_{\mathcal{B}(\mathfrak{E})} \le \gamma_2 \|\mathcal{F}_r g\|_{st}.$$

Proof. In the given setting, the generalizef transform $\mathcal{F}_r$ is a unitary operator from $\mathcal{B}(\mathfrak{E})$ onto $\mathcal{H}(A_{\mathfrak{E}})$ and $\mathfrak{E}(\lambda)$ is right strongly regular if and only if $A_{\mathfrak{E}} \in \mathcal{U}_{sR}(J_p)$. Thus the theorem is equivalent to Theorem 5.1. □

If $\mathfrak{E}(\lambda)$ is right strongly regular, then, as follows readily from the definition of $\mathcal{F}_r g$ and a well known theorem of Banach, there exist a pair of positive constants α_1 and α_2 such that

$$\alpha_1 \|g\|_{st} \le \|g\|_{\mathcal{B}(\mathfrak{E})} \le \alpha_2 \|g\|_{st}.$$

Thus, in this setting, $\mathcal{B}(\mathfrak{E})$ is a closed subspace of $L_2^p(\mathbb{R})$ (with respect to Lebesgue measure). In fact,

$$\mathcal{B}(\mathfrak{E}) = \mathcal{H}(b_3) \oplus \mathcal{H}_*(b_4), \tag{8.19}$$

with scalar product

$$\langle g, h \rangle_{\mathcal{B}(\mathfrak{E})} = \int_{-\infty}^{\infty} h(\mu)^* \Delta_{\mathfrak{E}}(\mu) f(\mu) d\mu, \tag{8.20}$$

where the pair $\{b_3, b_4\}$ of inner $p \times p$ mvf's may be obtained from the inner-outer and outer-inner factorizations

$$(E_-^{\#})^{-1} = b_3 \varphi_3 \quad \text{and} \quad E_+^{-1} = \varphi_4 b_4,$$

respectively.

We remark that if $\mathfrak{E}(\lambda)$ is a regular de Branges function, then the generalized transform $\mathcal{F}_l$ is a unitary operator from $\mathcal{B}(\mathfrak{E})$ onto $\mathcal{H}(A_{\mathfrak{E}}^{\sim})$ and

$$\mathfrak{E}(\lambda) \text{ is left strongly regular} \iff A_{\mathfrak{E}} \in \mathcal{U}_{lsR}(J_p) \iff A_{\mathfrak{E}}^{\sim} \in \mathcal{U}_{rsR}(J_p).$$

Thus, an application of Theorem 5.1 to $A_{\mathfrak{E}}^{\sim}$ yields the following result:

Theorem 8.4. *Let $\mathfrak{E}(\lambda)$ be a regular de Branges function. Then the following statements are equivalent:*

1. *$\mathfrak{E}(\lambda)$ is left strongly regular.*
2. *$\{\mathcal{F}_l g : g \in \mathcal{B}(\mathfrak{E})$ is a closed subspace of $L_2^m(\mathbb{R})$.*
3. *There exist a pair of positive constants γ_1 and γ_2 such that*

$$\gamma_1 \|\mathcal{F}_l g\|_{st} \le \|g\|_{\mathcal{B}(\mathfrak{E})} \le \gamma_2 \|\mathcal{F}_l g\|_{st}.$$

Theorem 8.5. *Let $\mathfrak{E}(\lambda)$ be a regular de Branges function. Then $\mathfrak{E}(\lambda)$ is left strongly regular if and only if the density $\Delta_{\mathfrak{E}}(\mu)$ satisfies the Treil-Volberg matrix version of the Muckenhoupt (A_2) condition.*

Proof. This is immediate from Theorem 7.5 in [ArD10]. □

9. Operator nodes with characteristic matrix functions of the class $\mathcal{U}_{sR}(J)$

Let $\mathcal{U}_0(J) = \mathcal{U}_0(J, \Omega_+)$ be the subclass of mvf's $U(\lambda)$ from $\mathcal{U}(J, \Omega_+)$ that are holomorphic at the point $\lambda = 0$ and,in addition, are normalized by the condition $U(0) = I_m$ when $\Omega_+ = \mathbb{C}_+$. It is known that every mvf $U(\lambda)$ of the class $\mathcal{U}_0(J)$ may be represented as the characteristic matrix function $U_\Sigma(\lambda)$ of a simple operator node Σ that is a J-unitary node if $\Omega_+ = \mathbb{D}_+$ and is an LB (Livsic-Brodskii) J-node, if $\Omega_+ = \mathbb{C}_+$.

We recall that a colligation $\Sigma = (A, C; X, Y)$ is called an LB J-node if $J \in \mathcal{L}(Y)$ is a signature operator, $A \in \mathcal{L}(X)$, $C \in \mathcal{L}(X, Y)$ and

$$A - A^* = iC^*JC. \tag{9.1}$$

Such a node Σ is called simple if

$$\bigcap_{n \geq 0} \text{Ker } CA^n = \{0\}. \tag{9.2}$$

The function

$$U_\Sigma(\lambda) = I + i\lambda C(I - \lambda A)^{-1}C^*J \tag{9.3}$$

is called the *characteristic function* of the LB J-node Σ. Two LB J-nodes $\Sigma_i = (A_i, C_i; X_i, Y_i)$ $(i = 1, 2)$ are said to be *unitarily equivalent* if

$$A_1 = R^{-1}A_2R \quad \text{and} \quad C_1 = C_2R, \tag{9.4}$$

where $R \in \mathcal{L}(X_1, X_2)$ is a unitary operator. It is known that if the characteristic functions $U_{\Sigma_1}(\lambda)$ and $U_{\Sigma_2}(\lambda)$ of two simple LB J-nodes Σ_1 and Σ_2 coincide in a neighbourhood of zero, then these nodes are unitarily equivalent. If $m = \dim Y < \infty$, then the operators from $\mathcal{L}(Y)$ may be defined in terms of an orthonormal basis in Y by $m \times m$ matrices and then the corresponding mvf $U_\Sigma(\lambda)$ is called the *characteristic matrix function* of Σ. In this case $\mathbb{C}^m$ may be considered instead of Y and then J and $U_\Sigma(\lambda)$ denote $m \times m$ matrices that define the corresponding operators from $\mathcal{L}(\mathbb{C}^m)$ in the standard basis.

An LB J-node is said to be *dissipative (accumulative)*, if $J = I$ $(J = -I$, respectively).

From now on we restrict attention to simple LB-J nodes Σ with characteristic function $U_\Sigma \in \mathcal{U}_0(J, \mathbb{C}_+)$.

The supplementary identities

$$K_\omega^{U_\Sigma}(\lambda) = \frac{1}{2\pi}C(I - \lambda A)^{-1}(I - \overline{\omega}A^*)^{-1}C^* \tag{9.5}$$

and

$$K_\omega^{\widetilde{U_\Sigma}}(\lambda) = \frac{1}{2\pi}C(I + \lambda A^*)^{-1}(I + \overline{\omega}A)^{-1}C^* \tag{9.6}$$

follow easily from the formulas (9.1) and (9.4). These identities imply that the two generalized Fourier transforms

$$(\mathcal{F}_r x)(\lambda) = \frac{1}{\sqrt{2\pi}} C(I - \lambda A)^{-1} x \quad \text{and} \quad (\mathcal{F}_l x)(\lambda) = \frac{1}{\sqrt{2\pi}} C(I + \lambda A^*)^{-1} x \tag{9.7}$$

are unitary maps from X onto $\mathcal{H}(U_\Sigma)$ and from X onto $\mathcal{H}(U_{\widetilde{\Sigma}})$, respectively. Moreover, the simple LB-J node $\Sigma = (A, C; X, \mathbb{C}^m)$ with characteristic function $U_\Sigma(\lambda) \in \mathcal{U}(J, \mathbb{C}_+)$ and the functional model $\overset{o}{\Sigma} = (R_0, C_0; \mathcal{H}(U_\Sigma), \mathbb{C}^m)$ based on the operators

$$R_0 : f \in \mathcal{H}(U_\Sigma) \quad \longrightarrow \quad \frac{f(\lambda) - f(0)}{\lambda} \tag{9.8}$$

$$C_0 : f \in \mathcal{H}(U_\Sigma) \quad \longrightarrow \quad \sqrt{2\pi} f(0) \in \mathbb{C}^m : \tag{9.9}$$

are unitarily equivalent:

$$A = (\mathcal{F}_r)^{-1} R_0 \mathcal{F}_r \quad \text{and} \quad C = C_0 \mathcal{F}_r.$$

In much the same way the transform $\mathcal{F}_l$ defines a unitary equivalence between the simple LB-J node $\widetilde{\Sigma} = (-A^*, C; X, \mathbb{C}^m)$ with characteristic function $U_{\widetilde{\Sigma}} \in \mathcal{U}(J, \mathbb{C}_+)$ and the functional model based on $\mathcal{H}(U_{\widetilde{\Sigma}})$.

If $J = J_p$ and the characteristic function $A_\Sigma(\lambda)$ of the simple LB-J_p node $\Sigma = (A, C; X, \mathbb{C}^m)$ belongs to the class $\mathcal{U}_0(J_p, \mathbb{C}_+)$ and is such that the mvf

$$c_0(\lambda) = (T_{A_\Sigma}[I_p])(\lambda) \tag{9.10}$$

admits a representation of the form

$$c_0(\lambda) = i\alpha + \frac{1}{\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{\mu - \lambda} - \frac{\mu}{1 + \mu^2} \right) \Delta_{\mathfrak{E}}(\mu) d\mu \tag{9.11}$$

for some $p \times p$ Hermitian matrix α, then

$$\mathfrak{E}(\lambda) = \sqrt{2}[I_p \quad 0] A_\Sigma(\lambda)$$

is a regular de Branges matrix and

$$A_\Sigma(\lambda) = A_{\mathfrak{E}}(\lambda),$$

where $A_{\mathfrak{E}}(\lambda)$ is the mvf that was constructed in the proof of Theorem 8.1. This in turn provides another functional model $\Sigma_{\mathfrak{E}} = (R_0, C; \mathcal{B}(\mathfrak{E}), \mathbb{C}^m)$ of the simple LB-J_p node Σ, where R_0 now designates the backwards shift operator acting in $\mathcal{B}(\mathfrak{E})$ and the operator C is defined by formulas (8.11)-(8.13). This functional model is easily obtained from the model $\overset{o}{\Sigma}$ since the map

$$f \longrightarrow \sqrt{2}[0 \quad I_p] f$$

is a unitary operator that carries the backwards shift R_0 in $\mathcal{H}(A_\Sigma)$ into the shift acting in $\mathcal{B}(\mathfrak{E})$. If the mvf $c_0(\lambda)$ that is defined in (9.10) meets the condition (9.11), then the generalized Fourier transform

$$\mathcal{F}_r^{(2)} x = \frac{1}{\sqrt{2\pi}} [0 \quad I_p] C(I - \lambda A)^{-1} x$$

is a unitary operator from X onto $\mathcal{B}(\mathfrak{E})$. This serves to establish the unitary equivalence of the simple LB-J_p node Σ with the functional model $\Sigma_{\mathfrak{E}}$.

We remark that

$$\text{(9.11) holds} \iff \bigcap_{n=0}^{\infty}[0 \quad I_p]C \ker A^n = \{0\}; \tag{9.12}$$

see [ArD8].

Theorem 9.1. *Let $\Sigma = (A, C; X, \mathbb{C}^m)$ be a simple LB J_p-node with a characteristic matrix function $U_\Sigma(\lambda)$ that is a right strongly regular J_p-inner mvf with respect to $\mathbb{C}_+$. Let*

$$\{b_3, b_4\} \in ap_{\mathrm{II}}(U_\Sigma) \quad \text{and} \quad b_2(0) = b_4(0) = I_p. \tag{9.13}$$

Let $\overset{o}{\Sigma}_+ = (\overset{o}{A}_+, \overset{o}{C}_+; \overset{o}{X}_+, \mathbb{C}^p)$ and $\overset{o}{\Sigma}_- = (\overset{o}{A}_-, \overset{o}{C}_-; \overset{o}{X}_-, \mathbb{C}^p)$ be simple dissipative and simple accumulative LB nodes with characteristic matrix functions $b_3(\lambda)$ and $b_4^{-1}(\lambda)$, respectively. Then

$$A = R^{-1}\begin{bmatrix} \overset{o}{A}_+ & 0 \\ 0 & \overset{o}{A}_- \end{bmatrix} R, \qquad [0_{p\times p} \; I_p]C = [\overset{o}{C}_+ \;\; \overset{o}{C}_-]R, \tag{9.14}$$

for some R that satisfies the conditions

$$R \in \mathcal{L}(X, \overset{o}{X}_+ \oplus \overset{o}{X}_-), \qquad R^{-1} \in \mathcal{L}(\overset{o}{X}_+ \oplus \overset{o}{X}_-, X). \tag{9.15}$$

Moreover,

$$\overset{o}{X}_- = \{0\} \iff b_4(\lambda) \equiv I_p \iff U_\Sigma \in \mathcal{N}_+^{m\times m}$$

and

$$\overset{o}{X}_+ = \{0\} \iff b_3(\lambda) \equiv I_p \iff U_\Sigma \in \mathcal{N}_-^{m\times m}.$$

Proof. The given simple LB nodes $\Sigma, \overset{o}{\Sigma}_+, \overset{o}{\Sigma}_-$ can be replaced by their functional models. In these models, $A, \overset{o}{A}_-, \overset{o}{A}_+$, are backwards shifts in the spaces $X = \mathcal{H}(U_\Sigma), \overset{o}{X}_+ = \mathcal{H}(b_3), \overset{o}{X}_- = \mathcal{H}_*(b_4)$, respectively and the operators $C, \overset{o}{C}_+, \overset{o}{C}_-$ map a vvf g from their respective domains $X, \overset{o}{X}_+, \overset{o}{X}_-$ into $\sqrt{2\pi}g(0)$. In the given setting, Theorem 5.4 is applicable to the mvf $A(\lambda) = U_\Sigma(\lambda)$ and the operator $R = (L_A)^{-1}$ satisfies the stated assertions. □

Theorem 9.2. [Ar]. *Let $\Sigma = (A, C; X, \mathbb{C}^m)$ be an LB j_{pq}-node with characteristic matrix function $U_\Sigma(\lambda) = W(\lambda)$ that is j_{pq}-inner with respect to $\mathbb{C}_+$. Let*

$$\{b_1, b_2\} \in ap\,(W), \quad b_1(0) = I_p, \quad \text{and} \quad b_2(0) = I_q. \tag{9.16}$$

1. *Let Σ be a simple node and $W \in \mathcal{U}_{sR}(j_{pq})$. Let $\overset{o}{\Sigma}_+ = (\overset{o}{A}_+, \overset{o}{C}_+; \overset{o}{X}_+, \mathbb{C}^p)$ be a simple dissipative LB node with characteristic matrix function $b_1(\lambda)$ and $\overset{o}{\Sigma}_- =$*

$(\overset{o}{A}_-, \overset{o}{C}_-; \overset{o}{X}_-, \mathbb{C}^q)$ *be a simple accumulative LB node with characteristic matrix function* $b_2(\lambda)^{-1}$. *Then*

$$A = R^{-1}\begin{pmatrix} \overset{o}{A}_+ & 0 \\ 0 & \overset{o}{A}_- \end{pmatrix} R, \qquad \overset{o}{C}_\pm = P_\pm C R^{-1}|\, \overset{o}{X}_\pm \tag{9.17}$$

for some R *such that*

$$R \in \mathcal{L}(X, \overset{o}{X}_+ \oplus \overset{o}{X}_-), \qquad R^{-1} \in \mathcal{L}(\overset{o}{X}_+ \oplus \overset{o}{X}_-, X), \tag{9.18}$$

where $P_+ = \mathrm{diag}\{I_p, 0_{q\times q}\}$ *and* $P_- = \mathrm{diag}\{0_{p\times p}, I_q\}$. *Moreover,*

$$\overset{o}{X}_- = \{0\} \Longleftrightarrow b_2(\lambda) \equiv I_q \Longleftrightarrow W \in \mathcal{N}_+^{m\times m}(\mathbb{C}_+)$$

and

$$\overset{o}{X}_+ = \{0\} \Longleftrightarrow b_1(\lambda) \equiv I_p \Longleftrightarrow W \in \mathcal{N}_-^{m\times m}(\mathbb{C}_-).$$

2. *Conversely, if* $\overset{o}{\Sigma}_+ = (\overset{o}{A}_+, \overset{o}{C}_+; \overset{o}{X}_+, \mathbb{C}^p)$ *is a dissipative LB node and if* $\overset{o}{\Sigma}_- = (\overset{o}{A}_-, \overset{o}{C}_-; \overset{o}{X}_-, \mathbb{C}^q)$ *is an accumulative LB node such that* (9.17) *and* (9.18) *hold, then* $W \in \mathcal{U}_{sR}(j_{pq})$.

Proof. A proof of the first statement may be based on the description of the space $\mathcal{H}(W)$ for $W \in \mathcal{U}_{sR}(j_{pq})$ that is furnished in [ArD4], in much the same way that the last theorem was verified.

The verification of the second assertion rests on the fact that the generalized right Fourier transforms based on the nodes $\overset{o}{\Sigma}_+$ and $\overset{o}{\Sigma}_-$ map into L_2^p and L_2^q, respectively and the fact that if

$$f = \begin{bmatrix} g \\ h \end{bmatrix} \in \mathcal{H}(W) \text{ and } W \in \mathcal{U}(j_{pq}),$$

then

$$f \in L_2^m \Longleftrightarrow g \in L_2^p \Longleftrightarrow h \in L_2^q.$$

□

If $U_\Sigma \in \mathcal{E} \cap \mathcal{U}(J)$, i.e., if the simple node Σ is a Volterra node, then the $b_j(\lambda)$ are entire inner functions and consequently the nodes $\overset{o}{\Sigma}_+$ and $\overset{o}{\Sigma}_-$ considered in the preceding two theorems are Volterra. If $m = 2$, then $b_1(\lambda) = b_3(\lambda) = e^{i\lambda\tau_3}$ and $b_2(\lambda) = b_4(\lambda) = e^{i\lambda\tau_4}$, where τ_3 and τ_4 are nonnegative numbers that may be computed by formula (7.5) with $b_j(\lambda)$ in place of $b_j^t(\lambda)$ for $j = 3$ and $j = 4$. The functional models based on the backwards shift acting in $\mathcal{H}(b_3)$ and $\mathcal{H}_*(b_4)$ are of course still applicable. However, since $\mathcal{H}(b_3) = L_2([0,\tau_3])^\wedge$ and $\mathcal{H}_*(b_4) = L_2([-\tau_4, 0])^\wedge$, the identities

$$h(\lambda) = \int_0^{\tau_3} e^{i\lambda a} h^\vee(a)da \Longrightarrow (R_0 h)(\lambda) = i\int_0^{\tau_3} e^{i\lambda b}\left(\int_b^{\tau_3} h^\vee(a)da\right)db$$

and

$$h(\lambda) = \int_{-\tau_4}^{0} e^{i\lambda a} h^{\vee}(a)da \Longrightarrow (R_0 h)(\lambda) = i\int_{-\tau_4}^{0} e^{i\lambda b}\left(\int_{-\tau_4}^{b} h^{\vee}(a)da\right)db$$

lead to well known functional models based on integration operators acting in the indicated subspaces of L_2. In particular, the functional model of a simple dissipative Volterra node with characteristic function $e^{i\lambda\tau_3}$ may be chosen equal to $\overset{o}{\Sigma}_+ = (\overset{o}{A}_+, \overset{o}{C}_+; L_2([0,\tau_3]), \mathbb{C})$, where

$$(\overset{o}{A}_+ u)(t) = i\int_t^{\tau_3} u(a)da \text{ and } \overset{o}{C}_+ u = \int_0^{\tau_3} u(a)da \text{ for } u \in L_2([0,\tau_3]).$$

In much the same way, the functional model of a simple accumulative Volterra node with characteristic function $e^{-i\lambda\tau_4}$ may be chosen equal to the node $\overset{o}{\Sigma}_- = (\overset{o}{A}_-, \overset{o}{C}_-; L_2([-\tau_4, 0])), \mathbb{C})$, where

$$(\overset{o}{A}_- u)(t) = i\int_{-\tau_4}^{t} u(a)da \text{ and } \overset{o}{C}_- u = \int_{-\tau_4}^{0} u(a)da \text{ for } u \in L_2([-\tau_4, 0])).$$

Correspondingly, the operator R considered in the preceding two theorems acts from X onto $L_2([0,\tau_3]) \oplus L_2([-\tau_4, 0]$.

There exist analogues of the preceding two theorems for the case $U_\Sigma \in \mathcal{U}_{lsR}(J)$ that may be obtained by applying the preceding results to $U_\Sigma^\sim$ and recalling that

$$U_\Sigma \in \mathcal{U}_{lsR}(J) \Longleftrightarrow U_\Sigma^\sim \in \mathcal{U}_{rsR}(J).$$

Analogs of Theorem 9.2 (see [Ar]) and of Theorem 9.1 may also be obtained for J-unitary nodes with characteristic matrix functions of the class $\mathcal{U}_{sR}(J, \mathbb{D}_+)$.

The properties of left strongly regular spaces $\mathcal{B}(\mathfrak{E})$ and of operators in these spaces related to the backwards shift R_0 were studied by other methods in the case that $E_+(\lambda)$ and $E_-(\lambda)$ are scalar entire functions by G.M. Gubreev [Gu], as an application of his theory of regular quasiexponentials. In particular, he noted the connection between the class $\mathcal{U}_{lsR}(J_p)$ and the class of left strongly regular de Branges spaces $\mathcal{B}(\mathfrak{E})$ when $p = 1$ and $\mathfrak{E}(\lambda)$ is entire. Some of his results may be obtained from the analysis in the last two sections.

Analogues of the last two theorems on LB J-nodes and J-unitary nodes with strongly regular J-inner characteristic matrix functions were obtained by Z. Arova in her PhD thesis, see also [Ara2]. Her characterizations of the class of simple operator nodes Σ with characteristic matrix function $U_\Sigma \in \mathcal{U}_{rsR}(J)$ used somewhat different nodes than were exhibited here. Thus, for example, in place of the relations (9.17), she used the relations

$$A = R^{-1}\begin{pmatrix} \overset{o}{A}_+ & 0 \\ 0 & \overset{o}{A}_- \end{pmatrix} R, \qquad \overset{o}{C}_\pm = CR^{-1}|\, \overset{o}{X}_\pm, \tag{9.19}$$

where $\overset{o}{\Sigma}_+ = (\overset{o}{A}_+, \overset{o}{C}_+; \overset{o}{X}_+, \mathbb{C}^m)$ and $\overset{o}{\Sigma}_- = (\overset{o}{A}_-, \overset{o}{C}_-; \overset{o}{X}_-, \mathbb{C}^m)$ are simple dissipative and simple accumulative LB nodes, respectively.

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