$L_2(\Sigma)$-REGULARITY OF THE BOUNDARY $\to$ BOUNDARY OPERATOR B^*L FOR HYPERBOLIC AND PETROWSKI PDES

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REPORT No. 05, 2002/2003, spring

ISSN 1103-467X
ISRN IML-R- -05-02/03- -SE+spring

INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

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Abstract

This paper takes up and thoroughly analyzes a technical mathematical issue in PDE theory, while—as a by-pass product—making a larger case. The technical issue is the $L_2(\Sigma)$-regularity of the boundary $\to$ boundary operator B^*L for (multidimensional) hyperbolic and Petrowski-type mixed PDEs problems, where L is the boundary input $\to$ interior solution operator, and B is the control operator from the boundary. Both positive and negative classes of distinctive PDE illustrations are exhibited and proved. The larger case to be made is that hard analysis PDE energy methods are the tools of the trade—not soft analysis methods. Not only to analyze B^*L, but also to establish three inter-related cardinal results: optimal PDE regularity, exact controllability, and uniform stabilization. Thus, the paper takes a critical view on a spate of 'abstract' results in "infinite-dimensional systems theory," generated by unnecessarily complicated and highly limited 'soft' methods, with no apparent awareness on both the high degree of restriction of the abstract assumptions made—far from necessary—as well as on how to verify them in the case of multidimensional dynamical systems such as PDEs.

Contents

1 An historical overview: Hard analysis beats soft analysis on regularity, exact controllability, and uniform stabilization of hyperbolic and Petrowski-type PDEs under boundary control

At first, naturally, PDEs boundary control theory for evolution equations tackled the most established of the PDE classes—parabolic PDEs—whose Hilbert space theory for mixed problems was already available in close to an optimal book-form [Lio.1], [L-M.1] since the early '70s.

Next, in the early '80s, when the study of boundary control problems for (linear) PDEs began to address hyperbolic and Petrowski-type systems on a multidimensional bounded domain [L-T.3], [F-L-T.1] (see books [B-D-D-M.1], [L-T.13], [L-T.21] for overview), it faced at the outset an altogether new and fundamental obstacle, which was bound to hamper any progress. Namely, that an optimal, or even sharp, theory on the preliminary, foundational questions of well-posedness and global regularity (both in the interior and on the boundary, for the relevant solution traces) was generally missing in the PDEs literature of Mixed (Initial and Boundary Value) Problems for hyperbolic and Petrowski-type systems [Lio.1]. Available results were often explicitly recognized as definitely non-optimal [L-M.1, vol. 2, p. 141].

Hard analysis energy methods. A happy and quite challenging exception was the optimal—both interior and boundary—regularity theory for mixed, non-symmetric, non-characteristic first-order hyperbolic systems, culminated through repeated efforts in the early '70s [Kr.1], [Ral.1], [Rau.1]. Its final, full success required eventually the use of pseudo-differential energy methods (Kreiss' symmetrizer). Apart from this isolated case, mathematical knowledge of global optimal regularity theory of hyperbolic and Petrowski-type mixed problems was scarce, save for some trivial one-dimensional cases. Thus, in this gloomy scenario, one may say that optimal control theory [L-T.3], [F-L-T.1], [Lio.1] provided a forceful impetus in seeking to attain an optimal global regularity theory for these classes of mixed PDEs problems. To this end, PDEs (hard analysis) energy methods—both in differential and pseudo-differential form—were introduced and brought to bear on these problems. The case of second-order hyperbolic equations under Dirichlet boundary control was tackled first. The resulting theory that turns out to be optimal and does not depend on the space dimension [L-T.1], [L-T.2], [L-T.20], [L-L-T.1], [Lio.2]. It was best achieved by the use of energy methods in differential form. The case of second-order hyperbolic equations, this time under Neumann boundary control, proved far more recalcitrant and challenging (in space dimension strictly greater than one), and was conducted in a few phases. The additional degree of difficulties for this mixed PDE class stems from the fact that the Lopatinski condition is not satisfied for it. Unlike the Dirichlet's, the Neumann boundary control case requires pseudo-differential analysis. Final results depend on the geometry [L-T.9], [L-T.11], [L-T.12], [L-T.20], [Ta.2].

Naturally, in investigative efforts which moved either in a parallel or in a serial mode, the conceptual and computational 'tricks' that had proved successful in obtaining an optimal, or sharp, regularity theory for second-order hyperbolic equations, were exported, with suitable variations and adaptations, to certain Petrowski-type systems, see, e.g., references below. The lessons learned with second-order equations served as a guide and a benchmark study for these other classes. To be sure, not all cases have been, to date, completely resolved. The problem of optimal regularity of some Petrowski-systems with "high" boundary operators is not yet fully solved. However, a large body of optimal regularity theory has by now emerged, dealing with systems such as: Schrödinger equations;

plate-like equations of both hyperbolic (Kirchhoff model) and non-hyperbolic type (Euler-Bernoulli model), etc. Subsequently, additional more complicated dynamics followed, such as: system of elasticity, Maxwell equations, dynamic shell equations, etc. Shared by all these endeavors, there is one common loud message: that hard analysis energy methods have been responsible for the resulting successes. A rather broad account of these issues under one cover may be found in [L-T.13], [Lio.3], [L-T.20], [L-T.21, vol. 2], etc.

Abstract models of PDEs mixed problems. Simultaneously, and in parallel fashion, the aforementioned investigative efforts since the mid-70's also produced 'abstract models' for mixed PDE problems, subject to control either acting on the boundary of, or else as a point control within, a multidimensional bounded domain: [Bal.1], [W.1-2] for parabolic problems; [Tr.3], [L-T.1], [L-T.2] for hyperbolic problems. Though, in particular, operators arising in the abstract model depend on both the specific class of PDEs and on its specific homogeneous and non-homogeneous B.C., one cardinal point reached in this line of investigation was the following discovery: that most of them—but by no means all [E-L-T.1], [L-L-P.1], [Tr.8]—are encompassed and captured by the abstract model:

$$\dot{y} = Ay + Bu, \text{ in } [\mathcal{D}(A^*)]', \qquad y(0) = y_0 \in Y, \tag{1.1}$$

where U and Y are, respectively, control and state Hilbert spaces, and where:

(i) the operator $A : Y \supset \mathcal{D}(A) \to Y$ is the infinitesimal generator of a strongly continuous (s.c.) semigroup e^{At} on Y, $t \geq 0$;

(ii) B is an 'unbounded' operator $U \to Y$ satisfying $B \in \mathcal{L}(U; [\mathcal{D}(A^*)]')$ or equivalently, $A^{-1}B \in \mathcal{L}(U;Y)$. Above, as well as in (1.1), $[\mathcal{D}(A^*)]'$ denotes the dual space with respect to the pivot space Y, of the domain $\mathcal{D}(A^*)$ of the Y-adjoint A^* of A. W.l.o.g. we take $A^{-1} \in \mathcal{L}(Y)$).

Many examples of these abstract models are given under one cover in [B-D-D-M.1], [L-T.13], [L-T.21, Vols. 1-2]; they include the case of first-order hyperbolic systems quoted before, where again the need for an abstract model came from boundary PDE control theory, and was not available in the purely PDE theory *per se*. See Section 4.1 below. Accordingly, having accomplished a first abstract unification of many dynamical PDEs mixed problems, it was natural to attempt to extract—wherever possible—additional, more in-depth, common 'abstract properties,' shared by sufficiently many classes of PDE mixed problems. For the purpose of this note, we shall focus on three 'abstract properties': (optimal) regularity, exact controllability, and uniform stabilization.

Regularity. The variation of parameter formula for (1.1) is

$$y(t) = e^{At}y_0 + (Lu)(t); \tag{1.2a}$$

$$(Lu)(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau; \quad L_T u = (Lu)(T) = \int_0^T e^{A(T-t)}Bu(t)dt. \tag{1.2b}$$

Per se, the abstract differential equation (1.1) is not the critical object of investigation. It is good to have it, inasmuch as it yields (1.2). The key element that defines the crucial feature of a particular PDE mixed problem, is, however, the regularity of the operators L and L_T. This is what was referred to above as 'interior regularity': the control u acts on the boundary, while Lu is the corresponding solution acting in the interior. Accordingly, this pursued line of investigation brought about a second, abstract realization [L-T.1], [L-T.2], [L-T.3], [L-T.20]: that of determining the “best” function space Y for each class of mixed hyperbolic and Petrowski-type problems, such that the following interior regularity property holds true:

$$L: \text{ continuous } L_2(0,T;U) \to C([0,T];Y), \tag{1.3}$$

for one, hence for all positive, finite T. Presently, such space Y is explicitly identified in most (but by no means all) of the mixed PDE problems of hyperbolic or Petrowski type. [The case $Y = [\mathcal{D}(A^*)]'$ is always true in the present setting, and not much informative, save for offering a back-up result for (1.1).] An equivalent (dual) formulation is given in (1.4) below [L-T.2], [L-T.3], [F-L-T.1].

Hard beats soft on regularity. It is hard analysis that delivers the soft-expressed interior regularity result (1.3). For the mixed PDEs classes under considerations, achieving the regularity property (1.3) with the “best” function space Y is, as amply stressed above, NOT an accomplishment of soft-analysis methods (say, semigroup theory or cosine operator theory, which instead gives the lousy result of (1.3) with $Y = [\mathcal{D}(A^*)]'$, and, in fact, something “better” such as $[\mathcal{D}(A^{*\alpha})]'$ for some $0 < \alpha < 1$ depending on the equation and the boundary conditions [L-M.1], [L-T.1], but far from optimal). On the contrary, it is the accomplishment of hard analysis PDE energy methods, tuned to the specific combination of PDE and boundary control, which first produce, for each such individual combination, a PDE-estimate for the corresponding dual PDE problem. The precursor was the multidimensional wave equation with Dirichlet control [L-T.1], [L-T.2], [L-L-T.1]. All such *a-priori* estimates thus obtained on an individual basis admit the following ‘abstract version’:

$$L_T^* \equiv B^* e^{A^* t}: \text{ continuous } Y \to L_2(0,T;U), \tag{1.4}$$

where L_T is defined by (1.2b) [L-T.1], [L-T.2], [L-L-T.1].

In PDE mixed problems, property (1.4) is a (sharp) ‘trace regularity property’ of the boundary homogeneous problem, which is dual to the corresponding map L_T in (1.2b): from the $L_2(0,T;U)$-boundary control to the PDE solution at time T, see many examples in the books [L-T.13], [L-T.21]. Indeed, such PDE estimate is both non-trivial and unexpected, and typically yields a finite *gain* (often $\frac{1}{2}$) *in the space regularity* of the solution trace, which does NOT follow even by a formal application of trace theory to the optimal interior regularity of the PDE solution. Some PDE circles have come to call it “hidden regularity,” and with good reasons. It was first discovered in the case of the wave equation with Dirichlet control [L-T.2].

Only after the fact, if one so wishes, soft methods can be brought into the analysis to show that, in fact, the abstract trace regularity (1.4) is equivalent to the interior regularity property (1.3) [L-T.2], [L-T.3], [F-L-T.1]. [Needless to say, this can actually be done also on a case-by-case basis for each PDE class.] Thus, one key message is clear: that for all such questions of regularity of mixed PDE problems, the slogan ‘hard beats soft’ holds definitely true. It is hard analysis PDE energy methods (differential or pseudo-differential) that produce the key—and unexpected—*a-priori* estimates which shine within (1.4). Soft analysis then takes advantage of these single *a-priori* estimates into a common abstract formulation only afterwards, for the purpose of unification; for instance, in carrying out the study of optimal control theory with quadratic cost, etc. This is the spirit of abstract, unifying treatments of optimal control problems for PDE subject to boundary (and point) control, that can be found in books such as [L-T.13], [B-D-D-M.1], [L-T.21, vol. 2]. As mentioned above, the regularity (1.4) is *equivalent* to the regularity (1.3) by a duality argument [L-T.2], [L-T.3], [F-L-T.1].

Surjectivity of L_T, or exact controllability. In a similar vein, we can describe the second abstract dynamic property of model (1.1) or (1.2); namely, the property that the input-solution operator L_T, defined in (1.2b), satisfies:

$$L_T \text{ be surjective : } L_2(0,T;U) \to \text{ onto } Y_1, \tag{1.5}$$

where $Y_1 \subset Y$. In the most desirable case Y_1 is the same space Y as in (1.3). This is, in fact, often the case with hyperbolic and Petrowski-type systems, but is by no means always true [example, second-order hyperbolic equations with Neumann control, Euler-Bernoulli plate equations with control in "high" boundary conditions]. For time reversible dynamics such as the hyperbolic and Petrowski-type systems under consideration, the functional analytic property (1.5) is re-labelled "exact controllability in Y_1 at $t = T$" in the PDE control theory literature. By a standard functional analysis result [T-L.1, p. 237], property (1.5) is equivalent by duality to the following so-called 'abstract continuous observability' estimate:

$$\|L_T^* z\| \geq C_T \|z\| \text{ or } \int_0^T \|B^* e^{A^* t} x\|_U^2 dt \geq C_T \|x\|_{Y_1}^2, \quad \forall\, x \in Y_1, \tag{1.6}$$

perhaps only for T sufficiently large in hyperbolic problems with finite speed of propagation, which we recognize as being the inverse inequality of (1.4), at least when $Y_1 = Y$, and T is large.

So far, so good: the abstract condition (1.6) shines for its unifying value (and for the utter simplicity by which it is obtained—just a duality step). But the crux of the matter begins now: How does one establish the validity of characterization (1.6) for exact controllability in the appropriate function spaces U and Y_1—in particular, if we can take $Y_1 = Y$—for the classes of multidimensional hyperbolic and Petrowski-type PDE with boundary control? The answer is the same as in the case of regularity of the operator L discussed before, except even more emphatically: again, for each single class, one establishes by appropriate PDE energy (hard analysis) methods, the *a-priori* concrete versions of the continuous observability inequality of which (1.6) is an abstract unifying reformulation. Thus, we can extract a second lesson, this time for the exact controllability problem. It is: "Hard beats Soft on exact controllability," an extension of the same slogan, now duplicated from global regularity to exact controllability as well. It is hard PDE analysis that permits one to obtain inverse-type inequalities such as (1.4), bounding the initial energy of the corresponding boundary homogeneous problem by the appropriate boundary trace.

Uniform stabilization. One may repeat the same set of considerations, in the same spirit, when it comes to establishing uniform stabilization of an originally conservative hyperbolic or Petrowski-type system, by means of a suitable boundary dissipation. The abstract characterization is an inverse-type inequality such as (1.6), except that it refers now to the boundary *dissipative* mixed PDE problem, not the boundary *homogeneous conservative PDE* problem. The particular abstract inequality will be given below in (2.12), in the context under discussion. However, the common lesson is duplicated once more. It is again the slogan 'Hard beats Soft,' this third time applied to the uniform stabilization problem. Indeed, this conclusion is even more acute in this case than in the preceding two cases, as—typically—establishing the uniform stabilization inequality for *the class* of hyperbolic, or Petrowski-type PDEs *under discussion* is more challenging, sometimes by much, than obtaining the corresponding specialization of the continuous observability inequality (1.6).

Enter 'infinite dimensional systems theory.' To repeat *ad nauseam*: The distinctive thrust described above in connection with the problems of regularity, exact controllability and uniform stabilization of hyperbolic and Petrowski-type mixed PDE problems is: one proves the concrete required estimates in each of the three issues by hard PDE analysis in the energy method, and only afterwards extracts and delivers the corresponding abstract version, for unification purposes.

One unfortunate consequence of all this is that a wanderer coming from outside may choose to see only the clean, shining abstract version, not the 'dirty' technical hard analysis that went into proving it in the first place. Thus, such a traveller may be tempted to move around only within the abstract level, in the comfort of some standard semigroup setting, and be induced to prove 'significant' results without descending into the arena of hard analysis. Indeed, in this way, while holding the neck above the Hilbert or Banach space clouds, one can show some results. The key is: under what assumptions? Consistently with the care to remain in lofty land, the assumptions will be 'abstract,' of course, meaning now 'soft.' And here is the key of this whole matter, the moral of the present introductory section:

(i) Are the 'abstract' soft assumptions introduced by an alternative, indirect approach ever true, hopefully at least for some non-trivial classes of multidimensional PDEs? How does one verify them? How does the effort to verify the assumptions of these indirect routes compare with the more gratifying effort of establishing directly the relevant, *a-priori* characterizing inequality, as already available in the literature of the past 20 years?

(ii) In case an hypothesis of the indirect route is indeed true at least for some classes of relevant PDEs, is it too strong for the final goal that is claimed? That is: How far is it from being necessary?

(iii) If the proposed 'new' route avoids the direct proof of the past literature to establish the desired result, by going around the circle instead of moving straight along the relevant diameter, is there anything gained in a detour offered as an alternative approach?

Infinite-dimensional systems theory offers many illustrations where the answer to the basic questions above is, overall and cumulatively, in the negative. A most recent case in point is displayed by paper [G-L.1]. It offers an eloquent opportunity to analyze and discuss the conceptual thrust of the present article, which is multifold. It includes, deliberately, a tutorial component for the purpose of enlightening and guiding those who are lured to the field, coming from (the smooth avenue of) Banach spaces, happily unaware of, and recalcitrant to learn, PDE techniques (save for the eigenfunctions, or at most standard Riesz basis, methods of one-dimensional domains, when applicable). How many times is the word 'semigroup' or the combination 'Riesz basis' ever used in Hörmander's volumes? Yet, the object of those volumes, a thorough description of dynamical properties of linear PDEs, though scarce on global properties of mixed PDE problems, should represent a preliminary setting for the most important and relevant classes of 'infinite-dimensional systems theory.'

2 A first analysis of the stabilization problem via B^*L in light of the content of Section 1

The recent paper [G-L.1] furnishes clear support for the analysis set forth in our present Section 1. To begin with, we shall point out in passing that:

(a) Theorem 1 of [G-L.1, p. 47] has been known for over 15 years, in fact in a much stronger *nonlinear and multivalued* version, see the 1989 paper [Las.1]. Moreover, a rather comprehensive treatment of this problem, including references and numerous applications can be found in the recent CMBS-NSF Lecture Notes [Las.3, Chapter 1]. For the linear model (which is the case considered in [G-L.1]) stronger results are given in the monograph [L-T.21, Theorem 7.6.2.2, p. 665, vol. II]. The fact that 'admissibility' of the control operator *has nothing to do* with the issue of generation (which seems surprising to [G-L.1]) has been known for 20 years of so.

(b) Theorem 2 of [G-L.1, p. 50] is also very well known: it is nothing but the so-called Russell's principle "controllability via stabilizability" for time reversible dynamics, put forward by Russell also for infinite-dimensional systems at least as far as 1974 [Ru.1], [Ru.2]. It has since been openly invoked in the literature of boundary control for PDE many many times, including the first case of a boundary controllability result of the wave equation with Neumann control, in the energy space $H^1(\Omega) \times L_2(\Omega)$, obtained in [C.1] in 1979.

And, by the way, in the spirit of the content of Section 1: this 'Principle' turned out to be a not so sound strategy, as it traded the generally easier exact controllability problem with the generally harder uniform stabilization result.

(c) The statement reported in [G-L.1, p. 46, 3rd paragraph] about the *lack of exact controllability* on any $[0, T]$ in the case of a bounded finite-dimensional control operator B has likewise long been known, and in a much stronger version since the U. of Minnesota, 1973 Ph.D. thesis by the second author, where the relevant topic was published in 1975 [Tr.1], [Tr.2], and has been reported widely also in book form. Indeed, various more demanding extensions motivated by boundary control of PDE have been later provided, still at least 10 years ago [Tr.4], [Tr.7]; see also the lack of uniform stabilization in [Tr.4], [Tr.6].

Rather, in light of Section 1 of the present note, we intend to concentrate on Theorem 3 of [G-L.1, p. 53], which, apparently, is also announced in Proposition 3.3 of [A-T.1]. Some background first. This is the setting of [Las.1], [L-T.21, Chapter 7, p. 663].

A second-order equation setting. Let H, U be Hilbert spaces and:
(h.1) $\mathcal{A} : H \supset \mathcal{D}(\mathcal{A}) \to H$ be a positive self-adjoint operator;
(h.2) $\mathcal{B} \in \mathcal{L}(U; [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')$; equivalently, $\mathcal{A}^{-\frac{1}{2}}\mathcal{B} \in \mathcal{L}(U; H)$.
We consider the open-loop control system

$$v_{tt} + \mathcal{A}v = \mathcal{B}u, \quad v(0) = v_0,\ v_t(0) = v_1, \tag{2.1}$$

as well as the corresponding closed-loop, dissipative feedback system

$$w_{tt} + \mathcal{A}w + \mathcal{B}\mathcal{B}^* w_t = 0, \quad w(0) = w_0,\ w_t(0) = w_1. \tag{2.2}$$

We rewrite (2.1) and (2.2) as first-order systems of the form (1.1) in the space $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times H$:

$$\frac{d}{dt}\begin{bmatrix} v(t) \\ v_t(t) \end{bmatrix} = A \begin{bmatrix} v(t) \\ v_t(t) \end{bmatrix} + Bu; \quad \frac{d}{dt}\begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} = A_F \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix}; \tag{2.3}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}; \ A_F = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\mathcal{B}\mathcal{B}^* \end{bmatrix} = A - BB^*,\ B = \begin{bmatrix} 0 \\ \mathcal{B} \end{bmatrix}, \tag{2.4}$$

with obvious domains. The operator A_F is maximal dissipative and thus the generator of a s.c. contraction semigroup $e^{A_F t}$, $t \geq 0$, on Y [L-T.21, Proposition 7.6.2.1, p. 664].

Setting $y(t) = [w(t), w_t(t)]$, $y_0 = [w_0, w_1]$, we have that the variation of parameter system for the w-problem is

$$\begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} = y(t) = e^{A_F t} y_0 = e^{At} y_0 - \int_0^t e^{A(t-\tau)} BB^* e^{A_F \tau} y_0 d\tau \tag{2.5a}$$

$$= e^{At} y_0 - \{L(B^* e^{A_F \cdot} y_0)\}(t), \tag{2.5b}$$

recalling the operator L defined in (1.2b).

A first-order equation setting. We now consider a first-order model with skew-adjoint generator. Let Y and U be two Hilbert spaces. The basic setting is now as follows:

(a.1) $A = -A^*$ is a skew-adjoint operator $Y \supset \mathcal{D}(A) \to Y$, so that $A = iS$, where S is a self-adjoint operator on Y, which (essentially without loss of generality) we take positive definite (as in the case of the Schrödinger equation of Section 4.2 below).

Accordingly, the fractional powers of S, A, A^* are well defined.

(a.2) B is a linear operator $U \to [\mathcal{D}(A^{*\frac{1}{2}})]'$, duality with respect to Y as a pivot space; equivalently, $Q \equiv A^{-\frac{1}{2}} B \in \mathcal{L}(U; Y)$ and $B^* A^{*-\frac{1}{2}} \in \mathcal{L}(Y; U)$.

Under assumptions (a.1), (a.2), we consider the operator $A_F : Y \supset \mathcal{D}(A_F) \to Y$ defined by

$$A_F x = [A - BB^*]x; \; x \in \mathcal{D}(A_F) = \{x \in Y : [A - BB^*]x \in Y\}. \tag{2.6}$$

Proposition 2.1. Under assumptions (a.1), (a.2) above, we have, with reference to (2.6):

(i)

$$\mathcal{D}(A_F) = A^{-\frac{1}{2}}[I - iQQ^*]^{-1} A^{-\frac{1}{2}} Y \subset \mathcal{D}(A^{\frac{1}{2}}) \subset \mathcal{D}(B^*); \tag{2.7a}$$

$$A_F^{-1} = A^{-\frac{1}{2}}[I - iQQ^*]^{-1} A^{-\frac{1}{2}} \in \mathcal{L}(Y). \tag{2.7b}$$

(ii) The operator A_F is dissipative; in fact, maximal dissipative, and hence the generator of a s.c. contraction semigroup $e^{A_F t}$ on Y, $t \geq 0$. [Similarly, the Y-adjoint A_F^* is the generator of a s.c. contraction semigroup on Y, with A_F^{*-1} given by the same expression (2.7b) with "+" sign rather than "−" sign for the operator in the middle.]

(iii) Hence, the abstract first-order, closed-loop equation

$$\dot{y} = (A - BB^*)y, \qquad y(0) = y_0 \in Y \tag{2.8a}$$

(obtained from the open-loop equation

$$\dot{\eta} = A\eta + Bu \tag{2.8b}$$

with feedback $u = -B^* y$) admits the unique solution $e^{A_F t} y_0, \quad t \geq 0$.

Proof. (i) Let $x \in \mathcal{D}(A_F)$. Then we can write

$$A_F x = [A - BB^*]x = A^{\frac{1}{2}}[I - (A^{-\frac{1}{2}}B)(B^* A^{-\frac{1}{2}})] A^{\frac{1}{2}} x$$

$$= A^{\frac{1}{2}}[I - iQQ^*] A^{\frac{1}{2}} x = f \in Y, \tag{2.9}$$

with $Q \equiv A^{-\frac{1}{2}}B \in \mathcal{L}(U;Y)$ by assumption, and $Q^* \equiv B^*A^{*-\frac{1}{2}} \in \mathcal{L}(Y;U)$, its dual or conjugate. Here, we have used (a.1): $A^* = -A$, so that $A^{*\frac{1}{2}} = iA^{\frac{1}{2}}$, hence $A^{*-\frac{1}{2}} = iA^{-\frac{1}{2}}$, finally $B^*A^{-\frac{1}{2}} = iB^*A^{*-\frac{1}{2}} = iQ^*$. It is clear that the operator $[I - iQQ^*]$, where $QQ^* \in \mathcal{L}(Y)$ is non-negative, self-adjoint on Y, is boundedly invertible on Y. Thus, (2.9) yields

$$x = A_F^{-1}f = A^{-\frac{1}{2}}[I - iQQ^*]^{-1}A^{-\frac{1}{2}}f \in \mathcal{D}(A_F), \; f \in Y, \tag{2.10}$$

and (2.7a-b) is proved. Then, the identity in (2.7a) plainly shows that $\mathcal{D}(A_F) \subset \mathcal{D}(A^{\frac{1}{2}})$, while $\mathcal{D}(A^{\frac{1}{2}}) \subset \mathcal{D}(B^*)$ by assumption (a.2). Part (i) is proved.

(ii) We next show that A_F is dissipative. Let $x \in \mathcal{D}(A_F)$. Thus, $x \in \mathcal{D}(A^{\frac{1}{2}}) = \mathcal{D}(A^{*\frac{1}{2}}) \subset \mathcal{D}(B^*)$ by part (i). Hence, we can write, if $(\quad , \quad)$ is the Y-inner product:

$$\begin{aligned} \text{Re}(A_F x, x) = \text{Re}([A - BB^*]x, x) &= \text{Re}(x, x) - \|B^*x\|^2 && (2.11a)\\ &\leq -\|B^*x\|^2 \leq 0, \quad \forall\, x \in \mathcal{D}(A_F), && (2.11b) \end{aligned}$$

since $\text{Re}(Ax, x) = \text{Re}\{-i\|A^{\frac{1}{2}}x\|^2\} = 0$, where each term in (2.11a-b) is well-defined. Thus, A_F is dissipative.

Finally, since $A_F^{-1} \in \mathcal{L}(Y)$ by part (i), then $(\lambda_0 - A_F)^{-1} \in \mathcal{L}(Y)$ as well for a suitable small $\lambda_0 > 0$, and then the range condition: range$(\lambda_0 - A_F) = Y$ is satisfied, so that A_F is maximal dissipative. By the Lumer-Phillips theorem [P.1, p. 14], A_F is the generator of a s.c. contraction semigroup on Y. The same argument shows that A_F^* is maximal dissipative. □

Remark 2.1. One can, of course, extend the range of Proposition 2.1, by adding to A a suitable perturbation P: either $P \in \mathcal{L}(Y)$ or else P relatively bounded dissipative perturbations as in known results [P.1, Corollary 3.3, Theorem 3.4, p. 82–83] for instance, and still obtain that $[(A+P) - BB^*]$ is the generator of a s.c. semigroup (of contractions in the last two cases). □

An extension of the key question in [G-L.1]. The question which follows was raised in [G-L.1, Theorem 3] only in connection with the second-order system (2.1), (2.2), subject to the assumptions (h.1), (h.2), that precede (2.1). However, in view of Proposition 2.1, we may likewise extend the same question to the first-order systems (2.8a-b) subject to the assumptions (a.1), (a.2) that precede Proposition 2.1. For both problems, we have $A^* = -A$, the skew-adjoint property of the free dynamics generator.

In [G-L.1], the following question has been asked with reference to systems (2.1), (2.2): Is it true that exact controllability of (2.1) on the state space $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times H$ by means of $L_2(0, T; U)$-controls is equivalent to uniform stabilization of (2.2) on the same space Y? Here we shall extend this question also in reference to systems (2.8a-b), in order to include, for instance, also the Schrödinger equation case of Section 4.2. Henceforth, $\{A, B, A_F, Y, U\}$ refer either to (2.5) or to (2.8), indifferently. Quantitatively, we may reformulate the above equation as follows: is the continuous observability inequality (1.6) [which characterizes exact controllability of (1.1) with A and B as in (2.4) or as in (2.6)] equivalent to the inequality

$$\int_0^T \|B^*e^{A_F t}x\|_U^2 dt \geq c_T \|e^{A_F T}x\|_Y^2, \qquad \forall\, x \in Y, \tag{2.12}$$

which characterizes the uniform stability of the w-problem (2.2) or the y-problem (2.8a)? In our case, A is skew-adjoint $A^* = -A$. Thus, exact controllability of $\{A, B\}$ (that is of (2.1) or (2.8a)) over $[0, T]$ is equivalent to exact controllability of $\{A^*, B\}$ over $[0, T]$. In other words, in our case, inequality (1.6) is equivalent to

$$\int_0^T \|B^* e^{At} x\|_U^2 dt \geq c_T \|x\|_Y^2, \qquad \forall\, x \in Y. \tag{2.13}$$

Thus, the present question is rephrased now as follows: is inequality (2.12) equivalent to inequality (2.13)?

In one direction, the implication: uniform stabilization of (2.1) or (2.8b) [that is, (2.12)] $\rightarrow$ exact controllability of (2.1) or (2.8b) [that is, (2.13)] was shown by D. Russell [Ru.1], [Ru.2] some 30 years ago, by virtue of a clean soft argument. This result is what paper [G-L.1] labels Theorem 2. The proof in [G-L.1] is exactly the same as the original well-known proof of Russell [Ru.1].

In the opposite direction, we have the following:

CLAIM 2.1. With reference to the second-order equations (2.1), (2.2) [respectively, the first-order equations (2.8a-b), assume the preceding assumptions (h.1), (h.2) [respectively, (a.1), (a.2)]. Then, the implication: exact controllability of (2.1) or (2.8b) [that is, (2.13)] $\Rightarrow$ uniform stabilization of (2.2) or of (2.8a) [that is, (2.12)] holds true, if one adds the assumption that

$$\text{the operator } B^*L\text{: continuous } L_2(0, T; U) \rightarrow L_2(0, T; U). \tag{2.14}$$

This result, which is almost trivial (see a standard short proof in Section 3 below), is *stronger* than what paper [G-L.1] labels Theorem 3, see Remark 2.2 below, even in connection with the second-order equations (2.1), (2.2) considered in [G-L.1].

Remark 2.2. We remark that if B is, in particular, a bounded operator, $B \in \mathcal{L}(U; Y)$, then [condition (1.3) and] condition (2.14) is, *a-fortiori*, satisfied. Thus, in this case, exact controllability of (2.1) or (2.8b) implies (and is implied by [Ru.1], [Ru.2]) uniform stabilization. We recover (with the simple proof of Section 3) a 30-year-old well-known result of [Sl.1] (based on the same finite-dimensional proof of the 1968-paper [Lu.1]). Yet, there are still contemporary papers that claim this as a new result [Tu.1]! $\square$

Remark 2.3. Actually Theorem 3 in [G-L.1] assumes, instead of (2.14) for B^*L, a property which amounts to a 'frequency domain' reformulation of property (2.14): the latter is less direct, less enlightening than the former, and at any rate unnecessary. Moreover, Theorem 3 in [G-L.1] assumes, in addition, the regularity property (1.3) for L, or its dual equivalent version (1.4), which the subsequent Remark 3 of [G-L.1] states that it may be dispensed with, as learned via the review process, but with no proof being presented. In Appendix A, we provide a proof that (2.14) for B^*L implies (1.4), or (1.3) for L: this is, in fact, a simple implication. Apparently, Theorem 3 of [G-L.1] was also announced in Proposition 3.3 of [A-T.1]. $\square$

At any rate, the statement of Claim 2.1 is also known to specialized PDE circles, and we shall provide several references below, where a result such as this, or technically comparable and very close to it, is *actually built-in* into existing proofs of regularity/exact controllability/uniform stabilization of *some* (surely not all) Petrowski-type systems, rather than singled out *per se*, and broadcast as a 'relevant' abstract result. There are very good reasons for this apparent lack of an explicit statement, which is due to a sensible choice of exposition and treatment in the literature of PDE boundary stabilization of the past 15 years. Here is a first preview:

(1) Claim 2.1 is very simple to prove within standard energy method settings, and thus is arguably unbecoming its elevation to the rank of "Theorem." See the short proof given in Section 3 below, which should be compared with the lengthier, more cumbersome time/frequency domain proof of [G-L.1, p. 54].

(2) The key assumption of the abstract Claim 2.1 is, of course, assumption (2.14) that $B^*L \in \mathcal{L}(L_2(0,T;U))$. How general is it? And how can one verify it? Only a one-dimensional Euler-Bernoulli beam is given in [G-L.1] as an illustrative example where assumption (2.14) is satisfied, and this after 6 pages of breathless eigenfunction computations for diagonal semigroups. Such *tour de force* in eigenfunction gimmickry can be spared, as we shall show below in Section 3.2 that a few lines detailing a standard energy argument will do it. More to the point: assumption (2.8) is, yes, satisfied in some serious multidimensional hyperbolic and Petrowski-type systems (identified below in Section 4, by essentially making reference to long-published PDE and PDE-control literature); though it is also restrictive, as it is *not fulfilled* in other hyperbolic/Petrowski problems, also identified below in Sections 5, 6, 7, 8. To add insult to injury, for these latter hyperbolic/Petrowski-type problems where assumption (2.14) fails, uniform boundary stabilization has been known to hold true for more than 15 years. In short: assumption (2.14) is far from being necessary, a further reason for dethroning Claim 2.1 from the rank of Theorem.

(3) We said above that assumption (2.14) is already known to hold true for some cases of hyperbolic/Petrowski-type systems, and just by relying on long-published literature. But then, how is it verified in this published literature? Here is the 'surprise': The validity of assumption (2.14) on Claim 2.1 for *some* hyperbolic/Petrowski systems is verified (see Section 4 below) by precisely the *same* hard analysis PDE energy methods that are used to prove *directly* the final sought-after result of regularity, exact controllability, and above all, uniform stabilization for these systems, save for the case of first-order hyperbolic systems, where the proof of regularity via pseudo-differential analysis is employed! Then: why does one need to go around the circle and artificially separate the desired conclusion on uniform stabilization into two sufficient building blocks—the properties of exact controllability (which is also necessary [Ru.1]) and the property (2.14) of regularity of B^*L (this second one, however, far from necessary)—if then the hard analysis PDE machinery that allows one to verify the assumption on B^*L is the very same that permits one to prove *directly* the sought-after uniform stabilization property in one shot?

No wonder that Claim 2.1 was not explicitly made in the PDE-control literature of the past 15 years! And no wonder if the actual proof of the soft Claim 2.1 is simple: the hard part to prove in order to reach the conclusion on uniform stabilization is buried in the hypotheses; one being far from necessary, but at any rate both verified by hard analysis energy methods. The lofty eyes of the traveller through Banach spaces do not wish to be perturbed by the hard machinery on the ground, where the serious computations take place.

3 The stabilization problem via B^*L revisited

3.1 A simple (alternative) proof to a non-linear generalization of Claim 2.1

We provide below a simple alternative proof of Claim 2.1, which, in fact, at no extra effort, yields a *new non-linear generalization* of Claim 2.1. In place of equation (2.8.a) (hence (2.2)) we consider the following nonlinear version

$$y_t = Ay - Bf(B^*y), \quad y(0) = y_0 \in Y. \tag{3.1.1}$$

under the same assumptions (a.1) for A and (a.2) for B, where f is a monotone increasing, continuous function on U. It is known [Las.1], [Las.3] that $A - Bf(B^*)$ generates a nonlinear semigroup of contractions - say $S_F(t)$ - which yields the following variation of parameter formula for (3.1.1)

$$y(t) = S_F(t)y_0 = e^{At}y_0 - \{L(f(B^*S_F(\cdot)y_0)\}(t) \tag{3.1.2}$$

and obeys the energy identity

$$||y(T)||_Y^2 = ||y(0)||_Y^2 - 2\int_0^T (f(B^*y), B^*y)_U dt \tag{3.1.3}$$

Proposition 3.1.1 In addition to the standing assumption we assume that :

(i) The operator B^*L is continuous $L_2(0,T;U) \to L_2(0,T;U)$, as in (2.14).

(ii) $m||u||_U^2 \leq (f(u), u)_U; \quad ||f(u)||_U \leq M||u||_U, \forall u \in U$

Then, exact controllability of (A, B) implies exponential stability of $S_F(t)$, i.e there exist positive constants $C, \omega > 0$ such that the solution of (3.1.1) satisfies

$$||y(t)||_Y^2 \leq Ce^{-\omega t}||y_0||_Y^2 \tag{3.1.4}$$

Proof: Step 1. We first show that for any $y_0 \in Y$, we have via assumption (i)= (2.14) and (ii)

$$\|B^*e^{A\cdot}y_0\|_{L_2(0,T;U)} \leq (1 + k_T M)\|B^*S_F(\cdot)y_0\|_{L_2(0,T;U)}, \tag{3.1.5}$$

where $k_T = |||B^*L|||$ in the uniform operator norm of $\mathcal{L}(L_2(0,T;U))$. Indeed, (3.1.5) stems readily from (3.1.2), which yields

$$B^*e^{At}y_0 = B^*S_F(t)y_0 + \{[B^*L]f(B^*S_F(\cdot)y_0\}(t) \tag{3.1.6}$$

Hence, invoking assumption (2.14) on B^*L, we see that (3.1.6) along with the bound on f in (ii) at once implies (3.1.5).

Step 2. The exact controllability assumption on the pair $\{A, B\}$, equivalently on the pair $\{A^*, B\}$, guarantees characterization (2.13). This combined with (3.1.5) yields then, for any $y_0 \in Y$:

$$\|y_0\|_Y^2 \leq C_T \int_0^T \|B^*e^{At}y_0\|_U^2 dt \leq C_T(1 + k_T M)\int_0^T \|B^*S_F(t)y_0\|_U^2 dt. \tag{3.1.7}$$

Step 3. The energy identity (3.1.3) when combined with (3.1.7) and (i) gives

$$\|S_F(T)y_0\|_Y^2 \le C_T(1+k_TM)\int_0^T \|B^*S_F(t)y_0\|_U^2 dt + 2\int_0^T (B^*S_F(t)y_0, f(B^*S_F(t)y_0))_U dt$$
$$\le (C_T(1+k_TM)m^{-1}+2)\int_0^T (B^*S_F(t)y_0, f(B^*S_F(t)y_0))_U dt =$$
$$(C_T(1+k_TM)m^{-1}+2)(||S_F(0)||_Y^2 - ||S_F(T)||_Y^2)$$

The above identity implies that $||S_F(T)||_Y \le \gamma < 1$ which, in turn implies exponential decays for the semigroup. The proof of Proposition 3.1.1 is complete. □

3.2 Example #2 in [G-L.1] revisited

In this section, we consider the 1-dimensional beam problem with boundary control, proposed by [G-L.1]. This reference spends six tight pages of dreadful eigenfunction computations for diagonal semigroups to conclude that, in the beam example, property (2.14): $B^*L \in L_2(0,T;L_2(\Gamma))$ holds true. However, the issue of exact controllability of this control problem is not addressed, or even mentioned. Thus, [G-L.1] cannot actually invoke Claim 2.1 or its (weaker) Theorem 3, p. 53, and conclude, as it does, that uniform stabilization holds true as well.

By contrast, we provide here an elementary, short, energy method proof that, within the same unified setting, will readily yield in one shot the following properties: (i) $B^*L \in \mathcal{L}(L_2(0,T;L_2(\Gamma))$, that is property (2.14)) [as well as the implied $L \in \mathcal{L}(L_2(0,T;L_2(\Gamma));C([0,T];Y))$, that is, property (1.3) with Y the space of finite energy defined below in (3.2.4)]; (ii) uniform stabilization of the corresponding boundary dissipative problem on the finite energy space Y. See Theorem 3.2.2 below.

Dynamics. Let $\Omega = (0,1)$, $\Sigma_i = (0,T] \times \{i\}$, $i = 0,1$; $Q = (0,T]\times\Omega$. We consider the following 1-dimensional beam problem with 'shear' boundary control at $x = 1$, and its corresponding dissipative version:

$$\begin{cases} v_{tt} + v_{xxxx} = 0; \\ v(0,\ \cdot\ ) = v_0,\ v_t(0,\ \cdot\ ) = v_1; \\ v|_{x=0} = v_x|_{x=0} \equiv 0; \\ v_{xx}|_{x=1} \equiv 0,\ v_{xxx}|_{x=1} = g; \end{cases} \qquad \begin{cases} w_{tt} + w_{xxxx} = 0 & \text{in } Q; & (3.2.1a) \\ w(0,\ \cdot\ ) = w_0,\ w_t(0,\ \cdot\ ) = w_1 & \text{in } \Omega; & (3.2.1b) \\ w|_{x=0} = w_x|_{x=0} \equiv 0 & \text{in } \Sigma_0; & (3.2.1c) \\ w_{xx}|_{x=1} \equiv 0,\ w_{xxx}|_{x=1} = w_t|_{x=1} & \text{in } \Sigma_1. & (3.2.1d) \end{cases}$$

Abstract model of v-problem. We introduce the operators

$$\mathcal{A}\psi = \Delta^2\psi,\ \psi \in \mathcal{D}(\mathcal{A}) = \{\psi \in H^4(\Omega) : \psi|_{x=0} = \psi_x|_{x=0} = \psi_{xx}|_{x=1} = w_{xxx}|_{x=1} = 0\}; \qquad (3.2.2)$$

$$\varphi = G_2 g \iff \{\Delta^2\varphi = 0 \text{ in } \Omega;\ \varphi|_{x=0} = \varphi_x|_{x=0} = \varphi_{xx}|_{x=1} = 0,\ \varphi_{xxx}|_{x=1} = g\}. \qquad (3.2.3)$$

The finite energy space of the above problems is

$$Y \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \equiv H^2(\Omega) \times L_2(\Omega); \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \{\psi \in H^2(\Omega) :\ \psi|_{x=0} = \psi_x|_{x=0} = 0\}. \qquad (3.2.4)$$

Then the abstract model of the v-problem is [L-T.21]

$$v_{tt} + \mathcal{A}v = \mathcal{A}G_2 g, \qquad \frac{d}{dt}\begin{bmatrix} v \\ v_t \end{bmatrix} = A\begin{bmatrix} v \\ v_t \end{bmatrix} + Bg; \tag{3.2.5}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}, \; Bg = \begin{bmatrix} 0 \\ \mathcal{A}G_2 g \end{bmatrix}, \; B^*\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = G_2^*\mathcal{A}x_2, \tag{3.2.6}$$

with obvious domains, where $*$ in B and G_2 actually refers to different topologies. With B^* defined by $(Bg, x)_Y = (g, B^*x)_{L_2(\Gamma)}$ with respect to the Y-topology defined by (3.2.4), we readily find the expression in (3.2.6).

The operator B^*L. With $y_0 = \{v_0, v_1\} = 0$, we have via (3.2.6) that

$$B^*Lg = B^*\begin{bmatrix} v(t; y_0 = 0) \\ v_t(t; y_0 = 0) \end{bmatrix} = G_2^*\mathcal{A}v_t(t; y_0 = 0) = -v_t|_{x=1}, \tag{3.2.7}$$

recalling the usual property $G_2^*\mathcal{A}\,\cdot\; = -\,\cdot\,|_{x=1}$ via [L-T.21], as well as the definition of L in (1.2b).

Regularity of L, B^*L; uniform stabilization. We introduce the PDE problem which is dual to the v-problem:

$$\begin{cases} \psi_{tt} + \psi_{xxxx} = 0 & \text{in } (0,T] \times \Omega; & (3.2.8a) \\ \psi(0,\,\cdot\,) = \psi_0,\; \psi_t(0,\,\cdot\,) = \psi_1 & \text{in } \Omega = (0,1); & (3.2.8b) \\ \psi|_{x=0} = \psi_x|_{x=0} = 0 & \text{in } (0,T] \times \{0\}; & (3.2.8c) \\ \psi_{xx}|_{x=1} = \psi_{xxx}|_{x=1} = 0 & \text{in } (0,T] \times \{1\}; & (3.2.8d) \end{cases}$$

$$\begin{bmatrix} \psi(t) \\ \psi_t(t) \end{bmatrix} = e^{At}\begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} \in C([0,T]; Y), \text{ if } \{\psi_0, \psi_1\} \in Y, \tag{3.2.9}$$

where e^{At} is a s.c. group on Y. [Actually, the dual problem requires initial conditions at $t = T$, not $t = 0$; but, equivalently for what follows below, we may take initial conditions at $t = 0$, since the ψ-problem is time reversible.] The above setting readily yields the following preliminary result.

Lemma 3.2.1. (i) With reference to the ψ-problem with $\{\psi_0, \psi_1\} \in Y$, we have:

Property (1.4) holds true: i.e., $B^*e^{A^*t}$: continuous $Y \to L_2(0,T)$ $\iff \displaystyle\int_0^T (\psi_t|_{x=1})^2 dt \le C_T \|\{\psi_0, \psi_1\}\|_Y^2$ (3.2.10)

$$\Updownarrow$$

$$L : g \to Lg = \{v, v_t\} : \text{ continuous } L_2(0,T) \to C([0,T]; Y \equiv H^2(\Omega) \times L_2(\Omega)), \tag{3.2.11}$$

where in (3.2.11) we have $\{v_0, v_1\} = 0$ for the v-problem (3.2.1).

(ii) With reference to the v-problem (3.2.1) again with $y_0 = \{v_0, v_1\} = 0$, we have

$$B^*L : \text{ continuous } L_2(0,T) \to L_2(0,T), \tag{3.2.12}$$

if and only if the v-problem in (3.2.1) (left) satisfies

$$\int_0^T (v_t|_{x=1})^2 dt = \mathcal{O}\left(\|g\|^2_{L_2(0,T)} \right). \tag{3.2.13}$$

(iii) With reference to the dissipative w-problem in (3.2.1), we have

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \text{Property (2.12) holds true:} \\ \displaystyle\int_0^T \|B^* e^{A_F t} x\|^2_U dt \geq C_T \|e^{A_F T} x\|^2_Y, \quad x \in Y \end{array} \right. \\ \Longleftrightarrow \\ \displaystyle\int_0^T (w_t|_{x=1})^2 dt \geq C_T \|\{w(T), w_t(T)\}\|^2_{Y=H^2(\Omega)\times L_2(\Omega)}. \end{array} \right. \tag{3.2.14}$$

Theorem 3.2.2. (i) The regularity of L in (3.2.11) holds true.
(ii) The regularity of B^*L in (3.2.12) holds true.
(iii) With reference to the w-problem (3.2.1), we have:

(iii1) the map $\{w_0, w_1\} \to \{w(t), w_t(t)\}$ defines a s.c. contraction semigroup $e^{A_F t}$ on $Y \equiv \mathcal{D}(A^{\frac{1}{2}}) \times L_2(\Omega)$, see (3.2.4);

(iii2) with reference to (3.2.1d):

$$w_{xxx}|_{x=1} = w_t|_{x=1} \in L_2(0,\infty) \text{ continuously in } \{w_0, w_1\} \in Y; \tag{3.2.15}$$

(iii3) Estimate (3.2.14) holds true: thus there exist constants $M \geq 1$, $\delta > 0$, such that

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|^2_Y = \left\| e^{A_F t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|^2_Y \leq M e^{-\delta t} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|^2_Y, \quad t \geq 0. \tag{3.2.16}$$

Proof. We shall show, equivalently, inequalities (3.2.10) and (3.2.13).

Step 1. Assume, at first, smooth data $\{v_0, v_1, g\}$. We multiply the v-problem (3.2.1) by the usual standard multiplier xv_x and integrate by parts in t and x. We obtain

$$\left[\int_0^1 v_t x v_x dx \right]_0^T - \int_0^T \int_0^1 v_t x v_{xt} dx\, dt + \int_0^T [v_{xxx} x v_x]_{x=0}^{x=1} dt$$
$$- \int_0^T \int_0^1 v_{xxx}(v_x + x v_{xx}) dx = 0. \tag{3.2.17}$$

Using the identities

$$v_t x v_{xt} = \frac{1}{2} \frac{d}{dx} (v_t^2 x) - \frac{1}{2} v_t^2; \quad v_{xxx} x v_{xx} = \frac{1}{2} \frac{d}{dx} (v_{xx}^2 x) - \frac{1}{2} v_{xx}^2; \tag{3.2.18}$$

$$\int_0^1 v_{xxx}v_x dx = [v_{xx}v_x]_0^1 - \int_0^1 v_{xx}^2 dx, \tag{3.2.19}$$

in (3.2.17) as well as the B.C. (3.2.1c-d), we obtain the preliminary desired identity

$$\frac{1}{2}\int_0^T (v_t|_{x=1})^2 dt = \frac{1}{2}\int_0^T \int_0^1 [v_t^2 + 3v_{xx}^2] dx\, dt + \left[\int_0^1 v_t x v_x dx\right]_0^T + \int_0^T g v_x|_{x=1} dt. \tag{3.2.20}$$

Step 2: Proof of (i). We take $g = 0$, i.e., we consider the corresponding specialization of the v-problem given by the ψ-problem (3.2.8) with I. C. $\{\psi_0, \psi_1\} \in Y$. Thus, specializing identity (3.2.20) to the ψ-problem (with $g = 0$) and using the generation result (3.2.9), we obtain

$$\begin{aligned}\frac{1}{2}\int_0^T (\psi_t^2|_{x=1})^2 dt &= \frac{1}{2}\int_0^T \int_0^1 [\psi_t^2 + 3\psi_{xx}^2] dx\, dt + \left[\int_0^1 \psi_t x \psi_x dx\right]_0^T \\ &= \mathcal{O}\left(\|\{\psi_0, \psi_1\}\|\right)_{Y = H^2(\Omega)\times L_2(\Omega)},\end{aligned} \tag{3.2.21}$$

and (3.2.10) is proved. Thus, (3.2.11) for L is established.

Step 3: Proof of (ii). Now we consider the v-problem (3.2.1) with $\{v_0, v_1\} = 0$ and regularity (3.2.11) for L just established. We return to identity (3.2.20) and using (3.2.11) we obtain

$$\frac{1}{2}\int_0^T (v_t|_{x=1})^2 dt = \mathcal{O}\left(\|g\|^2_{L_2(0,T)}\right) + \int_0^T g v_x|_{x=1} dt. \tag{3.2.22}$$

Next, we use here trace theory and again (3.2.11) to obtain

$$|v_x|_{x=1}| \le C\|v_x\|_{H^1(\Omega)} \le C\|v\|_{H^2(\Omega)} = \mathcal{O}\left(\|g\|_{L_2(0,T)}\right). \tag{3.2.23}$$

Finally, substituting (3.2.23) in (3.2.22) yields (3.2.13), as desired.

Step 4: Proof of (iii3). [Parts (iii1), (iii2) are very standard.]

Then, returning to identity (3.2.20) as specialized to the w-problem, hence with $g = w_t|_{x=1}$ as in (3.2.15) and thus with

$$E(t) = \|\{w(t), w_t(t)\}\|_Y^2; \quad \int_0^T g\, w_x|_{x=1} dt \ge -C_T \int_0^T (w_t|_{x=1})^2 dt, \tag{3.2.24}$$

recalling (3.2.23) with v replaced by w, and $g = w_t|_{x=1}$, we obtain

$$\int_0^T (w_t|_{x=1})^2 dt \;\ge\; c_1 \int_0^T E(t)dt - c_2[E(T) + E(0)] \tag{3.2.25}$$

$$\ge\; \tilde{c}_1 \int_0^T E(t)dt - \tilde{c}_2 E(T) \tag{3.2.26}$$

$$\ge\; [\tilde{c}_1 T - \tilde{c}_2]E(T), \tag{3.2.27}$$

and (3.2.27) is nothing but a rewriting of (3.2.14) with $C_T = \tilde{c}_1 T - \tilde{c}_2 > 0$ for T sufficiently large. To go from (3.2.25) to (3.2.26) and to (3.2.27), we have used the usual dissipativity identity. □

4 Classes of PDE satisfying the regularity property (2.14): $B^*L \in \mathcal{L}(L_2(0,T;U))$

Documenting, and reinforcing, the content of Section 1, our goal in the present paper is now two-fold.

(i) First, we provide (in the present section) several, multidimensional non-trivial hyperbolic and Petrowski-type mixed problems that indeed satisfy the regularity property (2.14) on B^*L. In this respect, our message is, in turn, that for each of the illustrations given below, the fact that B^*L fulfills property (2.14) was either already noted explicitly in the literature, or else is a *built-in block* in the proof of optimal regularity, exact controllability, and particularly, *uniform stabilization* of such systems—which is the ultimate goal in Claim 2.1.

(ii) Second, we document in Sections 5, 6, 7, 8 that property (2.14) *fails* to hold true for B^*L in the case of several other hyperbolic Petrowski-type PDE systems where, however, *uniform stabilization has long been proved*, by PDE energy methods, in the literature. This says that property (2.14) for B^*L is far from being necessary in Claim 2.1. That is to say, property (2.14) is *not* a pre-condition for either controllability or stabilization of these problems.

Points (i) and (ii) call into question the "usefulness" of a result such as Claim 2.1, as elaborated before.

Remark 4.1. Due to constraints on the overall length of the paper, to make our main point of the present Section 4—singling out relevant classes of PDEs where the regularity (2.14) for B^*L holds true—it will be expedient to de-emphasize generality. Thus, in our results below, we shall deal primarily with canonical PDEs and with control acting, possibly, on the whole boundary, even though a much greater degree of generality is well-known. In particular, we shall not necessarily insist on the case of variable coefficients PDEs, and refer instead to [B-L-R.1], [G-L-L-T.1], [L-T-Y.1], [Ta.1], [T-Y.1], [T-Y.2], etc. □

4.1 First-order hyperbolic systems with boundary control

This section considers a general first-order hyperbolic system, which may be non-symmetric and non-dissipative, and is defined on a sufficiently smooth bounded domain of arbitrary dimension. The control function acts through the boundary conditions. The treatment below follows closely [L-T.21, Chapter 10, Section 10.6].

The dynamics. Let $\Omega \subset R^m$ be an open bounded domain with smooth boundary Γ. In Ω, we consider a differential operator of the form

$$A(x,\partial)y \equiv \sum_{j=1}^{m} A_j(x)\partial_j y + A_0(x)y, \tag{4.1.1}$$

where $y(x)$ is a k-vector and $\partial_j = \partial/\partial x_j$. The coefficients $A_j,\ A_0$ are smooth $k \times k$ matrix-valued functions defined on the open bounded domain $\Omega \subset R^m$. We assume the following hypotheses throughout:

(h.1): $A(x,\partial)$ is strictly hyperbolic; i.e., the matrix $\sum_{j=1}^{m} A_j(x)\xi_j$ has k distinct real eigenvalues for all $\xi = [\xi_1,\dots,\xi_m] \in R^m \setminus \{0\}$ and $x \in \bar{\Omega}$;

(h.2): The boundary Γ is noncharacteristic; i.e., det $A_\nu(x) \neq 0$ for $x \in \Gamma$, where $A_\nu(x) \equiv \sum_{j=1}^m A_j \nu_j(x)$; $\nu = (\nu_1, \ldots, \nu_m)$ the inward unit normal.

It follows from (h.1) and (h.2) that after a smooth change of coordinates, we may assume that A_ν be of the following form

$$A_\nu = \begin{bmatrix} A_\nu^- & 0 \\ 0 & A_\nu^+ \end{bmatrix}, \quad A_\nu^- = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_\ell \end{bmatrix} < 0; \quad A_\nu^+ = \begin{bmatrix} a_{\ell+1} & & 0 \\ & \ddots & \\ 0 & & a_k \end{bmatrix} > 0. \quad (4.1.2)$$

Accordingly, any vector $v \in R^k$ will be split consistently as $v = [v^-, v^+]$, with $v^- = [v_1, \ldots, v_\ell]$ and $v^+ = [v_{\ell+1}, \ldots, v_k]$.

Boundary conditions are imposed with the aid of a boundary operator $M(x)$, which is a smooth $\ell \times k$ matrix-valued function, where ℓ stands for the number of negative eigenvalues of A_ν. We assume further the following hypotheses:

(h.3): rank $M(x) = \ell,\ x \in \Gamma$.

(h.4): (Kreiss condition) The frozen (at the boundary point) mixed problem has no eigenvalues or generalized eigenvalues with nonnegative real parts.

This means that after making a local change of coordinates which maps Ω into the half-space $\{x \in R^m;\ x_1 > 0\}$, the constant coefficient problem that arises by freezing $A_j,\ j = 1, \ldots, m$, and M at the boundary point and setting $A_0 = 0$, i.e.,

$$y_t - A_1 y_{x_1} - \sum_{j=2}^m A_j y_{x_j} = 0, \quad x_1 > 0; \tag{4.1.3a}$$

$$My = 0, \quad \text{at } x_1 = 0, \tag{4.1.3b}$$

has no eigenvalues or generalized eigenvalues with nonnegative real parts.

For the half-space problem (4.1.3), we have $A_\nu = A_1$, thus A_1 is invertible by (h.2). For a more detailed description of this condition we refer the reader to the fundamental papers [K.1], [Rau.1].

Convention. To streamline the notation, we shall write

$$\begin{aligned} &L_2(\Gamma),\ L_2(\Omega) \text{ to mean respectively, } L_2(\Gamma; R^\ell) \text{ and } L_2(\Omega; R^k), \text{ etc.,} \\ &\text{and } L_2(\Sigma),\ L_2(Q) \text{ to mean, respectively, } L_2(0,T; L_2(\Gamma; R^\ell)) \\ &\text{and } L_2(0,T; L_2(\Omega; R^k)), \end{aligned} \tag{4.1.4}$$

without further mention, where $\Sigma = (0,T] \times \Gamma;\ Q = (0,T] \times \Omega$, for a fixed $0 < T < \infty$.

The mixed problem for the first-order hyperbolic system which we consider is then

$$\begin{cases} y_t = A(x, \partial) y & \text{in } Q \equiv (0,T] \times \Omega & (4.1.5\text{a}) \\ y(0, \ \cdot\ ) = y_0(x) & \text{in } \Omega; & (4.1.5\text{b}) \\ [M(x)]y = g & \text{in } \Sigma \equiv (0,T] \times \Gamma, & (4.1.5\text{c}) \end{cases}$$

where the boundary control $g \in L_2(\Sigma) = L_2(0,T; L_2(\Gamma; R^\ell))$.

Regularity theory for problem (4.1.5) with $g \in L_2(\Sigma)$. A complete well-posedness theory for non-symmetric, non-characteristic first-order hyperbolic systems as in (6.1.5) has been provided by the fundamental paper [K.1], augmented by a note in [Ral.1], and completed by [Rau.2].

Theorem 4.1.1. [Rau.2, p. 272] Under the above given hypotheses of (h.1)–(h.4), for any $T > 0$, assume

$$y_0 \in L_2(\Omega); \quad g \in L_2(0,T; L_2(\Gamma)). \tag{4.1.6}$$

Then, the unique solution of problem (4.1.5) satisfies

$$y \in C([0,T]; L_2(\Omega));\ \ y|_\Gamma \in L_2(0,T; L_2(\Gamma)) \tag{4.1.7}$$

continuously. $\square$

Next, we single out the result of the homogeneous case $g \equiv 0$ for problem (4.1.5) in a form which will be useful in the sequel. To this end, we introduce the operator A, by setting

$$Ah = A(x,\partial)h :\ L_2(\Omega) \supset \mathcal{D}(A) \to L_2(\Omega) \tag{4.1.8a}$$

$$\mathcal{D}(A) = \{h \in L_2(\Omega) : A(x,\partial)h \in L_2(\Omega);\ Mh|_\Gamma = 0\}, \tag{4.1.8b}$$

where $A(x,\partial)$ is the differential operator in (4.1.1).

Corollary 4.1.2. Under the above hypotheses (h.1)–(h.4), we have that the operator A in (4.1.8), corresponding to problem (4.1.5) with $g \equiv 0$, is the generator of a s.c. semigroup e^{At} on $L_2(\Omega)$, $t \geq 0$.

Abstract Setting for Problem (4.1.5). To put problem (4.1.5), (4.1.6) into the abstract model (1.1), we need the following operators and spaces:

(i) The operator A defined by (4.1.8), which generates a s.c. semigroup e^{At} on the space

$$Y = L_2(\Omega). \tag{4.1.9}$$

(ii) Next, we introduce the "Dirichlet" map (natural extension from the boundary Γ into the interior Ω, which uniquely solve (a suitable translation of) the corresponding static problem), defined by

$$D_\lambda g = v \iff \begin{cases} A(x,\partial)v - \lambda v &= 0 \quad \text{in } \Omega; \\ Mv|_\Gamma &= g \quad \text{in } \Gamma, \end{cases} \tag{4.1.10}$$

for a suitably large constant $\lambda \geq 0$, as justified by the following result.

Lemma 4.1.3. With reference to problem (4.1.10), there exists a constant $\lambda \geq 0$, henceforth kept fixed, such that problem (4.1.10) admits a unique solution $v = D_\lambda g \in L_2(\Omega)$ for $g \in L_2(\Gamma)$. Moreover, the following estimate holds true: there is a constant $C_\lambda > 0$ depending on λ such that

$$\|D_\lambda g\|_{L_2(\Omega)} + \|D_\lambda g|_\Gamma\|_{L_2(\Gamma)} \leq C_\lambda \|g\|_{L_2(\Gamma)}. \tag{4.1.11}$$

Thus,

$$D_\lambda : \quad \text{continuous } L_2(\Gamma) \to L_2(\Omega), \tag{4.1.12}$$
$$D_\lambda^* : \quad \text{continuous } L_2(\Omega) \to L_2(\Gamma), \tag{4.1.13}$$

where D_λ^* is the adjoint $(D_\lambda g, v)_{L_2(\Omega)} = (g, D_\lambda^* v)_{L_2(\Gamma)}$.

(iii) We return to problem (4.1.5), and by virtue of the definition (4.1.10) of D_λ, λ henceforth as in Lemma 4.1.3, we rewrite it as

$$\begin{cases} y_t = (A(x,\partial) - \lambda)(y - D_\lambda g) + \lambda y & \text{in } (0,T] \times \Omega; \quad (4.1.14a) \\ y(0,x) = y_0(x) & \text{in } \Omega; \quad (4.1.14b) \\ M(y - D_\lambda g)|_\Gamma = 0 & \text{in } (0,T] \times \Gamma, \quad (4.1.14c) \end{cases}$$

or abstractly, by (4.1.8), as

$$y_t = (A - \lambda I)(y - D_\lambda g) + \lambda y \text{ in } L_2(\Omega), \quad y(0) = y_0 \in L_2(\Omega). \tag{4.1.15}$$

Moreover, extending the original operator A in (4.1.8) by $A: \ L_2(\Omega) \to [\mathcal{D}(A^*)]'$, i.e., extending the original A in (4.1.8) to its double adjoint A^{**}, we obtain from (4.1.15),

$$y_t = Ay - (A - \lambda I)D_\lambda g \text{ in } [\mathcal{D}(A^*)]', \quad y(0) = y_0 \in L_2(\Omega), \tag{4.1.16}$$

which is precisely the abstract model (1.1), with A as in (4.1.8), and

$$B = -(A - \lambda I)D_\lambda : \text{ continuous } U = L_2(\Gamma) \to [\mathcal{D}(A^* - \lambda I)]', \tag{4.1.17a}$$

equivalently,

$$(A - \lambda I)^{-1} B = -D_\lambda : \text{ continuous } L_2(\Gamma) \to L_2(\Omega), \tag{4.1.17b}$$

as guaranteed by (4.1.12).

Finally, with $B \in \mathcal{L}(U; [\mathcal{D}(A^* - \lambda I)]')$, and so $B^* \in \mathcal{L}(\mathcal{D}(A^*); U)$ after identifying $[\mathcal{D}(A^* - \lambda I)]''$ with $\mathcal{D}(A^*)$, we compute B^* as

$$B^* = -D_\lambda^*(A^* - \lambda I) : \text{ continuous } \mathcal{D}(A^*) \to U. \tag{4.1.18}$$

A more explicit representation of B^* is given by the next result.

Lemma 4.1.4. With reference to (4.1.18), we have

$$B^* y = -D_\lambda^*(A^* - \lambda I)y = [A_\nu^- y^-]_\Gamma, \quad y \in \mathcal{D}(A^*), \tag{4.1.19}$$

where A_ν^- is defined in (4.1.2) and the component y^- of y, consisting of the first ℓ coordinates is likewise defined below (4.1.2). □

The main result of the present section is

Theorem 4.1.5. With reference to the mixed problem (4.1.5) with $y_0 = 0$, we have that (recall the definition of L in (1.2b))

$$B^* Lg = B^* y(t; y_0 = 0) = [A_\nu^- y^-(t; y_0 = 0)]_\Sigma \in L_2(0, T; L_2(\Gamma))$$
$$\text{continuously in } g \in L_2(0, T; L_2(\Gamma)). \tag{4.1.20}$$

Proof. The regularity in (4.1.20) stems from (4.1.18) and (4.1.7) of Theorem 4.1.1. □

4.2 Schrödinger equation with Dirichlet boundary control

The present section deals with the (multidimensional) Schrödinger equation with Dirichlet-boundary control. The main goal is three-fold:

(i) to recall from the literature of 1992 the main results of (optimal) regularity, exact controllability and uniform stabilization;

(ii) to point out that such literature also essentially contains the result that the operator B^*L satisfies the required regularity assumption (2.14) which is, in fact, a *built-in block* into the process of studying the three related problems mentioned in point (i);

(iii) to conclude, accordingly, that the use of Claim 2.1—based on exact controllability of $\{A, B\}$ and regularity of B^*L—to obtain uniform stabilization of $\{A, B\}$ is neither enlightening nor technically and conceptually convenient.

Open-loop and closed-loop feedback dissipative systems. Let Ω be an open bounded domain in $\mathbb{R}^n$, with sufficiently smooth C^1-boundary Γ. We consider the following open-loop problem of the Schrödinger equation defined on Ω, with Dirichlet-control $u \in L_2(0, T; L_2(\Gamma)) \equiv L_2(\Sigma)$, and its corresponding boundary dissipative version

$$\begin{cases} y_t = -i\Delta y; \\ y(0, \cdot\,) = y_0; \\ y|_\Sigma = u \in L_2(\Sigma); \end{cases} \qquad \begin{cases} w_t = -i\Delta w & \text{in } Q; & (4.2.1\text{a}) \\ w(0, \cdot\,) = w_0 & \text{in } \Omega; & (4.2.1\text{b}) \\ w|_\Sigma = i\dfrac{\partial(A^{-1}w)}{\partial\nu} & \text{in } \Sigma, & (4.2.1\text{c}) \end{cases}$$

with $Q \equiv (0, T] \times \Omega$; $\Sigma \equiv (0, T] \times \Gamma$. Moreover, the operator A is defined below in (4.2.4) as $Aw = -\Delta w$, $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

Regularity, exact controllability of the y-problem; uniform stability of the w-problem. Paper [L-T.17] gives a full account of the (optimal) regularity and exact controllability of the open-loop y-problem in (4.2.1), as well as the uniform stabilization of the corresponding closed-loop w-problem. Regularity issues of interest here are also contained in [Las.2, pp. 175-177] and [L-T.21, Chapter 10].

Theorem 4.2.1. (Regularity [L-T.17, Theorem 1.2]) Regarding the y-problem (4.2.1) with $y_0 = 0$, for each $T > 0$, the following interior regularity holds true (recall the definition of L in (1.2b)):

$$\text{the map } L : u \to Lu = y \text{ is continuous } L_2(\Sigma) \to C([0, T]; H^{-1}(\Omega)). \tag{4.2.2}$$

Theorem 4.2.2. (Exact controllability, [L-T.17, Theorem 1.3]) Let $T > 0$. Given $y_0 \in H^{-1}(\Omega)$, there exists $u \in L_2(0, T; L_2(\Gamma))$ such that the corresponding solution to the y-problem (4.2.1) satisfies: $y(T) = 0$.

Theorem 4.2.3. (Uniform stabilization, [L-T.17, Theorems 1.4 and 1.5]) With reference to the w-problem in (4.2.1), we have that:

(i) the map $w_0 \in H^{-1}(\Omega) \to w(t)$ defines a s.c. contraction semigroup on $[\mathcal{D}(A^{\frac{1}{2}})]' \equiv H^{-1}(\Omega)$;

(ii) $w|_\Sigma \in L_2(0, \infty; L_2(\Gamma))$ continuously for $w_0 \in H^{-1}(\Omega)$;

(iii) There exist constants $M \geq 1$, $\delta > 0$ such that

$$\|w(t)\| \leq Me^{-\delta t}\|w_0\|, \ t \geq 0, \tag{4.2.3}$$

with $\| \quad \|$ the $H^{-1}(\Omega)$-norm.

Needless to say, in line with the content of Section 1, all three theorems above (as well as their generalizations alluded to in Remark 4.1) are obtained by PDE hard analysis energy methods (not by soft-analysis methods). The most challenging result to prove is Theorem 4.2.3 on uniform stabilization: this, in addition, requires a shift of topology from $H^{-1}(\Omega)$ (the space of the final result) to $H_0^1(\Omega)$ (the space where the energy method works). This shift of topology is implemented by a *change of variable*: this is the same change of variable that is noted below in (4.2.8), and that is needed to establish the desired regularity of B^*L.

Abstract model of y-problem. We let

$$A\psi = -\Delta\psi, \ \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega); \quad \varphi \equiv Dg \Longleftrightarrow \{\Delta\varphi = 0 \text{ in } \Omega; \ \varphi|_\Gamma = g \text{ on } \Gamma\}. \tag{4.2.4}$$

Then, the abstract model (in additive form) of the y-problem (4.2.1) is [L-T.17, Eqn. (1.2.2)]

$$\dot{y} = iAy - iADu, \ y(0) = y_0 \in Y \equiv [\mathcal{D}(A^{\frac{1}{2}})]' \equiv H^{-1}(\Omega). \tag{4.2.5}$$

Comparing with (1.1), we have

$$B = -iAD \quad \text{ hence } B^* = iD^*, \tag{4.2.6}$$

where the $*$ for B and D refer actually to different topologies, as the following computation yielding B^* in (4.2.6) shows: let $u, y \in Y$, then

$$(Bu, y)_Y = -i(ADu, y)_{[\mathcal{D}(A^{\frac{1}{2}})]'} = -i(Du, y)_{L_2(\Omega)} = -i(u, D^*y)_{L_2(\Gamma)} = (u, B^*y)_{L_2(\Gamma)}. \tag{4.2.7}$$

The operator B^*L. With reference to the y-problem in (4.2.1), we shall show that

$$B^*Lu = B^*y(t; y_0 = 0) = -i \left.\frac{\partial z}{\partial \nu}\right|_\Gamma, \tag{4.2.8a}$$

$$z(t) \equiv A^{-1}y(t; y_0 = 0) \in C([0, T]; \mathcal{D}(A^{\frac{1}{2}}) \equiv H_0^1(\Omega)), \tag{4.2.8b}$$

where z satisfies the following dynamics—abstract equation, and corresponding PDE-mixed problem:

$$\dot{z} = iAz - iDu \qquad \begin{cases} z_t = -i\Delta z - iDu & \text{in } Q; \quad (4.2.9a) \\ z(0, \cdot) = z_0 = 0 & \text{in } \Omega; \quad (4.2.9b) \\ z|_\Sigma \equiv 0 & \text{in } \Sigma. \quad (4.2.9c) \end{cases}$$

Indeed, to obtain (4.2.8)–(4.2.9), one uses the definitions in (4.2.8) and (4.2.6),

$$B^*Lu \equiv B^*y(t; y_0 = 0) = iD^*AA^{-1}y(t; y_0 = 0) = iD^*Az(t) = -i\frac{\partial z}{\partial \nu}, \tag{4.2.10}$$

as well as the usual property $D^*A = -\frac{\partial}{\partial \nu}$ on $\mathcal{D}(A^{\frac{1}{2}}) = H_0^1(\Omega)$ from [L-T.17, Eqn. (1.21)]. The absract z-equation in (4.2.9) follows from the abstract y-equation in (4.2.5) after applying A^{-1} and using the definition of $z(t)$ in (4.2.8b). Since $u(t) \in H_0^1(\Omega)$, then the abstract z-equation yields its PDE version in (4.2.9b).

Theorem 4.2.4. With reference to (4.2.8), we have

$$B^*L : \text{ continuous } L_2(0, T; L_2(\Gamma)) \to L_2(0, T; L_2(\Gamma)); \tag{4.2.11a}$$

equivalently, with reference to (4.2.10),

$$\text{the map } u \to \frac{\partial z}{\partial \nu} \text{ is continuous } L_2(0, T; L_2(\Gamma)) \to L_2(0, T; L_2(\Gamma)). \quad \Box \tag{4.2.11b}$$

This result (4.2.11) is explicitly stated and proved in [Las.2, Proposition 4.2; and ff., p. 175], where the regularity (4.2.8) for z is established in [Las.2, Eqn. (4.14)] by energy methods, (via the multiplier $h \cdot \nabla \bar{z}, h|_\Gamma = \nu$) without first establishing the y-regularity (4.2.2) in Theorem 4.2.1. This result (4.2.11) also follows from [L-T.l7, identity (2.1), Lemma 2.1] (built with the multiplier $h \cdot \nabla \bar{z}$) with $f = -iDu \in L_2(0, T; \mathcal{D}(A^{\frac{1}{4}-\epsilon}))$ AND the *a-priori regularity* $z \in C([0, T]; H_0^1(\Omega))$ in (4.2.8) for z: the latter uses, by contrast, the y-regularity (4.2.2) in Theorem 4.2.1. The two avenues chosen in [Las.2] and [L-T.17] are very closely related and based on the same energy method and duality. The expression "double duality" was used in [Las.2]" as duality was used twice.

Comparison between establishing Theorem 4.2.3(iii)—uniform stabilization—directly, or else via Claim 2.1.

(1) According to [L-T.17], in order to establish the exponential energy decay (4.2.3) *directly*, one needs the following ingredients:

(1a) (easier step) the properties of generation and feedback regularity listed in Theorem 4.2.3(i) and (ii): this is a readily accomplished application of the Lumer-Phillips Theorem;

(1b) (harder step) application of energy methods (multipliers $h \cdot \nabla \bar{p}$ and $\bar{p}$ div h to the p-problem, defined by $p \equiv A^{-1}w \in C([0, T]; H_0^1(\Omega))$ [L-T.17, Eqn. (4.6)], to obtain—in the end—the estimate [L-T.17, Eqn. (4.16)],

$$\int_0^T \int_\Gamma \left| \frac{\partial p}{\partial \nu} \right|^2 d\Sigma \ge c_T E_p(T), \tag{4.2.12}$$

$E_p(\,\cdot\,)$ being the 'energy' (square of $H^1(\Omega)$-norm) of p.

(2) In order to establish the exponential decay (4.2.3) *by virtue of Claim 2.1*, one needs the following ingredients:

(2a) proof of regularity property (2.8) for B^*L. According to [L-T.17] or [Las.2], this is accomplished as follows;

(2aI) [Las.2] either by applying energy methods (multiplier $h \cdot \nabla \bar{z}$) to the z problem (4.2.9) to obtain first the *a-priori* regularity $z \in C([0,T]; H_0^1(\Omega))$ and then the regularity trace inequality (specialization of (1.4)):

$$\int_0^T \int_\Gamma \left| \frac{\partial z}{\partial \nu} \right|^2 d\Sigma \leq c_T E_z(0), \tag{4.2.13}$$

(2aII) or else [L-T.17] by applying energy methods (multipliers $h \cdot \nabla \bar{\phi}, \bar{\phi}$ div h) to the dual homogeneous ϕ-problem

$$i\phi_t = \Delta \phi \quad \text{in } Q;\ \phi(0,\,\cdot\,) = \phi_0 \in H_0^1(\Omega),\ \phi|_\Sigma \equiv 0, \tag{4.2.14}$$

to obtain the same inequality (4.2.13) this time for ϕ, hence by duality $y \in C([0,T]; H^{-1}(\Omega))$ and hence $z(t) = A^{-1}y(t; y_0 = 0) \in C([0,T]; H_0^1(\Omega))$ (as in 2aI)); and then read off inequality (4.2.13) from identity [L-T.17, Eqn. (2.1)] in z, where one exploits the *a-priori* regularity of z;

(2b) establish exact controllability of the y-problem, that is continuous observability of the dual ϕ-problem (4.2.14), again by energy methods, to obtain

$$\int_0^T \int_\Gamma \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Sigma \geq c_T E_\phi(T), \tag{4.2.15}$$

(specialization of (1.6)) where $E_\phi(\,\cdot\,)$ is the energy (square of $H^1(\Omega)$-norm) of ϕ.

Conclusion. We submit that the direct approach in [L-T.17] is surely more desirable and amenable than application of Claim 2.1.

4.3 Euler-Bernoulli plate with clamped boundary controls. Case 1: Neumann control

The present subsection deals with the Euler-Bernoulli plate equation with 'clamped' boundary controls (in any dimension), while 'hinged' boundary controls will be considered in Subsection 4.4. In either case, the corresponding results of optimal regularity, exact controllability and uniform stabilization—all obtained by PDE energy methods!—have been known for over 10 years. Moreover, we claim that the regularity result $B^*L \in \mathcal{L}(L_2(0,T;U))$ is also true for each of the aforementioned E-B mixed problems. This result is contained in the treatments of the literature cited as a built-in block, rather than singled out in an explicit statement. Below we shall extract the necessary details from the literature. Ultimately, the message of the present, as well as of the next, subsection is the same as that of Section 4.2 dealing with the Schrödinger equation: that verifying the key assumptions of Claim 2.1—the regularity $B^*L \in \mathcal{L}(L_2(0,T;U))$ and the exact controllability of $\{A, B\}$—is not any easier—on the contrary!—than establishing uniform stabilization of $\{A, B\}$ *directly*. Thus, it pays off, possibly by much, to tackle uniform stabilization of $\{A, B\}$ directly, rather than attempting to apply the tortuous route of Claim 2.1. At any rate, in all of these results, PDE (hard analysis) energy methods are the key and critical tools; not soft methods.

For lack of space, and to limit repetitions, we shall state the three fundamental results of optimal regularity, exact controllability and uniform stabilization, and next establish the sought-after regularity of B^*L within the context of the treatments of the three aforementioned problems.

Open-loop and closed-loop feedback dissipative systems. Let Ω be an open bounded domain in $\mathbb{R}^n$ ($n = 2$, in the physical case of plates) with sufficiently smooth boundary Γ. We consider the following open-loop problem of the Euler-Bernoulli equation defined on Ω, with Neumann boundary control $g_2 \in L_2(0,T; L_2(\Gamma)) \equiv L_2(\Sigma)$, as well as its corresponding boundary dissipative version:

$$\begin{cases} v_{tt} + \Delta^2 v = 0; \\ v(0, \,\cdot\,) = v_0, \; v_t(0, \,\cdot\,) = v_1; \\ v|_\Sigma \equiv 0; \\ \left.\dfrac{\partial v}{\partial \nu}\right|_\Sigma = g_2; \end{cases} \qquad \begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } Q; \\ w(0, \,\cdot\,) = w_0, \; w_t(0, \,\cdot\,) = w_1 & \text{in } \Omega; \\ w|_\Sigma \equiv 0 & \text{in } \Sigma; \\ \left.\dfrac{\partial w}{\partial \nu}\right|_\Sigma = [\Delta(\mathcal{A}^{-1} w_t)]_\Sigma & \text{in } \Sigma, \end{cases} \qquad \begin{matrix} (4.3.1\text{a}) \\ (4.3.1\text{b}) \\ (4.3.1\text{c}) \\ (4.3.1\text{d}) \end{matrix}$$

with $Q = (0, T] \times \Omega$; $\Sigma = (0, T] \times \Gamma$. Moreover, the operator $\mathcal{A}$ is defined below in (4.3.6) as $\mathcal{A}w = \Delta^2 w$, $\mathcal{D}(\mathcal{A}) \equiv H^4(\Omega) \cap H_0^2(\Omega)$.

Regularity, exact controllability of the v-problem; uniform stabilization of the w-problem. References for this subsection include [Lio.3], [Lio.5], [L-T.6] for the v-problem and [O-T.1] for the w-problem. These references give a full account of these three problems. We begin by introducing the (state) space (of optimal regularity)

$$X \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'; \quad [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \equiv H^{-2}(\Omega); \quad \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^2(\Omega). \tag{4.3.2}$$

Theorem 4.3.1. (Regularity [Lio.3], [Lio.5]) Regarding the v-problem (4.3.1), with $y_0 = \{v_0, v_1\} = 0$, the following regularity result holds true for each $T > 0$ (recall the definition of L in (1.2b)): the map

$$L : g_2 \to Lg_2 = \{v, v_t\} \text{ is continuous } L_2(\Sigma) \to C([0,T]; X \equiv L_2(\Omega) \times H^{-2}(\Omega)). \tag{4.3.3}$$

Theorem 4.3.2. (Exact controllability [Lio.4], [Lio.5], [O-T.1]) Given any initial condition $\{v_0, v_1\} \in X$ and $T > 0$, there exists a $g_2 \in L_2(\Sigma)$, such that the corresponding solution of the v-problem (4.3.1) satisfies $\{v(T), v_t(T)\} = 0$.

Theorem 4.3.3. (Uniform stabilization [O-T.1]) With reference to the w-problem (4.3.1), we have that:

(i) the map $\{w_0, w_1\} \in X = L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \to \{w(t), w_t(t)\}$ defines a s.c. contraction semigroup e^{At} on X;

(ii)

$$\left.\frac{\partial w}{\partial \nu}\right|_{\Sigma} = [\Delta(\mathcal{A}^{-1}w_t)]_{\Sigma} \in L_2(0, \infty; L_2(\Gamma)) \text{ continuously in } \{w_0, w_1\} \in X; \tag{4.3.4}$$

(iii) there exists constants $M \geq 1$, $\delta > 0$, such that

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_X = \left\| e^{At} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_X \leq M e^{-\delta t} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_X, \quad t \geq 0. \tag{4.3.5}$$

Again, needless to say, in line with the content of Section 1, all three theorems above are obtained by PDE hard analysis energy methods (not by soft-analysis methods). As usual, the most challenging result to prove is Theorem 4.3.3 on uniform stabilization: this problem, in addition, requires a shift of topology from $X \equiv L_2(\Omega) \times H^{-2}(\Omega)$ (the space of the final result) to $H_0^2(\Omega) \times L_2(\Omega)$ (the space where the energy method works). This shift of topology is implemented by a *change of variable*: this is the same change of variable, noted below in (4.3.10), that is needed to establish the desired regularity of B^*L.

Abstract model of v-problem. We let

$$\mathcal{A}\psi = \Delta^2\psi,\ \mathcal{D}(\mathcal{A}) = H^4(\Omega) \cap H_0^2(\Omega);\ G_2 : H^s(\Gamma) \to H^{s+\frac{3}{2}}(\Omega),\ s \in \mathbb{R}; \tag{4.3.6a}$$

$$\varphi = G_2 g_2 \Longleftrightarrow \left\{ \Delta^2\varphi = 0 \text{ in } \Omega;\ \varphi|_\Gamma = 0,\ \left.\frac{\partial \varphi}{\partial \nu}\right|_\Gamma = g_2 \right\}. \tag{4.3.6b}$$

Then, the second order, respectively first order, abstract models (in additive form) of the v-problem (4.3.1) are [O-T.1], [L-T.6],

$$v_{tt} + \mathcal{A}v = \mathcal{A}G_2 g_2; \quad \frac{d}{dt}\begin{bmatrix} v \\ v_t \end{bmatrix} = A \begin{bmatrix} v \\ v_t \end{bmatrix} + Bg_2; \tag{4.3.7}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}; \quad Bg_2 = \begin{bmatrix} 0 \\ \mathcal{A}G_2 g_2 \end{bmatrix}; \quad B^* \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = G_2^* x_2, \tag{4.3.8}$$

where $*$ for B and G_2 refer actually to different topologies. With B^* defined by $(Bg_2, x)_X = (g_2, B^*x)_{L_2(\Gamma)}$ with respect to the X-topology, we readily find the expression in (4.3.8), since the second component of the space X is $[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$.

The operator B^*L. With $y_0 = \{v_0, v_1\} = 0$, we shall show that

$$B^*Lg_2 = B^* \begin{bmatrix} v(t; y_0 = 0) \\ v_t(t; y_0 = 0) \end{bmatrix} = G_2^* v_t(t; y_0 = 0) = -[\Delta z(t)]_\Gamma; \tag{4.3.9}$$

$$z(t) \equiv \mathcal{A}^{-1} v_t(t; y_0 = 0) \in C([0, T]; \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^2(\Omega)) \text{ continuously in } g_2 \in L_2(\Sigma). \tag{4.3.10}$$

The new variable $z(t)$ defined in (4.3.10) satisfies the following dynamics: abstract equation, and corresponding PDE-mixed problem

$$z_{tt} + \mathcal{A}z = G_2 g_{2t} \qquad \begin{cases} z_{tt} + \Delta^2 z = G_2 g_{2t} & \text{in } Q; & (4.3.11a) \\ z(0, \cdot) = z_0 = 0; \; z_t(0, \cdot) = z_1 & \text{in } \Omega; & (4.3.11b) \\ z|_\Sigma \equiv 0, \left.\dfrac{\partial z}{\partial \nu}\right|_\Sigma \equiv 0 & \text{in } \Sigma. & (4.3.11c) \end{cases}$$

Indeed, to establish (4.3.9) (right), (4.3.10), one uses the definition in (4.3.9) (left), followed by (4.3.8) for B^*, to obtain

$$B^* L g_2 = G_2^* v_t(t; y_0 = 0) = G_2^* \mathcal{A}\mathcal{A}^{-1} v_t(t; y_0 = 0) = G_2^* \mathcal{A} z(t) = -\Delta z(t)|_\Gamma, \tag{4.3.12}$$

where, in the last step, we have recalled the usual property $G_2^*\mathcal{A} = -\Delta \cdot |_\Gamma$ on $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^2(\Omega)$ [O-T.1, Eqn. (1.11)], [B-T.1, Eqn. (1.20), p. 49]. The abstract z-equation is readily obtained from the abstract v-equation, after applying throughout $\mathcal{A}^{-1}$ and $\frac{d}{dt}$ to it, and using the definition of $z(t)$ in (4.3.10), whose *a-priori* regularity in (4.3.10) follows from (4.3.3), (4.3.2). Since $z(t) \in H_0^2(\Omega)$, both B.C.'s are satisfied and the abstract z-equation leads to its corresponding PDE-version. By (4.3.19) below, and within the class (4.3.20), we can take $z_1 = 0$.

Remark 4.3.1. As already noted, the change of variable $v_t \to z$ in (4.3.10) and the resulting z-problems in (4.3.11) are precisely the same that were used in [O-T.1, Sect. 2.1] in obtaining the uniform stabilization, Theorem 4.3.3, *directly*; the only difference is the specific form of the right-hand side term (thus, the letter p was used in [O-T.1], Eqn. (2.11)], while the letter z is used now for a closely related, yet not identical system). In both cases, however, a time-derivative term occurs (in our case $G_2 g_{2t}$), which will require—in [O-T.1] as well as in Step 6 in the proof of Lemma 4.3.5 below, an integration by parts in t, to obtain the sought-after estimate. □

Theorem 4.3.4. With reference to (4.3.9), we have:

$$B^* L : \text{ continuous } L_2(0, T; L_2(\Gamma)) \to L_2(0, T; L_2(\Gamma)) \tag{4.3.13a}$$

equivalently, with reference to (4.3.11)

$$\text{the map } g_2 \to \Delta z|_\Sigma \text{ is continuous } L_2(0, T; L_2(\Gamma)) \to L_2(0, T; L_2(\Gamma)). \quad \square \tag{4.3.13b}$$

We shall see below in the proof that this result, though not explicitly stated, is built-in in the treatments of [O-T.1] of Theorem 4.3.3.

Proof. Step 1. (Basic energy identity) We return to the basic identity of the energy methods [O-T.1, Eqn. (2.24), p. 287], which we use with a vector field h satisfying (as usual in obtaining trace regularity results [L-L-T.1]) the additional condition $h|_\Gamma = \nu$. Thus, with $h \cdot \nu = 1$ on Γ, for the solution z of *a-priori* regularity $z \in C([0, T]; H_0^2(\Omega))$ as in (4.3.10), we have

$$\frac{1}{2} \int_\Sigma (\Delta z)^2 d\Sigma = \text{RHS}_1 + \text{RHS}_2 + b_{0,T}; \tag{4.3.14}$$

$$\text{RHS}_1 = \int_Q \Delta z \text{ div}[(H + H^T)\nabla z]dQ + \frac{1}{2}\int_Q z\Delta z\Delta(\text{div } h)dQ; \tag{4.3.15}$$

$$\text{RHS}_2 = -\int_Q G_2 g_{2t}\ h \cdot \nabla z\ dQ - \frac{1}{2}\int_Q G_2 g_{2t} z \text{ div } h\, dQ; \tag{4.3.16}$$

$$b_{0,T} = [(z_t, h \cdot \nabla z)_\Omega]_0^T + \frac{1}{2}[(z_t, z \text{ div } h)_\Omega]_0^T. \tag{4.3.17}$$

Step 2. (Estimate for RHS_1) From the *a-priori* regularity (4.3.10) for z, we immediately find that

$$\text{RHS}_1 = \mathcal{O}\left(\|g_2\|^2_{L_2(\Sigma)}\right), \quad \forall\ g_2 \in L_2(\Sigma). \tag{4.3.18}$$

Step 3. (Regularity of z_t) To handle RHS_2 (by integration by parts in t, precisely as in the proof of the uniform stabilization Theorem 4.3.3 given in [O-T.1, p. 283-289]), we need the regularity of z_t. By (4.3.10) and the v-equation (4.3.7), we obtain

$$\begin{aligned} z_t(t) &= \mathcal{A}^{-1}v_{tt} = \mathcal{A}^{-1}[-\mathcal{A}v + \mathcal{A}G_2 g_2] \\ &= -v + G_2 g_2 \in L_2(0,T; L_2(\Omega)) \text{ continuously in } g_2 \in L_2(\Sigma), \end{aligned} \tag{4.3.19}$$

by recalling that $v \in C([0,T]; L_2(\Omega))$ (see (4.3.3)) and that $G_2 g_2 \in L_2(0,T; H^{\frac{3}{2}}(\Omega))$, by virtue of (4.3.6a) with $s = 0$ on G_2 and $g_2 \in L_2(\Sigma)$.

Step 4. (Estimates for RHS_2 and $b_{0,T}$ for smoother g_2) Henceforth, to estimate both RHS_2 and $b_{0,T}$, we shall at first take g_2 within the smoother class

$$g_2 \in C([0,T]; L_2(\Gamma)), \quad g_2(0) = g_2(T) = 0. \tag{4.3.20}$$

This initial restriction is dictated by the fact that z_t in (4.3.19) is only in L_2 in time.

Lemma 4.3.5. In the present setting, we have

$$\text{RHS}_2 = \mathcal{O}\left(\|g_2\|^2_{L_2(\Sigma)}\right); \qquad b_{0,T} = \mathcal{O}\left(\|g_2\|^2_{L_2(\Sigma)}\right), \tag{4.3.21}$$

for all g_2 in the class (4.3.20). □

Step 5. (Proof of (4.3.21) for $b_{0,T}$) First from (4.3.10) and (4.3.3), (4.3.2), we have since $v_t(0) = v_1 = 0$:

$$z(0) = 0,\ z(T) = \mathcal{A}^{-1}v_t(T; y_0 = 0) \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^2(\Omega),\ \text{continuously in } g_2 \in L_2(\Sigma). \tag{4.3.22}$$

Next, for g_2 in the class (4.3.20) used in (4.3.19), we compute since $v(0) = v_0 = 0$:

$$z_t(0) = 0,\ z_t(T) = -v(T) \in L_2(\Omega),\ \text{continuously in } g_2 \in L_2(\Sigma), \tag{4.3.23}$$

where the regularity follows from (4.3.3). Using (4.3.22), (4.3.23) in (4.3.17), we readily obtain, as desired:

$$b_{0,T} = (z_t(T), h \cdot \nabla z(T))_\Omega + \frac{1}{2}(z_t(T), z(T) \text{ div } h)_\Omega = \mathcal{O}(\|g_2\|^2_{L_2(\Sigma)}), \tag{4.3.24}$$

for all g_2 in the class (4.3.20). Thus, (4.3.21) (right) is proved.

Step 6. (Proof of (4.3.21) for RHS_2) The most critical terms of RHS_2 to estimate is the first term in (4.3.16). As in the *direct* proof of the uniform stabilization Theorem 4.3.3 given in [O-T.1, p. 287], we integrate by parts in t, with g_2 in the class (4.3.20), thus obtaining

$$\int_Q G_2 g_{2t}\, h \cdot \nabla z\, dQ = \left[\int_\Omega G_2 g_2\, h \cdot \nabla z\, d\Omega\right]_0^T - \int_Q G_2 g_2\, h \cdot \nabla z_t dQ, \tag{4.3.25}$$

where the first term on the right side of (4.3.25) vanishes, since $g_2(0) = g_2(T) = 0$. Moreover, the usual divergence theorem [O-T.1, Eqn. (2.31), p. 288] yields with $h \cdot \nu = 1$:

$$\begin{aligned} &\int_0^T \int_\Omega G_2 g_2 h \cdot \nabla z_t d\Omega\, dt \\ &\quad = \int_0^T \int_\Gamma G_2 g_2 z_t h \cdot \nu\, d\Gamma\, dt - \int_0^T \int_\Omega z_t h \cdot \nabla(G_2 g_2) d\Omega\, dt \\ &\quad - \int_0^T \int_\Omega G_2 g_2 z_t \text{ div } h\, d\Omega\, dt = \mathcal{O}(\|g_2\|^2_{L_2(\Sigma)}), \end{aligned} \tag{4.3.26}$$

for all g_2 in the class (4.3.20). The indicated estimate in terms of g_2 in (4.3.26) follows by virtue of $z_t \in L_2(0, T; L_2(\Omega))$ (see (4.3.19)); $G_2 g_2 \in L_2(0, T; H^{\frac{3}{2}}(\Omega))$ by (4.3.6a) with $s = 0$ on G_2; and thus $|\nabla(G_2 g_2)| \in L_2(0, T; H^{\frac{1}{2}}(\Omega))$, all bounded by the $L_2(\Sigma)$-norm of g_2. A similar estimate as (4.3.26) holds true, *a-fortiori*, for the more regular second term in the definition of RHS_2 in (4.3.16). Accordingly, we obtain (4.3.21) for RHS_2. □

Step 7. We can then extend estimates (4.3.21) for RHS_2 and $b_{0,T}$ to all $g_2 \in L_2(\Sigma)$, by density, starting from the class (4.3.20). Using these extended estimates, as well as (4.3.18) in (4.3.14), we finally obtain

$$\int_\Sigma (\Delta z)^2 d\Sigma = \mathcal{O}(\|g_2\|^2_{L_2(\Sigma)}), \quad \forall\, g_2 \in L_2(\Sigma), \tag{4.3.27}$$

and (4.3.13b) is proved. The proof of Theorem 4.3.4 is complete. □

4.4 Euler-Bernoulli plate with clamped boundary controls. Case 2: Dirichlet control

Open-loop and closed-loop feedback dissipative systems. In the notation of Case 1 above, we consider the Euler-Bernoulli equation defined on Ω, with Dirichlet boundary control $g_1 \in$

$L_2(0,T;L_2(\Gamma))$, in both open-loop and closed-loop dissipative form:

$$\begin{cases} v_{tt} + \Delta^2 v = 0; \\ v(0,\ \cdot\ ) = v_0,\ v_t(0,\ \cdot\ ) = v_1; \\ v|_\Sigma = g_1; \\ \left.\dfrac{\partial v}{\partial \nu}\right|_\Sigma = 0; \end{cases} \qquad \begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } Q; & (4.4.1a) \\ w(0,\ \cdot\ ) = w_0,\ w_t(0,\ \cdot\ ) = w_1 & \text{in } \Omega; & (4.4.1b) \\ w|_\Sigma = -\left.\dfrac{\partial \Delta(\mathcal{A}^{-\frac{3}{2}} w_t)}{\partial \nu}\right|_\Sigma & \text{in } \Sigma; & (4.4.1c) \\ \left.\dfrac{\partial w}{\partial \nu}\right|_\Sigma = 0 & \text{in } \Sigma. & (4.4.1d) \end{cases}$$

Here the operator $\mathcal{A}$ is the same as in Section 4.3, Eqn. (4.3.6).

Regularity, exact controllability of the v-problem; uniform stabilization of the w-problem. References for this subsection are [Lio.4], [Lio.5], [L-T.6], [B-T.1]. These references give a full account of these three problems. We begin by introducing the (state) space (of optimal regularity)

$$Y \equiv [\mathcal{D}(\mathcal{A}^{\frac{1}{4}})]' \times [\mathcal{D}(\mathcal{A}^{\frac{3}{4}})]' \equiv H^{-1}(\Omega) \times V', \quad V = \left\{ f \in H^3(\Omega) :\ f|_\Gamma = \left.\frac{\partial f}{\partial \nu}\right|_\Gamma = 0 \right\}. \tag{4.4.2}$$

Theorem 4.4.1. (Regularity [Lio.5], [L-T.6, Theorem 1.0, p. 331] Regarding the v-problem (4.4.1), with $y_0 = \{v_0, v_1\} = 0$, the following regularity result holds true for each $T > 0$ (recall the definition of L in (1.2b)): the map

$$L : g_1 \to L g_1 = \{v, v_t\} \text{ is continuous } L_2(\Sigma) \to C([0,T]; Y \equiv H^{-1}(\Omega) \times V'). \qquad \Box \tag{4.4.3}$$

Theorem 4.4.2. (Exact controllability [L-T.6, Theorems 1.1 and 1.4]; [B-T.1, Theorem 1.3, Remark 1.1]) Assume that there exists a *coercive* vector field $h(x) \in [C^2(\overline{\Omega})]^n$ (in particular, a radial vector field $h(x) = x - x_0$, for some $x_0 \in \mathbb{R}^n$), such that

$$h \cdot \nu \geq 0 \qquad \text{on } \Gamma. \tag{4.4.4}$$

Given any initial condition $\{v_0, v_1\} \in Y$ and $T > 0$, there exists a $g_1 \in L_2(\Sigma)$ such that the corresponding solution of the v-problem (4.4.1) satisfies $\{v(T), v_t(T)\} = 0$. $\Box$

Theorem 4.4.3. (Uniform stabilization [B-T.1, Theorem 1.3, p. 51]) With reference to the w-problem (4.4.1), we have that:

(i) the map $\{w_0, w_1\} \in Y \equiv H^{-1}(\Omega) \times V' \to \{w(t), w_t(t)\}$ defines a s.c. contraction semigroup e^{At} on Y;

(ii)

$$w|_\Sigma = -\left.\frac{\partial \Delta(\mathcal{A}^{-\frac{3}{2}} w_t)}{\partial \nu}\right|_\Sigma \in L_2(0,\infty; L_2(\Gamma)) \text{ continuously in } \{w_0, w_1\} \in Y. \tag{4.4.5}$$

Moreover, assume the geometrical condition of Theorem 4.4.2. Then, there exist constants $M \geq 1$, $\delta > 0$, such that

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_Y = \left\| e^{At} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_Y \leq M e^{-\delta t} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_Y, \quad t \geq 0. \tag{4.4.6}$$

We stress again, in line with the content of Section 1, that all three theorems above are obtained by PDE hard analysis energy methods (not by soft-analysis methods). As usual, the most challenging result to prove is Theorem 4.4.3 on uniform stabilization: this problem, in addition, requires a shift of topology from $Y \equiv H^{-1}(\Omega) \times V' \equiv [\mathcal{D}(\mathcal{A}^{\frac{1}{4}})]' \times [\mathcal{D}(\mathcal{A}^{\frac{3}{4}})]'$ (the space of the final result) to $\mathcal{D}(\mathcal{A}^{\frac{3}{4}}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{4}})$ (the space where the energy method works). This shift of topology is implemented by a *change of variable*: this is the same change of variable, noted below in (4.4.10b), that is needed to establish the desired regularity of B^*L.

Abstract model of v-problem. We let

$$\mathcal{A}\psi = \Delta^2\psi, \ \mathcal{D}(\mathcal{A}) = H^4(\Omega) \cap H_0^2(\Omega); \ G_1 : H^s(\Gamma) \to H^{s+\frac{1}{2}}(\Omega), \quad s \in \mathbb{R}; \tag{4.4.7a}$$

$$\varphi = G_1 g_1 \iff \left\{ \Delta^2\varphi = 0 \text{ in } \Omega; \ \varphi|_\Gamma = g_1, \ \left.\frac{\partial\varphi}{\partial\nu}\right|_\Gamma = 0 \right\}. \tag{4.4.7b}$$

Then, the second order, respectively first order, abstract models (in additive form) of the v-problem (4.4.1) are [L-T.6], [B-T.1],

$$v_{tt} + \mathcal{A}v = \mathcal{A}G_1 g_1; \quad \frac{d}{dt}\begin{bmatrix} v \\ v_t \end{bmatrix} = A \begin{bmatrix} v \\ v_t \end{bmatrix} + Bg_1; \tag{4.4.8}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}; \quad Bg_1 = \begin{bmatrix} 0 \\ \mathcal{A}G_1 g_1 \end{bmatrix}; \quad B^* \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = G_1^* \mathcal{A}^{-\frac{1}{2}} x_2, \tag{4.4.9}$$

where $*$ for B and G_1 refer to different topologies. With B^* defined by $(Bg_1, x)_Y = (g_1, B^*x)_{L_2(\Gamma)}$ with respect to the Y-topology, we readily find the expression in (4.4.9), since the second component of the space Y is $[\mathcal{D}(\mathcal{A}^{\frac{3}{4}})]'$.

The operator B^*L. With $y_0 = \{v_0, v_1\} = 0$, we shall show that

$$B^*Lg_1 = B^* \begin{bmatrix} v(t; y_0 = 0) \\ v_t(t; y_0 = 0) \end{bmatrix} = G_1^* \mathcal{A}^{-\frac{1}{2}} v_t(t; y_0 = 0) = \left.\frac{\partial \Delta z(t)}{\partial \nu}\right|_\Gamma; \tag{4.4.10a}$$

$$z(t) \equiv \mathcal{A}^{-\frac{3}{2}} v_t(t; y_0 = 0) \in C([0,T]; \mathcal{D}(\mathcal{A}^{\frac{3}{4}}) \equiv V) \text{ continuously in } g_1 \in L_2(\Sigma). \tag{4.4.10b}$$

The new variable $z(t)$ defined in (4.4.10) satisfies the following dynamics: abstract equation, and corresponding PDE-mixed problem

$$z_{tt} + \mathcal{A}z = \mathcal{A}^{-\frac{1}{2}} G_1 g_{1t} \qquad \begin{cases} z_{tt} + \Delta^2 z = \mathcal{A}^{-\frac{1}{2}} G_1 g_{1t} & \text{in } Q; \quad (4.4.11a) \\ z(0, \cdot) = z_0 = 0; \ z_t(0, \cdot) = z_1 & \text{in } \Omega; \quad (4.4.11b) \\ z|_\Sigma \equiv 0, \left.\dfrac{\partial z}{\partial \nu}\right|_\Sigma \equiv 0 & \text{in } \Sigma. \quad (4.4.11c) \end{cases}$$

Indeed, to obtain (4.4.10a) (right), (4.4.11), one uses the definition in (4.4.9) (left), followed by (4.4.8) for B^*, to obtain

$$B^* L g_1 = G_1^* \mathcal{A}^{-\frac{1}{2}} v_t(t; y_0 = 0) = G_1^* \mathcal{A}\mathcal{A}^{-\frac{3}{2}} v_t(t; y_0 = 0) = G_1^* \mathcal{A} z(t) = \left.\frac{\partial \Delta z(t)}{\partial \nu}\right|_{\Gamma}, \tag{4.4.12}$$

where, in the last step, we have recalled the usual property $G_1^* \mathcal{A} = \left.\frac{\partial \Delta}{\partial \nu}\right|_{\Gamma}$ on V [B-T.1, Eqn. (1.19), p. 49], [L-T.6, Eqn. (2.4)]. The abstract z-equation is readily obtained from the abstract v-equation after applying throughout $\mathcal{A}^{-\frac{3}{2}}$ and $\frac{d}{dt}$ to it, and using the definition of $z(t)$ in (4.4.10b), whose *a-priori* regularity in (4.4.10b) follows from (4.4.3), (4.4.2). Since $z(t) \in \mathcal{D}(\mathcal{A}^{\frac{3}{4}}) = V$ (see (4.4.2)), both B.C. are satisfied and the abstract z-equation leads to its corresponding PDE-version. By (4.4.19) below, and within the class (4.4.20), we can take $z_1 = 0$.

Remark 4.4.1. As already noted, the change of variable $v_t \to z$ in (4.4.10) and the resulting z-problems in (4.4.11) are precisely the same that were used in [B-T.1] in obtaining the uniform stabilization, Theorem 4.4.4, *directly*; the only difference is the specific form of the right-hand side term (thus, the letter p was used in [B-T.1, Eqn. (3.12), p. 55], while the letter z is used now for a closely related, yet not identical system). In both cases, however, a time-derivative occurs (in our case $\mathcal{A}^{-\frac{1}{2}} G_1 g_{1t}$), which will require—in [B-T.1] as well as in Step 3 below, an integration by parts in t, to obtain the sought-after estimate. □

Theorem 4.4.4. With reference to (4.4.10), we have:

$$B^* L : \text{ continuous } L_2(0, T; L_2(\Gamma)) \to L_2(0, T; L_2(\Gamma)) \tag{4.4.13a}$$

equivalently, with reference to (4.4.11)

$$\text{the map } g_1 \to \left.\frac{\partial \Delta z}{\partial \nu}\right|_{\Gamma} \text{ is continuous } L_2(0, T; L_2(\Gamma)) \to L_2(0, T; L_2(\Gamma)). \quad \square \tag{4.4.13b}$$

We shall see below in the proof that this result, though not explicitly stated, is built-in in the treatments of [Lio.5], [L-T.6], [B-T.1] of Theorems 4.4.1, 4.4.2, and 4.4.3. This situation is the exact counterpart of what was noted in Section 4.3, in the paragraph just below Theorem 4.3.4.

Proof. Step 1. (Basic energy identity) We return to the basic identity of the energy method [L-T.6, Eqn. (2.24), p. 340], [B-T.1, Eqn. (3.31), p. 58, with $\beta = 0$ and values at $t = T$], which we use with a vector field h satisfying (as usual in obtaining trace regularity results [L-L-T.1]) the additional condition $h|_{\Gamma} = \nu$. Thus, with $h \cdot \nu = 1$ on Γ, for the solution z of *a-priori* regularity $z \in C([0, T]; \mathcal{D}(\mathcal{A}^{\frac{3}{4}}) \equiv V)$ as in (4.4.10):

$$\int_{\Sigma} \frac{\partial \Delta z}{\partial \nu} h \cdot \nabla(\Delta z) d\Sigma - \frac{1}{2} \int_{\Sigma} |\nabla(\Delta z)|^2 h \cdot \nu \, d\Sigma + \frac{1}{2} \int_{\Sigma} \frac{\partial \Delta z}{\partial \nu} \Delta z \text{ div } h \, d\Sigma$$
$$= \text{RHS}_1 + \text{RHS}_2 + \beta_{0,T}; \tag{4.4.14}$$

$$\mathrm{RHS}_1 = \int_Q H\nabla(\Delta z)\cdot\nabla(\Delta z)dQ + \int_Q H\nabla z_t\cdot\nabla z_t dQ; \tag{4.4.15}$$

$$\mathrm{RHS}_2 = \int_Q \mathcal{A}^{-\frac{1}{2}}G_1 g_{1t} h\cdot\nabla(\Delta z)dQ + \int_Q \mathcal{A}^{-\frac{1}{2}}G_1 g_{1t}\Delta z \text{ div } h\, dQ \tag{4.4.16}$$

$$\beta_{o,T} = \left[\frac{1}{2}\int_\Omega \text{div } h\,\nabla z\cdot\nabla z_t\, d\Omega\right]_0^T - \left[\int_\Omega z_t\ h\cdot\nabla(\Delta z)d\Omega\right]_0^T. \tag{4.4.17}$$

Step 2. (Estimate for RHS_1) From the *a-priori* regularity (4.4.10) for z, and V as in (4.4.2), we immediately find that

$$\mathrm{RHS}_1 = \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)}\right), \quad \forall\ g_1 \in L_2(\Sigma). \tag{4.4.18}$$

Step 3. (Regularity of z_t) To handle RHS_2 (by integration by parts in t, precisely as in the proof of the uniform stabilization Theorem 4.4.3 given in [B-T.1, p. 59], we need the regularity of z_t. By (4.4.10b) and the v-equation (4.4.8), we obtain

$$\begin{aligned} z_t(t) \quad &= \quad \mathcal{A}^{-\frac{3}{2}}v_{tt} = \mathcal{A}^{-\frac{3}{2}}[-\mathcal{A}v + \mathcal{A}G_1 g_1] = -\mathcal{A}^{-\frac{1}{2}}v + \mathcal{A}^{-\frac{1}{2}}G_1 g_1 \\ &\in L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{4}}) \equiv H_0^1(\Omega)), \text{ continuously in } g_1 \in L_2(\Sigma), \end{aligned} \tag{4.4.19}$$

by recalling that $v \in C([0,T];[\mathcal{D}(\mathcal{A}^{\frac{1}{4}})]')$ (see (4.4.3), (4.4.2)) and that $G_1 g_1 \in L_2(0,T;H^{\frac{1}{2}}(\Omega))$, by virtue of (4.4.6a) with $s=0$ on G_1, hence (conservatively) $\mathcal{A}^{-\frac{1}{2}}G_1 g_1 \in L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^2(\Omega))$ for $g_1 \in L_2(\Sigma)$.

Step 4. (Estimates for RHS_2 and $b_{0,T}$ for smoother g_1) Henceforth, to estimate both RHS_1 and $\beta_{0,T}$, we shall at first take g_1 within the smoother class

$$g_1 \in C([0,T];L_2(\Gamma)), \quad g_1(0) = g_1(T) = 0. \tag{4.4.20}$$

This initial restriction is dictated by the fact that z_t in (4.4.19) is only in L_2 in time.

Lemma 4.4.5. In the present setting, we have

$$\mathrm{RHS}_2 = \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)}\right);\ \beta_{0,T} = \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)}\right), \tag{4.4.21}$$

for all g_1 in the class (4.4.20). $\square$

Step 5. (Proof of (4.4.21) for $\beta_{0,T}$) First from (4.4.10) and (4.4.3), (4.4.2), we have since $v_t(0) = v_1 = 0$:

$$z(0) = 0,\ z(T) = \mathcal{A}^{-\frac{3}{2}}v_t(T;y_0 = 0) \in \mathcal{D}(\mathcal{A}^{\frac{3}{4}}) \equiv V, \\ \text{continuously in } g_1 \in L_2(\Sigma). \tag{4.4.22}$$

Next, for g_1 in the class (4.4.20) used in (4.4.19), we compute since $v_t(0) = v_1 = 0$:

$$z_t(0) = 0, \; z_t(T) = -\mathcal{A}^{-\frac{1}{2}} v_t(T; y_0 = 0) \in \mathcal{D}(\mathcal{A}^{\frac{1}{4}}) \equiv H_0^1(\Omega), \text{ continuously in } g_1 \in L_2(\Sigma), \quad (4.4.23)$$

where the regularity follows from (4.4.3), (4.4.2). Using (4.4.22), (4.4.23) in (4.4.17), we readily obtain, as desired:

$$\begin{aligned} \beta_{0,T} &= \frac{1}{2}\int_\Omega \text{div } h \nabla z(T) \cdot \nabla z_t(T) d\Omega - \int_\Omega z_t(T) h \cdot \nabla(\Delta z(T)) d\Omega \\ &= \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)}\right), \end{aligned} \quad (4.4.24)$$

for all g_1 in the class (4.4.20). Thus, (4.4.21) (right) is proved.

Step 6. (Proof of (4.4.21) for RHS_2) The most critical terms of RHS_2 to estimate is the first term in (4.4.16). As in the *direct* proof of the uniform stabilization Theorem 4.4.3 in [B-T.1, p. 59], we integrate by parts in t, with g_1 in the class (4.4.20), thus obtaining

$$\int_Q \mathcal{A}^{-\frac{1}{2}} G_1 g_{1t} h \cdot \nabla \Delta z \, dQ = \left[\int_\Omega \mathcal{A}^{-\frac{1}{2}} G_1 g_1 h \cdot \nabla \Delta z \, d\Omega\right]_0^T - \int_Q \mathcal{A}^{-\frac{1}{2}} G_1 g_1 h \cdot \nabla \Delta z_t dQ, \quad (4.4.25)$$

where the first term on the right side of (4.4.25) vanishes, since $g_1(0) = g_1(T) = 0$. Moreover, we shall see that

$$\int_Q \mathcal{A}^{-\frac{1}{2}} G_1 g_1 h \cdot \nabla \Delta z_t dQ = \int_0^T (G_1 g_1, \mathcal{A}^{-\frac{1}{2}} h \cdot \nabla(\Delta z_t))_\Omega dt = \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)}\right). \quad (4.4.26)$$

Since, by [B-T.1, Lemma 3.5, p. 50] the second term in the inner product satisfies (as $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^2(\Omega)$):

$$\begin{aligned} \|\mathcal{A}^{-\frac{1}{2}} h \cdot \nabla \Delta z_t\|_{L_2(\Omega)} &= \|h \cdot \nabla \Delta z_t\|_{[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} \leq C_1 \|h \cdot \nabla \Delta z_t\|_{H^{-2}(\Omega)} \\ &\leq C_h \|\nabla(\Delta z_t)\|_{H^{-2}(\Omega)} \leq C_h \|z_t\|_{H^1(\Omega)} = C_h \|\mathcal{A}^{\frac{1}{4}} z_t\|_{L_2(\Omega)}, \end{aligned} \quad (4.4.27)$$

where, in the last step, we have used $z_t|_\Gamma = 0$. Recalling (4.4.19),

$$\|\mathcal{A}^{\frac{1}{4}} z_t\|_{L_2(0,T;L_2(\Omega))} = \mathcal{O}\left(\|g_1\|_{L_2(\Sigma)}\right), \quad (4.4.28)$$

we then see that (4.4.27) and (4.4.28), used in the integral term of (4.4.26), produce the indicated estimates. From (4.4.26) used in (4.4.25), we conclude that

$$\int_Q \mathcal{A}^{-\frac{1}{2}} G_1 g_{1t} \; h \cdot \nabla \Delta z \, dQ = \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)}\right), \quad (4.4.29)$$

for all g_1 in the class (4.4.20), as desired. A similar estimate as the one in (4.4.29) holds true, *a-fortiori* for the more regular second term in the definition of RHS_2 in (4.4.16). Accordingly, we obtain (4.4.21) for RHS_2. □

Step 7. We can then extend estimates (4.4.21) for RHS_2 and $\beta_{0,T}$ to all $g_1 \in L_2(\Sigma)$, by density, starting from the class (4.4.20). Using these extended estimates, as well as (4.4.18) in (4.4.14), we obtain for its Right-Hand Side:

$$\text{RHS of (4.4.14)} = \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)}\right), \quad \forall g_1 \in L_2(\Sigma). \tag{4.4.30}$$

Step 8. It remains to handle the Left-Hand Side (boundary terms) of identity (4.4.14). We first note that since $h|_\Gamma = \nu \perp \Gamma$, then as usual,

$$\text{on } \Gamma: \; h \cdot \nabla(\Delta z) = \frac{\partial \Delta z}{\partial \nu}; \; |\nabla(\Delta z)|^2 = \left|\frac{\partial \Delta z}{\partial \nu}\right|^2 + |\nabla_\sigma(\Delta z)|^2, \tag{4.4.31}$$

where ∇_σ denotes the tangential gradient on Γ. Hence, regarding the first two terms on the LHS of (4.4.14) we have by (4.4.31):

$$\text{on } \Gamma: \; \frac{\partial \Delta z}{\partial \nu} h \cdot \nabla(\Delta z) - \frac{1}{2}|\nabla(\Delta z)|^2 h \cdot \nu = \frac{1}{2}\left|\frac{\partial \Delta z}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla_\sigma(\Delta z)|^2. \tag{4.4.32}$$

Hence, (4.4.32) yields for the Left-Hand Side of (4.4.14),

$$\text{LHS of (4.4.14)} = \frac{1}{2}\int_\Sigma \left|\frac{\partial \Delta z}{\partial \nu}\right|^2 d\Sigma - \frac{1}{2}|\nabla_\sigma(\Delta z)|^2 + \frac{1}{2}\int_\Sigma \frac{\partial \Delta z}{\partial \nu}\, \Delta z \text{ div } h \, d\Sigma \tag{4.4.33}$$

$$\geq \left(\frac{1}{2} - \frac{\epsilon}{4}\right)\int_\Sigma \left|\frac{\partial \Delta z}{\partial \nu}\right|^2 d\Sigma - \frac{C_h}{4\epsilon}\int_\Sigma |\Delta z|^2 d\Sigma - \frac{1}{2}\int_\Sigma |\nabla_\sigma(\Delta z)|^2 d\Sigma; \tag{4.4.34}$$

$$\int_0^T \int_\Gamma |\Delta z|^2 d\Sigma \leq C \int_0^T \|z\|^2_{H^3(\Omega)} dt = \mathcal{O}\left(\|z\|^2_{L_2(0,T;V)}\right) \tag{4.4.35}$$

$$\text{(by (4.4.10))} \qquad = \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)}\right). \tag{4.4.36}$$

In the last step in (4.4.35) we have recalled that z satisfies the two B.C. (4.4.11c), as well as the space V in (4.4.2). To go from (4.4.35) to (4.4.36), we have invoked (4.4.10). Finally, substituting estimate (4.4.36) in (4.4.34) and recalling (4.4.30), we obtain

$$\left(\frac{1}{2} - \frac{\epsilon}{4}\right)\int_\Sigma \left|\frac{\partial \Delta z}{\partial \nu}\right|^2 d\Sigma = \mathcal{O}_\epsilon\left(\|g_1\|^2_{L_2(\Sigma)}\right) + \frac{1}{2}\int_\Sigma |\nabla_\sigma(\Delta z)|^2 d\Sigma. \tag{4.4.37}$$

Step 9. We now estimate in terms of $g_1 \in L_2(\Sigma)$ the last integral term in the RHS of (4.4.37).

Lemma 4.4.6. With reference to problem (4.4.11) and (4.4.37) we have

$$\int_\Sigma |\nabla_\sigma(\Delta z)|^2 d\Sigma = \mathcal{O}(\|g_1\|^2_{L_2(\Sigma)}), \; g_1 \in L_2(\Sigma). \tag{4.4.38}$$

Proof. As in [L-L-T.1], [L-T.6], [L-T.21, p. 970], we introduce the following operator

$$\mathcal{B} \equiv \text{first-order differential operator on } \overline{\Omega},\ \textit{tangential} \text{ to } \Gamma \text{ (i.e., without transversal derivatives to } \Gamma\text{, when expressed in local coordinates) and with smooth coefficients on } \overline{\Omega}. \tag{4.4.39}$$

We next define a new variable

$$y \equiv \mathcal{B}z \in C([0,T]; H^2(\Omega)),\ y_t \equiv \mathcal{B}z_t \in L_2(0,T; L_2(\Omega)) \quad \text{continuously in } g_1 \in L_2(\Sigma); \tag{4.4.40a}$$

$$y_t \in C([0,T]; L_2(\Omega)) \text{ for } g_1 \text{ in the class (4.4.20)} \quad \text{continuously in the } L_2(\Sigma)\text{-norm of } g_1, \tag{4.4.40b}$$

where the indicated regularity of $\{y, y_t\}$ in (4.4.40a) stems from (4.4.10b) and (4.4.19), respectively. Moreover, (4.4.19) yields (4.4.40b) if g_1 belongs to the class (4.4.20).

Thus, applying $\mathcal{B}$ to the PDE z-problem (4.4.11) yields the corresponding y-problem

$$\begin{cases} y_{tt} + \Delta^2 y = F & \text{in } (0,T] \times \Omega \equiv Q; & (4.4.41a) \\ y(0,\ \cdot\ ) = 0;\ y_1(0,\ \cdot\ ) = y_1 = \mathcal{B}z_1 & \text{in } \Omega; & (4.4.41b) \\ y|_\Sigma \equiv 0,\ \left.\dfrac{\partial y}{\partial \nu}\right|_\Sigma = u & \text{in } (0,T] \times \Gamma \equiv \Sigma, & (4.4.41c) \end{cases}$$

where

$$F \equiv [\Delta^2, \mathcal{B}]z + \mathcal{A}^{-\frac{1}{2}} G_1 g_{1t};\ K_I z \equiv [\Delta^2, \mathcal{B}]z \in C([0,T]; H^{-1}(\Omega)); \tag{4.4.42}$$

$$u \equiv \left.\left[\frac{\partial}{\partial \nu}, \mathcal{B}\right] z\right|_\Gamma \in C([0,T]; H^{\frac{3}{2}}(\Gamma)). \tag{4.4.43}$$

Both regularity properties in (4.4.42), (4.4.43), are continuous in $g_1 \in L_2(\Sigma)$. Moreover, if g_1 is in the class (4.4.20), we can take $y_1 = 0$. The regularity of the 4^{th}-order commutator in (4.4.42) and of the 1^{st}-order commutator in (4.4.43) follows from the regularity for z in (4.4.10b) as well as trace theory in the former case. Further, we notice that by (4.4.39), (4.4.40a), we have

$$\begin{aligned} \int_\Gamma |\nabla_\sigma(\Delta z|_\Gamma)|^2 d\Gamma &= \int_\Gamma |\mathcal{B}(\Delta z|_\Gamma)|^2 d\Gamma = \int_\Gamma |[\Delta(\mathcal{B}z)]_\Gamma|^2 d\Gamma + l.o.t. \\ &= \int_\Gamma |\Delta y|_\Gamma|^2 d\Gamma + l.o.t. \end{aligned} \tag{4.4.44}$$

Thus, by (4.4.44), instead of establishing (4.4.38), we seek to prove equivalently that

$$\int_\Sigma |\Delta y|_\Gamma|^2 d\Sigma = \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)}\right), \quad g_1 \in L_2(\Sigma). \tag{4.4.45}$$

Furthermore, since u in (4.4.41c) is smooth, see (4.4.43), we replace the y-problem (4.4.41) with the following boundary homogeneous η-problem:

$$\begin{cases} \eta_{tt} + \Delta^2 \eta = F & \text{in } Q; \\ \eta(0, \ \cdot \ ) = 0, \ \eta_t(0, \ \cdot \ ) = y_1 & \text{in } \Omega; \\ w|_\Sigma \equiv 0, \ \left.\dfrac{\partial \eta}{\partial \nu}\right|_\Sigma \equiv 0 & \text{in } \Sigma, \end{cases} \qquad \begin{matrix} (4.4.46a) \\ (4.4.46b) \\ (4.4.46c) \end{matrix}$$

where F is defined by (4.4.42) and where η is subject to the same *a-priori* regularity as y (compare with (4.4.40a–b)):

$$\eta \in C([0,T]; H_0^2(\Omega)); \ \eta_t \in L_2(0,T; L_2(\Omega)) \text{ continuously in } g_1 \in L_2(\Sigma); \tag{4.4.47a}$$

$$\begin{aligned} \eta_t \in C([0,T]; L_2(\Omega)) \text{ for } g_1 \text{ in the class (4.4.20) continuously in the} \\ L_2(\Sigma)\text{-norm of } g_1, \text{ in which case we can take } y_1 = 0. \end{aligned} \tag{4.4.47b}$$

Accordingly, we now seek to establish that

$$\int_\Sigma |\Delta \eta|_\Gamma|^2 d\Sigma = \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)} \right), \ g_1 \in L_2(\Sigma), \tag{4.4.48}$$

which is equivalent to (4.4.45), hence to the original sought-after estimate (4.4.38).

Proof of (4.4.48). We take, at first, g_1 in the class (4.4.20), prove estimate (4.4.48), and then extend it to all $g_1 \in L_2(\Sigma)$. Thus, below, we may assume the regularity (4.4.47b). To establish (4.4.48), we recall the energy method based on the multiplier $h \cdot \nabla \eta$ for problem (4.4.46), where h is a smooth vector field such that $h = \nu$ on Γ, and hence $h \cdot \nu = 1$ on Γ. We can thus invoke the usual identity, see, e.g., [O-T.1, Eqn. (2.20), p. 286], for the η-problem (4.4.46):

$$\frac{1}{2} \int_\Sigma (\Delta \eta)^2 h \cdot \nu \, d\Sigma = \text{RHS}_1 + \text{RHS}_2 + b_{0,T}; \tag{4.4.49}$$

$$\begin{aligned} \text{RHS}_1 \quad &= \quad \frac{1}{2} \int_Q [\eta_t^2 - (\Delta \eta)^2] \text{div } h \, dQ + \int_Q \Delta \eta \, \text{div } [(H + H^T) \nabla \eta) dQ \\ &- \int_Q \Delta \eta \nabla \eta \cdot \nabla (\text{div } h) dQ \end{aligned} \tag{4.4.50}$$

$$\text{RHS}_2 \quad = \quad - \int_Q F h \cdot \nabla \eta \, dQ; \ b_{0,T} = [(\eta_t(t), h \cdot \nabla \eta(t))_\Omega]_0^T. \tag{4.4.51}$$

¿From the *a-priori* regularity of $\{\eta, \eta_t\}$ in (4.4.47a-b), we have

$$\text{RHS}_1 \quad = \quad \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)} \right), \quad \forall \, g_1 \in L_2(\Sigma); \tag{4.4.52}$$

$$b_{0,T} \quad = \quad \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)} \right), \text{ for } g_1 \text{ in the class (4.4.40)}. \tag{4.4.53}$$

[We are taking g_1 in the class (4.4.40), since $b_{0,T}$ requires continuity in time of η_t as in (4.4.47b), which is not available in (4.4.47a). Alternatively, as in [L-L-T.1], we could apply the multiplier $(T-t)h \cdot \nabla \eta$ to problem (4.4.46) to eliminate the terns in $[\quad]_0^T$.] It remains to show that

$$\text{RHS}_2 = -\int_Q Fh \cdot \nabla \eta \, dQ \equiv \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)} \right), \; g_1 \in L_2(\Sigma). \tag{4.4.54}$$

We now establish (4.4.54). Since $F = K_I z + \mathcal{A}^{-\frac{1}{2}} G_1 g_{1t}$ by (4.4.42), where K_I is the interior commutator in (4.4.42), we proceed for each term separately. We have:

$$\int_Q K_I z \, h \cdot \nabla \eta \, dQ = \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)} \right), \quad g_1 \in L_2(\Sigma). \tag{4.4.55}$$

This is so for the following reasons. First, we have $K_I z \in C([0,T]; H^{-1}(\Omega))$ continuously in $g_1 \in L_2(\Sigma)$ by (4.4.42), while preliminary $|\nabla \eta| \in C([0,T]; H^1(\Omega))$. Next, the latter combined with $\eta|_\Sigma = 0$, hence $\nabla \eta \perp \Gamma$, and $\frac{\partial \eta}{\partial \nu} = \nabla \eta \cdot \nu = 0$ on Σ, hence $|\nabla \eta| = 0$ on Σ, yields finally $|\nabla \eta| \in C([0,T]; H_0^1(\Omega))$ continuously in $g_1 \in L_2(\Sigma)$, and (4.4.55) is proved. [We could also use the divergence theorem [O-T.1, Eqn. (2.3.1), p. 288] to reach the same conclusion.] Similarly,

$$\int_\Omega \int_0^T \mathcal{A}^{-\frac{1}{2}} G_1 g_{1t} h \cdot \nabla \eta \, dt \, d\Omega = \left[\int_\Omega \mathcal{A}^{-\frac{1}{2}} G_1 g_1 \, h \cdot \nabla \eta \, d\Omega \right]_0^T$$

$$- \int_Q \mathcal{A}^{-\frac{1}{2}} G_1 g_1 \, h \cdot \nabla \eta_t dQ = \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)} \right), \tag{4.4.56}$$

since $\mathcal{A}^{-\frac{1}{2}} G_1 g_1 \in L_2(0,T; \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^2(\Omega))$ for $g_1 \in L_2(\Sigma)$, as noted below (4.4.19) and $|\nabla \eta_t| \in L_2(0,T; H^{-1}(\Omega))$ for $g_1 \in L_2(\Sigma)$ by (4.4.47a). Thus, (4.4.56) is proved. Then, estimates (4.4.55) and (4.4.56) as well as $F \equiv K_I z + \mathcal{A}^{-\frac{1}{2}} G_1 g_{1t}$ yield estimate (4.4.54), as desired. Thus, estimate (4.4.48) is proved. Equivalently, estimate (4.4.45) and the sought-after estimate (4.4.38) are established as well. $\square$

Step 10. We use (4.4.38) in (4.4.37) and obtain

$$\int_\Sigma \left| \frac{\partial \Delta z}{\partial \nu} \right|^2 d\Sigma = \mathcal{O}\left(\|g_1\|^2_{L_2(\Sigma)} \right), \; \forall \, g_1 \in L_2(\Sigma), \tag{4.4.57}$$

and Theorem 4.4.4 is finally proved. $\square$

4.5 Euler-Bernoulli plate with hinged boundary controls. Case 1: Control in the 'moment' B.C.

Open-loop and closed-loop feedback dissipative systems. We let, again, Ω be an open bounded domain in $\mathbb{R}^n$ ($n = 2$ in the physical case of plates) with sufficiently smooth $C^?$-boundary Γ. We consider the following open-loop problem of the Euler-Bernoulli equation defined on Ω), with boundary control $g_2 \in L_2(0,T;L_2(\Gamma)) \equiv L_2(\Sigma)$, in the 'moment' B.C., as well as its corresponding boundary dissipative version:

$$\begin{cases} v_{tt} + \Delta^2 v = 0; \\ v(0,\ \cdot\ ) = v_0,\ v_t(0,\ \cdot\ ) = v_1; \\ v|_\Sigma \equiv 0; \\ \Delta v\Big|_\Sigma = g_2; \end{cases} \qquad \begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } Q; \\ w(0,\ \cdot\ ) = w_0,\ w_t(0,\ \cdot\ ) = w_1 & \text{in } \Omega; \\ w|_\Sigma \equiv 0 & \text{in } \Sigma; \\ \Delta w\Big|_\Sigma = \frac{\partial}{\partial\nu}(\mathcal{A}^{-1}w_t) & \text{in } \Sigma, \end{cases} \qquad \begin{matrix} (4.5.1a) \\ (4.5.1b) \\ (4.5.1c) \\ (4.5.1d) \end{matrix}$$

with $Q = (0,T] \times \Omega$; $\Sigma = (0,T] \times \Gamma$. Moreover, the operator $\mathcal{A}$ is defined below in (4.5.6) as $\mathcal{A}f = -\Delta f$; $\mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega)$.

Regularity, exact controllability of the v-problem; uniform stabilization of the w-problem. References for this subsection include [L-T.7], [L-T.10], [L-T.14], [Lio.4], [Lio.5], [Li.1], [Le.1], [Las.2]. We begin by introducing the (state) space of optimal regularity

$$Y \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \equiv H_0^1(\Omega) \times H^{-1}(\Omega). \tag{4.5.2}$$

Theorem 4.5.1. (Regularity [L-T.7, Theorem 1.3, Eqns. (1.22), (1.23), p. 203]) Regarding the v-problem (4.5.1) with $y_0 = \{v_0, v_1\} = 0$, the following regularity result holds true for each $T > 0$ (recall the definition of L in (1.2b)): the map

$$L : g_2 \to Lg_2 = \{v, v_t\} \text{ is continuous } L_2(\Sigma) \to C([0,T];\ H_0^1(\Omega) \times H^{-1}(\Omega)); \tag{4.5.3a}$$

$$\to v_{tt} \text{ continuous } L_2(\Sigma) \to L_2(0,T;[\mathcal{D}(\mathcal{A}^{\frac{3}{2}})]' \equiv V'); \tag{4.5.3b}$$

$$V = \mathcal{D}(\mathcal{A}^{\frac{3}{2}}) = \{h \in H^3(\Omega) :\ h|_\Gamma = \Delta h|_\Gamma = 0\} \tag{4.5.4}$$

[warning: the operator A in [L-T.7, Theorem 1.3] is $A = \mathcal{A}^2$ in our present notation for $\mathcal{A}$, see [L-T.7, Eqns. (1.5), (1.6)]].

Theorem 4.5.2. (Exact controllability [Las.2], [Le.1]) Given any initial condition $\{v_0, v_1\} \in Y$ and $T > 0$, there exists a $g_2 \in L_2(\Sigma)$ such that the corresponding solution of the v-problem (4.5.1) satisfies $\{v(T), v_t(T)\} = 0$.

Remark 4.5.1. Exact controllability of the v-problem (4.5.1) with two boundary controls: $v|_\Sigma = g_1$ and $\Delta v|_\Sigma = g_2$, $g_1 \in H_0^1(0,T;L_2(\Gamma))$, $g_2 \in L_2(\Sigma)$ was previously obtained in [L-T.10, Theorem 1.2], [Lio.4], [Lio.5]. A different exact boundary controllability result with $g_1 = 0$ and $g_2 \in L_2(0,T;H^{\frac{1}{2}}(\Gamma))$, however, in the space $[H^2(\Omega) \cap H_0^1(\Omega)] \times L_2(\Omega)$ was obtained in [L-T.14, Theorem 1.1]. □

Theorem 4.5.3. (Uniform stabilization [Las.2]) With reference to the w-problem (4.5.1), we have that:

(i) the map $\{w_0, w_1\} \in Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \to \{w(t), w_t(t)\}$ defines a s.c. contraction semigroup e^{At} on Y;

(ii)

$$\Delta w|_{\Sigma} = \frac{\partial \mathcal{A}^{-1} w_t}{\partial \nu} \in L_2(0, \infty; L_2(\Gamma)) \tag{4.5.5}$$

continuously in $\{w_0, w_1\} \in Y$.

(iii) There exist constants $M \geq 1$, $\delta > 0$, such that

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_Y = \left\| e^{At} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_Y \leq M e^{-\delta t} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_Y, \ t \geq 0. \tag{4.5.6}$$

As in past Sections 4.1–4.4, and in line with the content of Section 1, we stress once more that all three theorems above are obtained by PDE hard analysis energy methods (not by soft-analysis methods). As usual, the most challenging result to prove is Theorem 4.5.3 on uniform stabilization:

Abstract model of v-problem. We let

$$\mathcal{A}\psi = -\Delta\psi, \ \mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega); \ G_2 : H^s(\Gamma) \to H^{s+\frac{5}{2}}(\Omega), \ s \in \mathbb{R}, \tag{4.5.7}$$

$$\varphi = G_2 g_2 \Longleftrightarrow \{\Delta^2 \varphi = 0 \text{ in } \Omega; \ \varphi|_\Gamma = 0, \ \Delta\varphi|_\Gamma = g_2 \text{ on } \Gamma\}, \tag{4.5.8}$$

and we recall the Dirichlet map $D : H^s(\Gamma) \to H^{s+\frac{1}{2}}(\Omega)$ defined in (4.2.4):

$$\varphi = D g_2 \Longleftrightarrow \{\Delta\varphi = 0 \text{ in } \Omega; \ \varphi|_\Gamma = g_2 \text{ on } \Gamma\}; \ G_2 = -\mathcal{A}^{-1} D, \tag{4.5.9}$$

where the last relationship is taken from [L-T.7, Remark 3.2, p. 211]. Then, the second-order, respectively first-order, abstract models (in additive form) of the v-problem (4.5.1) are [L-T.7], [L-T.10],

$$v_{tt} + \mathcal{A}^2 v = \mathcal{A}^2 G_2 g_2 = -\mathcal{A} D g_2; \ \frac{d}{dt} \begin{bmatrix} v \\ v_t \end{bmatrix} = A \begin{bmatrix} v \\ v_t \end{bmatrix} + B g_2; \tag{4.5.10}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A}^2 & 0 \end{bmatrix}; \ B g_2 = \begin{bmatrix} 0 \\ \mathcal{A}^2 G_2 g_2 \end{bmatrix}; \ B^* \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = G_2^* \mathcal{A} x_2 = -D^* x_2, \tag{4.5.11}$$

where $*$ for B, and G_2 and D, refer to different topologies. With B^* defined by $(B g_2, x)_Y = (g_2, B^* x)_{L_2(\Gamma)}$ with respect to the Y-topology defined in (4.5.2), we readily find the expression in (4.5.11) also by virtue of $G_2 = -\mathcal{A}^{-1} D$.

The operator $B^* L$. With $y_0 = \{v_0, v_1\} = 0$, we shall show that

$$B^* L g_2 = B^* \begin{bmatrix} v(t; y_0 = 0) \\ v_t(t; y_0 = 0) \end{bmatrix} = G_2^* \mathcal{A} v_t(t; y_0 = 0) = -D^* v_t(t; y_0) \tag{4.5.12a}$$

$$= \frac{\partial}{\partial \nu} \mathcal{A}^{-1} v_t(t; y_0 = 0) = \frac{\partial}{\partial \nu} z_t(t); \tag{4.5.12b}$$

$$z(t) = \mathcal{A}^{-1} v(t; y_0 = 0) \in C([0, T]; \mathcal{D}(\mathcal{A}^{\frac{3}{2}}) \equiv V) \text{ continuously in } g_2 \in L_2(\Sigma). \tag{4.5.13}$$

Indeed, to obtain (4.5.12a–b), one uses the definition in (4.5.11) for B^*, followed by the usual property that $G_2^*\mathcal{A}^2 = \frac{\partial}{\partial\nu}$ on $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ [L-T.7, Lemma 3.1, Eqn. (3.7), p. 212] or $D^*\mathcal{A} = -\frac{\partial}{\partial\nu}$ on $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H_0^1(\Omega)$ [L-T.17, Eqn. (1.21)].

The regularity of $z(t)$ noted in (4.5.13) follows from (4.5.3a) for v, and $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^1(\Omega)$. The new variable $z(t)$ defined in (4.5.13) satisfies the following dynamics: abstract equation, and corresponding PDE-mixed problem

$$z_{tt} + \mathcal{A}^2 z = \mathcal{A}G_2 g_2 = -Dg_2 \qquad \begin{cases} z_{tt} + \Delta^2 z = \mathcal{A}G_2 g_2 = -Dg_2 & \text{in } Q; & (4.5.14a) \\ z(0, \ \cdot\ ) = 0, \ z_t(0, \ \cdot\ ) = 0 & \text{in } \Omega; & (4.5.14b) \\ z|_\Sigma \equiv 0, \Delta z|_\Sigma \equiv 0 & \text{in } \Sigma. & (4.5.14c) \end{cases}$$

The abstract z-equation in (4.5.14) (left) is readily obtained from the abstract v-equation in (4.5.10), after applying $\mathcal{A}^{-1}$ and using the definition of $z(t)$ in (4.5.13). Since $z(t) \in \mathcal{D}(\mathcal{A}^{\frac{3}{2}}) \equiv V$ (see (4.5.4)), both B.C. are satisfied and the abstract z-equation leads to its corresponding PDE-version.

Remark 4.5.2. As already noted, the change of variable $v \to z$ in (4.5.13) and the resulting z-problems in (4.5.14) are precisely the same that were used in [Las.2, Eqns. (2.7), (2.8), (4.3)] in obtaining there the uniform stabilization, Theorem 4.5.3, *directly*; the only difference is that, in [Las.2, Eqns. (2.8), (4.3)] g_2 is expressed in feedback form: $g_2 = D^*\mathcal{A}p_t = \frac{\partial}{\partial\nu} p_t \in L_2(0,\infty; L_2(\Gamma))$ in the notation of [Las.2]. Thus, the letter p was used in [Las.2], while the letter z is used now. Thus, the techniques in the proof of the next, sought-after result are contained in [Las.2] and indeed in [L-T.10], [Lio.5]. □

Theorem 4.5.4. With reference to (4.5.12), we have

$$B^*L: \text{ continuous } L_2(0,T;L_2(\Gamma)) \to L_2(0,T;L_2(\Gamma)), \tag{4.5.15}$$

equivalently, with reference to (4.5.14),

$$\text{the map } g_2 \to \left.\frac{\partial z_t}{\partial\nu}\right|_\Sigma \text{ is continuous } L_2(0,T;L_2(\Gamma)) \to L_2(0,T;L_2(\Gamma)). \tag{4.5.16}$$

We shall see below in the proof that this result, though not explicitly stated, is built-in in the treatments of [Las.2], [L-T.7], [L-T.10], [Lio.4], [Lio.5] of Theorem 4.5.1.

Proof. Step 1. (Basic energy identity) We return to the basic identity of the energy method [Las.2], [L-T.7], [L-T.10], [Lio.5], which we use with a vector field h satisfying (as usual in obtaining trace regularity results [L-L-T.1]) the additional condition $h|_\Gamma = \nu$. Thus, with $h \cdot \nu = 1$ on Γ, for the solution z of *a-priori* regularity $z \in C([0,T]; \mathcal{D}(\mathcal{A}^{\frac{3}{2}}) \equiv V)$ as in (4.5.13), we have (e.g., [L-T.10, Eqns. (2.29), (2.32)], [L-T.7, Eqns. (2.1), (2.4)]):

$$\frac{1}{2}\int_\Sigma \left[\left(\frac{\partial\Delta z}{\partial\nu}\right)^2 + \left(\frac{\partial z_t}{\partial\nu}\right)^2\right] d\Sigma = \text{RHS}_1 + \text{RHS}_2 + b_{0,T}; \tag{4.5.17}$$

$$\begin{aligned}
\text{RHS}_1 &= \int_Q H\nabla\Delta z\cdot\nabla\Delta z\,dQ + \int_Q H\nabla z_t\cdot\nabla z_t\,dQ \\
&\quad +\frac{1}{2}\int_Q \left(|\nabla z_t|^2 - |\nabla\Delta z|^2\right)\ \text{div}\ h\,dQ + \int_Q z_t\nabla(\text{div}\ h)\cdot\nabla z_t dQ; \qquad (4.5.18)\\
\text{RHS}_2 &= -\int_Q Dg_2\nabla\Delta z\,dQ; \qquad (4.5.19)\\
b_{0,T} &= -\left[(z_t, h\cdot\nabla\Delta z)_{L_2(\Omega)}\right]_0^T. \qquad (4.5.20)
\end{aligned}$$

Step 2. (Regularity of z_t) To handle RHS_1, we need the *a-priori* regularity of z_t,

$$z_t = \mathcal{A}^{-1}v_t(t; y_0 = 0) \in C([0,T]; \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^1(\Omega)) \text{ continuously in } g_2 \in L_2(\Sigma), \qquad (4.5.21)$$

as it follows from (4.5.13), (4.5.3a) and $H^{-1}(\Omega) = [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, see (4.5.2).

Step 3. (Estimate of RHS_1) By (4.5.13) for z and (4.5.21) for z_t, we obtain

$$|\nabla\Delta z|,\ |\nabla z_t| \in C([0,T]; L_2(\Omega)),\ \text{continuously in } g_2 \in L_2(\Sigma). \qquad (4.5.22)$$

Using (4.5.22) in (4.5.18) readily yields

$$\text{RHS}_1 = \mathcal{O}\left(\|g_2\|^2_{L_2(\Sigma)}\right), \quad \forall\ g_2 \in L_2(\Sigma). \qquad (4.5.23)$$

Step 4. (Estimates of RHS_2 and $b_{0,T}$) From (4.5.19) and (4.5.20), by virtue of (4.5.21), (4.5.22), we readily obtain

$$\text{RHS}_2 + b_{0,T} = \mathcal{O}\left(\|g_2\|^2_{L_2(\Sigma)}\right), \quad \forall\ g_2 \in L_2(\Sigma). \qquad (4.5.24)$$

Step 5. (Final estimate) Using (4.5.23)–(4.5.24) in (4.5.17) yields

$$\frac{1}{2}\int_\Sigma \left[\left(\frac{\partial\Delta z}{\partial\nu}\right)^2 + \left(\frac{\partial z_t}{\partial\nu}\right)^2\right] d\Sigma = \mathcal{O}\left(\|g_2\|^2_{L_2(\Sigma)}\right), \quad \forall\ g_2 \in L_2(\Sigma), \qquad (4.5.25)$$

and (4.5.25) *a-fortiori* proves (4.5.16), as desired. The proof of Theorem 4.5.4 is complete. □

Remark 4.5.3. In this case, the proof of Theorem 4.5.4 is easier than the proof of uniform stabilization in [Las.2]. But Claim 2.1 requires also exact controllability.

4.6 Euler-Bernoulli plate with hinged boundary controls. Case 2: Control in the Dirichlet B.C.

Open-loop and closed-loop feedback dissipative systems.

In the notation for Ω, Γ, $\mathcal{A}$ of Section 4.5, we consider now the following open-loop problem of the Euler-Bernoulli equation with boundary control $g_1 \in L_2(0,T;L_2(\Gamma)) \equiv L_2(\Sigma)$ and its corresponding boundary dissipative version

$$\begin{cases} v_{tt} + \Delta^2 v = 0; \\ v(0,\ \cdot\ ) = v_0,\ v_t(0,\ \cdot\ ) = v_1; \\ w|_\Sigma = g_1; \\ \Delta w|_\Sigma = 0; \end{cases} \qquad \begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } Q; & (4.6.1a) \\ w(0,\ \cdot\ ) = w_0,\ w_t(0,\ \cdot\ ) = w_1 & \text{in } \Omega; & (4.6.1b) \\ w|_\Sigma = \dfrac{\partial}{\partial\nu}(\mathcal{A}^{-2} w_t) & \text{in } \Sigma; & (4.6.1c) \\ \Delta w|_\Sigma \equiv 0 & \text{in } \Sigma. & (4.6.1d) \end{cases}$$

Regularity, exact controllability of the v-problem; uniform stabilization of the w-problem. References for this subsection include [Las.2], [L-T.7], [L-T.10]. We begin by introducing the (state) space of optimal regularity

$$X \equiv [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \times [\mathcal{D}(\mathcal{A}^{\frac{3}{2}})]' \equiv H^{-1}(\Omega) \times V', \tag{4.6.2}$$

with the space V defined in (4.5.4).

Theorem 4.6.1. (Regularity [L-T.7, Theorem 1.3, Eqns. (1.20), (1.21), p. 203]) Regarding the v-problem (4.6.1) with $y_0 = \{v_0, v_1\} = 0$, the following regularity result holds true for each $T > 0$ (recall (1.2b)): the map

$$L : g_1 \to Lg_1 = \{v, v_t\} \text{ is continuous } L_2(\Sigma) \to C([0,T]; X \equiv H^{-1}(\Omega) \times V'). \tag{4.6.3}$$

Theorem 4.6.2. (Exact controllability [Las.2]) Given any initial condition $\{v_0, v_1\} \in X$ and $T > 0$, there exists a $g_1 \in L_2(\Sigma)$ such that the corresponding solution of the v-problem (4.6.1) satisfies $\{v(T), v_t(T)\} = 0$.

Remark 4.6.1. Exact controllability of the v-problem (4.6.1) with two boundary controls: $v|_\Sigma = g_1 \in L_2(\Sigma)$ and $\Delta v|_\Sigma = g_2 \in [H^1(0,T;L^2(\Gamma))]'$ was previously obtained in [L-T.10, Theorem 1.1], [Lio.5]. $\square$

Theorem 4.6.3. (Uniform stabilization [Las.2]) With reference to the w-problem (4.6.1), we have that:

(i) the map $\{w_0, w_1\} \in X \equiv [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \times [\mathcal{D}(\mathcal{A}^{\frac{3}{2}})]' \to \{w(t), w_t(t)\}$ defines a s.c. contraction semigroup e^{At} on X;

(ii)

$$w|_\Sigma = \frac{\partial \mathcal{A}^{-2} w_t}{\partial\nu} \in L_2(0,\infty; L_2(\Gamma)) \tag{4.6.4}$$

continuously in $\{w_0, w_1\} \in X$;

(iii) there exists constants $M \geq 1$, $\delta > 0$, such that

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_X \equiv \left\| e^{At} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_X \leq M e^{-\delta t} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_X, \quad t \geq 0. \quad \Box \tag{4.6.5}$$

Abstract model of the v-problem. In addition to the operator $\mathcal{A}$ in (4.5.7), we need now the Green map

$$G_1 : H^s(\Gamma) \to H^{s+\frac{1}{2}}(\Omega),\ s \in \mathbb{R};\ \varphi = G_1 g_1 \Longleftrightarrow \{\Delta^2 \varphi = 0 \text{ in } \Omega,\ \varphi|_\Gamma = g_1,\ \Delta\varphi|_\Gamma = 0 \text{ on } \Gamma\}; \tag{4.6.6a}$$

$$G_1 = D, \text{ where } D \text{ is defined by (4.5.9) [L-T.7, Remark 3.1, p. 211].} \tag{4.6.6b}$$

Then, the second-order, respectively the first-order, abstract models (in additive form) of the v-problem (4.6.1) are [L-T.7]

$$v_{tt} + \mathcal{A}^2 v = \mathcal{A}^2 G_1 g_1 = \mathcal{A}^2 D g_1;\ \frac{d}{dt} \begin{bmatrix} v \\ v_t \end{bmatrix} = A \begin{bmatrix} v \\ v_t \end{bmatrix} + B g_1; \tag{4.6.7}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A}^2 & 0 \end{bmatrix};\quad B g_1 = \begin{bmatrix} 0 \\ \mathcal{A}^2 G_1 g_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \mathcal{A}^2 D g_1 \end{bmatrix};\quad B^* \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = D^* \mathcal{A}^{-1} x_2, \tag{4.6.8}$$

where $*$ for B and D refer to different topologies. With B^* defined by $(Bg_1, x)_X = (g_1, B^* x)_{L_2(\Gamma)}$ with respect to the X-topology defined in (4.6.2), we readily find the expression in (4.6.8).

The operator $B^* L$. With $y_0 = \{v_0, v_1\} = 0$, we shall show that

$$\begin{aligned} B^* L g_1 = B^* \begin{bmatrix} v(t; y_0 = 0) \\ v_t(t; y_0 = 0) \end{bmatrix} &= D^* \mathcal{A}^{-1} v_t(t; y_0 = 0) \\ &= D^* \mathcal{A} \mathcal{A}^{-2} v_t(t; y_0 = 0) = -\frac{\partial}{\partial \nu} z_t(t); \end{aligned} \tag{4.6.9}$$

$$z(t) = \mathcal{A}^{-2} v(t; y_0 = 0) \in C([0, T]; \mathcal{D}(\mathcal{A}^{\frac{3}{2}}) \equiv V) \text{ continuously in } g_1 \in L_2(\Sigma). \tag{4.6.10}$$

Indeed, to obtain (4.6.9), one uses the definition in (4.6.8) for B^*, followed by the usual property that $D^* \mathcal{A} = -\frac{\partial}{\partial \nu}$ on $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H_0^1(\Omega)$ [as below (4.2.10)]. The regularity of $z(t)$ noted in (4.6.10) follows from (4.6.3) with $[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' = H^{-1}(\Omega)$ with V defined by (4.5.4). The new variable $z(t)$ defined in (4.6.10) satisfies the following dynamics: abstract equation and corresponding PDE-mixed problem

$$z_{tt} + \mathcal{A}^2 z = G_1 g_1 = D g_1 \qquad \begin{cases} z_{tt} + \Delta^2 z = D g_1 & \text{in } Q; \quad (4.6.11a) \\ z(0, \cdot) = 0,\ z_t(0, \cdot) = 0 & \text{in } \Omega; \quad (4.6.11b) \\ z|_\Sigma \equiv 0,\ \Delta z|_\Sigma \equiv 0 & \text{in } \Sigma, \quad (4.6.11c) \end{cases}$$

which is essentially the same problem (4.15.14). Since now $g_1 \in L_2(\Sigma)$ [while in (4.5.14), $g_2 \in L_2(\Sigma)$], Theorem 4.5.4 yields at once

Theorem 4.6.4. With reference to (4.6.9), we have

$$B^*L: \text{ continuous } L_2(0,T;L_2(\Gamma)) \to L_2(0,T;L_2(\Gamma)), \tag{4.6.12}$$

equivalently, with reference to (4.6.11),

$$\text{the map } g_1 \to \frac{\partial z_t}{\partial \nu} \text{ is continuous } L_2(0,T;L_2(\Gamma)) \to L_2(0,T;L_2(\Gamma)). \quad \Box \tag{4.6.13}$$

4.7 Wave equation with Dirichlet boundary control: The 1-dimensional case

In this section, let $\Omega = (0,1)$. Consider the 1-dimensional wave equation

$$\begin{cases} v_{tt} = v_{xx} & \text{in } (0,T] \times \Omega; & (4.7.1a) \\ v(0, \cdot) = 0, \ v_t(0, \cdot) = 0 & \text{in } \Omega; & (4.7.1b) \\ v|_{x=0} = g(t), \ w|_{x=1} \equiv 0 & \text{in } (0,T], & (4.7.1c) \end{cases}$$

with Dirichlet boundary control $g \in L_2(0,T)$. We extend g to vanish for $t < 0$. Then, the well-known solution of problem (4.7.1) is [L-T.1, p. 52], [L-T.21, p. 966],

$$(Lg)(t,x) = v(t,x) = \sum_{\substack{k=0 \\ k \text{ even}}}^{K} g(t-k-x) - \sum_{\substack{k=1 \\ k \text{ odd}}}^{K} g(t-(k+1)+x),$$
$$\text{a.e. in } t, K \le t \le (K+1), \tag{4.7.2}$$

in agreement with the physical fact that the input g applied at $x = 0$ travels with speed equal to 1, and is reflected at $x = 1$ in such a way as to satisfy the zero boundary condition. It is shown in Section 5, see Eqn. (5.1.8) below—in the multidimensional case—that for problem (4.7.1) we have

$$B^*Lg = D^*v_t, \tag{4.7.3}$$

where D is the Dirichlet map defined in (4.5.9), and D^* its adjoint. In our present, 1-dimensional problem (4.7.1), we have

$$(Dg)(x) = -gx + g, \ g \in \mathbb{R}; \ D^*\varphi = \int_0^\pi (1-x)\varphi(x)dx, \ \varphi \in L_2(0,1). \tag{4.7.4}$$

Goal. With reference to (4.7.3), our goal is to show that

$$B^*L: \ L_2(0,T) \to L_2(0,T); \tag{4.7.5a}$$

equivalently, that

$$D^*v_t \in L_2(0,T), \text{ continuously in } g \in L_2(0,T). \tag{4.7.5b}$$

Because of the solution formula (4.7.2), it will suffice to take

$$v(t,x) = g(t-x),\ v_t(t,x) = \dot{g}(t-x),\ 0 \le t \le 1, \tag{4.7.6}$$

and, in view of (4.7.4), show that

$$D^* v_t = D^* \dot{g}(t - \ \cdot\ ) = \int_0^1 (1-x)\dot{g}(t-x)dx \in L_2(0,T), \tag{4.7.7}$$

for $g \in L_2(0,T)$, $T \le 1$. We obtain

$$\begin{aligned} D^* v_t = D^* \dot{g}(t - \ \cdot\ ) &= (1+t)[g(t) - g(t-1)] + (t-1)g(t-1) \\ &\quad - t\, g(t) - \int_t^{t-\pi} g(r)dr \in L_2(0,T), \end{aligned} \tag{4.7.8}$$

and thus (4.7.7) is established in this case. The proof is similar for the other terms of (4.7.2) for a general T fixed. Thus, the regularity property (4.7.5) is proved for problem (4.7.1).

4.8 Wave equation with Neumann boundary control: The 1-dimensional case

In this section, let $\Omega = (0,1)$. Consider the 1-dimensional wave equation

$$\begin{cases} v_{tt} = v_{xx} & \text{in } (0,T] \times \Omega; & (4.8.1a) \\ v(0,\ \cdot\ ) = 0,\ v_t(0,\ \cdot\ ) = 0 & \text{in } \Omega; & (4.8.1b) \\ v_x|_{x=0} = g(t),\ v|_{x=1} = 0 & \text{in } (0,T], & (4.8.1c) \end{cases}$$

with Neumann boundary control $g \in L_2(0,T)$. Define the function

$$U(r) = \begin{cases} -\displaystyle\int_0^r g(\sigma)d\sigma & r \ge 0; \\ 0 & r < 0. \end{cases}$$

Then, the solution of problem (4.8.1) is [L-T.21, p. 882],

$$\begin{cases} (Lg)(t,x) = v(t,x) = \sum_{\substack{k=0 \\ k \text{ even}}}^K a_k U(t-k-x) - \sum_{\substack{k=1 \\ k \text{ odd}}}^K a_k U(t-(k+1)+x), \\ a_k \equiv 1, \text{ for } k = 0,3,4,7,8,\dots \\ a_k \equiv -1, \text{ for } k = 1,2,5,6,9,10,\dots, \qquad K \le t \le K+1. \end{cases} \tag{4.8.2}$$

It is shown in Section 6, Eqn. (6.1.9) below—in the multidimensional case—that for problem (4.8.1) we have

$$B^* Lg = v_t|_{\Sigma_0}, \quad \Sigma_0 = (0,T] \times \Gamma_0, \tag{4.8.3}$$

Γ_0 being the controlled portion of the boundary Γ. In our present 1-dimensional case (4.8.1), we have $\Gamma_0 = \{x = 0\}$, the point $x = 0$.

Goal. With reference to (4.8.3), our goal is to show that

$$B^*L: \; L_2(0,T) \to L_2(0,T), \tag{4.8.4a}$$

equivalently that

$$v_t|_{x=0} \in L_2(0,T), \text{ continuously in } g \in L_2(0,T). \tag{4.8.4b}$$

Because of the solution formula (4.8.2), it will suffice to take

$$v(t,x) = U(t-x) = \begin{cases} -\displaystyle\int_0^{t-x} g(\sigma)d\sigma & 1 \geq t \geq x; \\ 0 & 0 \leq t < x. \end{cases} \tag{4.8.5}$$

Therefore (4.8.5) yields

$$v_t(t,x)|_{x=0} = \dot{U}(t-x)|_{x=0} = \begin{cases} -g(t) & 1 \geq t \geq x; \\ 0 & 0 \leq t < x. \end{cases} \tag{4.8.6}$$

and (4.8.4b) is trivially verified in this case. The proof can be repeated for the other terms in (4.8.2) for a general T fixed. Thus, the regularity property (4.8.4) is proved for problem (4.8.1).

4.9 One-dimensional Kirchhoff equation with 'moments' boundary control

Let $\Omega = (0,1)$. Consider the open-loop Kirchhoff equation in Ω, with boundary control acting in the 'moments' B.C.

$$\begin{cases} v_{tt} - \gamma v_{xxtt} + v_{xxxx} = 0 & \text{in } (0,T] \times \Omega; & (4.9.1a) \\ v(0, \,\cdot\,) = v_0, \; v_t(0, \,\cdot\,) = v_1 & \text{in } \Omega; & (4.9.1b) \\ v|_{x=0} = v|_{x=1} \equiv 0 & \text{in } (0,T] \times \{0\}; & (4.9.1c) \\ v_{xx}|_{x=0} = 0; \; v_{xx}|_{x=1} = g & \text{in } (0,T] \times \{1\}. & (4.9.1d) \end{cases}$$

We shall see in Section 7 that the Kirchhoff equation in any dimension with boundary controls in the 'moments' B.C. can be reduced, modulo lower-order terms, to the wave equation with Dirichlet boundary control, treated in Section 4.7. Accordingly, the results of this section imply:

Theorem 4.9.1. With reference to problem (4.9.1) with $v_0 = v_1 = 0$, we have that the corresponding B^*L operator is defined, via (7.1.10) of Section 7, by

$$B^*Lg = v_{tx}|_{x=1} \tag{4.9.2}$$

and satisfies

$$B^*L: \text{ continuous } L_2(0,T) \to L_2(0,T). \tag{4.9.3}$$

Remark 4.9.1. By contrast, Section 7 will show that the regularity property (4.9.3) for B^*L is *false* in the multidimensional version (dim $\Omega \geq 2$) of problem (4.9.1).

5 First hyperbolic class where (2.14) fails: $B^*L \notin \mathcal{L}(L_2(0,T;U))$. The multidimensional wave equation with Dirichlet boundary control

The present section complements Section 4.7. In the latter, we showed that $B^*L \in \mathcal{L}(L_2(0,T;L_2(\Gamma))$ in the 1-dimensional wave equation case with Dirichlet boundary control. In the present section we show that this result is false if dim $\Omega \geq 2$. Thus, Claim 2.1 in Section 2—the key theoretical result in [G-L.1]—is *not applicable.* Yet, *uniform stabilization* of the multidimensional wave equation with suitable (dissipative) feedback in the Dirichlet B.C. *does hold true*: see Theorem 5.1.3 below. It was first established, for strictly convex domains Ω, in [L-T.4]. This geometrical restriction was later removed in [L-T.16]. These results show that the assumption $B^*L \in \mathcal{L}(L_2(0,T;U))$ in Claim 2.1 in the paper [G-L.1] is *far from necessary in critical PDE problems.*

This negative fact, combined with the considerations made throughout this paper, that proving uniform stabilization directly is *preferable, conceptually and technically,* over proving exact controllability and $B^*L \in \mathcal{L}_2(0,T;U)$, document that Claim 2.1 is *not* the right tool, or approach, to seek uniform stabilization of physically significant PDE problems. This program was emphasized in Section 1.

5.1 Preliminaries. The operator B^*L

Open-loop and closed-loop dissipative systems. In this section, let Ω be an open bounded domain in $\mathbb{R}^n$, $n \geq 1$, with sufficiently smooth boundary Γ. We consider the open loop wave equation on Ω with Dirichlet boundary control $g \in L_2(0,T;L_2(\Gamma)) \equiv L_2(\Sigma)$, and its corresponding closed loop dissipative system:

$$\begin{cases} v_{tt} = \Delta v; \\ v(0,\,\cdot\,) = v_0, v_t(0,\,\cdot\,) = v_1; \\ v|_\Sigma = g; \end{cases} \qquad \begin{cases} w_{tt} = \Delta w & \text{in } Q; & (5.1.1a) \\ w(0,\,\cdot\,) = w_0,\ w_t(0,\,\cdot\,) = w_1 & \text{in } \Omega; & (5.1.1b) \\ w|_\Sigma = \dfrac{\partial(\mathcal{A}^{-1}w_t)}{\partial\nu} & \text{in } \Sigma, & (5.1.1c) \end{cases}$$

with $Q = (0,T] \times \Omega$; $\Sigma = (0,T] \times \Gamma$. Moreover, the operator $\mathcal{A}$ is defined by (5.1.6) below: $\mathcal{A}\psi = -\Delta\psi$, $\mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega)$.

Regularity, exact controllability of the v-problem; uniform stabilization of the w-problem. References for this subsection include [Ho.1], [L-T.1], [L-T.2], [L-T.4], [L-T.16], [Lio.4], [Lio.5], [L-L-T.1].

We begin by introducing the (state) space of optimal regularity

$$Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \equiv L_2(\Omega) \times H^{-1}(\Omega). \tag{5.1.2}$$

Theorem 5.1.1. (Regularity [L-T.1-2], [L-L-T.1]) Regarding the v-problem (5.1.1), with $y_0 = \{v_0, v_1\} = 0$, the following regularity result holds true for each $T > 0$ (recall the definition of L in

(1.2b)): the map

$$L: g \to Lg \equiv \{v, v_t\} \text{ is continuous } L_2(\Sigma) \to C([0,T]; Y \equiv L_2(\Omega) \times H^{-1}(\Omega)). \tag{5.1.3}$$

Theorem 5.1.2. (Exact controllability [Ho.1], [L-T.4], [Tr.5], [Lio.5]) Given any initial condition $\{v_0, v_1\} \in Y$ and $T > 0$ sufficiently large, there exists a $g \in L_2(\Sigma)$ such that the corresponding solution of the v-problem (5.1.1) satisfies $\{v(T), v_t(T)\} = 0$.

Theorem 5.1.3. (Uniform stabilization [L-T.4], [L-T.15]) With reference to the w-problem (5.1.1), we have that:

(i) the map $\{w_0, w_1\} \in Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \to \{w(t), w_t(t)\}$ defines a s.c. contraction semigroup e^{At} on Y;

(ii)

$$w|_\Sigma = \frac{\partial(\mathcal{A}^{-1}w_t)}{\partial\nu} \in L_2(0,\infty; L_2(\Gamma)) \tag{5.1.4}$$

continuously in $\{w_0, w_1\} \in Y$;

(iii) there exist constants $M \geq 1$, $\delta > 0$, such that

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_Y = \left\| e^{At} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_Y \leq M e^{-\delta t} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_Y, \quad t \geq 0. \tag{5.1.5}$$

Again, needless to say, in line with the content of Section 1, all three theorems above are obtained by PDE hard analysis energy methods (not by soft analysis methods). As usual, the most challenging result to prove is Theorem 4.3.3 on uniform stabilization: this, in addition, requires a shift of topology from $L_2(\Omega) \times H^{-1}(\Omega)$ (the space of the final result) to $H_0^1(\Omega) \times L_2(\Omega)$ (the space where the energy method works). This shift of topology is implemented by a *change of variable*: this is the same change of variable that is noted below in (5.1.10).

Abstract model of v-problem. We let

$$\begin{aligned} \mathcal{A}f &= -\Delta f, \ \mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega); \ D: H^s(\Gamma) \to H^{s+\frac{1}{2}}(\Omega), \ s \in \mathbb{R}, \\ \varphi &= Dg \Longleftrightarrow \{\Delta\varphi = 0 \text{ in } \Omega; \ \varphi|_\Gamma = g \text{ in } \Gamma\}, \end{aligned} \tag{5.1.6}$$

as in (4.5.7), (4.5.9). The abstract model for the v-problem in (5.1.1) is [L-T.4], [L-T.1], [L-T.2], [Tr.3]

$$v_{tt} = -\mathcal{A}v + \mathcal{A}Dg; \qquad \frac{d}{dt}\begin{bmatrix} v \\ v_t \end{bmatrix} = A \begin{bmatrix} v \\ v_t \end{bmatrix} + Bg; \tag{5.1.7a}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}; \quad Bg = \begin{bmatrix} 0 \\ \mathcal{A}Dg \end{bmatrix}, \ B^* \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = D^* x_2, \tag{5.1.7b}$$

where $*$ for B and D refer to different topologies, and where the Dirichlet map D is defined in (5.1.6). Moreover, with B^* defined by $(Bg, x)_Y = (g, B^*x)_{L_2(\Gamma)}$, with respect to the Y-topology in (5.1.2), we readily find the expression in (5.1.7).

The operator B^*L. With $y_0 = \{v_0, v_1\} = 0$, we shall show that

$$B^*Lg = B^* \begin{bmatrix} v(t; y_0 = 0) \\ v_t(t; y_0 = 0) \end{bmatrix} = D^* v_t(t; y_0 = 0) = D^* \mathcal{A}\mathcal{A}^{-1} v_t(t; y_0 = 0) \tag{5.1.8}$$

$$= -\frac{\partial}{\partial \nu} \mathcal{A}^{-1} v_t(t; y_0 = 0) = -\frac{\partial z(t)}{\partial \nu}; \tag{5.1.9}$$

$$z(t) \equiv \mathcal{A}^{-1} v_t(t; y_0 = 0) \in C([0,T]; \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^1(\Omega)) \text{ continuously in } g \in L_2(\Sigma). \tag{5.1.10}$$

Indeed, to obtain (5.1.8), (5.1.9), one uses the definition of L in (5.1.3) followed by the definition of B^* in (5.1.7) and the usual property $D^*\mathcal{A} = -\frac{\partial}{\partial \nu}$ on $H_0^1(\Omega)$ [L-T.4, Eqn. (1.10)]. Finally, the regularity of z in (5.1.10) follows from the regularity (5.1.3) on v_t with $H^{-1}(\Omega) = [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$. The new variable $z(t)$ defined in (5.1.10) satisfies the following dynamics: abstract equation, and corresponding PDE-mixed problem

$$z_{tt} = -\mathcal{A}z + Dg_t \qquad \begin{cases} z_{tt} = \Delta z + Dg_t & \text{in } Q; \quad (5.1.11a) \\ z(0, \cdot\,) = 0, \;\; z_t(0, \cdot\,) = z_1 & \text{in } \Omega; \quad (5.1.11b) \\ z|_\Sigma \equiv 0 & \text{in } \Sigma. \quad (5.1.11c) \end{cases}$$

Indeed, the abstract z-equation in (5.1.11) (left) is readily obtained from the abstract v-equation in (5.1.7), after applying throughout $\mathcal{A}^{-1}$ and $\frac{d}{dt}$ to it, and using the definition of $z(t)$ in (5.1.10). Moreover, since $z(t) \in H_0^1(\Omega)$ from (5.1.10), then z satisfies the Dirichlet B.C. in (5.1.11c). For g in the class (5.1.15), we can take $z_1 = 0$, see (5.1.13).

The energy method on the mixed PDE problem (5.1.11) fails to show that $\frac{\partial z}{\partial \nu} \in L_2(0,T; L_2(\Gamma))$, continuously in $g \in L_2(0,T; L_2(\Gamma))$, except in the 1-dimensional case. As in [L-L-T.1], multiplying the PDE problem (5.1.11) by $h \cdot \nabla z$, with h a C^2-vector field on $\overline{\Omega}$, with $h|_\Gamma = \nu$ on Γ, and using the B.C. (5.1.11c), we obtain the identity [L-L-T.1, Eqn. (2.27), p. 157]:

$$\frac{1}{2}\int_\Sigma (T-t)\left(\frac{\partial z}{\partial \nu}\right)^2 d\Sigma$$

$$= \int_Q (T-t) H \nabla z \cdot \nabla z \, dQ + \frac{1}{2}\int_Q (T-t)[z_t^2 - |\nabla z|^2] \text{div } h \, dQ$$

$$+ \int_Q z_t h \cdot \nabla z \, dQ - \int_Q (T-t) Dg_t h \cdot \nabla z \, dQ. \tag{5.1.12}$$

Moreover, in addition to the *a-priori* regularity for z in (5.1.10), we also have that for z_t:

$$z_t = \mathcal{A}^{-1} v_{tt} = \mathcal{A}^{-1}[-\mathcal{A}v + \mathcal{A}Dg] = -v + Dg \in L_2(0,T; L_2(\Omega)) \text{ continuously in } g \in L_2(\Sigma), \tag{5.1.13}$$

as it follows from $v \in C([0,T]; L_2(\Omega))$ by (5.1.3) and $Dg \in L_2(0,T; H^{\frac{1}{2}}(\Omega))$ by (5.1.6) with $s = 0$. [Since z_t is only L_2 in time, we have used the multiplier $(T-t)h \cdot \nabla z$, to eliminate the terms at

$t = 0$ and $t = T$. Otherwise, one takes preliminarily g in the class (5.1.15) below, and uses just the multiplier $h \cdot \nabla z$.] Thus, the *a-priori* regularity of $\{z, z_t\}$ in (5.1.10) and (5.1.13) guarantee that all first three integral terms on the RHS of (5.1.12) are well-defined, continuously in $g \in L_2(\Sigma)$. Hence, we obtain from (5.1.12)

$$\frac{1}{2}\int_\Sigma (T-t)\left(\frac{\partial z}{\partial \nu}\right)^2 d\Sigma = \mathcal{O}\left(\|g\|^2_{L_2(\Sigma)}\right) - \int_Q (T-t) D g_t \, h \cdot \nabla z \, dQ. \tag{5.1.14}$$

Letting now g be (temporarily) in the class

$$g \in C([0,T]; L_2(\Gamma)) \qquad g(T) = g(0) = 0, \tag{5.1.15}$$

dense in $L_2(\Sigma)$, we see by integration by parts in t with use of (5.1.15), followed by the usual divergence theorem, that

$$-\int_Q (T-t) D g_t \, h \cdot \nabla z \, dQ = \int_0^T \int_\Omega Dg \, h \cdot \nabla z_t \, d\Omega \, dt + l.o.t. \tag{5.1.16}$$

$$= \int_0^T \int_\Gamma Dg \, z_t h \cdot \nu \, d\Gamma \, dt - \int_0^T \int_\Omega z_t h \cdot \nabla(Dg) d\Omega \, dt$$

$$- \int_0^T \int_\Omega Dg \, z_t \text{div } h \, d\Omega \, dt + l.o.t., \tag{5.1.17}$$

in view of $z_t|_\Gamma = 0$ by (5.1.11c). The last integral term in the RHS of (5.1.17) is well-defined continuously in $g \in L_2(\Sigma)$, by (5.1.13) on z_t and $Dg \in L_2(0,T; H^{\frac{1}{2}}(\Omega))$. Thus, from (5.1.14) we obtain via (5.1.17)

$$\int_\Sigma \left(\frac{\partial z}{\partial \nu}\right)^2 d\Sigma = \mathcal{O}\left(\|g\|^2_{L_2(\Sigma)}\right) + \int_0^T \int_\Omega z_t h \cdot \nabla(Dg) d\Omega \, dt. \tag{5.1.18}$$

One-dimensional case. In the one-dimensional case, $(Dg)(x)$ is a linear function of x, see (4.7.4), thus $\nabla(Dg) \equiv 0$ and we get

$$\int_\Sigma \left(\frac{\partial z}{\partial \nu}\right)^2 d\Sigma = \mathcal{O}\left(\|g\|^2_{L_2(\Sigma)}\right), \tag{5.1.19}$$

thus re-proving—in a more complicated way!—the result of Section 4.7.

Multidimensional case: dim $\Omega \geq 2$. In this case, the *a-priori* regularity of $z_t \in L_2(0,T; L_2(\Omega))$ and $Dg \in L_2(0,T; H^{\frac{1}{2}}(\Omega))$, hence $|\nabla(Dg)| \in L_2(0,T; (H_{00}^{\frac{1}{2}}(\Omega))')$ [L-M.1, p. 85] show that, roughly speaking, "$\frac{1}{2}$" *space derivative is apparently missing* in order to have the integral term on the RHS of (5.1.18) well-defined. This will be confirmed by the actual counterexample in Section 5.2.

5.2 Counterexample to (2.14): $B^*L \notin \mathcal{L}(L_2(0,T;U))$. Wave equation with Dirichlet boundary control in dimension ≥ 2

It will suffice to consider the wave equation defined on a 2-dimensional half-space, with Dirichlet boundary control. So let

$$\Omega \equiv \mathbb{R}_2^+ = \{(x,y) : x \geq 0, \ y \in \mathbb{R}\}, \ \Gamma = \{(0,y) : y \in \mathbb{R}\} = \Omega|_{x=0}. \tag{5.2.1}$$

On Ω we consider the wave equation with Dirichlet boundary control:

$$\begin{cases} v_{tt} = v_{xx} + v_{yy} & \text{in } Q \equiv (0,\infty] \times \Omega; & (5.2.2a) \\ v(0, \ \cdot\ ) = 0, \ v_t(0, \ \cdot\ ) = 0 & \text{in } \Omega; & (5.2.2b) \\ v|_\Sigma = g & \text{in } \Sigma \equiv (0,\infty) \times \Gamma, & (5.2.2c) \end{cases}$$

where $g \in L_2(0,\infty; L_2(\Gamma))$. We have seen in Section 5.1, Eqn. (5.1.8), that for problem (5.2.2) we have

$$B^*Lg = D^*v_t. \tag{5.2.3}$$

Goal. We want to show that: given $T > 0$, there exists some $g \in L_2(0,T;L_2(\Gamma))$ such that

$$B^*Lg \notin L_2(0,T;L_2(\Gamma)). \tag{5.2.4}$$

To this end, it will suffice to show that there exists $g \in L_2(0,\infty;L_2(\Gamma))$, such that

$$e^{-\gamma t}(B^*Lg)(t) \notin L_2(0,\infty;L_2(\Gamma)), \tag{5.2.5}$$

no matter which constant $\gamma > 0$ we choose.

Proof of (5.2.5). Our proof is inspired by [L-T.11, Counterexample, p. 294], for a different type of result.

Step 1. Let $\hat{v}(\tau,x,\eta)$ denote the Laplace-Fourier transform of $v(t,x,y)$: Laplace in time $t \to \tau = \gamma + i\sigma$, $\gamma > 0$, $\sigma \in \mathbb{R}$, and Fourier in $y \to i\eta$, $\eta \in \mathbb{R}$, leaving $x \geq 0$ as a parameter. We then obtain for the solution of (5.2.2) vanishing at $x = \infty$:

$$\tau^2\hat{v} = \hat{v}_{xx} - \eta^2\hat{v}; \qquad \text{or } \hat{v}(\tau,x,\eta) = \hat{g}(\tau,\eta)e^{-\sqrt{\tau^2+\eta^2}x}, \ x \geq 0;$$

$$\tau^2 + \eta^2 = (\gamma^2 + \eta^2 - \sigma^2) + 2i\gamma\sigma. \tag{5.2.6}$$

Step 2. Let $\varphi \in L_2(0,\infty;L_2(\Gamma))$. We consider the Laplace equation in Ω, with Dirichlet B.C. on Γ given by φ a.e. in t, that is, in the notation for D in (5.1.6):

$$u = D\varphi, \ \text{where } u_{xx} + u_{yy} = 0 \text{ in } \Omega; \ u|_\Gamma = \varphi \text{ in } \Gamma. \tag{5.2.7}$$

The solution $u = D\varphi$ of problem (5.2.7) is given by the well-known formula in the transformed variables [Hab.1, Sect. 9.7.3, p. 375]:

$$\hat{u}(\tau, x, \eta) = \widehat{D\varphi}(\tau, x, \eta) = \hat{\varphi}(\tau, \eta)e^{-|\eta|x}, \ \forall \ \tau, \eta \in \mathbb{R}, \ x \geq 0. \tag{5.2.8}$$

Step 3. To establish the negative result expressed in (5.2.5), it suffices to show that: there exists $g \in L_2(0, \infty; L_2(\Gamma))$, such that

$$\left(e^{-2\gamma t} B^* L g, g\right)_{L_2(0,\infty;L_2(\Gamma))} = \infty. \tag{5.2.9}$$

We prove (5.2.9) in a few steps.

Step 3(i). First, we establish that: for all $g \in L_2(0, \infty; L_2(\Gamma))$, we have

$$\begin{aligned} &\left(e^{-2\gamma t} B^* L g, g\right)_{L_2(0,\infty;L_2(\Gamma))} \\ &\quad = \frac{1}{2\pi} \iint\limits_{\mathbb{R}^2_{\sigma\eta}} \tau |\hat{g}(\tau, \eta)|^2 \int_0^\infty e^{-\sqrt{\tau^2+\eta^2}x} e^{-|\eta|x|} dx \, d\sigma \, d\eta, \end{aligned} \tag{5.2.10}$$

where $\mathbb{R}^2_{\sigma\eta}$ denotes the 2-dimensional Euclidean space in the variable σ and η.

Proof of (5.2.10). Recalling (5.2.3), the Parseval identity for Laplace transforms [D.1, Theorem 31.8, p. 212] and (5.2.6), (5.2.8), we compute ($\sim$ indicates the Laplace transform in (5.2.13)), where $\tau = \gamma + i\sigma$:

$$\left(e^{-2\gamma t}(B^* L g)(t), g(t)\right)_{L_2(0,\infty;L_2(\Gamma))} = \int_0^\infty e^{-2\gamma t}(B^* L g, g)_{L_2(\Gamma)} dt \tag{5.2.11}$$

$$\text{(by (5.2.3))} \quad = \int_0^\infty e^{-2\gamma t}(D^* v_t, g)_{L_2(\Gamma)} dt = \int_0^\infty e^{-2\gamma t}(v_t, Dg)_{L_2(\Omega)} dt \tag{5.2.12}$$

$$\text{(by [D.1, p. 212])} \quad = \frac{1}{2\pi} \int_{-\infty}^\infty (\tilde{v}_t(\tau, x, y), \widetilde{Dg}(\tau, x, y))_{L_2(\Omega)} d\sigma \tag{5.2.13}$$

$$= \frac{1}{2\pi} \iint\limits_{\mathbb{R}^2_{\sigma\eta}} \int_0^\infty \tau \hat{v}(\tau, x, \eta) \widehat{Dg}(\tau, x, \eta) dx \, d\sigma \, d\eta \tag{5.2.14}$$

$$\text{(by (5.2.6), (5.2.8))} \quad = \frac{1}{2\pi} \iint\limits_{\mathbb{R}^2_{\sigma\eta}} \int_0^\infty \tau \hat{g}(\tau, \eta) e^{-\sqrt{\tau^2+\eta^2}x} \ \overline{\hat{g}(\tau, \eta)} e^{-|\eta|x} dx \, d\sigma \, d\eta \tag{5.2.15}$$

$$= \frac{1}{2\pi} \iint\limits_{\mathbb{R}^2_{\sigma\eta}} \tau |\hat{g}(\tau, \eta)|^2 \int_0^\infty e^{-\sqrt{\tau^2+\eta^2}x} e^{-|\eta|x} dx \, d\sigma \, d\eta, \tag{5.2.16}$$

and (5.2.16) establishes (5.2.10), as desired. In (5.2.13), (5.2.14), we have invoked Parseval formula for Laplace $t \to \tau$ [D.1, p. 212] and Fourier transform $y \to i\eta$; while in (5.2.15), we have recalled (5.2.6) and (5.2.8) with $\varphi = g$.

Step 3(ii). Define the (bad) region in the (σ, η)-plane by

$$\mathcal{B}_{\sigma\eta} = \{\sigma > 0,\ \eta > 0,\ \sigma^2 + \eta^2 \geq 1;\ \eta^2 \leq \sigma \leq 4\eta^2\}, \tag{5.2.17}$$

so that $\mathcal{B}_{\sigma\eta}$ is the set in the first quadrant comprised between two parabolas.

In view of identity (5.2.10), in order to establish the negative result (5.2.9), it is sufficient to show that: there exists $g \in L_2(0, \infty; L_2(\Gamma))$ such that

$$\iint\limits_{\mathcal{B}_{\sigma\eta}} \sigma |\hat{g}(\sigma, \eta)|^2 \int_0^\infty e^{-\mathrm{Re}\sqrt{\tau^2+\eta^2}x} e^{-|\eta|x} dx\, d\sigma\, d\eta = \infty. \tag{5.2.18}$$

Proof of (5.2.18). First, we write recalling $\tau^2 + \eta^2$ below (5.2.6):

$$\begin{cases} z \equiv \tau^2 + \eta^2,\ \sqrt{z} = A + iB,\ A = \mathrm{Re}\sqrt{z} = \mathrm{Re}\sqrt{\tau^2 + \eta^2}; & (5.2.19\text{a}) \\ A^2 - B^2 = \gamma^2 + \eta^2 - \sigma^2,\ AB = 2\gamma\sigma. & (5.2.19\text{b}) \end{cases}$$

Solving the system in (5.2.19b) by elementary computations, we obtain

$$A^2 = \frac{8\gamma^2\sigma^2}{\{(\sigma^2 - \eta^2 - \gamma^2)^2 + 16\gamma^2\sigma^2\}^{\frac{1}{2}} + (\sigma^2 - \eta^2 - \gamma^2)}. \tag{5.2.20}$$

Next, restricting to $(\sigma, \eta) \in \mathcal{B}_{\sigma\eta}$ where $\sigma \sim \eta^2$, we obtain

$$\text{in } \mathcal{B}_{\sigma\eta}:\ A^2 \sim \frac{\sigma^2}{\eta^4} \sim 1;\ A = \mathrm{Re}\sqrt{\tau^2 + \eta^2} \sim 1,\ \mathrm{Re}\sqrt{\tau^2 + \eta^2} > 0. \tag{5.2.21}$$

By use of (5.2.17), (5.2.21), we then have for $(\sigma, \eta) \in \mathcal{B}_{\sigma\eta}$:

$$\int_0^\infty e^{-\mathrm{Re}\sqrt{\tau^2+\eta^2}x} e^{-|\eta|x} dx = \frac{1}{\mathrm{Re}\sqrt{\tau^2 + \eta^2} + \eta} \sim \frac{1}{\eta}. \tag{5.2.22}$$

Using (5.2.22) in (5.2.18) yields

$$\iint\limits_{\mathcal{B}_{\sigma\eta}} \sigma |\hat{g}(\sigma, \eta)|^2 \int_0^\infty e^{-\mathrm{Re}\sqrt{\tau^2+\eta^2}x} e^{-|\eta|x} dx\, d\sigma\, d\eta = \iint\limits_{\mathcal{B}_{\sigma\eta}} \frac{\sigma}{\eta} |\hat{g}(\sigma, \eta)|^2 d\sigma\, d\eta \tag{5.2.23}$$

$$(\text{by } (5.2.17)) \qquad \sim \iint\limits_{\mathcal{B}_{\sigma\eta}} \eta |\hat{g}(\sigma, \eta)|^2 d\sigma\, d\eta \sim \iint\limits_{\mathcal{B}_{\sigma\eta}} \sigma^{\frac{1}{2}} |\hat{g}(\sigma, \eta)|^2 d\sigma\, d\eta, \tag{5.2.24}$$

where in (5.2.24) we have invoked (5.2.17). Thus, it suffices to take a function $\hat{g}(\sigma, \eta)$ which is $L_2(\mathcal{B}_{\sigma\eta})$, and no better, on $\mathcal{B}_{\sigma\eta}$ and zero elsewhere, to obtain the sought-after function producing the negative conclusion (5.2.9).

6 Second hyperbolic class where (2.14) fails: $B^*L \notin \mathcal{L}(L_2(0,T;U)$. The multidimensional wave equation with Neumann boundary control

The present section complements Section 4.8. In the latter, we showed that $B^*L \in \mathcal{L}(L_2(0,T;L_2(\Gamma))$ in the 1-dimensional wave equation case with Neumann boundary control. In the present section, we show that this result is false if $\dim \Omega \geq 2$.

Remark 6.1. In all the hyperbolic or Petrowski-type PDE problems considered in the present paper, we always have that the generator A in Eqn. (1.1) is skew-adjoint, modulo a scalar multiplication of the identity; that, $A = iS + kI$, S self-adjoint in Y and k a real constant (equal to zero in the conservative case). In this case, in view of Proposition A.1 in the Appendix, the following result holds true

$$B^*L \in \mathcal{L}(L_2(0,T;U)), \text{ i.e., property (2.14)} \quad \Rightarrow \quad L \in \mathcal{L}(L_2(0,T;U);C([0,T];Y)), \text{ i.e., property (1.3); equivalent to property (1.4)}, \tag{6.1}$$

where B^* is defined with respect to the Y-topology. Several PDE hyperbolic/Petrowski-type are known [L-T.19, Section 1.2] where:

(i) Property (6.1) (right) for L *fails to be true when Y is the desirable space of finite energy in dim $\Omega \geq 2$; a fortiori, property (6.1) (left) for B^*L also fails to be true.*

(ii) Yet, *uniform stabilization* with boundary dissipation, say, active on the whole boundary (or a portion of the boundary, under suitable geometric conditions) *does hold true* in the space of finite energy: a fact that has been known for over 20 years. We list two physically significant cases:

Ex. #1: The wave equations with Neumann boundary control in $\dim \Omega \geq 2$, as in Eqn. (6.1.1) of Section 6.1 below.

Ex. #2: The Euler-Bernoulli plate model in $\dim \Omega = 2$, with free B.C.:

$$\begin{cases} v_{tt} + \Delta^2 v + v = 0 & \text{in } (0,T] \times \Omega \equiv Q; & (6.2a) \\ v(0,\,\cdot\,) = v_0,\; v_t(0,\,\cdot\,) = v_1 & \text{in } \Omega; & (6.2b) \\ [\Delta v + (1-\eta) B_1 v]_\Sigma = 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma; & (6.2c) \\ \left[\dfrac{\partial \Delta v}{\partial \nu} + (1-\eta) B_2 v\right]_\Sigma = g & \text{in } \Sigma, & (6.2d) \end{cases}$$

where $0 < \eta < 1$ is the Poisson's modulus and B_1 and B_2 are the usual boundary operators, defined, say, in [Lag.1], [L-L.1], [L-T.21, Vol. 1, p. 249].

Regarding Ex. #1. Here, with reference to problem (6.1.1) below, the space of finite energy is $Y \equiv H^1(\Omega) \times L_2(\Omega)$ as in (6.1.2) below. Yet, for $\dim \Omega \geq 2$, the map: $g \to Lg \equiv \{v, v_t\}$ defined by problem (6.1.1) is *not* continuous: $L_2(\Sigma) \to C([0,T]; H^1(\Omega) \times L_2(\Omega))$. See counterexample in [L-T.11, p. 294]. Nevertheless, uniform stabilization of the multidimensional wave equation with suitable (dissipative) feedback in the Neumann B.C. does hold true in the finite energy space: see Theorem 6.1.2 below. It was first established with progressively more relaxed geometrical conditions in [C.1] in 1979, [Lag.1]. Geometrical conditions were later further relaxed [L-T.16], [B-L-R.1].

Regarding Ex. #2. Here, with reference to problem (6.2a-d), the space of finite energy is $Y \equiv H^2(\Omega) \times L_2(\Omega)$. Yet, for dim $\Omega \geq 2$, the map $g \to Lg = \{v, v_t\}$ defined by problem (6.2a-d) is *not* continuous $L_2(\Sigma) \to C([0,T]; H^2(\Omega) \times L_2(\Omega))$. Nevertheless, exact controllability/uniform stabilization results for the corresponding dissipative problem on such space $H^2(\Omega) \times L_2(\Omega)$ of finite energy are given in [Lag.1], [L-L.1], with geometrical conditions relaxed or eliminated by virtue of the sharp trace results in [L-T.18].

Thus, Claim 2.1 in Section 2—a stronger result than the key theoretical result in [G-L.1]—is *not applicable.* This shows that the assumption $B^*L \in \mathcal{L}(L_2(0,T;U))$ in Claim 2.1 and in [G-L.1]-paper, is, once more, *far from necessary in critical PDE problems.* These negative facts, combined with the considerations above document that Claim 2.1 is not the right tool, or approach, to seek uniform stabilization of physically significant PDE problems.

Notwithstanding the considerations made above in Ex. #1 (via Proposition A.1 of the Appendix), in the next Subsection 6.1.1 we are going to show *directly*, by means of an explicit counterexample in dim $\Omega \geq 2$, that $B^*L \notin \mathcal{L}(L_2(0,T;U))$. The analysis of the present counterexample for B^*L is a modification of that in [L-T.11, p. 294] for L.

6.1 Preliminaries. The operator B^*L

Open-loop and closed-loop, feedback dissipative systems. In this section, let Ω be an open bounded domain in $\mathbb{R}^n$, $n \geq 1$, with sufficiently smooth boundary Γ. We consider the open-loop wave equation in Ω with Neumann boundary control $g \in L_2(0,T;L_2(\Gamma_1)) \equiv L_2(\Sigma_1)$, and its corresponding closed-loop dissipative system:

$$
\begin{cases} v_{tt} = \Delta v \\ v(0,\,\cdot\,) = v_0, \;\; v_t(0,\,\cdot\,) = v_1 \\ v|_{\Sigma_0} = 0 \\ \left.\dfrac{\partial v}{\partial \nu}\right|_{\Sigma_1} = g \end{cases}
\qquad
\begin{cases} w_{tt} = \Delta w & \text{in } Q; \\ w(0,\,\cdot\,) = w_0, \;\; w_t(0,\,\cdot\,) = w_1 & \text{in } \Omega; \\ w|_{\Sigma_0} = 0 & \\ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} = -w_t & \text{in } \Sigma, \end{cases}
\begin{matrix} (6.1.1a) \\ (6.1.1b) \\ (6.1.1c) \\ (6.1.1d) \end{matrix}
$$

with $Q = (0,T] \times \Omega$, $\Sigma_i = (0,T] \times \Gamma_i$, $i = 0,1$; $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \neq \phi$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \phi$; $h \cdot \nu \leq$ on Γ_0 for a coercive smooth vector field h on Ω.

For the treatment of the present section, we shall *not* need to invoke the theory of sharp/optimal regularity of the mixed v-problem, for which we refer to [L-T.9], [L-T.11], [L-T.12], [L-T.20], [L-T.21, Sect. 9.4, p. 857 for dim $\Omega = 1$], [Ta.2].

Exact controllability of the v-problem; uniform stabilization of the w-problem. We begin by introducing the finite energy (state) space (which is *not*, however, the space of optimal regularity [L-T.11, Counterexample, p. 294 in dim $\Omega \geq 2$])

$$Y \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \equiv H^1(\Omega) \times L_2(\Omega); \tag{6.1.2}$$

$$\mathcal{A}f = \Delta f,\ \mathcal{D}(\mathcal{A}) = \left\{ f \in H^2(\Omega) : f|_{\Gamma_0} = 0;\ \left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_1} = 0 \right\}. \tag{6.1.3}$$

Theorem 6.1.1. (Exact controllability [L-T.8], [Lio.4], [Lio.5], [B-L-R.1], [Ta.1], [L-T-Y.1], [L-T-Y.2]) Given any finite energy initial condition $\{v_0, v_1\} \in Y$ and $T > 0$ sufficiently large, there exists a $g \in L_2(\Sigma)$ such that the corresponding solution of the v-problem (6.1.1) satisfies $\{v(T), v_t(T)\} = 0$. □

Theorem 6.1.2. (Uniform stabilization [C.1], [Lag.1], [L-T.16], [B-L-R.1], [Ta.1], [L-T-Y.1], [L-T-Y.2]) With reference to the w-problem in (6.1.1), we have that:

(i) the map $\{w_0, w_1\} \in Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \to \{w(t), w_t(t)\}$ defines a s.c. contraction semigroup e^{At} on Y;

(ii)

$$\left.\frac{\partial w}{\partial \nu}\right|_{\Sigma_1} \equiv -w_t \in L_2(0, \infty; L_2(\Gamma_1)) \tag{6.1.4}$$

continuously in $\{w_0, w_1\} \in Y$;

(iii) there exist constants $M \geq 1$, $\delta > 0$, such that

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_Y^2 = \left\| e^{At} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\| \leq M e^{-\delta t} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_Y, t \geq 0. \quad \square \tag{6.1.5}$$

Remark 6.1.1. (i) Let $\Gamma_0 = \phi$. Then, instead of $Y = H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, one has to take the proper subspace $Y_0 = \{[u_1, u_2] \in Y : \int_{\Gamma_1} u_1 d\Gamma + \int_\Omega u_2 d\Omega = 0\}$ [L-T-Y.2, p. 32] for uniform stabilization.

(ii) We also refer to [Tr.5, Section 5], [L-T-Z.1], [T-Y.2] for the more demanding case of the *purely* Neumann B.C., i.e., with $\frac{\partial w}{\partial \nu}|_{\Sigma_0}$ in (6.1.1c), including the variable coefficient case. □

Again, in line with the content of Section 1, both theorems above are obtained by PDE hard analysis, possibly pseudo-differential, methods (not by soft analysis methods).

Abstract model of v-problem. The abstract model for the v-problem in (6.1.1) is [Tr.1], [L-T.1], [L-T.2]:

$$v_{tt} = -\mathcal{A}v + \mathcal{A}Ng \qquad \frac{d}{dt}\begin{bmatrix} v \\ v_t \end{bmatrix} = A \begin{bmatrix} v \\ v_t \end{bmatrix} + Bg; \tag{6.1.6}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix};\ Bg = \begin{bmatrix} 0 \\ \mathcal{A}Ng \end{bmatrix},\ B^* \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = N^*\mathcal{A}x_2 = x_2|_{\Gamma_1}; \tag{6.1.7}$$

$$N : H^s(\Gamma) \to H^{\frac{3}{2}}(\Omega),\ s \in \mathbb{R};\ u = Ng \iff \left\{ \Delta u = 0 \text{ in } \Omega, u|_{\Gamma_0} = 0,\ \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_1} = g \right\}, \tag{6.1.8}$$

where $*$ of B and N refers to different topologies. With B^* defined by $(Bg, x)_Y = (g, B^*x)_{L_2(\Gamma)}$ with respect to the Y-topology in (6.1.2), we readily find the expression in (6.1.7).

The operator B^*L**.** With $y_0 = \{v_0, v_1\} = 0$, we shall show that

$$B^*Lg = B^* \begin{bmatrix} v(t; y_0 = 0) \\ v_t(t; y_0 = 0) \end{bmatrix} = N^*\mathcal{A}v_t(t; y_0 = 0) = v_t|_{\Sigma_1}, \tag{6.1.9}$$

recalling $N^*\mathcal{A} = \ \cdot\ |_\Gamma$ [L-T.8], [L-T.21, Vol. 1].

6.2 Counterexample to (2.14): $B^*L \notin \mathcal{L}(L_2(0,T;U))$. Wave equation with Neumann boundary control in dimension ≥ 2

It will suffice to consider the 2-dimensional half-space setting of Section 5.2, however, now with Neumann boundary control:

$$\begin{cases} v_{tt} = v_{xx} + v_{yy} & \text{in } Q \equiv (0,\infty) \times \Omega; & (6.2.1\text{a}) \\ v(0,\ \cdot\ ) = 0,\ v_t(0,\ \cdot\ ) = 0 & \text{in } \Omega; & (6.2.1\text{b}) \\ v_x|_{x=0} = g & \text{in } \Sigma \equiv (0,\infty) \times \Gamma, & (6.2.1\text{c}) \end{cases}$$

where $g \in L_2(0,\infty; L_2(\Gamma))$, see [L-T.11, Counterexample, p. 294]. We have seen in (6.1.9) that

$$B^*Lg = v_t|_\Sigma. \tag{6.2.2}$$

Goal. We want to show that: given $T > 0$, there exists some $g \in L_2(0,T; L_2(\Gamma))$ such that

$$B^*Lg \notin L_2(0,T; L_2(\Gamma)). \tag{6.2.3}$$

To this end, it will suffice to show that there exists $g \in L_2(0,\infty; L_2(\Gamma))$ such that

$$e^{-\gamma t}(B^*Lg)(t) \notin L_2(0,\infty; L_2(\Gamma)), \tag{6.2.4}$$

no matter which constant $\gamma > 0$ we choose.

Proof of (6.2.4). We follow closely [L-T.11, p. 294-5].

Step 1. Let $\hat{v}(\tau, x, \eta)$ denote the Laplace-Fourier transform of $v(t, x, y)$: Laplace in time $t \to \tau = \gamma + i\sigma$, $\gamma > 0$, $\sigma \in \mathbb{R}$, and Fourier in $y \to i\eta$, $\eta \in \mathbb{R}$, leaving $x \geq 0$ as a parameter. We then obtain for the solution of (6.2.1)

$$\begin{cases} \tau^2\hat{v} = \hat{v}_{xx} - \eta^2\hat{v} \\ \hat{v}_x(\tau, 0, \eta) = \hat{g}(\tau, \eta) \end{cases} \quad \text{or } \hat{v}(\tau, x, \eta) = -\frac{e^{-\sqrt{\tau^2+\eta^2}x}}{\sqrt{\tau^2 + \eta^2}}\,\hat{g}(\tau, \eta), \tag{6.2.5}$$

$\tau^2 + \gamma^2 = (\gamma^2 + \eta^2 - \sigma^2) + 2i\gamma\sigma$ as in (5.2.6).

Step 2. We shall show that

$$\int_0^\infty e^{-2\gamma t}\|(B^*Lg)(t)\|^2_{L_2(\Gamma)}dt = \int_0^\infty e^{-2\gamma t}\|v_t(t,0,\ \cdot\ )\|^2_{L_2(R_y)}dt$$

$$= \frac{1}{2\pi}\iint\limits_{\mathbb{R}^2_{\sigma\eta}} |\tau|^2\frac{|\hat{g}(\tau,\eta)|^2}{|\tau^2+\eta^2|}\,d\sigma\,d\eta, \qquad (6.2.6)$$

where $\mathbb{R}^2_{\sigma\eta}$ is the 2-dimensional Euclidean space in the variables σ and η. In fact, recalling (6.2.2), the Parseval identity for Laplace transforms $t \to \tau$ [D.1, Theorem 31.8, p. 212] and for the Fourier transform $y \to i\eta$, as well as (6.2.5) for $x = 0$, we compute ($\sim$ denotes the Laplace transform in (6.2.8)):

$$\int_0^\infty e^{-2\gamma t}\|(B^*Lg)(t)\|^2_{L_2(\Gamma)}dt = \int_0^\infty e^{-2\gamma t}\|v_t(t,0,\ \cdot\ )\|^2_{L_2(\Gamma)}dt \qquad (6.2.7)$$

$$= \int_{-\infty}^\infty\int_0^\infty e^{-2\gamma t}|v_t(t,0,y)\|^2dt\,dy$$

$$\text{(by [D.1, p. 212])} \quad = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty |\tilde{v}_t(\tau,0,y)|^2d\sigma\,dy \qquad (6.2.8)$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty |\tau|^2|\tilde{v}(\tau,0,y)|^2d\sigma\,dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty |\tau|^2|\hat{v}(\tau,0,\eta)|^2d\sigma\,d\eta \qquad (6.2.9)$$

$$\text{(by (6.2.5))} \quad = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty |\tau|^2\frac{|\hat{g}(\tau,\eta)|^2}{|\sqrt{\tau^2+\eta^2}|^2}\,d\sigma\,d\eta \qquad (6.2.10)$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty |\tau|^2\frac{|\hat{g}(\tau,\eta)|^2}{|\tau^2+\eta^2|}\,d\sigma\,d\eta, \qquad (6.2.11)$$

and (6.2.11) establishes (6.2.6). In (6.2.8) and (6.2.9) we have invoked the Parseval identity for the Laplace transform $t \to \tau$ [D.1, p. 212] and for the Fourier transform $y \to i\eta$; while in (6.2.10), we have recalled $\hat{v}(\tau,0,\eta)$ from (6.2.5) with $x = 0$.

Step 3. For fixed $\gamma > 0$, we define, as in [L-T.11, Eqn. (2.18)], the (bad) region $\mathcal{B}^\gamma_{\sigma\eta}$ of the first quadrant of the (σ,η)-plane by

$$\mathcal{B}^\gamma_{\sigma\eta} \equiv \{(\sigma,\eta) \in \mathbb{R}^2 : 2\gamma\sigma \geq 1,\ \eta \geq 0 : |\gamma^2+\eta^2-\sigma^2| \leq 1\}, \qquad (6.2.12)$$

comprised between the equilateral hyperbolas $\gamma^2+\eta^2-\sigma^2 = \pm 1$, around the equilateral hyperbola $\mathrm{Re}(\tau^2+\eta^2-\sigma^2) = 0$ for $\sigma \geq \frac{1}{2\gamma}$. We note that in $\mathcal{B}^\gamma_{\sigma\eta}$ we have

$$\text{in } \mathcal{B}^\gamma_{\sigma\eta} : \sigma \sim \eta;\ |\tau^2+\eta^2| \sim \sigma \sim \eta. \qquad (6.2.13)$$

In view of identity (6.2.6), in order to establish the negative result (6.2.4), it is sufficient to show that: there exists $g \in L_2(0,\infty; L_2(\Gamma))$ such that

$$\iint_{\mathcal{B}^{\gamma}_{\sigma\eta}} \sigma^2 \frac{|\hat{g}(\tau,\eta)|^2}{|\tau^2+\eta^2|}\, d\sigma\, d\eta = \infty. \tag{6.2.14}$$

Indeed, (6.2.14) holds true, since by (6.2.13) we have

$$\iint_{\mathcal{B}^{\gamma}_{\sigma\eta}} \frac{\sigma^2 |\hat{g}(\tau,\eta)|^2}{|\tau^2+\eta^2|}\, d\sigma\, d\eta \sim \iint_{\mathcal{B}^{\gamma}_{\sigma\eta}} \sigma |\hat{g}(\tau,\eta)|^2 d\sigma\, d\eta. \tag{6.2.15}$$

Thus, it suffices to take a function $\hat{g}(\sigma,\eta)$ which is in $L_2(\mathcal{B}^{\gamma}_{\sigma\eta})$ and no better on $\mathcal{B}^{\gamma}_{\sigma\eta}$ and zero elsewhere, to obtain the sought-after function producing the negative conclusion (6.2.14).

7 A third hyperbolic class where (2.14) fails: $B^*L \notin \mathcal{L}(L_2(0,T;U))$. The multidimensional Kirchhof equation with 'moments' boundary control

Section 4.9 stated that, when *dim* $\Omega = 1$, the Kirchhof equation with moments boundary control does satisfy property (2.14) on B^*L by reducing this problem to the one-dimensional wave equation with Dirichlet-boundary control. The same reduction shows that, when dim $\Omega \geq 2$, the Kirchhof equation with moments controls fails to satisfy property (2.14) on B^*L.

In this section we consider the hyperbolic Kirchhof equation on an open bounded domain Ω, dim $\Omega \geq 2$, with boundary control acting on the 'moment' boundary conditions. Because of the special nature of the boundary conditions, this mixed PDE problem can be converted into a wave equation problem—more precisely, the z-problem (5.1.11) in Section 5.1—modulo lower-order terms. Thus, the results of Section 5.1 can be invoked, in particular, the counterexample in Section 5.2. As a result, we likewise obtain that $B^*L \notin \mathcal{L}(L_2(0,T;U))$ for the present class of Kirchhof equations.

7.1 Preliminaries. The operator B^*L

Open-loop and closed-loop dissipative systems. In this section we let Ω be an open bounded domain in $\mathbb{R}^n$, $n \geq 2$, with sufficiently smooth boundary Γ. We consider the open-loop Kirchhoff equation in Ω, with boundary control acting in the 'moment' B.C. (actually, the physical moment, in dim $\Omega \geq 2$, is a slight modification of our B.C.), and its corresponding closed-loop dissipative system:

$$\begin{cases} v_{tt} - \gamma\Delta v_{tt} + \Delta^2 v = 0; \\ v(0,\;\cdot\;) = v_0, v_t(0,\;\cdot\;) = v_1; \\ v|_{\Sigma} \equiv 0,\ \Delta v|_{\Sigma} = g; \end{cases} \qquad \begin{cases} w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w = 0 & \text{in } Q; \quad (7.1.1a) \\ w(0,\;\cdot\;) = w_0, w_t(0,\;\cdot\;) = w_1 & \text{in } \Omega; \quad (7.1.1b) \\ w|_{\Sigma} \equiv 0, \Delta w|_{\Sigma} = -\dfrac{\partial w_t}{\partial \nu} & \text{in } \Sigma, \quad (7.1.1c) \end{cases}$$

with $Q \equiv (0,T] \times \Omega$; $\Sigma \equiv (0,T] \times \Gamma$. In (7.1.1a), γ is a positive constant, $\gamma > 0$ (this is critical to make (7.1.1) hyperbolic).

Regularity, exact controllability of the y-problem; uniform stabilization of the w-problem. References for this subsection include [L-T.15], [H-L.1]. We begin by introducing the (state) space of optimal regularity

$$Y \equiv \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega), \tag{7.1.2}$$

where $\mathcal{A}\psi = -\Delta\psi$ as in (5.1.6). For the stabilization result, we shall topologize Y with an *equivalent* norm, in which case we use the notation

$$Y_\gamma \equiv \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}}); \tag{7.1.3a}$$

$$(f_1, f_2)_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})} = ((I + \gamma\mathcal{A}^{\frac{1}{2}})f_1, f_2)_{L_2(\Omega)}, f_1, f_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H_0^1(\Omega). \tag{7.1.3b}$$

Theorem 7.1.1. (Regularity [L-T.15]) Regarding the v-problem (7.1.1), with $y_0 = \{v_0, v_1\} = 0$, the following regularity result holds true for each $T > 0$ (recall (1.2b)): the map

$$L : g \to Lg \equiv \{v, v_t\} \text{ is continuous } L_2(\Sigma) \to C([0,T];\ Y \equiv [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)). \tag{7.1.4}$$

Theorem 7.1.2. (Exact controllability [L-T.15], [H-L.1]) Given any initial condition $\{v_0, v_1\} \in Y$ and $T > 0$ sufficiently large, then there exists a $g \in L_2(\Sigma)$ such that the corresponding solution of the v-problem (7.1.1) satisfies $\{v(T), v_t(T)\} = 0$.

Theorem 7.1.3. (Uniform stabilization [L-T.15], [H-L.1]) With reference to the w-problem (7.1.1), we have that:

(i)

$$\text{the map } \{w_0, w_1\} \in Y_\gamma \equiv \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}}) \to \{w(t), w_t(t)\} \tag{7.1.5}$$

defines a s.c. contraction semigroup e^{At} on Y_γ;

(ii)

$$\Delta w|_\Sigma = -\frac{\partial w_t}{\partial \nu} \in L_2(0, \infty; L_2(\Gamma)) \tag{7.1.6}$$

continuously in $\{w_0, w_1\} \in Y_\gamma$;

(iii) there exist constants $M \geq 1$, $\delta > 0$, such that

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_{Y_\gamma} = \left\| e^{At} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_{Y_\gamma} \leq M e^{-\delta t} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_{Y_\gamma}, \quad t \geq 0. \quad \Box \tag{7.1.7}$$

This result was first shown in [L-T.15] for Ω strictly convex. Then this geometrical condition was eliminated in [H-L.1].

Again, in line with the content of Section 1, all three theorems above are obtained by PDE hard analysis energy methods (not by soft analysis methods). As usual, the most challenging result to prove is Theorem 7.1.3 on uniform stabilization.

Abstract model of v-problem. [L-T.15] We let $\mathcal{A}$ and D be the operators in (5.1.6). Then, the abstract model for the v-problem in (7.1.1) is [L-T.15, Eqn. (2.7), (2.9), p. 70]

$$v_{tt} = -(I+\gamma\mathcal{A})^{-1}\mathcal{A}^2[v + \mathcal{A}^{-1}Dg]; \quad \frac{d}{dt}\begin{bmatrix} v \\ v_t \end{bmatrix} = A\begin{bmatrix} v \\ v_t \end{bmatrix} + Bg; \tag{7.1.8}$$

$$A = \begin{bmatrix} 0 & I \\ -(I+\gamma\mathcal{A})^{-1}\mathcal{A}^2 & 0 \end{bmatrix}; \quad Bg = \begin{bmatrix} 0 \\ -(I+\gamma\mathcal{A})^{-1}\mathcal{A}Dg \end{bmatrix}; \quad B^*\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = D^*\mathcal{A}x_2. \tag{7.1.9}$$

With B^* defined by $(Bg_2, x)_{Y_\gamma} = (g_2, B^*x)_{L_2(\Gamma)}$ with respect to the Y_γ-topology in (7.1.3), we readily find the expression in (7.1.9).

Reduction of v-model to a wave equation model, modulo lower-order terms.

The operator B^*L. With $y_0 = \{v_0, v_1\} = 0$, we see that

$$B^*Lg_2 = B^*\begin{bmatrix} v(t; y_0 = 0) \\ v_t(t; y_0 = 0) \end{bmatrix} = -D^*\mathcal{A}v_t(t; y_0 = 0) = \frac{\partial v_t}{\partial\nu}(t; y_0 = 0), \tag{7.1.10}$$

recalling the standard property that $D^*\mathcal{A} = -\frac{\partial}{\partial\nu}$ on $H_0^1(\Omega)$.

Goal. Our goal in this section is to show that for the v-problem (7.1.1), we have

$$B^*L \notin \mathcal{L}(L_2(0,T; L_2(\Gamma)). \tag{7.1.11}$$

Reduction of v-model to a wave-model. Using [L-T.15, (C.3), p. 100]

$$(I+\gamma\mathcal{A})^{-1}\mathcal{A}^2 = \frac{\mathcal{A}}{\gamma} - \frac{1}{\gamma^2}I + \frac{1}{\gamma^2}(I+\gamma\mathcal{A})^{-1} \text{ on } \mathcal{D}(\mathcal{A}) \tag{7.1.12}$$

in the v-equation (7.1.8), yields

$$v_{tt} = -\frac{\mathcal{A}v}{\gamma} - \frac{Dg}{\gamma} + \left[\frac{I}{\gamma^2} - \frac{(I+\gamma\mathcal{A})^{-1}}{\gamma^2}\right](v + \mathcal{A}^{-1}Dg), \tag{7.1.13}$$

where $v|_\Sigma \equiv 0$ by (7.1.4). Motivated by (7.1.13), we then introduce the abstract equation

$$u_{tt} = -\frac{\mathcal{A}u}{\gamma} - \frac{Dg}{\gamma}, \text{ or } \begin{cases} u_{tt} = \frac{1}{\gamma}\Delta u - \frac{1}{\gamma}Dg & \text{in } Q; \quad (7.1.14a) \\ u(0, \cdot) = 0, \; u_t(0, \cdot) = 0 & \text{in } \Omega; \quad (7.1.14b) \\ u|_\Sigma = 0 & \text{in } \Sigma. \quad (7.1.14c) \end{cases}$$

We note that the u-problem in (7.1.14) differs from the v-problem in (7.1.13) only by lower-order terms in v, and smoother terms in g. Thus, the u-problem and the v-problem *possess the same regularity.* In particular, recalling (7.1.4), we have

$$\{u, u_t\} \in C([0,T]; [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)) \text{ continuously in } g \in L_2(\Sigma). \tag{7.1.15}$$

Thus, in light of (7.1.10), in order to prove (7.1.11), we shall equivalently establish that: with reference to the u-problem (7.1.14), we have

$$\text{the map } g \to \frac{\partial u_t}{\partial \nu} \text{ is } \textit{not} \text{ continuous } L_2(\Sigma) \to L_2(\Sigma). \tag{7.1.16}$$

Indeed, statement (7.1.16) follows at once, if we introduce the new variable $z = u_t \in C([0,T]; H_0^1(\Omega))$, continuously in $g \in L_2(\Sigma)$. Then, the u-PDE problem in (7.1.14) becomes essentially the z-PDE problem in (5.1.11) with same *a-priori* regularity as in (5.1.10). For this z-problem, the statement

$$\text{the map } g \to \frac{\partial z}{\partial \nu} \text{ is } \textit{not} \text{ continuous } L_2(\Sigma) \to L_2(\Sigma), \tag{7.1.17}$$

equivalent to (7.1.16) has been proved by virtue of the counterexample in Section 5.2. Hence, the desired conclusion (7.1.11) is established.

8 A fourth Petrowski's class where (2.14) fails: $B^*L \notin \mathcal{L}(L_2(0,T;U))$. The multidimensional Schrödinger equation with Neumann boundary control

8.1 Exact controllability/uniform stabilization in $H^1(\Omega)$, $\dim \Omega \geq 1$

Here, to make our point, it suffices to consider the canonical case of the multidimensional Schrödinger equation:

$$\begin{cases} iy_t - \Delta y = 0; \\ y(0, \cdot) = y_0; \\ y|_{\Sigma_0} \equiv 0; \\ \left.\dfrac{\partial y}{\partial \nu}\right|_{\Sigma_1} = u \in L_2(\Sigma_1); \end{cases} \qquad \begin{cases} iw_t - \Delta w = 0 & \text{in } Q; \\ w(0, \cdot) = w_0 & \text{in } \Omega; \\ w|_{\Sigma_0} \equiv 0 & \text{in } \Sigma_0; \\ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} = -w_t & \text{in } \Sigma_1, \end{cases} \qquad \begin{matrix} (8.1.18a) \\ (8.1.18b) \\ (8.1.18c) \\ (8.1.18d) \end{matrix}$$

where $\Gamma = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \phi$, $\Gamma_0 \neq 0$, $h \cdot \nu \leq 0$ in Γ_0 for a coercive smooth vector field $h(x)$ on Ω. We then leave more general situations (variable coefficients in the principal part; energy level $H^1(\Omega)$-terms with variable coefficients, etc.) to the literature [Tr.9], [T-Y.1], etc. We shall focus on the exact controllability/uniform stabilization results.

Theorem 8.1.1. (Exact controllability [M.1], [Le.1], [Tr.9], [T-Y.1]) Let $T > 0$ be arbitrary. Then, the y-problem in (8.1.1) is exactly controllable on the state space $H^1_{\Gamma_0}(\Omega)$, with $L_2(\Sigma_1)$-controls, $\Sigma_1 = (0,T] \times \Gamma_1$.

Theorem 8.1.2. (Uniform stabilization [M.1], [Le.1], [Tr.9], [T-Y.1]. (i) The w-problem in (8.1.1) is well-posed in the semigroup sense on the space $H^1_{\Gamma_0}(\Omega)$; i.e., the map $w_0 \to w(t) = e^{A_F t} w_0$ defines a s.c. semigroup $e^{A_F t}$ on $H^1_{\Gamma_0}(\Omega)$, which is contraction in the equivalent norm of $\mathcal{D}((-A_F)^{\frac{1}{2}})$.

(ii) Moreover, the w-problem is uniformly stable on $H^1_{\Gamma_0}(\Omega)$: there exist constants $M \geq 1$, $\delta > 0$ such that $\|e^{A_F t}\| \leq M e^{-\delta t}$, $t \geq 0$, in the uniform operator norm. □

Remark 8.1.1. First, [Le.1] shows the result under more general 'geometric optics' conditions. Next, the case where $\cdot|_{\Sigma_0} = 0$ is replaced by $\frac{\partial \cdot}{\partial \nu}|_{\Sigma_0} = 0$ for both the y and the w-problem is much more challenging, it requires an additional geometrical condition [L-T-Z.2]. $\square$

The regularity result is considered (at least in the negative sense for $\dim \Omega \geq 2$) in Section 8.2 below.

8.2 Counterexample for the multidimensional Schrödinger equation with Neumann boundary control: $L \notin \mathcal{L}(L_2(0,T;U); H^{\epsilon}(\Omega))$, $\epsilon > 0$. *A-fortiori*: $B^*L \notin \mathcal{L}(L_2(0,T;U))$

The present section complements Section 8.1. Here, the focus will be on the multidimensional case $\dim \Omega \geq 2$. Two main results of negative character are given, with the second being implied by the first by virtue of Proposition A.1 in Appendix A:

(1) With reference to the boundary $\to$ interior map L defined in (1.3), we shall show by means of a counterexample that: $L \notin \mathcal{L}(L_2(\Sigma);\ L_2(0,T;H^1(\Omega))$, though $H^1(\Omega)$ is the space of exact controllability/uniform stabilization, as seen in Section 8.1. Even more drastically, we shall show that

$$L \notin \mathcal{L}(L_2(\Sigma);\ L_2(0,T;H^{\epsilon}(\Omega)), \quad \forall\, \epsilon > 0. \tag{8.2.1}$$

This negative result is the counterpart of the negative result for wave equations with $L_2(\Sigma)$-Neumann control given in [L-T.11, Counterexample, p. 294], which was already invoked in Section 6. The present proof is an adaptation of that given in [L-T.11].

(2) As a consequence of part (1) via Proposition A.1 of Appendix A (see also the implication (6.1), we deduce that $B^*L \in \mathcal{L}(L_2(0,T;U))$ in the present case.

Counterexample. It will suffice to consider the Schrödinger equation on a 2-dimensional half-space, the setting in Sections 5.2 and 6.2, with Neumann boundary control. Hereafter, we let $\Omega \equiv \mathbb{R}^+_2$ and $\Gamma = \Omega|_{x=0}$ as in (5.2.1). On Ω we consider the problem

$$iv_t = v_{xx} + v_{yy} \quad \text{in } Q \equiv (0,\infty) \times \Omega; \tag{8.2.2a}$$
$$v(0,\ \cdot\ ) = 0 \quad \text{in } \Omega; \tag{8.2.2b}$$
$$v_x|_{x=0} = g \quad \text{in } \Sigma \equiv (0,\infty) \times \Gamma. \tag{8.2.2c}$$

Goal. We want to show that: given $T > 0$, there exists some $g \in L_2(0,T;L_2(\Gamma))$ such that

$$Lg = v \notin L_2(0,T;H^{\epsilon}(\Omega)), \quad \forall\, \epsilon > 0. \tag{8.2.3}$$

To this end, it will suffice to show that there exists $g \in L_2(0,\infty;L_2(\Gamma))$ such that

$$e^{-\gamma t}(Lg)(t) = e^{-\gamma t}v(t) \notin L_2(0,\infty;H^{\epsilon}(\Omega)), \tag{8.2.4}$$

no matter which constant $\gamma > 0$ we choose.

Proof of (8.2.4.) **Step 1.** Let $\hat{v}(\tau, x, \eta)$ be the Laplace-Fourier transform of $v(t, x, y)$: Laplace in time $t \to \tau = \gamma + i\sigma$, $\gamma > 0$, $\sigma \in \mathbb{R}$, and Fourier in $y \to i\eta$, $\eta \in \mathbb{R}$, leaving $x \geq 0$ as a parameter. We then obtain for the solution of (8.2.8), where $\eta^2 + i\tau = (\eta^2 - \sigma) + i\gamma$:

$$\begin{cases} i\tau\hat{v} = \hat{v}_{xx} - \eta^2\hat{v} \\ \hat{v}_x(\tau, 0, \eta) = \hat{g}(\tau, \eta) \end{cases} \quad \text{or } \hat{v}(\tau, x, \eta) = -\frac{\hat{g}(\tau, \eta)}{\sqrt{(\eta^2 - \sigma) + i\gamma}} e^{-\sqrt{(\eta^2-\sigma)+i\gamma}\ x}. \tag{8.2.5}$$

Step 2. For fixed $\gamma > 0$, we define (by adaptation of [L-T.11, Eqn. (2.18)], or (6.2.12)) the (bad) region $\mathcal{B}^{\gamma}_{\sigma\eta}$ of the first quadrant of the (σ, η)-plane by

$$\mathcal{B}^{\gamma}_{\sigma\eta} \equiv \{(\sigma, \eta) \in \mathbb{R}^2 : \sigma \geq 1, \eta \geq 0 : |\eta^2 - \sigma| \leq 1\} \tag{8.2.6}$$

comprised between the two parabolas $\eta^2 - \sigma = \pm 1$ in the first quadrant, around the parabola $\eta^2 = \sigma$. We note that in $\mathcal{B}^{\gamma}_{\sigma\eta}$ we have

$$\text{in } \mathcal{B}^{\gamma}_{\sigma\eta} : \ \sigma \sim \eta^2; \ |(\eta^2 - \sigma) + i\gamma| \sim 1 \quad [\gamma \leq |(\eta^2 - \sigma) + i\gamma| \leq \sqrt{1 + \gamma^2} \tag{8.2.7}$$

$$\text{Re}\sqrt{(\eta^2 - \sigma) + i\gamma} \sim 1.$$

Step 3. In order to establish the negative result (8.2.4), it is sufficient to prove that: there exists $g \in L_2(0, \infty; L_2(\Gamma))$ such that, recalling (8.2.5), we have

$$|\eta|^{\epsilon}|\hat{v}| = |\eta|^{\epsilon} \frac{|\hat{g}(\tau, \eta)|}{|\sqrt{(\eta^2 - \sigma) + i\gamma}|} \ e^{-\text{Re}\sqrt{(\eta^2-\sigma)+i\gamma}\, x} \notin L_2(0, \infty; L_2(\Omega)). \tag{8.2.8}$$

To this end, we compute

$$\iint_{\mathcal{B}^{\gamma}_{\sigma\eta}} \int_0^{\infty} |\eta|^{2\epsilon} \frac{|\hat{g}(\tau, \eta)|^2}{|(\eta^2 - \sigma) + i\gamma|} \ e^{-\text{Re}\sqrt{(\eta^2-\sigma)+i\gamma}\ x} dx\, d\sigma\, d\eta \tag{8.2.9}$$

$$= \iint_{\mathcal{B}^{\gamma}_{\sigma\eta}} |\eta|^{2\epsilon} \frac{|\hat{g}(\sigma, \eta)|^2}{|(\eta^2 - \sigma) + i\gamma|} \ \frac{1}{\text{Re}\sqrt{(\eta^2 - \sigma) + i\gamma}} \ d\sigma\, d\eta \tag{8.2.10}$$

$$\text{(by (8.2.7))} \quad \sim \iint_{\mathcal{B}^{\gamma}_{\sigma\eta}} |\eta|^{2\epsilon} |\hat{g}(\sigma, \eta)|^2 d\sigma\, d\eta, \tag{8.2.11}$$

where in the last step we have invoked (8.2.7). Thus, it suffices to take a function $\hat{g}(\sigma, \eta)$ which is $L_2(\mathcal{B}^{\gamma}_{\sigma\eta})$, and no better, on $\mathcal{B}^{\gamma}_{\sigma\eta}$, and zero elsewhere, to obtain the sought-after function producing the negative conclusion (8.2.4).

Appendix: From the regularity (2.14) of B^*L to the regularity of (1.3) L

Proposition A.1. Consider system (1.1) under the assumptions stated there in (i) and (ii) on A and B. Assume further that

(i)

$$B^*L \in \mathcal{L}(L_2(0,T;U)); \text{ i.e., property (2.8);} \tag{A.1}$$

(ii) A is of the form $A = iS + kI$, with S a self-adjoint operator on Y and $k \in \mathbb{R}$, so that $A^* = -A + 2kI$, and

$$e^{A^*s} = e^{-As}e^{2ks}, \quad s \in \mathbb{R}. \tag{A.2}$$

Then, with reference to (1.2b), we have that

$$L: \text{ continuous } L_2(0,T;U) \to C([0,T];Y). \tag{A.3}$$

Proof. First, since A is the generator of a s.c. group on Y, we can invoke the lifting theorem from [L-T.5], [L-T.21, Chapter 7]: accordingly, in order to establish (A.3), it is sufficient (and necessary) to prove that

$$L: \text{ continuous } L_2(0,T;U) \to L_2(0,T;Y). \tag{A.4}$$

We thus show (A.4). To this end, let $u \in L_2(0,T;U)$. Then, the following inner product on $L_2(0,T;U)$ is well-defined:

$$\int_0^T \left(\{B^*Lu\}(t), \int_t^{2t} e^{-k(t-\tau)} u(2t-\tau)d\tau \right)_U dt = \text{well-defined} \tag{A.5}$$

$$= \int_0^T \left(B^* \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \int_t^{2t} e^{-k(t-\tau)}u(2t-\tau)d\tau \right)_U dt \tag{A.6}$$

$$= \int_0^T \left(\int_0^t e^{A\frac{(t-\tau)}{2}}Bu(\tau)d\tau, \int_t^{2t} e^{A^*\frac{(t-\tau)}{2}} e^{-k(t-\tau)}Bu(2t-\tau)d\tau \right)_Y dt \tag{A.7}$$

(use (A.2) with $s = \frac{t-\tau}{2}$)

$$= \int_0^T \left(\int_0^t e^{A\frac{(t-\tau)}{2}}Bu(\tau)d\tau, \int_t^{2t} e^{A\frac{(\tau-t)}{2}}Bu(2t-\tau)d\tau \right)_Y dt \tag{A.8}$$

(change of variable $\tau - t = t - \sigma$, or $\sigma = 2t - \tau$)

$$= \int_0^T \left(\int_0^t e^{A\frac{(t-\tau)}{2}}Bu(\tau)d\tau, \int_0^t e^{A\frac{(t-\sigma)}{2}}B(\sigma)d\sigma \right)_Y dt \tag{A.9}$$

$$= \int_0^T \left\| \int_0^t e^{A\frac{(t-\tau)}{2}}Bu(\tau)d\tau \right\|_Y^2 dt \tag{A.10}$$

$$= \int_0^{\frac{T}{2}} \left\| \int_0^r e^{A(r-\xi)} Bu(2\xi) 2\, d\xi \right\|_Y^2 2dr \tag{A.11}$$

$$= 8 \int_0^{\frac{T}{2}} \|\{L\mu\}(r)\|_Y^2 dr, \tag{A.12}$$

after setting $\frac{t}{2} = r$, $\frac{\tau}{2} = \xi$ in going from (A.10) to (A.11), and after recalling L in (1.2b) and setting $\mu(\,\cdot\,) = u(2\,\cdot\,)$ in going from (A.11) to (A.12). Next, making (A.5) more precise by virtue of assumption (A.1), the identity from (A.5) to (A.12) yields via Schwarz inequality

$$\int_0^{\frac{T}{2}} \|\{L\mu\}(r)\|_Y^2 dr \leq \|B^* Lu\|_{L_2(0,T;U)} \left\| \int_t^{2t} e^{-k(t-\tau)} u(2t-\tau) d\tau \right\|_{L_2(0,T;U)} \tag{A.13}$$

(invoking (A.1) and using the change of variable $2t - \tau = s$)

$$\leq |||B^* L|||\, \|u\|_{L_2(0,T;U)} \left\| e^{kt} \int_0^t u(\sigma) d\sigma \right\|_{L_2(0,T;U)}, \tag{A.14}$$

where $|||\quad|||$ denotes the norm in $\mathcal{L}(L_2(0,T;U))$. Thus, since $\|\mu\|_{L_2(0,T/2;U)} = \|u\|_{L_2(0,T;U)}$, then (A.14) leads to

$$\|L\mu\|_{L_2\left(0,\frac{T}{2};U\right)} \leq C_T |||B^* L|||\, \|\mu\|_{L_2\left(0,\frac{T}{2};U\right)}, \tag{A.15}$$

e.g., with $C_T = e^{2kT}\sqrt{T}$ and (A.15) proves (A.3) since T is arbitrary. □

Corollary A.2. Proposition A.1 applies to the v-system (2.1), with A and B defined in (2.4), $A^* = -A$ (hence $k = 0$), on $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times H$. □

ACKNOWLEDGMENT: This work was carried out while the authors were visiting the Scuola Normale Superiore, Pisa, Italy, whose generous hospitality is acknowledged and greatly appreciated. Our particular thanks go to Professor G. Da Prato.

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