# EQUIVALENT CHARACTERIZATION OF INVARIANT SUBSPACES OF $H^2$ AND APPLICATIONS TO OPTIMAL SENSITIVITY PROBLEM

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REPORT No. 07, 2002/2003, spring

ISSN 1103-467X
ISRN IML-R- -07-02/03- -SE+spring

INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

# Equivalent Characterization of Invariant Subspaces of $H^2$ and Applications to Optimal Sensitivity Problem

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**Abstract**

This paper gives some equivalent characterizations for invariant subspaces of $H^2$, when the underlying structure is specified by the so-called pseudorational transfer functions. This plays a fundamental role in computing the optimal sensitivity for a certain important class of infinite-dimensional systems, including delay systems. A closed formula, easier to compute than the well-known Zhou-Khargonekar formula, is given for optimal sensitivity for such systems. An example is given to illustrate the result.

## 1 Introduction

The characterization of shift-invariant subsapces of $H^2$ by the Beurling-Lax theorem has been of great theoretical interest [7]. In particular, it has played a key role in $H^\infty$ control theory, especially in the optimal sensitivity problem. This is usually formulated as follows: Given a rational and stable weighting function $W(s)$ and an inner function $m(s)$, find the optimal sensitivity

$$\gamma_{\rm opt} := \inf_{\psi \in H^\infty} \|W - m\psi\|_\infty. \tag{1}$$

When $W$ is proper and rational (which covers all practically interesting cases), a beautiful solution is known as follows [19, 13, 18]: Let $(A, B, C)$ be a minimal realization of $W$. Define its Hamiltonian $H_\gamma$ by

$$H_\gamma := \begin{bmatrix} A & BB^T/\gamma \\ -C^TC/\gamma & -A^T \end{bmatrix}. \tag{2}$$

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Then the optimal sensitivity $\gamma_{\rm opt}$ is the maximum of those that satisfy

$$\det m\tilde{}(H_\gamma)|_{22} = 0$$

where $M|_{22}$ denotes the $(2,2)$-block of matrix $M$ and $m\tilde{}(s) = \overline{m(-\overline{s})}$. It also has a natural extension to the two-block case [6]. Furthermore, the case of a simple delay $m(s) = e^{-Ls}$ has played a key role in obtaining the solution to the $H^\infty$ problem for sampled-data systems [1, 9].

It should however be noted that when $m$ is a general inner function this computation is not necessarily easy to execute. For in general, the plant is not specified in terms of such an inner function $m$. For example, it is not trivial to obtain such $m$ for a delay-differential system

$$\begin{aligned} \dot{x}(t) &= \alpha x(t-1) + u(t) \\ y(t) &= x(t) - \beta x(t-1), \quad 0 < \alpha, \beta < 1. \end{aligned}$$

The plant transfer function is $(e^s - \beta)/(se^s - \alpha)$ but the associated $m$ is hard to compute.

In view of this, we attempt to derive a more direct formula for the computation of the optimal sensitivity. We deal with the class of pseudorational transfer functions (impulse responses) introduced by the latter author [16]. The advantage here is that such an impulse response $f$ admits a fractional representation over a ring of distributions of compact support as $f = q^{-1} * p$. The associated transfer function $\hat{f}(s)$ is then written as $\hat{f}(s) = \hat{p}(s)/\hat{q}(s)$, where $\hat{p}(s)$ and $\hat{q}(s)$ are entire functions of exponential type, according to the Paley-Wiener theorem [12]. This representation is particularly amenable to inner functions and also in close connection with delay systems. In this regard, we give a characterization of all left shift invariant subspaces of $H^2$, equivalent to the Beurling-Lax theorem, and then apply the result to derive a computable condition for the optimal sensitivity problem equivalent to (1). An example is given to illustrate the effectiveness of the result.

# 2 Pseudorational impulse responses and their relizatons

## 2.1 Preliminaries

We need some notions from distribution theory [12]. For a given distribution $\alpha$, $\operatorname{supp}\alpha$ denotes its support. The following quantities are of importance; we allow them to be $\pm\infty$, but when $\operatorname{supp}\alpha$ is bounded, they are both finite:

$$\begin{aligned} \ell(\alpha) &:= \inf\,\{t : t \in \operatorname{supp}\alpha\}, \\ r(\alpha) &:= \sup\,\{t : t \in \operatorname{supp}\alpha\}. \end{aligned}$$

$\mathcal{E}'(\mathbb{R}_-)$ denotes the space of distributions having compact support in $(-\infty, 0]$; the Dirac delta distribution $\delta$ at the origin, its derivative $\delta'$, delta distribution $\delta_a (a < 0)$ at point $a$, etc., are elements in $\mathcal{E}'(\mathbb{R}_-)$. $\mathcal{D}'_+(\mathbb{R})$ is the space of distributions having support bounded on the left. Clearly $\mathcal{E}'(\mathbb{R}_-)$ is a subspace of $\mathcal{D}'_+(\mathbb{R})$. Both $\mathcal{E}'(\mathbb{R}_-)$ and $\mathcal{D}'_+(\mathbb{R})$ constitute a covolution algebra. Concerning the suport of convolution of distributions with bounded support, the following lemma is obtained in [17]:

**Lemma 2.1** *For distributions $\alpha, \beta$ having support bounded on the right,*

$$r(\alpha * \beta) = r(\alpha) + r(\beta).$$

This is a consequence of Titchmarsh's theorem on convolution [3]. Similarly, it follows that for $\alpha,\ \beta \in \mathcal{D}'_+(\mathbb{R})$,

$$\ell(\alpha * \beta) = \ell(\alpha) + \ell(\beta).$$

The class of pseudorational transfer functions (impulse responses) plays a crucial role in realization, modeling, and control of infinite-dimensional systems, especially delay-differential systems [16, 17]:

**Definition 2.2** An impulse response $f$ is said to be *pseudorational* if there exist $q, p \in \mathcal{E}'(\mathbb{R}_-)$ such that

1. $q^{-1}$ exists over $\mathcal{D}'_+(\mathbb{R})$,
2. $\operatorname{ord} q^{-1} = -\operatorname{ord} q$,
3. $f$ can be written as

$$f = q^{-1} * p,$$

where $q^{-1}$ is taken with respect to convolution and $\operatorname{ord} q$ denotes the order of a distribution $q$ [12].

For a distribution $\alpha$, its Laplace transform is denoted by $\hat{\alpha}(s)$. Similarly, for a set $X$ of distributions, $\hat{X} := \{\hat{f} : f \in X\}$ provided that every $f \in X$ is Laplace transformable. From the Paley-Wiener-Schwartz theorem [12], a distribution $q$ belongs to $\mathcal{E}'(\mathbb{R}_-)$ if and only if its Laplace transform $\hat{q}$ is an entire function satisfying

$$\begin{aligned} |\hat{q}(s)| &\le C(1+|s|)^k e^{a \operatorname{Re} s}, \quad && \operatorname{Re} s \ge 0, \\ &\le C(1+|s|)^k, && \operatorname{Re} s < 0, \end{aligned}$$

for some $a,\ C,\ k > 0$. In the Laplace domain, pseudorational transfer functions are thus the ratio of entire functions of exponential type—the simplest extension of polynomials.

As usual $H^p$ and $H^p_-$ denote the Hardy spaces on the open right- and left-half complex plain, respectively. As is well known ([7]), $H^2$ and $H^2_-$ are the spaces of the Laplace transform of functions in $L^2[0, \infty)$ and $L^2(-\infty, 0]$, respectively. The Hilbert space of functions square integrable on the imaginary axis will be denoted by $L^2(j\mathbb{R})$. Clearly $L^2(j\mathbb{R}) = H^2 \oplus H^2_-$.

Let $q^{\sim}$ denote the paraconjugate of $q$, i.e.,

$$q^{\sim}(s) := \overline{q(-\overline{s})}.$$

Then $q^{\sim}$ is in $H^p_-$ if and only if $q$ belongs to $H^p$.

## 2.2 Pseudorational impulse responses and their realizations

In this subsection, we outline realization theory of pseudorational transfer functions introduced in [16].

Let $\Omega := \lim\limits_{\rightarrow} L^2[-n, 0]$ denote the *inductive limit* of the spaces $\{L^2[-n, 0]\}_{n>0}$; it is the union of all these spaces endowed with the finest topology that makes all injections $j_n : L^2[-n, 0] \to \Omega$ continuous; see, e.g., [16]. Dually, $\Gamma := L^2_{loc}[0, \infty)$ is the space of all *locally* Lebesgue square integrable functions with obvious family of seminorms:

$$\|\phi\|_n := \left\{ \int_0^n |\phi(t)|^2 dt \right\}^{1/2}.$$

This is the *projective limit* of spaces $\{L^2[0, n]\}_{n>0}$. $\Omega$ is the space of past inputs, and $\Gamma$ is the space of future outputs, with the understanding that the present time is 0. These spaces are equipped with the natural *left shift* semigroups $\sigma_t$ [16].

An *input/output* or a *Hankel operator* associated with an impulse response function $f$ is defined to be the continuous linear mapping $\mathcal{H}_f : \Omega \to \Gamma$ defined by

$$\mathcal{H}_f(\omega)(t) := \int_{-\infty}^0 f(t - \tau)\omega(\tau)d\tau.$$

For a pseudorational impulse response $f = q^{-1} * p$, one can always associate with it a realization $\Sigma^{q,p}$ as follows [16]:

Define $X^q$ as follows:

$$X^q := \{x \in \Gamma \mid \pi(q * x) = 0\}$$

where $\pi$ is the truncation to $(0, \infty)$. It is easy to see that $X^q$ is a $\sigma_t$-invariant closed subspace of $\Gamma$. To define $\Sigma^{q,p}$, take this $X^q$ as the state space with

$\sigma_t$ (restricted to $X^q$) as its transition semigroup. Then define $g : \Omega \to X^q$ and $h : X^q \to \Gamma$ as follows.

$$\begin{aligned} g(\omega) &:= \pi(q^{-1} * p * \omega) \\ h(x) &= x \text{ (injection).} \end{aligned}$$

The mappings $g$ and $h$ are called *reachability map* and *observability map* respectively, and satisfy $\mathcal{H}_f = hg$.

Related to this realization of pseudorational impulse response, the following facts are known [14, 16, 17].

**Facts 2.3**

1. If the pair $(q, p)$ is approximately coprime, i.e.,

$$\lim_{n\to\infty} (q * r_n + p * s_n) = \delta$$

for some sequences $\{r_n\}$ and $\{s_n\}$ in $\mathcal{E}'(\mathbb{R}_-)$, the realization $\Sigma^{q,p}$ is *canonical* [16].

2. The spectrum of the infinitesimal generator $A$ of $\sigma_t$ in $X^q$ is given by

$$\sigma(A) = \{\lambda \,|\, \hat{q}(\lambda) = 0\}.$$

Furthermore, every point in $\sigma(A)$ is an eigenvalue with finite multiplicity.

3. For each $\lambda \in \sigma(A)$, the generalized eigenfunctions are of the form $\{e^{\lambda t}, te^{\lambda t}, \ldots, t^{n-1}e^{\lambda t}\}$, where $n$ is the geometric multiplicity.

4. The state space $X^q$ is decomposed as

$$X^q \cong L^2[0, T] \oplus \overline{X_0}$$

where $X_0$ is the linear subspace spanned by the generalized eigenfunctions given as above.

5. $\Sigma^{q,p}$ is *exponentially stable*, i.e., there exist positive constant $C, \beta$ such that

$$\|\sigma_t x\| \le Ce^{-\beta t}\|x\|,$$

if and only if

$$\sup\{\operatorname{Re}\lambda \,:\, \lambda \in \sigma(A)\} < 0.$$

In other words, there exists $c > 0$ such that

$$\operatorname{Re}\lambda \le -c \text{ for all } \lambda \text{ such that } \hat{q}(\lambda) = 0.$$

# 3 Equivalent Characterization of Invariant Subspaces of $H^2$

As seen in Section 2, $X^q$ plays an important role to construct the state space. The space $X^q$ consists of the function $x \in \Gamma$ such that $q * x$ belongs to $\mathcal{E}'(\mathbb{R}_-)$ [14]. In this section, confining ouuselves to stable transfer functions, we give yet another characterization for $X^q$.

It is known ([15]) that $X^q \subset L^2[0,\infty)$ by stability, and $\hat{X}^q$ is a left shift invariant closed subspace of $H^2$. Dually, its orothogonal complement $H^2 \ominus \hat{X}^q$ is a right shift invariant closed subspace in $H^2$. Hence we can characterize $H^2 \ominus \hat{X}^q$ by an inner function according to the Beurling-Lax theorem [7]: There exists a unique inner function $\hat{m}$ modulo constants such that $\hat{X}^q = H(\hat{m}) := H^2 \ominus \hat{m}H^2$. It is known ([7]) that

$$H(m) = \left\{ \hat{x} \in H^2 : \hat{m}\tilde{\ } \hat{x} \in H^2_- \right\}.$$

In the case of rational functions, this inner function is clearly given as the finite Blaschke product consisting of poles of $1/\hat{q}$. In Theorem 3.2, we give the associated inner function for all stable pseudorational transfer functions.

The inner fuction $\hat{m}$ satisfying $X^q = H(\hat{m})$ is much related to eigenfunction completeness of $X^q$. In [17], $r(q) = 0$ is shown to be necessary and sufficient for $X^q \cong \overline{X_0}$. Corresponding to 4 of Facts 2.3, such an inner function is given as a form consisting of a delay and an infinite Blaschke product. Here the pseudorationality is crucial to prove the convergence of this infinite product.

The following lemma asserts that if $q^{-1}$ is pseudorational with Laplace transform in $H^\infty$, then the left-shifted (by $-\ell(q)$) transfer function $1/(e^{\ell(q)s}\hat{q})$ also belongs to $H^\infty$.

**Lemma 3.1** *Let $q \in \mathcal{E}'(\mathbb{R}_-)$ and $1/\hat{q}(s) \in H^\infty$. Then $1/(e^{\ell(q)s}\hat{q}(s))$ also belongs to $H^\infty$.*

**Proof** Recall that $\hat{f}$ is in $H^\infty$ if and only if convolution with $f$ defines a bounded linear operator on $L^2[0,\infty)$. Take an arbitrary $x \in L^2[0,\infty)$, and we show $(\delta_{\ell(q)} * q^{-1} * x) \in L^2[0,\infty)$. First $q^{-1} * x \in L^2[0,\infty)$ since $\hat{q}^{-1}$ belongs to $H^\infty$. Then it remains to show that the support of $(q^{-1} * x)$ is contained in $[-\ell(q),\infty)$. Notice that $\ell(q^{-1}) + \ell(q) = \ell(\delta) = 0$ and

$$\ell(q^{-1} * x) = \ell(q^{-1}) + \ell(x) = -\ell(q) + \ell(x) \geq -\ell(q),$$

by Lemma 2.1, since $x$ is in $L^2[0,\infty)$. ∎

For example take $\hat{q}(s) = se^s - c$ and the left-shifted (by 1) transfer function $e^s/(se^s - c)$ is indeed causal. The following theorem gives the inner function $\hat{m}$ satisfying $\hat{X}^q = H(\hat{m})$ in a simple form for all stable pseudorational transfer functions.

**Theorem 3.2** *Let $1/\hat{q}(s)$ be stable. Then $\hat{X}^q = H(\hat{m})$ where $\hat{m}$ is given by*

$$\hat{m} = e^{-\ell(q)s}\frac{\hat{q}^{\tilde{}}(s)}{\hat{q}(s)}. \qquad (3)$$

**Proof** First we show that $\hat{m}$ defined by (3) is indeed an inner function. Since clearly $|\hat{m}| = 1$ on the imaginary axis, it suffices to prove that $\hat{m}$ is in $H^\infty$. Take an arbitrary $x \in L^2[0,\infty)$, i.e., $\hat{x} \in H^2$ and we show $m * x \in L^2[0,\infty)$. From the property above $\hat{m}\hat{x} \in L^2(j\mathbb{R})$ and this implies $m * x \in L^2(-\infty,\infty)$. Since $q^{\tilde{}}$ is the mirror image of the distribution $q$, the support of $q^{\tilde{}}$ is entirely contained in $[0, -\ell(q)]$. Therefore we have

$$\ell(m * x) = \ell(q) + \ell(q^{-1}) + \ell(q^{\tilde{}}) + \ell(x) \geq 0$$

by Lemma 2.1. Then $m * x \in L^2[0,\infty)$ and $\hat{m}$ is inner.

Now let us show $\hat{X}^q \subset H(\hat{m})$. Take any $\hat{\omega} \in \hat{X}^q \subset H^2$, i.e., $q * \omega \in \mathcal{E}'(\mathbb{R}_-)$. Then $\hat{m}^{\tilde{}}\hat{\omega}$ is in $L^2(j\mathbb{R})$, because $\hat{m}$ is inner. It follows from Lemma 2.1 that $r((q^{\tilde{}})^{-1}) = -r(q^{\tilde{}}) = \ell(q)$ and

$$r(m^{\tilde{}} * \omega) = r(\delta_{-\ell(q)} * (q^{\tilde{}})^{-1}) + r(q * \omega) \leq 0.$$

This yields $m^{\tilde{}} * \omega \in L^2(-\infty, 0]$, i.e., $\hat{m}^{\tilde{}}\hat{\omega} \in H^2_-$ and we have $\hat{X}^q \subset H(\hat{m})$.

Conversely, suppose that $\hat{x} \in H^2$ and that $\hat{m}^{\tilde{}}\hat{x} \in H^2_-$. Hence

$$\hat{m}^{\tilde{}}\hat{x} = \frac{\hat{q}\hat{x}}{e^{-\ell(q)s}\hat{q}^{\tilde{}}} =: \hat{\psi} \in H^2_-.$$

This yields $\hat{q}\hat{x} = (e^{-\ell(q)s}\hat{q}^{\tilde{}})\hat{\psi}$. Since $r(q * x) = \ell(q) + r(q^{\tilde{}}) + r(\psi) \leq 0$ and $\ell(q * x)$ is bounded, $q * x$ belongs to $\mathcal{E}'(\mathbb{R}_-)$. This implies $H(m) \subset \hat{X}^q$. ∎

For example, we take $W(s) = 1/(se^s - c)$ and

$$X^{(\delta'_{-1} - c\delta)} = \left\{ x \in \Gamma : x(t) = x(1) + c\int_0^{t-1} x(t)dt,\ t \geq 1 \right\}. \qquad (4)$$

This implies $X^q \cong L^2[0,1] \times \mathbb{R}$ [16]. Applying this theorem to (4), we have

$$\hat{X}^{(\delta'_{-1} - c\delta)} = H\left(e^s \frac{-se^{-s} - c}{se^s - c}\right) = H\left(\frac{ce^s + s}{se^s - c}\right).$$

**Remark 3.3** Recall that $\hat{q}(s)$ is expressed as infinite product [2]

$$\hat{q}(s) = e^{as} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\lambda_n}\right) \exp\left(\frac{s}{\lambda_n}\right)$$

where

$$\sum_{n=1}^{\infty} \frac{|\operatorname{Re}\lambda_n|}{|\lambda_n|^2} < \infty.$$

This condition of the zeros is sufficient for the convergence of the corresponding infinite Blaschke product [11]:

$$\sum_{n=1}^{\infty} \frac{|\operatorname{Re}\lambda_n|}{1 + |\lambda_n|^2} < \infty.$$

This readily implies that $e^{-\ell(q)s}\hat{q}^{\sim}(s)/\hat{q}(s)$ is of the form $e^{bs}m_b(s)$ where $m_b(s)$ is the Blaschke product consisting of the zeros of $\hat{q}$.

## 4 Optimal Sensitivity Problem

In this section, we consider the optimal weighted sesitivity problem to find

$$\gamma_{\text{opt}} := \inf_{C(s):\text{stabilizing}} \|W(s)(1 + P(s)C(s))^{-1}\|_\infty.$$

We assume that the given stable weighting function $W(s)$ is rational and strictly proper. We also assume that the given plant $P(s)$ is in $H^\infty$ and pseudorational. By the Youla parametrization, this problem is known to be equivalent to finding $\Gamma_W(WP)$, where $\Gamma_W(\cdot)$ is defined by

$$\Gamma_W\left(\frac{\hat{p}}{\hat{q}}\right) := \inf_{\psi \in H^\infty} \left\| W - \frac{\hat{p}}{\hat{q}}\psi \right\|_\infty$$

for a given pseudorational transfer function $\hat{p}/\hat{q} \in H^\infty$. Our goal in this section is to derive a formula for $\Gamma_W(\hat{p}/\hat{q})$ in a computable form.

For this problem we can applly the method given in [19, 13, 18], if $WP$ is inner. As seen in Section 1, for an inner function $\hat{m}$

$$\Gamma_W(\hat{m}) = \max\{\gamma : \det \hat{m}^{\sim}(H_\gamma)|_{22} = 0\}. \tag{5}$$

Furthermore if $\hat{m} = e^{-hs}$ $(h > 0)$, the optimal sensitivity can be computed by a bisection algorithm [5]. However, $WP$ is in general not necessarily inner. For a given $\hat{p}/\hat{q} \in H^\infty$, we obtain an inner function $\hat{m}$ such that $\Gamma_W(\hat{p}/\hat{q}) = \Gamma_W(\hat{m})$ via the inner-outer factorization. However it is not trivial to obtain the inner part of a general pseudorational function. In what follows, we find such an inner function $\hat{m}$ for a given psudorational transfer function $\hat{p}/\hat{q} \in H^\infty$ having no zeros on the imaginary axis.

## 4.1 Pseudorational function without unstable zeros

We start with a relatively simple case. Consider a pseudorational transfer function $\hat{p}/\hat{q} \in H^\infty$ with no unstable zeros. In the case of a rational function, the stable zeros are irrelevant, i.e., its inner part is given by $\hat{m} = 1$. For example,

$$\Gamma_W\left(\frac{e^{-hs}}{as+1}\right) = \Gamma_W\left(e^{-hs}\right)$$

for any $a, h > 0$ [8]. Here we extend the result to general pseudorational functions without unstable zeros. We show that the inner function associated with such plants is given as a simple delay.

First the following theorem deals with the transfer function $1/\hat{q}$, containing the above example $\hat{q}(s) = (as+1)e^{hs}$.

**Theorem 4.1** *Let $1/\hat{q} \in H^\infty$ be pseudorational. Then*

$$\Gamma_W\left(\frac{1}{\hat{q}}\right) = \Gamma_W\left(e^{\ell(q)s}\right).$$

**Proof** Suppose that there exist $\psi \in H^\infty$ and $\gamma > 0$ such that

$$\left\| W(s) - \frac{1}{q(s)}\psi(s) \right\|_\infty < \gamma.$$

From Lemma 3.1, $\check{\psi} := e^{-\ell(q)s}\psi/q$ belongs to $H^\infty$ and satisfies

$$\left\| W(s) - e^{\ell(q)s}\check{\psi}(s) \right\|_\infty < \gamma. \tag{6}$$

This implies $\Gamma_W(1/\hat{q}) \geq \Gamma_W\left(e^{\ell(q)s}\right)$.

Conversely, we prove $\Gamma_W(1/\hat{q}) \leq \Gamma_W\left(e^{\ell(q)s}\right)$. First we show that for every $\varepsilon > 0$ there exist positive constants $k$ and $\tau$ such that

$$e^{\ell(q)s}\frac{\hat{q}}{(1+\tau s)^k} \in H^\infty \tag{7}$$

and

$$\left\| W(s)\left(1 - \frac{1}{(1+\tau s)^k}\right) \right\|_\infty < \varepsilon. \tag{8}$$

Since $q$ is in $\mathcal{E}'(\mathbb{R}_-)$, $\hat{q}(s)$ is an entire function and there exist $C_1 > 0$ and $k_1 > 0$ that satisfies

$$|\hat{q}(s)| \leq e^{-\ell(q)s}C_1(1+|s|)^{k_1}, \ \operatorname{Re} s \geq 0$$

from the Paley-Wiener theorem [12]. Therefore it follows that

$$\left|\frac{e^{\ell(q)s}\hat{q}(s)}{(1+\tau s)^k}\right| \leq \frac{C_1(1+|s|)^{k_1}}{|1+\tau s|^k}, \ \operatorname{Re} s \geq 0.$$

This implies that $k > k_1$ leads to (7). Furthermore for an arbitrary given interger $k > 0$ we can find $\tau > 0$ that satisfy (8), since $W(s)$ is stable and strictly proper [4].

Then suppose that there exist $\check{\psi} \in H^\infty$ and $\gamma > 0$ satisfying (6). Here define $\varepsilon := \gamma - \left\|W(s) - e^{\ell(q)s}\check{\psi}(s)\right\|_\infty > 0$ and

$$\psi := \frac{e^{\ell(q)s}\hat{q}}{(1+\tau s)^k}\check{\psi} \in H^\infty.$$

by (7). Then we have

$$\begin{aligned}
& \left\|W(s) - \frac{1}{q(s)}\psi(s)\right\|_\infty \\
= & \left\|W(s) - e^{\ell(q)s}\frac{1}{(1+\tau s)^k}\check{\psi}(s)\right\|_\infty \\
\leq & \left\|\left(W(s) - e^{\ell(q)s}\check{\psi}(s)\right)\frac{1}{(1+\tau s)^k}\right\|_\infty + \left\|W(s)\left(1 - \frac{1}{(1+\tau s)^k}\right)\right\|_\infty < \gamma.
\end{aligned}$$

This completes the proof. ∎

This implies that for the plant $1/\hat{q} \in H^\infty$, the optimal sensitivity problem is equivalent to that of the delay with $-\ell(q) > 0$. This theorem can be extended to general pseudorational transfer function $\hat{p}/\hat{q}$ without unstable zeros. The following corollary shows that the optimal sensitivity $\Gamma_W(\hat{p}/\hat{q})$ depends not on the stable zeros of $\hat{p}$ as in the case of rational functions, but on the support of $p$.

**Corollary 4.2** *Let $\hat{p}/\hat{q},\ 1/\hat{p} \in H^\infty$ be pseudorational. Then*

$$\Gamma_W\left(\frac{\hat{p}}{\hat{q}}\right) = \Gamma_W\left(e^{(\ell(q)-\ell(p))s}\right).$$

**Proof** For brevity we show only $\Gamma_W(\hat{p}/\hat{q}) \geq \Gamma_W\left(e^{(\ell(q)-r(p))s}\right)$. The converse $\Gamma_W(\hat{p}/\hat{q}) \leq \Gamma_W\left(e^{(\ell(q)-r(p))s}\right)$ follows similarly. There exist positive constants $\tau$ and $k$ satisfying (7) and (8), as shown in the proof of Theorem 4.1. Now suppose that there exist $\check{\psi} \in H^\infty$ and $\gamma > 0$ satisfying $\left\|W - \hat{p}\check{\psi}/\hat{q}\right\|_\infty < \gamma$. Notice that (7) is sufficient for

$$\psi := e^{(\ell(q)-\ell(p))s}\frac{\hat{q}}{\hat{p}(1+\tau s)^k}\check{\psi} \in H^\infty,$$

since $1/e^{\ell(p)}\hat{p}$ belongs to $H^\infty$ by Lemma 3.1. We can check that $\psi$ satisfies $\left\|W - e^{(\ell(q)-\ell(p))s}\psi\right\|_\infty < \gamma$, by defining $\varepsilon := \gamma - \left\|W(s) - \hat{p}\check{\psi}(s)/\hat{q}\right\|_\infty$. ∎

This result shows that if a given stable plant $\hat{p}/\hat{q}$ has no unstable zeros, the associated inner function is a simple delay. Therefore the optimal value for this case can be computed with arbitrary accuracy [5].

### 4.2 Pseudorational function with unstable zeros

Let us now consider the case with (possibly infinitely many) unstable zeros. Several problems arise: we need to check whether the corresponding Blaschke product converges. Even if this is guaranteed, computing the optimal sensitivity according to the Zhou-Khargonekar formula (5) is not necessarily easy, when we only know such an inner function as an infinite product. The objective of the present subsection is that by Theorem 3.2 we can obtain a computable form equivalent to the Zhou-Khargonekar formula.

**Corollary 4.3** *Let $\hat{p}/\hat{q},\ 1/\hat{p}^{\check{}} \in H^\infty$ be pseudorational. Then*

$$\Gamma_W\left(\frac{\hat{p}}{\hat{q}}\right) = \Gamma_W\left(e^{(\ell(q)+r(p))s}\frac{\hat{p}}{\hat{p}^{\check{}}}\right).$$

**Proof** The proof of the equality is analogous to Corolarly 4.2. We only show that $e^{(\ell(q)+r(p))s}\hat{p}/\hat{p}^{\check{}}$ is a inner function. We have

$$e^{(\ell(q)+r(p))s}\frac{\hat{p}}{\hat{p}^{\check{}}} = \left(e^{-\ell(\acute{p})s}\frac{(\hat{\acute{p}})^{\check{}}}{\hat{\acute{p}}}\right)e^{(\ell(q)-\ell(p))s}$$

where $\acute{p} := \delta_{\ell(p)} * p^{\check{}} \in \mathcal{E}'(\mathbb{R}_-)$ and $\ell(\acute{p}) = \ell(p) - r(p)$. Since $1/\hat{\acute{p}}$ is in $H^\infty$, $e^{-\ell(\acute{p})s}(\hat{\acute{p}})^{\check{}}/\hat{\acute{p}}$ is inner from the proof of Theorem 3.2. Then the desired result follows. ∎

Let us check the consistency between this corollary and Theorem 3.2. Take a pseudorational transfer function $\hat{p}/\hat{q} \in H^\infty$ such that $1/\hat{p}^{\check{}} \in H^\infty$. We assume that $\ell(q) = \ell(p)$ and $r(p) = 0$ for simplicity. Since the zeros of $\hat{p}/\hat{q}$ are unstable, we can obtain the inner function $\hat{m}_1$ satisfying $H(\hat{m}_1) = X^{(p^{\check{}})}$ from Theorem 3.2 as follows:

$$\hat{m}_1 = e^{-\ell(p^{\check{}})s}\frac{(\hat{p}^{\check{}})^{\check{}}}{\hat{p}^{\check{}}} = e^{\ell(p)s}\frac{\hat{p}}{\hat{p}^{\check{}}}$$

This is exactly the same as the inner function obtained in Corollary 4.3.

As in Theorem 3.2, the associated inner function is of form $e^{as}m_b(s)$ where $m_b(s)$ is the Blaschke product consisting of the zeros of $\hat{p}$. However, the following theorem guarantees that we do not need to compute $\hat{m}^{\check{}}(H_\gamma)$ according to the Blaschke product form:

**Theorem 4.4** *Let $\hat{p}_1\hat{p}_2/\hat{q}$, $1/\hat{p}_1$ and $1/\hat{p}_2{}^{\sim} \in H^\infty$ be pseudorational and $(A, B, C)$ be a minimal realization of $W$. Define the Hamiltonian $H_\gamma$ of $W$ by (2). Then*

$$\Gamma_W\left(\frac{\hat{p}_1\hat{p}_2}{\hat{q}}\right) = \max\{\gamma : \det\left(e^{LH_\gamma}\hat{p}_2{}^{\sim}(H_\gamma)\hat{p}_2(H_\gamma)^{-1}|_{22}\right) = 0\}$$

*where $L := -\ell(q) + \ell(p_1) - r(p_2)$.*

**Proof** According to Corollaries 4.2 and 4.3, we have

$$\Gamma_W\left(\frac{\hat{p}_1\hat{p}_2}{\hat{q}}\right) = \Gamma_W\left(\frac{\hat{p}_2}{e^{(-\ell(q)+\ell(p_1))s}}\right) = \Gamma_W\left(e^{-Ls}\frac{\hat{p}_2}{\hat{p}_2{}^{\sim}}\right).$$

Then the desired result follows from the Zhou-Khargonekar formula (5). ∎

## 4.3 Example

Let us compute the optimal sensitivity by usign Theorem 4.4. Consider the weighting function $W(s)$ and the plant $P(s)$ as follows,

$$\begin{aligned} W(s) &= \frac{1}{s+1}, \\ P(s) &= \frac{e^s - \alpha}{2e^{2s} - 1} \quad (\alpha > 0,\ \alpha \neq 1). \end{aligned}$$

Observe that $P(s)$ has infinitely many zeros. Define the Hamiltonian of $W$ as

$$H_\gamma := \begin{bmatrix} -1 & \gamma^{-1} \\ -\gamma^{-1} & 1 \end{bmatrix}.$$

If $0 < \alpha < 1$, $P(s)$ has no unstable zeros. Hence the optimal sensitivity is given by

$$\Gamma_W(WP) = \max\{\gamma : \det\left(e^{H_\gamma}|_{22}\right) = 0\}$$

independent of $\alpha$. On the contrary, in the case of that $\alpha > 1$, $P(s)$ has infinitely many unstable zeros, i.e., $s = \ln\alpha + 2n\pi j$ $(n \in \mathbb{Z})$. Then

$$e^{2s}\cdot\frac{e^{-s} - \alpha}{e^s - \alpha} = e^s \cdot \frac{1 - \alpha e^s}{e^s - \alpha}$$

and we have

$$\Gamma_W(WP) = \max\{\gamma : \det\left(e^{H_\gamma}(I - \alpha e^{H_\gamma})(e^{H_\gamma} - \alpha I)^{-1}|_{22}\right) = 0\}.$$

Figure 1 shows the absolute value of determinant of the corresponding matrices above. When the plant has no unstable zeros, the optimal sensitivity is about 0.44. On the other hand in the case of $\alpha = 2$ the optimal value is around 0.80.

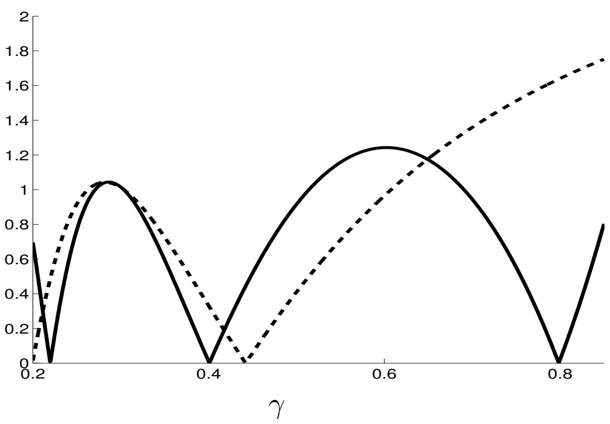


Figure 1: $0 < \alpha < 1$(dash) and $\alpha = 2$ (solid)

# 5 Conclusion

This paper gave some equivalent characterizations for invariant subspaces of $H^2$, when the underlying structure is specified by the so-called pseudorational transfer function. We have derived a simple closed formula for optimal sensitivity for such systems.

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