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Abstract. In this paper it is shown that given a finite dimensional linear system with polynomial output that there exists a linear system with the same outputs. This linearization is a subsystem of the classical Carleman linearization. It is further shown that there is a relationship between the observability of the linearization and the observability of the polynomial system.

1. Introduction

In this note we consider the problem of the output behavior of linear systems with polynomial output. The system that is considered is

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \tag{1}$$

$$p(x_1, \cdots, x_n) = \sum_{\sum k_i \le k} a_{k_1 k_2 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n} \tag{2}$$

The arguments are restricted to the case of a single scalar output. Our main theorem is

Theorem 1.1. *Let $\dot{x} = Ax$ and $y = p(x)$ where $x \in R^n$ and $p(x)$ is a polynomial of n variables and is the sum of homogeneous polynomials p_j of degree $m_j \le k$. Then there exists an observable linear system of degree M where $M \le N = \sum_{m_j} \begin{pmatrix} n+m_j-1 \\ m_j \end{pmatrix}$ with linear observation, $\dot{z} = \mathcal{H}z$, $y = cz$, and a polynomial map F from R^n to R^N such that*

(1) *$p(x) = cz$ whenever $z \in$ Image(F)*

and further more

(2) *If $\dot{z} = \mathcal{H}z$ $y = cz$ is observable and F is one to one then $\dot{x} = Ax$ $y = p(x)$ is observable.*

We also determine the dimension of the minimal linear system that has the same output as the polynomial system. From this theorem we conclude that polynomial output does not yield any behavior not realizable from a linear system with linear output.

Key words and phrases. Carleman linearization, linear systems, polynomial observability .

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2. Realization

We first write $p(x_1, \cdots, x_n)$ as a sum of homogeneous polynomials of degree at most k, i.e. that for $m_j \leq k$

$$p_{m_j}(x_1, \cdots, x_n) = \sum_{\sum k_i = m_j} a_{k_1 k_2 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n}. \tag{3}$$

and

$$p(x_1, \cdots, x_n) = \sum_{m_j} p_m(x_1, \cdots, x_n) \tag{4}$$

We, at this point, do not assume any special structure on the matrix A.

Let

$$S^n_{m_j} = \{x_1^{k_1} \cdots x_n^{k_n} : \sum_{i=1}^{n} k_i = m_j\} \tag{5}$$

and let

$$W^n_{m_j} = \text{span } S^n_{m_j}. \tag{6}$$

Define

$$W^n_k = \oplus \sum_{m_j} W^n_{m_j}. \tag{7}$$

Lemma 2.1. *W^n_k is closed under differentiation along the trajectories of $\dot{x} = Ax$.*

Proof: It suffices to show that the derivative of each element of S_{m_j} is in W_{m_j}. Let $r(x) \in S_{m_j}$, then we have

$$\frac{d}{dt} r(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} r(x) \frac{d}{dt} x_i(t) \tag{8}$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} r(x) e_i A x \tag{9}$$

where e_i is the i^{th} unit vector. Now $\frac{\partial}{\partial x_i} r(x)$ is homogeneous of degree $m_j - 1$ and $e_i A x$ is homogeneous of degree 1 and the lemma is proved.

Now assume that each $S^n_{m_j}$ is ordered using the lexicographic ordering and let the $\cup_{m_j} S^n_{m_j}$ be ordered with lexicographic ordering extended by letting the first element of the ordering be the m_js. Let z be the vector of the totally ordered $\cup S^n_{m_j}$. We have shown in the lemma that there exists a matrix $\mathcal{H}$ so that

$$\dot{z}(t) = \mathcal{H} z(t) \tag{10}$$

It is well known, [3], that the cardinality of $S^n_{m_j}$ is

$$\begin{pmatrix} m_j + n - 1 \\ m_j \end{pmatrix}$$

We see that the dimension of the space W^n_k is

$$N = \sum_{m_j} \begin{pmatrix} m_j + n - 1 \\ m_j \end{pmatrix}$$

and grows at a non-polynomial rate.

Now since the observation is the sum of homogenous polynomials we can write

$$p(x_1, x_2, \cdots, x_n) = cz \tag{11}$$

where the coefficients are the coefficients of the monomials in p. So we have constructed a linear system

$$\dot{z} = \mathcal{H}z \tag{12}$$
$$y = cz \tag{13}$$

and for initial conditions for z that correspond to the initial conditions for the original system

$$\dot{x} = Ax \tag{14}$$
$$y = p(x) \tag{15}$$

we have that

$$cz(t) = p(x(t)).$$

We state this as the following theorem.

Theorem 2.2. *Let the system $\dot{x} = Ax,\ y = p(x)$ be given. There exists a finite dimensional linear system, $\dot{z} = \mathcal{H},\ y = cz$ that linearizes the above system in sense that the output trajectories of $\dot{x} = Ax,\ y = p(x)$ are a subset of the output trajectories of $\dot{z} = \mathcal{H},\ y = cz$.*

In order to make the above statement about initial data precise we define a function F with domain R^n and range R^N by

$$F(a) = z(a)$$

where this notation is intended to indicate that each monomial is evaluated at the point a. Note that F is the sum of homogeneous polynomial maps.

Lemma 2.3. *The function F is one to one if and only if there exists an m_j such that $m_j = 2r + 1$.*

Proof: First assume that each of the m_js is even. Then we have $F(-x) = F(x)$ and so F is not one to one. Now suppose that there exists an m_j that is odd and assume that $F(a) = F(b)$. then since all monomials of degree m_j are present we have $a_i^{m_j} = b_i^{m_j}$ for each i. Since m_j is odd and a and b are real we have that $a_i = b_i$ and the lemma is proved.

Thus we have the two systems

$$\dot{x} = Ax,\quad x(0) = a \qquad \dot{z} = \mathcal{H}z \quad z(0) = F(a) \tag{16}$$
$$y = p(x) \qquad y = c(x) \tag{17}$$

with identical outputs.

Remark 2.4. *Recall how the initial data is recovered from a linear system, $\dot{x} = Ax$ with linear output $y = cx$. The standard approach is to differentiate y $n-1$ times at 0 and then solve the resulting system of linear equations. With polynomial output the above procedure is quite similar. We are in effect differentiating $p(x)$ N times and eliminating the monomials in order to obtain a differential equation in the output y. The Carleman linearization,* [2]*,provides a systematic way to do the elimination.*

3. Observability

In this section we consider the relation between the observability of the system with polynomial output with the observability of the linearization. Dayawansa and Martin examined this problem form another point of view in [1]. There they were concerned with the existence of a polynomial out put that observes the system. In this paper we are concerned with the observability of a given polynomial output. Starkov and Helmke in [5] considered the problem of discrete observability of such system and used similar decomposition results as we use in this paper.

The following proposition is straightforward.

Proposition 3.1. *If $\dot{z} = \mathcal{H}z,\quad y = cz$ is observable and F is one to one then $\dot{x} = Ax,\quad y = p(x)$ is observable.*

Proof: The fact that the large system is observable says that each monomial is recovered and if the initial value is in the image of F then F^{-1} essentially solves the monomials for the values of the x_i, i.e. $p(x)$ and its derivatives along trajectories can used to solve the nonlinear equations.

3.1. **A is diagonal.** Let

$$\dot{x} = Ax, \quad y = p(x)$$

with $x \in R^n$ be such that A has distinct eigenvalues, real or complex. Assume that the system is observable and that A has been diagonalized. Also assume that $p(x)$ is homogeneous of degree k Let

$$\dot{z} = \mathcal{H}z, \quad y = cz$$

$z \in R^N$ be the Carleman linearization of the small system.

Lemma 3.2. *The Carleman linearization is observable if and only if 1) every monomial occurs in $p(x)$ and 2) the mapping*

$$(i_1, \cdots, i_n) \mapsto \sum_{j=1}^{n} \lambda_j i_j$$

is one to one.

Proof: Let $m_j(x) = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ be any monomial of degree k. Assume that the monomials have been totaly ordered. We construct the derivative of $m(x)$ along trajectories of $\dot{x} = Ax$. Since A is diagonal we have that

$$Dm_j(x) = i_1\lambda_1 m_j(x) + \cdots + i_n\lambda_n m_j(x) = (\sum_{k=1}^{n} i_k\lambda_k) m_j(x) = a_j m_j(x).$$

Thus $\mathcal{H}$ is diagonal. So we can construct the observability Grammian for the the system.

$$F = \begin{pmatrix} c_1 & c_2 & \cdots & c_{N-1} & c_N \\ c_1a_1 & c_2a_2 & \cdots & c_{N-1}a_{N-1} & c_Na_N \\ & & \vdots & & \\ c_1a_1^N & c_2a_2^N & \cdots & c_{N-1}a_{N-1}^N & c_Na_N^N \end{pmatrix}$$

We see that F can be factored as

$$F = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_{N-1} & a_N \\ & & \vdots & & \\ a_1^N & a_2^N & \cdots & a_{N-1}^N & a_N^N \end{pmatrix} \begin{pmatrix} c_1 & 0 & \cdots & 0 & 0 \\ 0 & c_2 & \cdots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & 0 & c_N \end{pmatrix}$$

The matrix F is nonsingular if and only if both of the factors are nonsingular. The matrix on the left is a Vandermonde matrix so it is nonsingular if and only if the a_i are distinct and the matrix on the right is nonsingular if and only if the c_i are all nonzero. We state this as a proposition.

Proposition 3.3. *Let the matrix A be $n \times n$ diagonal with distinct eigenvalues λ_i and let $y = p(x)$ where p is homogenous of degree k. Then the Carleman linearization of the system with the state vector being the vector of all monomials of degree k has state matrix $\mathcal{H}$ diagonal with the elements $n_1\lambda_1 + \cdots + n_n\lambda_n$ where $n_1 + n_2 + \cdots + n_n = k$. The system $\dot{z} = \mathcal{H}$, $y = cx$ is observable if and only if each $c_i \neq 0$ and the mapping*

$$(n_1, n_2, \cdots, n_n) \rightarrow \lambda_1 n_1 + \cdots + \lambda_n n_n$$

is one to one over the set of all partitions of k into nonnegative parts.

The following corollary is of some interest.

Corollary 3.4. *The Carleman linearization of $\dot{x} = Ax$ $y = p(x)$ where p is a homogeneous polynomial of degree k, is observable for every k if and only if the eigenvalues of A are linearly independent over the rational numbers and the polynomial is the nonzero sum of every monomial of degree k .*

This condition has arisen in the paper of Dayawansa and Martin, [1], as a condition for the existence of a polynomial output that makes the system observable, in Strakov and Helmke as a condition for discrete observability and in Sternberg, [6] for the linearization of nonlinear system around an equilibrium.

3.2. **A has a Single Jordan Block.** We assume that

$$A = \begin{pmatrix} a & 1 & 0 & 0 & \cdots & 0 \\ 0 & a & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & & \\ 0 & \cdots & & & a & 1 \\ 0 & & & & 0 & a \end{pmatrix}$$

and that $x = (x_1, \cdots, x_n)^T$. We define the vector of monomials of degree k to be $z = (x_1^k, x_1^{k-1}x_2, \cdots, x_n^k)$ where the ordering of the monomials is the lexicographic ordering on the exponents. We now construct the Carleman linearization of the system $\dot{x} = Ax$. Let $m(x) = x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$ where $k_1 + \cdots + k_n = k$. Now

$$\begin{aligned} D(m(x)) &= k_1am(x) + k_1x_1^{k_1-1}x_2^{k_2+2}\cdots x_n^{k_n} \\ &\quad +k_2am(x) + k_2x_1^{k_1}x_2^{k_2-1}x_3^{k_3+1}\cdots x_n^{k_n} \\ &\quad + \cdots + k_nam(x) \\ &= ka\, m(x) + \text{ terms lower in the lexicographic ordering} \end{aligned}$$

We can now write the matrix $\mathcal{H} = kaI + N$ where the matrix N is strictly upper triangular. The question is now to determine the Jordan structure. We can assume that $a = 0$. The total Jordan structure is not easy to calculate and the question we want to answer is "When are there at least two Jordan blocks?". This is easy to answer. Let $y(t) = x_1^k$. solving the equation $\dot{x} = Ax$ we have that $x_1(t) = \sum_{i=1}^n b_it^{i-1}/(i-1)!$ and hence is a polynomial of degree at most $n-1$. Therefore $\frac{d^{nk-k+1}}{dt^{nk-k+1}}x_1^k(t) = 0$ thus we conclude that there is no Jordan block of dimension greater than $nk - k + 1$. Now the dimension of $\mathcal{H}$ is $\begin{pmatrix} n+k-1 \\ k \end{pmatrix}$. We must determine when

$$nk - k + 1 \leq \begin{pmatrix} n+k-1 \\ k \end{pmatrix}.$$

When $n = 1$ we have that $1 \leq 1$ for arbitrary k. When $n = 2$ we have $k+1 = k+1$. However when $n = 3$ we have $2k + 1 < \frac{(k+1)(k+2)}{2}$ unless $k = 0$ or $k = 1$. For larger n the right hand side increasing factorially and the left hand side linearly. A simple induction argument suffices to formally prove the statement. Thus only for $n = 1$ and $n = 2$ do we have a single Jordan block when $k > 1$. We state these facts as a proposition.

Proposition 3.5. *If A has a single Jordan block with eigenvalue a then for monomials of degree k, $\mathcal{H} = akI + N$ where N is strictly upper triangular. $\mathcal{H}$ has a single Jordan block if and only if $n = 1$ or $n = 2$ and k is arbitrary or $k = 0$ or $k = 1$ and n is arbitrary.*

Now we consider the observability of $\dot{z} = \mathcal{H}z$, $y = cz$. We first note that if $n > 2$ then the system cannot be observable for any c since there are at least two Jordan blocks with the same eigenvalue. So we suppose that $n = 2$. In this case

$$z = (x_1^k, x_1^{k-1}x_2, x_1^{k-2}x_2^2, \cdots, x_2^k)^T$$

and hence

$$\mathcal{H} = \begin{pmatrix} ak & 1 & 0 & 0\cdots & 0 & 0 \\ 0 & ak & 2 & 0 & \cdots & 0 \\ & & & & & \\ \vdots & & & \ddots & \ddots & \\ 0 & & & 0 & ak & k \\ 0 & & & & 0 & ak \end{pmatrix}$$

Now this system is observable if and only if the the system with $a = 0$ is observable. Thus we can construct the observability matrix as

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 & \cdots & c_{k+1} \\ 0 & c_1 & 2c_2 & 3c_3 & \cdots & kc_k \\ 0 & 0 & 2c_1 & 2!c_2 & \cdots & (k-1)kc_{k-1} \\ \\ 0 & & & \cdots & & k!c_1 \end{pmatrix}.$$

Thus the system is observable if and only if $c_1 \neq 0$. We state this as a proposition.

Proposition 3.6. *If $n = 2$ and $k > 0$ and the matrix A is a single Jordan block then the system $\dot{z} = \mathcal{H}$, $y = cx$ is observable if and only if $c_1 \neq 0$.*

3.3. **The general case.** Let the matrix A be in Jordan canonical form and let the state vector x be partitioned accordingly, i.e.,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Let

$$x_i = (x_{i1}, \cdots, x_{in_i}).$$

We assume that the maximal degree of $p(x)$ is k. Let M be the space spanned by the set of all monomials of degree less than or equal to k in the variables of x. Define an action of the ring of polynomials in one variable, $R[x]$, on M by defining for every monomial $m(x)$

$$Dm(x) = \sum_{i=1}^{m} \frac{\partial}{\partial x_{ij}} \frac{d}{dt} x_{ij}.$$

This is of course just the Carleman linearization. Now extend this action to any polynomial in D. Thus with this action M becomes an $R[x]$ module.

Define

$$M_i = \text{span}\{m(x_i) : \ m(x_i) \text{ a monomial of degree less than or equal to } k \text{ in the variables } x_i\}$$

Lemma 3.7. *The space M_i is a submodule of M.*

Define

$$M_i^j = \text{span}\{m(x_i) \in M_i : \ \text{degree}(m(x_i)) = j\}.$$

Again we have that M_i^j is a submodule of M and in fact $M_j = \bigoplus M_i^j$ where $\bigoplus$ indicates direct sum. Note that $M \neq \bigoplus M_i$ but $\bigoplus M_i$ is a direct summand of M. We see immediately that if A has a Jordan block of dimension greater than 2 then the Carleman linearization cannot be observable. For the rest of this section we assume that the Jordan form of A consists of blocks of dimension 1 and 2 and we further assume that 0 occurs as an eigenvalue in at most one Jordan block.

Now let

$$m(x) = (x_1^{k_1} \cdots x_r^{k_r})(x_{11}^{k_{11}} x_{12}^{k_{12}}) \cdots (x_{s1}^{k_{s1}} x_{s2}^{k_{s2}})$$

where $Dx_i = \lambda_i x_i$, $Dx_{i1} = \lambda_{i1}x_{i1} + x_{i2}$ and $Dx_{i2} = \lambda_{i1}x_{i2}$. Now we have

$$\begin{aligned} Dm(x) &= (\sum_{i=1}^{r} k_i\lambda_i)m(x) + (\sum_{i=1}^{s} \lambda_{i1}k_{i1})m(x) \\ &\quad + \sum_{i=1}^{s} x_1^{k_1}\cdots x_r^{k_r}x_{11}^{k_{11}}x_{12}^{k_{12}}\cdots x_{i1}^{k_{i1}-1}x_{i2}^{k_{i2}+1}\cdots x_{s1}^{k_{s1}}x_{s2}^{k_{s2}} \\ &\quad +(\sum_{i=1}^{s}\lambda_{i1}k_{i2})m(x) \\ &= (\sum_{i=1}^{r} k_i\lambda_i + \sum_{i=1}^{s}\lambda_{i1}(k_{i1}+k_{i2}))m(x) \\ &\quad + \sum_{i=1}^{s} k_{i1}x_1^{k_1}\cdots x_r^{k_r}x_{11}^{k_{11}}x_{12}^{k_{12}}\cdots x_{i1}^{k_{i1}-1}x_{i2}^{k_{i2}+1}\cdots x_{s1}^{k_{s1}}x_{s2}^{k_{s2}} \end{aligned}$$

Let $S_{n,m} = \{x_1^{k_1}\cdots x_r^{k_r}x_{11}^{k_{11}}x_{12}^{k_{12}}\cdots x_{s1}^{k_{s1}}x_{s2}^{k_{s2}} :$ for every i $k_i = n_i,\ k_{i1}+k_{i2} = m_i\}$ Let

$$S = \text{span}_{R[x]}S_{nm}$$

i.e., S is R[x]-module generated by S_{nm}.

Lemma 3.8. *The module S is cyclic if and only if $s = 1$.*

Proof: From the above construction it is clear that S is an R[x]-module. If $s = 1$ then $m_0(x) = x_1^{n_1}\cdots x_r^{n_r}x_{11}^{m_1}$ is a generator for S. If $s > 1$ then consider a simple example. Let $n_i = 0$ and let $m_1 = m_2 = 1$. Then there are four monomials: $x_{11}x_{21}$, $x_{11}x_{21}$, $x_{12}x_{21}$ and $x_{12}x_{22}$. In this order they are in lexicographic order and the matrix that corresponds to differentiation is

$$H = \begin{pmatrix} \lambda_1+\lambda_2 & 1 & 1 & 0 \\ 0 & \lambda_1+\lambda_2 & 0 & 1 \\ 0 & 0 & \lambda_1+\lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_1+\lambda_2 \end{pmatrix}$$

Now $\dim(\ker(H-(\lambda_1+\lambda_2)I) = 2$, $\dim(\ker(H-(\lambda_1+\lambda_2)I)^2 = 3$, $\dim(\ker(H-(\lambda_1+\lambda_2)I)^3 = 4$ and hence the matrix has two Jordan blocks–one of dimension 1 and one of dimension three. This example can be extended to k Jordan blocks of arbitrary dimension.

This lemma is rather discouraging for it says that in general the module M is simply to large to observable by any linear output map. Let $S_{n,m} = \{x_1^{k_1}\cdots x_r^{k_r} : k_1+\cdots+k_r = n\}\cup\{x_{i1}^{k_{i1}}x_{i2}^{k_{i2}} : i = 1,\cdots,s,\ k_{i1}+k_{i2} = m_i\}$ Let

$$S = \text{span}_{R[x]}S_{nm}$$

i.e., S is R[x]-module generated by S_{nm}. Now the matrix

$$\mathcal{H} = \begin{pmatrix} H_0 & 0 & 0 & \cdots & 0 \\ 0 & H_1 & 0 & \cdots & 0 \\ & & & & \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & H_s \end{pmatrix}.$$

The matrix H_0 is diagonal with diagonal elements $k_1\lambda_1+\cdots+k_r\lambda_r$ taken over all choices of k_i. The matrix H_i is a single Jordan block with eigenvalue $(k_{i1}+k_{i2})\lambda_{r+i}$. Thus we have the following theorem.

Theorem 3.9. *If the eigenvalues of the matrix A are linearly independent over the rational numbers then the system $\dot{z} = \mathcal{H}z,\ y = cz$ is observable for all c except for a set of c algebraic co-dimension 1.*

Attacking the question of observability in this manner is rather delicate. Linearizing spaces are built up and then cut down. This is close to the manner of realization. A realization is constructed and then is reduced by factoring out the unobservable subspace. Much of the time it is difficult to find the unobservable state explicitly. There is another way to attack the problem.

Theorem 3.10. *Let M be the minimal R[x]-module that contains the tuple of monomials found in $p(x)$. This module linearizes the system $\dot{x} = Ax,\ y = p(x)$ and is observable if and only if $\dot{x} = Ax,\ y = p(x)$ is observable.*

The theorem is basically a tautology. The problem with applying this theorem is that it is difficult to construct M.

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