THE SYMPLECTIC FORM IN THREE DIMENSIONAL SPACES OF CONSTANT CURVATURE AND THE EQUATIONS OF FLUID MECHANICS

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The symplectic structure of curves in three dimensional spaces of constant curvature and the equations of fluid mechanics

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1 Introduction

This paper grew out of two somewhat independent mathematical interests. The first interest was triggered by two extraordinary observations of H. Hasimoto. The first observation of Hasimoto was that the complex function

$$\psi(t,s) = \kappa(t,s)e^{i\int_0^s \tau(t,x)\,dx}$$

where $\kappa(t,s)$ and $\tau(t,s)$ are the curvature and the torsion of a curve $\gamma(t,s))$ that evolves according to to the filament equation

$$\frac{\partial\gamma}{\partial t}(t,s) = \kappa(t,s)B(t,s) \tag{1}$$

is a solution of the non-linear Schroedinger equation

$$-i\frac{\partial\psi}{\partial t}(t,s) = \frac{\partial^2\psi}{\partial s^2}(t,s) + 1/2|\psi(t,s)|^2(\psi(t,s)+c) \tag{2}$$

for some complex constant c. The vector $B(t,s)$ that appears in the filament equation is the binormal vector associated with curve γ.

Hasimoto's second observation was that the curves that minimize the functional $1/2\int_0^L \kappa^2(s)\,ds$, i.e., the elastic curves, correspond to the soliton solutions of the filament equation([8], [9]).These results of Hasimoto suggest that both the filament equation, which is principally known in the literature on gas dynamics([2]), and the non-linear Schroedinger equation, famous in fluid mechanics and physics([18]), have geometric interpretations independent of their applied context, and call for further investigations into the basic phenomena.

Our second interest is inspired by a remarkable paper of J.Millson and B.Zombro ([15]) which provides a symplectic structure on the infinite dimensional Fréchet space of smooth, closed curves that pass through a common point and have a fixed length L in a Euclidean space $\mathbb{E}^3$. Although written with different goals in mind, the symplectic form of Millson and Zombro, nevertheless, reveals interesting connections to the previous equations, in the sense that the Heisenberg magnetic equation

$$\frac{\partial S}{\partial t}(t,s) = S(t,s) \times \frac{\partial S^2}{\partial t^2} \tag{3}$$

is induced by the Hamiltonian flow associated with the function $f(\gamma) = 1/2 \int_0^L ||\frac{d^2\gamma}{ds^2}||\, ds$. Since there was already an estabilished Hamiltonian formalism in the literature aimed at the theory of solitons in which the non-linear Schroedinger equation plays a central role (L. Faddeev and L. Takhtajan [5]), it occured to us to look for a geometric setting that would illuminate the connections between the Millson-Zombro symplectic structure and that of Faddeev-Takhtajan.

With these motives in mind we decided to investigate the symplectic structure of smooth arc-length parametrized curves γ of fixed length L that pass through a common initial point and have fixed initial orientation in each 3-dimensional non - Euclidean space of constant curvature , i.e. on the sphere S^3 and on the hyperboloid $\mathbb{H}^3$. On the basis of Hasimoto's results it became clear that the basic formalism should also inlude the orthonormal frames associated each curve under consideration. So in addition to the above assumptions we consider curves γ that admit smoothly periodic orthonormal frames along $\gamma(s)$ that are adapted to γ by requiring that the first leg of the frame coincides with the tangent vector $\frac{d\gamma}{ds}$ for all $s \in [0, L]$. We call such curves anchored curves with periodic frames. Smoothly closed anchored curves generate periodic Serret-Frenet frames, but not all curves with smoothly periodic frames are closed, for instance, helices or hyperbolic geodesics.

We show that the space of anchored curves with periodic frames form an infinite dimensional symplectic Fréchet manifold. We then consider the Hamiltonian flow that corresponds to the function $f(\gamma) = 1/2 \int_0^L \kappa^2\, ds$, where κ denotes the geodesic curvature of the loop γ. The flow defines a family of curves $\gamma(t,s)$, whose tangents are represented family by the curves $\Lambda(t,s)$ in the space of Hermitian matrices in the case of the hyperboloid, and in su_2, the Lie algebra of the unitary group SU_2, in the case of the sphere. The curves $\Lambda(t,s)$ are the solutions of the following Landau-Lifschitz type equations (in the terminology of [2])

$$\frac{\partial \Lambda}{\partial t}(t,s) = [\Lambda(t,s), \frac{\partial^2 \Lambda}{\partial s^2}(t,s)], \tag{4}$$

on the sphere, and

$$\frac{\partial \Lambda}{\partial t}(t,s) = \frac{1}{i}[\Lambda(t,s), \frac{\partial^2 \Lambda}{\partial s^2}(t,s)] \tag{5}$$

in the case of the hyperboloid. Each of the preceding equations can be represented in form (3) in terms of the coordinates of Λ relative to the basis defined by the appropriate Pauli matrices (the ones in su_2 differ from the Hermitian ones by an i factor).

To get to the non-linear Schroedinger equation, we need to consider orthonormal frames along the curves $\gamma(t,s)$. Rather than take the Serret-Frenet frames, as it is usually done in the literature on this subject, we consider instead any choice of smoothly periodic orthonormal frames $(v_1(t,s), v_2(t,s), v_3(t,s))$ adapted to γ by requiring that $\frac{d\gamma}{dt}(t,s) = v_1(t.s)$. In each of the above cases, the frames can be identified with a curve $\phi(t,s)$ in SU_2, which in turn defines curves $U(t,s)$ and $V(t,s)$ in su_2 through the relations

$$\frac{\partial\phi}{\partial s}(t,s) = \phi(t,s)U(t,s), \text{ and } \frac{\partial\phi}{\partial t}(t,s) = \phi(t,s)V(t,s)$$

It then follows that the curves $U(t,s)$ and $V(t,s)$ are smoothly periodic and are related by the zero curvature equation (in the terminology of([5])):

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial s} + [U,V] = 0 \tag{6}$$

In contrast to ([5]), where the zero curvature equation was used in an ad-hoc manner, we show that (6) is a consequence of the equality between the covariant derivatives $\frac{D}{dt}\frac{\partial\phi}{\partial s}$ and $\frac{D}{ds}\frac{\partial\phi}{\partial t}$ along the curve ϕ in SU_2 relative to the standard Riemannian structure on S^3.

We then show that the function $\psi(t,s) = u(t,s)e^{i\int_0^s u_1(t,x\,dx}$ satisfies the non-linear Schroedinger equation with the constant $c = 0$, where $u(t,s)$ and $u_1(t,s)$ are the entries of the matrix U, i.e.,

$$U = \begin{pmatrix} iu_1 & u \\ -\bar{u} & -iu_1 \end{pmatrix}$$

The above complex function coincides with the Hasimoto function described earlier when the frames $\phi(t,s)$ coincide with the Serret-Frenet frames.

We conclude the paper with a brief account of non-Euclidean elastic curves, and show that they generate travelling waves for the non-linear Schroedinger equation, i.e., that they correspond to its soliton solutions. We also relate the integrals of motion discovered by C.Shabat-V.Zakharov ([18]) to the integrals of motion associated with the elastic curves as reported by J. Langer and R.Perline in ([14])

2 Geometric background and notations

There are two Lie groups which figure prominently in the geometry of the sphere S^3 and the hyperboloid $\mathbb{H}^3$. The first, and perhaps more fundamental of these groups is $SL_2(C)$, the group of 2×2 complex matrices whose determinant is equal to 1.

The second group is SU_2, the subgroup of $SL_2(C)$ consisiting of matrices of the form $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$.

The group $SL_2(C)$ is the double cover of $SO_4(R)$, the orthonormal frame bundle of $\mathbb{H}^3$, and will also be considered as the principal SU_2 bundle with $\mathbb{H}^3$ as the base space. As such, it provides a natural setting for a discussion of framed curves in the hyperboloid $\mathbb{H}^3$.

The sphere S^3 is equal to SU_2 via the quaternionic representation. Apart from this fact, SU_2 will also be identified with the frames in either of our spaces and consequently will provide for easy transition from one space to another. With these remarks in mind then it might be expedient to begin the paper by discussing the basic geometry of thes two Lie groups. We shall use $\mathfrak{g}$ to denote its Lie algebra, i.e., the algebra of 2×2 complex matrices of trace zero. For any matrix A in $\mathfrak{g}$, A^* will denote its Hermitian transpose. The vector space of all Hermitian matrices will be denoted by $\mathfrak{h}$. The Lie subalgebra of all skew-Hermitian matrices will be denoted by $\mathfrak{k}$. Then any A in $\mathfrak{g}$ can be uniquely written as $A = A_h + A_s$ with $A_h \in \mathfrak{h}$ and $A_s \in \mathfrak{k}$. In fact,

$$A_h = \frac{1}{2}(A + A^*), \quad \text{and} \quad A_s = \frac{1}{2}(A - A^*)$$

It follows by an easy verification that the Lie bracket $[A_1, A_2]$ is in $\mathfrak{k}$ for any pair of Hermitian matrices, and that moreover, the following Lie algebraic relations hold

$$[\mathfrak{h}, \mathfrak{h}] = \kappa, \quad [\mathfrak{h}, \mathfrak{k}] = \mathfrak{h} \quad \text{and} \quad [\kappa, \mathfrak{k}] = \mathfrak{k} \tag{7}$$

The relation $[\mathfrak{h}, \mathfrak{k}] = \mathfrak{h}$ easily implies that $\varphi A \varphi^{-1} \in \mathfrak{h}$ for any $\varphi \in S\mathbf{U}_2$ and any A in $\mathfrak{h}$, which further implies that the same holds for absolutely continuous curves

$$\varphi(x) A(x) \varphi^{-1}(x) \tag{8}$$

where $\varphi(x)$ is a curve in SU_2 and $A(x)$ is a curve in $\mathfrak{h}$.

Definition 2.1 *For any matrices A, B in $\mathfrak{g}$, $\langle A, B\rangle$ will denote the trace of $2AB$. We shall refer to this quadratic form as the trace form.*

It follows that

$$\langle A, B\rangle = 4a_1b_1 + 2(a_2b_3 + a_3b_2)$$

for any $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}$, and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}$. In particular if A and B are Hermitian, then

$$\langle A, B\rangle = 4a_1b_1 + 2(a_2\overline{b}_2 + \overline{a}_2b_2) = 4(a_1b_1 + \operatorname{Re} a_2\overline{b}_2), \tag{9}$$

from which it follows that $\langle A, A\rangle = 4(a_1^2 + |a_2|^2)$, and when A and B are skew symmetric, then $\langle A, B\rangle = -4(a_1b_1 + \mathrm{Re}\, a_2\overline{b}_2)$. Consequently, the trace form is negative definite on $\mathfrak{k}$, and positive definite on $\mathfrak{h}$.

Throughout the paper $A_1,\ A_2,\ A_3$ will denote the standard basis for $\mathfrak{k}$:

$$A_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad A_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

The preceding matrices are often refered to as the Pauli matrices. We shall use $B_1,\ B_2,\ B_3$ to denote the Hermitian Pauli matrices,which are related to the preceding matrices as follows:

$$B_1 = \frac{1}{i}A_1, \quad B_2 = \frac{1}{i}A_2, \quad B_3 = \frac{1}{i}A_3\,.$$

The Hermitian Pauli matrices form an orthonormal basis for $\mathfrak{h}$ relative to the trace form, while the Pauli matrices form an orthonormal basis for $\mathfrak{k}$ relative to the negative of the trace form.

The following Lie bracket table will be useful in some calculations further on in the paper..

[,]	A_1	A_2	A_3	B_1	B_2	B_3
A_1	0	$-A_3$	A_2	0	$-B_3$	B_2
A_2	A_3	0	$-A_1$	B_3	0	$-B_1$
A_3	$-A_2$	A_1	0	$-B_2$	B_1	0
B_1	0	B_3	$-B_2$	0	A_3	$-A_2$
B_2	$-B_3$	0	B_1	$-A_3$	0	A_1
B_3	B_2	$-B_1$	0	A_2	$-A_1$	0

Table 1

Nota Bene Our convention of the Lie bracket is $[A, B] = BA - AB$, and not the usual commutator of matrices.

It may also be helpful for some of the subsequent calculations to note the following relations:

i. If $A = \sum_{i=1}^{3} a_iA_i$ and $B = \sum_{i=1}^{3} b_iA_i$, then $[A, B] = \sum_{i=1}^{3} c_iA_i$, where

$$c_1 = b_2a_3 - b_3a_2, \quad c_2 = a_1b_3 - a_3b_1, \quad c_3 = a_2b_1 - b_2a_1$$

Thus,

$$c = b \times a \quad \text{where} \quad c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

where a, b, c, where $\times$ denotes the vector product in $\mathbf{R}^3$.

ii. If $A = \sum_{i=1}^{3} a_i A_i$ and $B = \sum_{i=1}^{3} b_i B_i$, then $[A, B] = \sum_{i=1}^{3} c_i B_i$, and again $c = b \times a$.

iii. If $A = \sum_{i=1}^{3} a_i B_i$ and $B = \sum_{i=1}^{3} b_i B_i$ then $[A, B] = \sum_{i=1}^{3} c_i A_i$, and $c = a \times b$. Thus, for Hermitian matrices the order of coordinates is reversed in the cross product.

Finally, it should be noted that $\langle A, B\rangle = i(a_1 b_1 + Im\ (a_2 + ia_3)(b_3 + ib_2))$ whenever $A = a_1 A_1 + a_2 A_2 + a_3 A_3$, and $B = b_1 B_1 + b_2 B_2 + b_3 B_3$. The last relation implies that $\mathfrak{h} \cong \mathfrak{k}^*$, where $\mathfrak{k}^*$ denotes the dual of $\mathfrak{k}$. This fact is of central importance for the results that follow.

2.1 Riemannian metric

a. The hyperboloid. In what follows $\mathbf{H}^3$ will denote the connected component of the hyperboloid $x_0^2 - x_1^2 + x_2^2 + x_4^2 = 1$, $x_0 > 0$, which will further be identified with the space of positive definite Hermitian matrices

$$P = \begin{pmatrix} x_0 - x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 + x_1 \end{pmatrix}$$

of determinant 1. As is well known, $SL_2(\mathbf{C})$ acts on the space of positive definite matrices through the mapping $\pi(g) = gPg^*$, for P in $\mathbf{H}^3$, and g in $SL_2(C)$ where g^* denotes the Hermitian transpose of g. The isotropy subgroup K that fixes the identity is equal to SU_2, and hence the orbit of $SL_2(C)$ through the identity is equal to $SL_2(C)/SU_2$. Since the action is transitive, this orbit is equal to $\mathbf{H}^3$. This recognition turns $SL_2(C)$ into a principal SU_2 bundle with $\mathbf{H}^3$ as the base space.

Let now $\mathcal{H}$ denote the distribution consisting of all left invariant vector fields in $SL_2(C)$ which take values in $\mathfrak{h}$ at the identity. Thus X is in $\mathcal{H}$ if and only if $X(g) = gA$ for some Hermitian matrix A. Such a distribution itself is often called horizontal,a terminology which we will adopt in this paper. We shall presently show that $\mathcal{H}$ is a connection on $SL_2(C)$, in the language of the principal bundles ([18]).

Let R_a denote the right translation on $SL_2(C)$ by the elements a in SU_2, and let R_{a*} and π_* denote the tangent maps associated with with R_a and the projection map π. Then,

1). $\pi_\star(gA) = 2gAg^*$ for any Hermitian matrix A. Thus, $\pi_*\mathcal{H}(g) = T_{\pi(g)}\mathbf{H}^3$, since vectors gAg^* fill the tangent space of $\mathbf{H}^3$ at each point gg^*. Secondly,

2). $R_{a*}\mathcal{H}(g) = \mathcal{H}(ga)$, because a^*Aa is Hermitian for any a in SU_2 and any Hermitian matrix A.

Even though it follows from the general theory of connections on a principal G bundle that every vector field on the base manifold can be uniquely lifted to the horizontal distribution $\mathcal{H}$, it might be instructive to proceed constructively.

For these purposes it will be also useful to identify each positive definite matrix $P = \begin{pmatrix} x_0 - x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 + x_1 \end{pmatrix}$ of determinant 1 with a Borel matrix $b = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$

where $\alpha > 0$ and $\beta \in \mathbf{C}$. This identification is done via the relation $bb^* = P$ which yields:

$$\begin{pmatrix} \alpha^2 + |\beta|^2 & \beta\alpha^{-1} \\ \alpha^{-1}\overline{\beta} & (\alpha^{-1})^2 \end{pmatrix} = \begin{pmatrix} x_0 - x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 + x_1 \end{pmatrix}$$

Hence, $(\alpha^{-1})^2 = x_0 + x_1$, and since $x_0 + x_1 > 0$, it follows that $\alpha = \frac{1}{\sqrt{x_0+x_1}}$. Then, $\beta = \frac{x_2+ix_3}{\sqrt{x_0+x_1}}$, and consequently,

$$b = \begin{pmatrix} \frac{1}{\sqrt{x_0+x_1}} & \frac{x_2+ix_3}{\sqrt{x_0+x_1}} \\ 0 & \sqrt{x_0+x_1} \end{pmatrix}$$

Suppose now that $\gamma(s)$ is a curve in $\mathbf{H}^3$ which, as we have just seen, can be represented by a Borel matrix

$$b(s) = \begin{pmatrix} \frac{1}{\sqrt{x_0(s)+x_1(s)}} & \frac{x_2(s)+ix_3(s)}{\sqrt{x_0(s)+x_1(s)}} \\ 0 & \sqrt{x_0(s)+x_1(s)} \end{pmatrix}$$

in the sense that $\gamma(s) = b(s)b^*(s)$. Then, $\frac{d\gamma}{ds}(s) = b(s)((B(s) + B^*(s)b^*(s))$, where $B(s)$ is a matrix in $\mathfrak{g}$ defined by $\frac{db}{ds}(s) = b(s)B(s)$.

Let $B(s) = B_h(s) + B_a(s)$ where $B_h(s) = \frac{1}{2}(B(s) + B^*(s))$, and where $B_a(s) = \frac{1}{2}(B(s) - B^*(s))$, i.e., B_h and B_a are the Hermitian and the skew-Hermitian part of B respectively.

Then $X(ba) = ba(a^*B_ha)$ is the lift of the tangent vector $\frac{d\gamma}{dt}$ to the horizontal distribution $\mathcal{H}$ at each point ba in the fiber over γ. Since $\langle a^*Aa, a^*Ba\rangle = \langle A, B\rangle$, the trace form evaluated on the lifted vector is equal to $\langle B_h, B_h\rangle$. We shall now show that

$$\langle B_h, B_h\rangle = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 - \dot{x}_0^2 \tag{10}$$

If $b = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$, then

$$B(s) = b^{-1}\frac{db}{ds} = \begin{pmatrix} \frac{1}{\alpha} & -\beta \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} \frac{d\alpha}{ds} & \frac{d\beta}{ds} \\ 0 & -\frac{1}{\alpha^2}\frac{d\alpha}{ds} \end{pmatrix} = \begin{pmatrix} \frac{\dot{\alpha}}{\alpha} & \frac{\dot{\beta}}{\alpha} + \frac{\beta\dot{\alpha}}{\alpha^2} \\ 0 & -\frac{\dot{\alpha}}{\alpha} \end{pmatrix}$$

When $\alpha(s) = \frac{1}{\sqrt{x_0(s)+x_1(s)}}$, and $\beta(s) = \alpha(s)u(s)$ with $u(s) = x_2(s) + ix_3(s)$, then $\dot{\alpha}(s) = -\frac{1}{2}\frac{\dot{x}_0+\dot{x}_1}{x_0+x_1)^{3/2}}$, and $\frac{\dot{\beta}}{\alpha} + \frac{\beta}{\alpha^2}\dot{\alpha} = \dot{u} + 2u\frac{\dot{\alpha}}{\alpha}$. Thus,

$$B_h = \frac{1}{2}(B + B^*) = \begin{pmatrix} \frac{\dot{\alpha}}{\alpha} & \frac{\dot{u}}{2} + u\frac{\dot{\alpha}}{\alpha} \\ \frac{\dot{\overline{u}}}{2} + \overline{u}\frac{\dot{\alpha}}{\alpha} & -\frac{\dot{\alpha}}{\alpha}, \end{pmatrix}$$

It follows that

$$\langle B_h, B_h\rangle = 4((\frac{\dot\alpha}{\alpha})^2 + (u\frac{\dot\alpha}{\alpha} + \frac{\dot u}{2})^2) = 4((\frac{\dot\alpha}{\alpha})^2(1+|u|^2) + \frac{1}{4}|\dot u|^2 + \frac{1}{2}\frac{\dot\alpha}{\alpha}(u\dot{\overline{u}} + \overline{u}\dot u))$$

Because $1 = x_0^2 - x_1^2 - x_2^2 - x_3^3$,

$$1 + |u|^2 = x_0^2 - x_1^2 \text{ and } x_2\dot x_2 + x_3\dot x_3 = x_0\dot x_0 - x_1\dot x_1.$$

Moreover,

$$\frac{\dot\alpha}{\alpha} = -\frac{1}{2}\frac{\dot x_0 + \dot x_1}{\sqrt{x_0+x_1}^3}\sqrt{x_0+x_1} = -\frac{1}{2}\frac{\dot x_0 + \dot x_1}{x_0+x_1}$$

. Therefore,
$4\left(\frac{\dot\alpha}{\alpha}\right)^2(1+|u|^2) = (\dot x_0+\dot x_1)^2\frac{(x_0+x_1)^2}{(x_0^2-x_1^2)} = (\dot x_0+\dot x_1)^2\frac{x_0-x_1}{x_0+x_1}$,
and

$$\begin{aligned} 2\frac{\dot\alpha}{\alpha}(\overline{u}\dot u + u\dot{\overline{u}}) &= = -\frac{(\dot x_0+\dot x_1)}{x_0+x_1}((x_2 - ix_3)(\dot x_2 + i\dot x_3) + (x_2+ix_3)(\dot x_2 - i\dot x_3)) \\ &= -2\frac{(\dot x_0+\dot x_1)}{x_0+x_1}(x_2\dot x_2 + x_3\dot x_3) \end{aligned}$$

After the substitutions,

$$\langle B_h, B_h\rangle = (\dot x_0+\dot x_1)^2\frac{(x_0-x_1)}{x_0+x_1} - 2\frac{(\dot x_0+\dot x_1)(x_0\dot x_0 - x_1\dot x_1)}{x_0+x_1} + \dot x_2^2 + \dot x_3^2$$

But,

$$(\dot x_0+\dot x_1)^2(x_0-x_1) - 2(\dot x_0+\dot x_1)(x_0\dot x_0 - x_1\dot x_1) = (x_0+x_1)(\dot x_1^2 - \dot x_0^2)$$

and therefore

$$\langle B_h, B_h\rangle = \dot x_1^2 + \dot x_2^2 + \dot x_3^2 - \dot x_0^2$$

■

The preceding calculations show that the hyperbolic length of any curve $\gamma(s)$ is defined by the trace form on the space of Hermitian matrices through the formula

$$\int_0^L \langle\Lambda,\Lambda\rangle^{1/2}\, ds = \int_0^L \sqrt{\dot x_1^2 + \dot x_1^2 + \dot x_3^2 - \dot x_0^2}\, ds \qquad (11)$$

where Λ is a Hermitian matrix associated with the horizontal curve $g(s)$ that projects onto the curve γ. The above formula also shows that the Lorentzian metric is invariant under SU_2.

More importantly this formalism shows that the Lorentzian metric could have been defined entirely in terms of the horizontal distribution $\mathcal{H}$ on $SL_2(C)$ as follows:

Recall that horizontal curves are curves in $SL_2(C)$ whose tangents lie in the horizontal distribution$\mathcal{H}$. Since $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{k}$, it follows from Chow's theorem that for any two points g_0 and g_1 in G there exists a horizontal curve $g(s),\ 0 \le s \le 1$ such that $g(0) = g_0$ and $g(1) = g_1$.

The trace form is positive definite on $\mathfrak{h}$ and defines a natural length on the space of horizontal curves. The length of any such curve $g(s)\ 1 \le s \le 1$ is given by $\int_0^1 \langle \Lambda(s), \Lambda(s) \rangle \, ds$ where $\Lambda(s) = g^{-1}(s)\frac{dg}{ds}(s)$.

Horizontal curves which minimize the length are called the sub-Riemannian geodesics. It is known that for any pair of points g_0 and g_1 there is a sub-Riemannian geodesic that connects them.

The length of a horizontal curve is equal to the length of the projected curve $\gamma \in \mathbf{H}^3$.

It is known that the sub-Riemannian geodesics which originate at the identity are of the form

$$g(s) = e^{(A+B)s}e^{-As}$$

where $A \in \mathfrak{k}$, and $B \in \mathfrak{h}$.

The hyperbolic geodesics correspond to the Sub-Riemannian geodssics for which $B = 0$, i.e., they are the exponentials of the Hermitian matrices. This can be seen as a consequence of the symmetry: they are invariant under the action of SU_2, and hence the extremal curves must be transversal to the elements of $\mathfrak{k}$.

It follows that curves $\gamma(s) \in \mathbf{H}^3$ are parametrized by hyperbolic length if and only if the corresponding horizontal curves are parametrized by their sub-Riemannian length, i.e., whenever $\langle \Lambda(s), \Lambda(s) \rangle = 1$ for all$s \in [0, L]$.

b. The Sphere. For the purposes of this paper it will be most convenient to think of the sphere S^3 as the group SU_2 via the correspondence:

Points $x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$ of S^3 are identified with the matrix $X = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}$ whose determinant is equal to 0. We may also think of points of S^3 as the unit quaternions $q = x_0 + ix_1 + jx_2 + kx_3$.

Then , as is well known, the quaternionic multiplication is isomorphic to the group multiplication in SU_2.

We shall represent curves $\gamma(s) = \begin{pmatrix} x_0(s) \\ x_1(s) \\ x_2(s) \\ x_3(s) \end{pmatrix}$ by curves $\Gamma(s)$ in SU_2. The tangent vectors of curves in SU_2 will be represented by matrices $\Lambda \in \mathfrak{k}$, i.e.,$\frac{d\Gamma}{ds}$ will be represented by $\frac{d\Gamma}{ds}(s) = \Gamma(s)\Lambda(s)$. The entries of matrix Λ are given by the following expression:

$$\Lambda = \begin{pmatrix} \bar{z}\dot{z} + w\dot{\bar{w}} & \bar{z}\dot{w} - w\dot{\bar{z}} \\ \bar{w}\dot{z} - z\dot{\bar{w}} & \bar{w}\dot{w} + z\dot{\bar{z}} \end{pmatrix}$$

where $z(s)$ and $w(s)$ are the complex numbers defined by $z(s) = x_0(s) + ix_1(s)$ and $w(s) = x_2(s) + ix_3(s)$. The calculation follow easily from $\Lambda = \Gamma^* \frac{d\Gamma}{ds}$. It also follows from an easy calculation, which we shall omit, that

$$\langle\Lambda, \Lambda\rangle = \dot{x}_0^2 + \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2$$

where $\langle\Lambda, \Lambda\rangle$ now denotes the negative of the trace form introduced earlier. Therefore,

$$\text{length of } \gamma = \int_0^L \langle\Lambda(s), \Lambda(s)\rangle \, ds \tag{12}$$

This metric coincides with the standard Riemannian metric on the sphere. It easily follows from above that the metric is invariant under the left action by SU_2, which then implies by Noether's theorem that the geodesics are the exponentials of elements in $\mathfrak{k}$. Curves $\gamma(s)$ are parametrized by the spherical length if and only if $\langle\Lambda(s), \Lambda(s)\rangle = 1$ for all $s \in [0, L]$. The recognition of the Riemannian metric through the trace form reveals fundamental similarities between the spherical and the hyperbolic geometries: the circles in the spherical case are replaced by the hyperbolas in the hyperbolic case, and each space is homogeneous with respect to the appropriate group action.

2.2 The Covariant Derivative

Since both $\mathbf{H}^3$ and SU_2 are Riemannian manifolds it makes sense to talk about the covariant derivative of a vector $W(s)$ along a curve $\gamma(s)$. We shall use $\frac{D_\gamma W}{ds}$ to denote such a derivative. Normally, in the literature on differential geometry, the covariant derivative is induced by the Levi-Civita connection, but in this context, we may proceed more directly and take advantage of the fact that both $\mathbf{H}^3$ and SU_2 are submanifolds of $\mathbf{R}^4$ with their Riemannian structure inherited, either from the Euclidean, or from the Lorentzian structure of $\mathbf{R}^4$, and hence, the covariant derivative is equal to the orthogonal projection of the ordinary derivative in $\mathbf{R}^4$ onto the tangent spaces of the underlying manifold along the curve γ. Then,

$$\frac{D_\gamma W}{ds}(s) = \frac{DW}{ds} + (W(s) \cdot \frac{d\gamma(s)}{ds})\gamma(s) \tag{13}$$

where $(W \cdot \frac{d\gamma}{ds)} = w_0\dot{\gamma}_0 \mp (w_1\dot{\gamma}_1 + w_2\dot{\gamma}_2 + w_3\dot{\gamma}$ depending whether γ is in $\mathbf{H}^3$ or in S^3, i.e., where $(W_1 \cdot W_2)$ denotes either the Lorentzian inner product in R^4 in the case of the hyperboloid, or the Euclidean inner product in R^4 in the case of the sphere.

For the purposes of this paper it will be convenient to express the covariant derivative in terms of the representations described earlierr, namely, in the space of positive definite matrices for the hyperboloid and in SU_2 for the sphere. Let us begin with the hyperboloid, and let us assume that $\gamma(s) = g(s)g^*(s)$ for some curve $g(s)$ in $SL_2(C)$ such that $\frac{dg}{ds}(s) = g(s)\Lambda(s)$ with $\Lambda(s)$ a curve of Hermitian matrices. Then, $\frac{d\gamma}{ds}(s) = 2g(s)\Lambda(s)g^*(s]$. Suppose that $W(s)$ is a tangent vector along γ which we shall express as $W(s) = 2g(s)V(s)g^*(s)$, for some curve of Hermitian matrices $V(s)$. According to the formula (13),

$$\frac{D_\gamma}{ds}(W)(s) = 2g(s)(\Lambda(s)V(s) + V(s)\Lambda(s))g^* + g(s)\frac{dV}{ds}(s) + (W(s) \cdot \frac{d\gamma}{ds}(s))\gamma(s)$$

It is an easy calculation that shows that

$$\frac{1}{2}(\Lambda V + V\Lambda) = v_1\lambda_1 + Re(\bar{v}\lambda))I$$

where

$$V = \begin{pmatrix} v_1 & v \\ \bar{v} & -v_1 \end{pmatrix}, \text{ and } \Lambda = \begin{pmatrix} \lambda_1 & \lambda \\ \bar{\lambda} & -\lambda_1 \end{pmatrix}$$

The Lorentzian inner product can also be expressed in terms of the Hermitian matrices as follows:

$$\frac{1}{2}(XY^\dagger + YX^\dagger) = (X \cdot Y)I$$

where points x in R^4 correspond to the Hermitian matrices $X = \begin{pmatrix} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{pmatrix}$. In this notation $X^\dagger$ stands for the matrix $\begin{pmatrix} x_0 - x_1 & -x_2 - ix_3 \\ -x_2 + ix_3 & x_0 + x_1 \end{pmatrix}$.

It follows easily from the above representation that the Lorentzian inner product is invariant under $SL_2(C)$, in the sense that $X \cdot Y = (gXg^*) \cdot (gYg^*)$. Therefore, the covariant derivative is given by the following expression

$$\frac{D_\gamma W}{ds}(s) = 2g(s)\frac{dV}{ds}(s)g^*(s) \tag{14}$$

The preceding results can be paraphrased as follows: if $g(s)$ is the horizontal curve that projects onto the curve γ, then each tangent vector $W(s)$ along γ can be represented by a Hermitian matrix $V(s)$ and $\frac{dV}{ds}(s)$ then corresponds to the covariant derivative of W along γ.

We shall now express the covariant derivative for the sphere in terms of the SU_2 representation.

Let the tangent vector of a curve $\Gamma(s)$ in $S\mathbf{U}_2$ be written as $\frac{d\Gamma}{ds}(s) = \Gamma(s)\Lambda(s)$ for some skew-Hermitian matrix $\Lambda(s)$. Then any vector field $W(s)$ along $\Gamma(s)$ can be represented by $\Gamma(s)V(s)$ for some skew-Hermitian matrix $V(s)$. Analogous to the Lorentzian inner product, the Euclidean inner product can be expressed as follows:

$$\frac{1}{2}(XY^* + YX^*) = (X \cdot Y)I$$

When X and Y are skew-Hemitian, $XY^* + YX^* = -(XY + YX)$, and moreover, $\frac{1}{2}((gX)(gY)^* + (gY)(gX)^*) = (X \cdot Y)I$ for any g in SU_2.

According to the expression (13),

$$\frac{D_\Gamma W}{ds}(s) = \Gamma(s)(\Lambda(s)V(s) + \frac{dV}{ds}(s) - \frac{1}{2}(\Lambda(s)V(s) + V(s)\Lambda(s))$$

from which it follows that

$$\frac{D_\Gamma W}{ds}(s) = \Gamma(s)(\frac{dV}{ds} + \frac{1}{2}[V(s), \Lambda(s)]) \tag{15}$$

2.3 The Orthonormal Frame Bundle

The group SU_2 acts on the space of tangent vectors at $P = gg^*$ by the formula $(\varphi, g\Lambda g^*) \to g(\varphi\Lambda\varphi^*)g^*$. for each $\varphi \in SU_2$. This action extends to the orthonormal frame bundle such that the frame $F = (F_1, F_2, F_3)$ at $P = gg^*$ goes into the frame φF at P consisting of vectors $(\varphi F_1\varphi^{-1}, \varphi F_2\varphi^{-1}, \varphi F_3\varphi^{-1})$ for every φ in SU_2.

It is easy to verify that the action of SU_2 on the orthonormal frames is transitive with the kernel $\{\pm I\}$. We shall identify positively orthonormal frame with the orbit of SU_2 through the frame of the Hermitian Pauli matrices $F_0 = (B_1, B_2, B_3)$. Recall that,

$$B_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad B_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and so each frame $F = (F_1, F_2, F_3)$ is identified with $\varphi \in SU_2$ by the formula

$$F_1 = \varphi B_1\varphi^{-1}, \quad F_2 = \varphi B_2\varphi^{-1}, \quad F_3 = \varphi B_3\varphi^{-1}$$

It is well known that the orthonormal frame bundle of $\mathbf{H}^3$ is equal to the isometry group $SO(1,3)$ and that the positively oriented component coincides with the connected component $SO_0(1,3)$ of $SO(1,3)$ that contains the group identity. Since SU_2 is a double cover of $SO_3(R)$, $SL_2(C)$ is a double cover of the orthonormal frame bundle of $\mathbf{H}^3$.

It follows from the preceding observations that curves in the orthonormal frame bundle of $\mathbf{H}^3$ can be represented by curves $g(s) \in SL_2(C)$. It may be instructive to relate this representation to the horizontal distribution $\mathcal{H}$ defined earler.

Suppose that $g(s)$ is any curve in $SL_2(C)$. Then,

$$\frac{dg}{ds}(s) = g(s)(B(s) + A(s)), \text{ where } A(s) \in \mathfrak{k} \text{ and } B(s) \in \mathfrak{h}$$

Let $\phi(s)$ denote any solution in SU_2 of the equation $\frac{d\phi}{ds}(s) = \phi(s)A(s)$. Then, $g_0(s) = g(s)\phi^{-1}(s)$ is the horizontal curve that projects onto the same base curve $\gamma(s)$ as $g(s)$. In fact, $\frac{dg_0}{ds} = g_0(s)\Lambda(s)$ where $\Lambda = \phi B \phi^*$.

Let $F(s) = (V_1, V_2, V_3)$ denote the frame defined by ϕ, i.e., let $V_j = \phi B_j \phi^{-1}, 1 \leq j \leq 3$. If the matrix B is written as $B(s) = b_1(s)B_1 + b_2(s)B_2 + b_3(s)B_3$, then

$$\frac{d\gamma}{ds} = 2g_0(s)\Lambda(s)g_0^*(s) = 2\sum_{j=1}^{j=3} b_j(g_0(s)V_j g_0^*(s))$$

is the representation of the tangent vector in terms of the basis defined by the vectors of the frame.

The frame is said to be adapted to the curve parametrized by arc length if the first leg of the frame coincides with the tangent vector of the curve. It follows from above that F is adapted to γ if and only if $B = B_1$ in which case, $\Lambda = \phi B_1 \phi^*$.

Conversely, if $g_0(s)$ is the horizontal curve that projects onto a curve γ and if F is any frame along γ then the frame induces a curve $\phi \in SU_2$ which together with γ defines a framed curve $g(s)$ in $SL_2(C)$ through the formula $g(s) = g_0\phi$.

Suppose now that $\phi(s)$ is any curve in SU_2 that defines a frame which is adapted to the curve γ. Let $U(s) = \frac{1}{2}\begin{pmatrix} iu_1 & u \\ \bar{u} & -iu_1 \end{pmatrix}$ denote the curve in su_2 such that $\frac{d\varphi}{ds}(s) = \varphi(s)U(s)$. Since $\Lambda = \phi B_1 \phi^*$,

$$\frac{d\Lambda}{ds} = \varphi[B_1, U]\varphi^* = \frac{1}{2}\varphi\begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix}\varphi^*$$

Therefore the geodesic curvature $k(s)$ is given by the following expression

$$k^2 = \left\| \frac{D_\gamma}{ds}\frac{d\gamma}{ds} \right\|^2 = \left\langle \frac{d\Lambda}{ds}, \frac{d\Lambda}{ds} \right\rangle = |u|^2 \tag{16}$$

We now recall the Serret- Frenet frames defined by

$$\frac{D_\gamma v_1}{ds} = \kappa v_2, \; \frac{D_\gamma v_2}{ds} = -\kappa v_1 + \tau v_3, \text{ and } \frac{D_\gamma v_3}{ds} = -\tau v_2$$

For such frames the following must hold

$$[B_1, U] = \kappa B_2, \; [B_2, U] = -\kappa B_1 + \tau B_3, \text{ and } [B_3, U] = -\tau B_2 \tag{17}$$

which then implies that $u_3 = \kappa$, $u_2 = 0$ and $u_1 = \tau$.

The situation is similar on the sphere. As already mentioned earlier, we identify curves γ on S^3 with curves Γ in SU_2.The orthonormal frame bundle of S^3 is equal to $SO_4(R)$, and $SU_2 \times SU_2$ is a double cover of $SO_4(R)$. For the purposes of this paper however, it will be advantageous to think of framed curves as curves in $SU_2 \times SU_2$ rather than curves in $SO_4(R)$, and for that reason it will be necessary to think of $SU_2 \times SU_2$ as the principal SU_2 bundle with SU_2 as the base space. The construction is parallel to the hyperbolic case and goes as follows.

Let $G = SU_2 \times SU_2$. Then G acts on points $X \in SU_2$ by pXq^* for each $(p, q) \in G$, and SU_2 is equal to the orbit through the identity. Let π denote the projection map from G ontoSU_2, i.e., $\pi(p, q) = pq^*$. The group $H = SU_2$ acts on G diagonally on the right, and this construction realizes G as the principal H bundle with the base space SU_2.

The tangent map π_* induces an action of SU_2 on the tangent vectors pAq^* at any point $\gamma = pq^*$ given by $\{aAa^*, a \in SU_2\}$. This action extends to the orthonormal frames in the usual way, which enables us to identify an orthonormal frame $F = (f_1, f_2, f_3)$ with an element q in SU_2 through the formula

$$f_1 = qA_1q^*, f_2 = qA_2q^*, f_3 = qA_3q^*$$

. There are exactly two such choices q and $-q$. Any curve $(p(s), q(s))$ in G will be considered as a framed curve over the base curve $\gamma(s) = p(s)q^*(s)$ with the frames given by the preceding formula. If $A(s)$ and $B(s)$ are the matrices defined by $\frac{dp}{ds}(s) = p(s)A(s)$ and $\frac{dq}{ds}(s) = q(s)B(s)$, then the tangent vector of γ is given by $\frac{d\gamma}{ds}(s) = p(s)(A(s) - B(s))q^*(s)$, and the frame $q(s)$ is adapted to γ whenever $A - B = A_1$, because $\frac{d\gamma}{ds} = pA_1q^* = pq^*(qA_1q^*) = f_1$.

Suppose now that $\frac{d\varphi}{ds} = \varphi U$, with $U = u_1A_1 + u_2A_2 + u_3A_3$ is any choice of frames adapted to the curve γ. Then the geodesic curvature of γ conforms to the following relation

$$k^2 = \|(\frac{D_\Gamma}{ds}\frac{d\Gamma}{ds})\| = u_2^2 + u_3^2$$

We also note that when φ corresponds to the Serret-Frenet frame, then

$$u_1 = \tau - \frac{1}{2}, \, u_2 = 0, \text{ and } u_3 = k \tag{18}$$

3 The symplectic structure of anchored curves with smoothly periodic frames

We now consider the family of curves $\gamma : [0, L] \to M^3$, where M^3 denotes either $\mathbf{H}^3$, or S^3, , called the family of anchored curves with periodic frames, that is defined by the following conditions:

(1). Each curve γ is parametrized by arc-length, i.e., $\|\frac{d\gamma}{ds}(s)\| = 1$ for all s in $[0, L]$.

(2). The length L of each curve is fixed, and each curve γ satisfies $\gamma(0) = x_0$, where x_0 is a fixed point in M^3. It will be convenient to take $x_0 = I$, and we shall do so.

To define curves with periodic frames we need to consider each case separately. On $\mathbf{H}^3$ any curve $\gamma(s)$ can be represented by a curve $g(s)$ in $SL_2(C)$ that satisfies

$$gg^*(s) = \gamma(s),\ g(0) = I, \text{ and } \frac{dg}{ds}(s) = g(s)\Lambda(s)$$

with $\Lambda(s) \in \mathfrak{h}$ for all $s \in [0, L]$.It follows from (11) that $\langle \Lambda(s), \Lambda(s) \rangle = 1$ whenever γ satisfies condition (1). If $g_1(s)$ and $g_2(s)$ are any curves in $SL_2(C)$ that project on the same curve γ that satisfies conditions (1) and (2) then $g_1(s) = g_2(s)a$ for some $a \in SU_2$ which implies that $\Lambda_1(s) = a\Lambda_2(s)a^{-1}$. If we assume that the initial orientation of γ is fixed then $\Lambda(0)$ is fixed which implies that the curve $g(s)$ is unique. We shall assume that in addition to conditions (1) and (2) the curve γ also satisfies

(3) $\Lambda(s)$ is smoothly periodic in the interval $[0, L]$ and $\Lambda(0) = B_1$.

By smoothly periodic here it is understood that Λ has a smooth extension to an open interval that contains the interval $[0, L]$ in which $\Lambda^{(k)}(0) = \Lambda^{(k)}(L)$ for each derivative $\Lambda^{(k)}$ of Λ.

We now have the following lemma.

Lemma 1 *Each curve γ that satisfies conditions (1),(2) and (3) admits a smoothly periodic orthonormal frame $v_1(s), v_2(s), v_3(s)$ adapted to γ by $\frac{d\gamma}{ds}(s) = v_1(s)$ for all $s \in [0, L]$.*

Proof. Let $g(s)$ denote any curve in $SL_2(C)$ that projects onto the curve γ. Suppose that

$$\frac{dg}{ds} = g(s)(B(s) + A(s)) \text{ where } B(s) \in \mathfrak{h} \text{ and } A(s) \in su_2$$

Let $\phi(s)$ denote the solution in SU_2 of the equation $\frac{d\phi}{ds}(s) = \phi(s)A(s)$ with $\phi(0) = I$, and let $g_0(s) = g(s)\phi^{-1}(s)$. Then, $\frac{dg_0}{ds}(s) = g_0(s)(\phi(s)B(s)\phi^*(s)) = g_0(s)\Lambda(s)$, hence $g_0(s)$ is the horizontal curve that projects onto γ. When γ satisfies the conditions of the lemma, then Λ is smoothly periodic and satisfies $\Lambda(0) = B_1$.

The curve ϕ defines an orthonormal frame $F = (v_1, v_2, v_3)$ along γ defined by $v_i(s) = \phi(s)B_i\phi^*(s), 1 \leq i \leq 3$. The frame F is adapted to γ whenever $\Lambda = v_1$ which holds if and only if $\Lambda(s) = \phi(s)B_1\phi^*(s)$, i.e., whenever $B(s) = B_1$.

Let $A(s) = u_1(s)A_1 + u_2(s)A_2 + u_3(s)A_3$, and suppose that γ is not a geodesic. Then the Serret-Frenet frame along γ is well defined and is given by equations (17). It follows that the geodesic curvature $k(s) = u_3(s) = |\Lambda(s)|$ is periodic, and therefore, $\phi B_2\phi^*$ is periodic since $\frac{d\Lambda}{ds}(s) = u_3(s)\phi(s)B_2\phi^*(s)$. Moreover,

$$\frac{1}{i}[\Lambda, \frac{d\Lambda}{ds}] = k(s)\phi(s)B_3\phi^*(s)$$

and therefore, $\phi B_3\phi^*$ is smoothly periodic. It follows that ϕ must be smoothly periodic since each $\phi)B_i\phi^*), 1 \leq i \leq 3$ is smoothly periodic. For geodesic curves $\frac{d\Lambda}{ds} = 0$, and therefore Λ is constant. Condition (3) implies that $\Lambda = B_1$, and then $\phi(s) = I$ meets the requirements of the lemma. ■

Remark. In general, the choice of periodic orthonormal frames is not unique. The orthonormal frames that are adapted to the curve revolve in the plane that is perpendicular to the tangent vector and they can be all described in terms of an angle Θ that defines the rotation relative to the Serret-Frenet frames. A frame that is adapted to a non-geodesic curve is smoothly periodic whenever Θ is a smooth periodic function of period L.

On S^3 curves γ are represented by matrices $\Gamma \in SU_2$. Here, condition (3) is analogous to condition (3) of the hyperbolic case, except that the matrix Λ is now defined by $\frac{d\Gamma}{ds}(s) = \Gamma(s)\Lambda(s)$. Since Lemma1 is valid here as well, we then have the following

Definition 3.1 *The space of curves on M^3 that satisfy conditions (1), (2), (3) is called the space of anchored curves with periodic frames and will be denoted by $\mathcal{L}(L)$. Anchored curves with periodic frames in $\mathbf{H}^3$ will be denoted by $\mathcal{L}_h(L)$, while $\mathcal{L}_s(L)$ will denote the anchored curves on the sphere S^3.*

The space of anchored curves with periodic frames includes smoothly closed curves for the following reason. The Serret-Frenet frame of a smoothly periodic curve is smoothly periodic, which then implies that the Hermitian matrix Λ associated with the horizontal curve $g(s)$ is smoothly periodic (we leave the details to the reader to prove), but not every curve in it is necessarily closed (such as, for instance, a helix or a hyperbolic geodesic)

.

3.1 Fréchet spaces

In what follows we shall consider $\mathcal{L}(L)$ as an infinite dimensional Fréchet manifold. We first recall the basic definitions.

A vector space V is called a Fréchet space if it is Hausdorff, is topologized by a countable family of semi-norms p_n, and is complete relative to the semi-norms in $\{p_n\}$ A Fréchet manifold is a topological Hausdorff space equipped with an atlas whose charts take values in open subsets of a Fréchet space V such that any change of coordinate charts is smooth.

Any locally convex topological space that is defined by a countable number of semi-norms becomes a quasi-normed space with the quasi-norm $\| \quad \|$ defined by

$$\|f\| = \sum_{n=0}^{\infty} \frac{p_n(f)}{(1 + p_n(f))}$$

A quasi-norm $\| \quad \|$ in a vector space X satisfies

i. $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$

ii. $\|x + y\| \leq \|x\| + \|y\|$

iii. $\| - x\| = \|x\|$ and

iv. $\lim \|\alpha_n x\| = 0$ for every $x \in X$ and every sequence of scalars $\{\alpha_n\}$ such that $\lim \alpha_n = 0$

v. $\lim_{n\to\infty} \|\alpha x_n\| = 0$ for any scalar α and any sequence $\{x_n\} \subset X$ such that $\lim x_n = 0$.

Let $C^\infty(I_L, \mathbf{E}^3)$ denote the vector space of C^∞ maps from the interval $I_L = [0, L]$ into $\mathbf{E}^3$ This space is equipped with a countable family of semi-norms p_n defined by

$$p_n(f) = \sup\{\|f^{(n)}(x)\|, x \in I_L\}$$

. It follows by standard arguments that $C^\infty(I_L, \mathbf{E}^3)$ is a Fréchet space. If $C^\infty(I_L, M^3)$ denotes the space of smooth maps from I_L into M^3, then $C^\infty(I_L, \mathbf{H}^3)$ is a Fréchet manifold, as it is locally diffeomorphic to $C^\infty(I_L, \mathbf{E}^3)$. Since M^3 is finite dimensional, both $C^\infty(I_L, \mathbf{E}^3)$ and $C^\infty(I_L, \mathbf{H}^3)$ are tame Fréchet spaces, and hence the implicit function theorem holds ([6]). This fact implies that $\mathcal{L}(L)$ is a Fréchet submanifold of $C^\infty(I_L, M^3)$. All of these details follow from the general discussion in ([6]) and will not be repeated here..

3.2 Vector fields and Differential Forms

Let $\mathcal{M}$ denote a Fréchet manifold with charts in a Fréchet space V.

A tangent vector at a point x in $\mathcal{M}$ is an equivalence class of curves $\sigma(t)$ in $\mathcal{M}$ such that $\sigma(0) = x$ and all have the same tangent vector $\frac{d\sigma}{dt}$ at $t = 0$. The tangent space at each point x, denoted by $T_x\mathcal{M}$. A vector field X on $\mathcal{M}$ is a smooth mapping from $\mathcal{M}$ into the tangent bundle $T\mathcal{M}$ (which is a Fréchet manifold also) such that $X(s) \in T_x\mathcal{M}$ for each $x \in \mathcal{M}$.

Let $\mathcal{X}(\mathcal{M})$ denote the set of all vector fields on $\mathcal{M}$. A differential form ω of degree n is a map

$$\omega : \underbrace{\mathcal{X}(\mathcal{M}) \times \cdots \times \mathcal{X}(\mathcal{M})}_{n} \to C^\infty(\mathcal{M})$$

which is $C^\infty(\mathcal{M})$ multilinear and skew-symmetric. It is known ([3]) that forms could also be defined locally on each chart $U \subset V$ as smooth R-multilinear and skew symmetric mappings

$$\omega : U \times V^n \to R^n$$

Definition 3.2 *By the exterior derivative $d\omega$ of an n-form we shall mean the $n+1$ form defined by*

$$d\omega(X_1,\ldots,X_{n+1}) = \sum_{i=1}^{n+1}(-1)^{i+1}X_i(\omega(X_1,\ldots,\hat{X}_1,\ldots,X_n)) - \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],\ldots\hat{X}_i,\ldots\hat{X}_j,X_{n+1})$$

where the roof sign above an entry indicates its absence from the expression (i.e., $w(\hat{X}_1,X_2)=w(X_2)$ and $w(X_1,\hat{X}_2)=w(X_1)$)

A differential form ω is said to be closed if its exterior derivative $d\omega$ is equal to zero.

Definition 3.3 *A differential form w of degree 2 is said to be symplectic whenever it is closed and non-degenerate. Here non-degenerate means that for each non-zero vector field X, the induced 1-form $(i_X\omega)(Y)=\omega(X,Y)$ is non-zero.*

For a function $f\in C^\infty(\mathcal{M})$ its differential df is a 1-form defined $df(v)=\frac{d}{dt}f\circ\sigma(t)|_{t=0}$ for any smooth curve in $\mathcal{M}$ such that $\sigma(0)=x$, and $\frac{d\sigma}{dt}(0)=v$.

In finite dimensional symplectic manifolds there is a unique vector field X_f such that $df=i_{X_f}\omega$. X_f is called the Hamiltonian vector field induced by f and f is called the Hamiltonian of X_f. In infinite dimensional symplectic manifolds it may happen that the form df is not equal to i_Xw for any $X\in\mathcal{X}(\mathcal{M})$. This is due to the fact that the cotangent bundle of an infinite dimensional Fréchet space is never a Fréchet manifold. Nevertheless,

Definition 3.4 *We shall call a vector field X_f Hamiltonian if there exists a smooth function f such that*

$$df(Y)=\omega(X_f,Y)$$

for all vector fields Y on $\mathcal{M}$.

If f and g are functions which admit Hamiltonian vector fields then their Poisson bracket $\{f,g\}$ is defined by the usual formula

$$\{f,g\}(x)=w(X(x),X_g(x)) \quad \text{for all } x\in\mathcal{M}$$

With these generalities in mind we now return to the space of anchored curves with periodic frames.

3.3 The symplectic structure of anchored curves with periodic frames

Our first task is to describe the tangent spaces of anchored curves with periodic frames. Let us begin with curves in $\mathcal{L}_h^0(L)$.

Lemma 2 *Suppose that $g_0(s) \in SL_2(C)$ is the horizontal curve that projects onto a curve γ_0 in $\mathcal{L}_h^0(L)$, i.e., suppose that*

$$\gamma(s) = g_0(s)g_0^*(s), \frac{dg_0}{ds}(s) = g_0(s)\Lambda_0(s) \text{ with } \Lambda_0(s) \in \mathfrak{h}$$

Then the tangent space of $\mathcal{L}'_\langle$ at γ is equal to $2g_0(s)W(s)g_0^*(s)$ where $W(s)$ is a smooth curve on the interval $[0, L]$ with values in the space of Hermitian matrices such that $W(0) = 0$, $\frac{dW}{ds}$ is smoothly periodic, and $\langle \frac{dW}{ds}(s), \Lambda(s)\rangle = 0$ for all $s \in [0, L]$.

Proof. Let $g(s, t)$ denote a smooth family of horizontal curves in $SL_2(C)$ that project onto a family of curves $\gamma(s, t)$ in $\mathcal{L}_h^0(L)$ subject to the condition that $g(s, 0) = g_0(s)$. It then follows that $g(0, t) = I$ for all t, and if $\Lambda(s, t)$ denote the Hermitian matrices that satisfy $\frac{\partial g}{\partial s}(s, t) = g(s, t)\Lambda(s, t)$, then $\Lambda(s, t)$ is smoothly periodic in s for each t and satisfies the initial boundary condition $\Lambda(0, t) = B_1$.

It then follows from the definition of tangent spaces in Fréchet manifolds that $\frac{\partial \gamma}{\partial t}(s, 0)$ is a tangent vector at γ_0. Let $\Omega(s, t)$ denote the Hermitian matrices defined by the relation $\frac{\partial g}{\partial t}(s, t) = g(s, t)\Omega(s, t)$. Then,

$$\frac{\partial \gamma}{\partial t}(s, t) = 2g(s, t)\Omega(s, t)g^*(s, t)$$

If $W(s) = \Omega(s, 0)$, then $2g_0(s)W(s)g_0^*(s)$ is a tangent vector at γ_0. the fact that the curves $g(s, 0)$ are anchored at I implies tha $W(0) = 0$. As in any Riemannian manifold,

$$\frac{D_\gamma}{\partial s}\frac{\partial \gamma}{\partial t}(s, t) = \frac{D_\gamma}{\partial t}\frac{\partial \gamma}{\partial t}(s, t)$$

where $\frac{D}{\partial s}$ and $\frac{D}{\partial t}$ denote the appropriate covariant derivative along the curves $\gamma(s, t)$. It now follows from formula (14) that

$$\frac{\partial}{\partial s}(\Omega(s, t)) = \frac{\partial}{\partial t}\Lambda(s, t) \tag{19}$$

Since $\frac{\partial}{\partial t}\Lambda(s, t)$ is smoothly periodic for each t, the same can be said of $\frac{\partial}{\partial s}\Omega(s, t)$.

Let $C(s) = \frac{\partial \Lambda}{\partial t}(s, 0)$. It follows from equation (20) that $\frac{dW}{ds}(s) = C(s)$. Since the perturbations $\gamma(s, t)$ are in the space of arc length parametrized curves $\langle \Lambda(s, t), \Lambda(s, t)\rangle = 1$ for all t and s. Therefore, $\left\langle \frac{\partial \Lambda}{\partial t}(s, t), \Lambda(s, t)\right\rangle = 0$, and hence, $\frac{dW}{ds}$ is orthogonal to

$C(s)$. Apart from this condition, the curve $C(s)$ is an arbitrary smoothly periodic curve in $\mathfrak{h}$.

It remains to show that any curve $W(s)$ of Hermitian matrices that satisfies the conditions of the lemma can be realized by the perturbations of the preceding kind. So let $W(s)$ be any such curve, and let $C(s) = \frac{dW}{ds}(s)$. We need to define a family of Hermitian curves $\Lambda(s,t)$ smoothly periodic in s for each t such that

$$\Lambda(0,t) = 0,\ \langle \Lambda(s,t), \Lambda(s,t)\rangle = 1,\ \frac{\partial \Lambda}{\partial t}(s,0) = C(s), \text{ and } \Lambda(s,0) = \Lambda_0(s))$$

Let $\phi(t)$ be any smooth function such that $\phi(0) = 0$ and $\frac{d\phi}{dt}(0) = 1$. Define

$$\Lambda(s,t) = \frac{1}{1 + \phi^2(t)\langle C(s), C(s)\rangle}(\Lambda_0(s) + \phi(t)C(s))$$

It is easy to verify that $\Lambda(s,t)$ satisfies all of the preceding conditions, and therefore our proof is finished,

■

We now come to the main part of the paper the symplectic form, which we define as follows:

$$\omega_\gamma(W_1, W_2) = \frac{1}{i}\int_0^L \left\langle \Lambda(s), [\frac{dW_1}{ds}(s), \frac{dW_2}{ds}(s)] \right\rangle ds, \tag{20}$$

where W_1 and W_2 are any tangent vectors to γ.

We remind the reader that the Lie bracket of Hermitian matrices is skew-Hermitian, and that $\langle A, B\rangle$ is imaginary for any Hermitian matrix A and any skew-Hermitian matrix B,hence, the above form is real. We also remind the reader that the trace form is invariant, in the sense that

$$\langle A, [B, C]\rangle = \langle [A, B], C\rangle$$

for any matrices A, B, and C in $\mathfrak{g}$).

Theorem 1 *The space of anchored hyperbolic curves with periodic frames is symplectic relative to (20).*

Proof. Evidently the above form is skew-symmetric. To show that it is non-degenerate, assume that for some tangent vector W at γ, $\omega_\gamma(W, V) = 0$ for all tangent vectors V at γ. Then take $V(x) = i\int_0^x [\Lambda, \frac{dW}{ds}]\, ds$. It follows that

$$\omega_\gamma(W, V) = \int_0^L \|[\Lambda, \frac{dW}{ds}]\|\, ds$$

Therefore, $[\Lambda, \frac{dW}{ds}] = 0$, and consequently Λ and $\frac{dW}{ds}$ must be colinear. Since they are also orthogonal to each other, it follows that $W = 0$. This argument proves that ω is non-degenerate. To show that ω is closed we will use definition (3.2).

Let W_i, $0 \le i \le 3$ be any tangent vectors to γ. Let X_i, $0 \le i \le 3$ denote the vector fields such that at each point σ in $\mathcal{L}_h^0(L)$, $X_i(\sigma)(s) = \int_0^s [\Sigma(x), A_i]\, dx$ for some fixed matrices A_i in $\mathfrak{k}2$. Here, Σ denotes the Hermitian matrix associated with the tangent vector of σ. The matrices A_i can be chosen so that at $\sigma = \gamma$, $X_i(\gamma) = W_i$ for each index i, in which case, $[\Lambda, A_i] = \frac{dW_i}{ds}$.

Suppose now that $\sigma(s,t)$ is any family of curves in $\mathcal{L}_h^0(L)$ parametrized by t such that $\sigma(0,s) = \gamma(s)$. Then, $X_i(\omega_\sigma(X_j(\sigma), X_k(\sigma)))$ is equal to

$$\frac{1}{i}\int_0^L \left\langle \frac{dW_i}{ds}, [\frac{dW_j}{ds}, \frac{dW_k}{ds}] \right\rangle ds + \frac{1}{i}\int_0^L \left\langle \Lambda, ([[\frac{dW_i}{ds}, A_j], [\Lambda, A_k]]) + ([[\Lambda, A_j], [\frac{dW_i}{ds}, A_k]] \right\rangle ds \tag{21}$$

as can be easily verified by differentiating $\frac{1}{i}\int_0^L \langle \Sigma(t,s), [[\Sigma(t,s), A_j], [\Sigma(t,s), A_k]] \rangle\, ds$ with respect to t at $t = 0$.

Vectors $\frac{dW_i}{ds}$, $0 \le i \le 3$ are orthogonal to Λ, and therefore linearly dependent. Hence the volume spanned by them, which is equal to $\frac{1}{i}\left\langle \frac{dW_i}{ds}, [\frac{dW_j}{ds}, \frac{dW_k}{ds}] \right\rangle$ must be 0. Therefore, the first integral in (22) is 0. To show that the remaining integral is also equal to 0, we shall need the following auxiliary formulas:

$$\text{a. } [A,[B,C]] = \langle A, C\rangle B - \langle A, B\rangle C \text{ for any } A, B, C \text{ in } \mathfrak{h} \tag{22}$$

$$\text{b. } [A,[B,C]] = \langle A, B\rangle C - \langle A, C\rangle B \text{ for any } B \text{ in } \mathfrak{h}, \text{ and any } A, C \text{ in } su_2. \tag{23}$$

These relations can be easily verified either by using Table 1, or by the cross product relations at the end of the section on generalities. We leave these verifications to the reader (the reader should note however, that in (24) $\langle A, B\rangle$ is imaginary, and that therefore, $\langle A, B\rangle C$ is in su_2).

It now follows from (24) that

$$[[\frac{dW_i}{ds}, A_j], [\Lambda, A_k]] = \left\langle [A_k, \Lambda], \frac{dW_i}{ds} \right\rangle A_j - \langle [A_k, \Lambda], A_j\rangle \frac{dW_i}{ds}$$

However, the second term in the above expression does not add to the integral above, because Λ is orthogonal to $\frac{dW_i}{ds}$, so we can ignore it. Similarly can be said for

$$[[\Lambda, A_j], [\frac{dW_i}{ds}, A_k]] = \left\langle [\Lambda, A_j], \frac{dW_i}{ds} \right\rangle A_k - \langle [\Lambda, A_j], A_k\rangle \frac{dW_i}{ds}$$

Therefore,the second integral above will be zero whenever,

$$\left\langle [A_k, \Lambda], \frac{dW_i}{ds} \right\rangle \langle A_j, \Lambda\rangle - \left\langle [A_j, \Lambda], \frac{dW_i}{ds} \right\rangle \langle A_k, \Lambda\rangle = 0$$

. It is easy to verify that the preceding difference of terms is equal to

$\langle A_k, A_i\rangle \langle A_j, \Lambda\rangle - \langle A_j, A_i\rangle \langle A_k, \Lambda\rangle$ because $\frac{dW_i}{ds} = [\Lambda, A_i]$, and then, it is equally easy to verify that the cyclic sum of the these terms is equal to 0. We shall omit these calculations.

Since it follows from the Jacobi's identity that the cyclic sum of the terms

$$\int_0^L \left\langle \Lambda, [[\frac{dW_i}{ds}, \frac{dW_j}{ds}], \frac{dW_k}{ds}] \right\rangle ds$$

is equal to 0, $d\omega = 0$. Therefore, $\mathcal{L}_h^0$ is symplectic. ■

The symplectic structure of anchored curves with periodic derivatives on the sphere can be defined quite analogously to the hyperbolic case as follows. Let γ denote a curve in $\mathcal{L}_s^0$, which we shall represent by a curve Γ in SU_2. Let $\Lambda(s)$ denote the curve in $\mathfrak{k}$ such that $\frac{d\Gamma}{ds}(s) = \Gamma(s)\Lambda(s)$. Tangent vectors at γ will be represented by matrices W in su_2 defined by $\frac{d\Sigma}{dt}(0,s) = \Gamma(s)W(s)$, where $\Sigma(t,s)$ is a curve in SU_2 such that $\Sigma(0,s) = \Gamma(s)$.

Let $U(t,s)$ and $V(t,s)$ denote the matrices such that

$$\frac{d\Sigma}{ds}(t,s) = \Sigma(t,s)U(t,s) \text{ ,and } \frac{d\Sigma}{dt}(t,s) = \Sigma(t,s)V(t,s)$$

It then follows that

$$\frac{\partial U}{\partial t}(t,s) - \frac{\partial V}{\partial s}(t,s) + [U(t,s), V(t,s)] = 0$$

as a consequence of $\frac{D_\Sigma}{dt}(\frac{d\Sigma}{ds}) = \frac{D_\Sigma}{dt}(\frac{d\Sigma}{ds})$.

The above implies that

$$\frac{dW}{ds}(s) = [\Gamma(s), W(s)] + C(s), \text{ where } C(s) = \frac{\partial U}{\partial t}(0,s) \tag{24}$$

We shall anchor our curves at the identity, which then implies that the tangent vectors $W(s)$ at Γ are the unique solutions of eq (25) such that $W(0) = 0$. As in the hyperbolic case, $\frac{dW}{ds}(s)$ is orthogonal to $\Lambda(s)$, which in turn implies that $C(s)$ is orthogonal to Λ. Apart from this constraint, $C(s)$ can be an arbitrary curve in $\mathfrak{k}$.

The symplectic form in the spherical case is then defined as follows:

$$\omega_\gamma(W_1, W_2) = -\int_0^L \langle \Lambda, [C_1, C_2]\rangle \, ds \tag{25}$$

where $C_i(s) = \frac{dW_i}{ds}(s) - [\Lambda(s), W(s)]$, for $i = 1, 2$.

Remark. As in the case of the canonical symplectic form on the cotangent bundles of manifolds, there is a choice of sign to be made, both ω and $-\omega$ are equally valid. Our choice is made in conformity with the hyperbolic case: the symplectic form in the spherical ase is obtained from the hyperbolic symplectic form by multiplying all quantities by i.

The proof that the above $2 - form$ is non-degenerate and closed is completely analogous to the one done for the hyperbolic case, and will therefore, be omitted.

As a way of summary for this section, it may be worthwhile to point out that in each of our cases the symplectic form is an extension of the standard symplectic form ω on the sphere S^2 , given explicitly by

$$\omega_\gamma(a, b) = \gamma \cdot (a \times b)$$

where a an b are tangent vectors at a point γ on S^2, to the space of curves. In fact, the symplectic structure of anchored curves with periodic frames is isomorphic to to the symplectic structure of anchored loops on the sphere.

The symmetry of the sphere ascends to the anchored curves via the right action of SU_2, a recognition that is important for some applications. This symmetry can be described as follows.

The right action of SU_2 extends to the curves with the action $a\gamma a^*$ for each curve γ and each $a \in SU_2$. This action is symplectic relative to the symplectic form above, and it is easy to show that $J(\gamma) = \int_0^L \Lambda(s)\, ds$ is the moment map associated with the action. More precisely, the dual $\mathfrak{g}^*$ of the Lie algebra of $SL_2(C))$ is identified with $\mathfrak{g}$ via the trace form. For each element $A \in \mathfrak{g}$ the moment map induces a function $J(\gamma) = \int_0^L \langle \Lambda(s), A \rangle\, ds$ on the space of anchored curves. The Hamiltonian vector field induced by this function coincides with the infinitesimal generator of the action induced one-parameter group of transformations $\{e^{tA}\gamma e^{-tA}\}$. Then it is well known ([1]) that J is an integral of motion for each Hamiltonian function which is invariant for the action. We will see an application of this fact further down in the text.

There is another symplectic form on the space of anchored curves, analogous to the volume form in R^3, that is given by the following expression:

$$\omega_\gamma(W_1, W_2) = \int_0^L \langle \Lambda, [W_1, W_2] \rangle\, ds$$

Such a form is mentioned elsewhere in the literature (see for instance [2],[3], and [13]). These two forms are compatible in the sense of Magri ([15]), and Magri's scheme can be used to get the integrability results for systems which are bi-Hamiltonian. However, we shall not pursue such directions here, as they would take us away from the main theme of this paper stated in the introduction.

4 The Hamiltonian flow of $\frac{1}{2}\int_0^L k^2(s)\, ds$

We now consider the Hamiltonian flow associated with the function $f(\gamma) = \frac{1}{2}\int\limits_0^L k^2(s)\, ds$ where $k(s)$ denotes the geodesic curvature of the curve γ. Recall that the geodesic curvature is defined by $k^2(s) = \|\frac{D_\gamma}{\partial s}\frac{d\gamma}{ds}\|^2$ whenever γ is parametrized by arc length. We also recall formula (16) which states that $k^2 = \langle \frac{d\Lambda}{ds}, \frac{d\Lambda}{ds} \rangle$, hence

$$f(\gamma) = \frac{1}{2}\int_0^L \left\langle \frac{d\Lambda}{ds}, \frac{d\Lambda}{ds} \right\rangle ds.$$

The Hamiltonian vector field $\vec{f}$associated with the function f is defined through the usual relation $df_\gamma(W) = \omega_\gamma(\vec{f}, W)$, where $df_\gamma(W)$ is the directional derivative of f at γ in the direction of a tangent vector W.For simplicity of exposition we shall consider each of our cases separately.

4.1 The hyperbolic case

To calculate the directional derivative of f at γ in the direction $\mathbb{W}$ we need to consider a family of curves $\hat{\gamma}(\epsilon, s)$ whose tangent vectors are described by the Hermitian matrices $\hat{\Lambda}(\epsilon, s)$, in the sense that $\frac{d\hat{\gamma}}{ds}(\epsilon, s) = 2\hat{g}\hat{\Lambda}\hat{g}^*$ for a curve $\hat{g} \in SL_2(C)$ that satisfies $\hat{g}\hat{g}^* = \hat{\gamma}$ and where, in addition we assume that $\hat{\Lambda}(0, s) = \Lambda(s)$, and $\frac{\partial\hat{\Lambda}}{\partial\epsilon}(0, s) = W(s)$.

$$\begin{aligned}
\frac{\partial}{\partial\epsilon} f(\hat{\gamma}(\epsilon, s)_{\epsilon=0} &= \frac{1}{2}\int_0^L \left\langle \frac{\partial\hat{\Lambda}}{\partial s}(\epsilon, s), \frac{\partial\hat{\Lambda}}{\partial s}(\epsilon, s) \right\rangle ds_{\epsilon=0} \\
&= \int_0^L \left\langle \frac{\partial\hat{\Lambda}}{\partial s}(\epsilon, s), \frac{\partial}{\partial s}\frac{\partial\hat{\Lambda}}{\partial\epsilon}(\epsilon, s) \right\rangle ds|_{\epsilon=0} \\
&= \int_0^L \left\langle \frac{d\Lambda}{ds}, \frac{d}{ds}\left(\frac{dW}{ds}\right) \right\rangle ds \\
&= -\int_0^L \left\langle \frac{d^2\Lambda}{ds^2}, \frac{dW}{ds} \right\rangle ds + \left\langle \frac{d\Lambda}{ds}, \frac{dW}{ds} \right\rangle |_{s=0}^{s=L}
\end{aligned}$$

The boundary terms vanish because of periodicity and consequently,

$$df_\gamma(W) = -\int_0^L \left\langle \frac{d^2\Lambda}{ds^2}, \frac{dW}{ds} \right\rangle ds = \frac{1}{i}\int_0^L \left\langle \Lambda(s), [\frac{d\vec{f}}{ds}, \frac{dW}{ds}] \right\rangle ds$$

Since W is arbitrary, $-\frac{d^2\Lambda}{ds^2} + \left\langle \frac{d^2\Lambda}{ds^2}, \Lambda \right\rangle \Lambda = \frac{1}{i}\left[\Lambda(s), \frac{d\vec{f}}{ds}\right]$, which further yields

$$-\left[\frac{d^2\Lambda}{ds^2}, \Lambda(s)\right] = \frac{1}{2}\left[\left[\Lambda(s), \frac{d\vec{f}}{ds}\right], \Lambda(s)\right]$$

The right-hand side of the above equation is equal to $-\frac{1}{i}\frac{d\vec{f}}{ds}$, as can be easily verified through the equation (23). Hence,

$$\frac{1}{i}\frac{d\vec{f}(s)}{ds} = \left[\frac{d^2\Lambda}{ds^2}, \Lambda(s)\right] \tag{26}$$

The Hamiltonian flow $\vec{f}$ then satisfies

$$\frac{\partial\gamma}{\partial t}(t,s) = 2g(t,s)\vec{f}(t,s)g^*(t,s), \quad \text{where} \tag{27}$$

$$\frac{\partial\gamma}{\partial s}(t,s) = 2g(t,s)\Lambda(t,s)g^*(t,s) \tag{28}$$

But then $\frac{D_\gamma}{\partial s}\frac{\partial\gamma}{\partial t}(t,s) = \frac{D_\gamma}{\partial t}\frac{\partial\gamma}{\partial s}$ and consequently, $g(t,s)\frac{\partial\vec{f}}{\partial s}(t,s) = g(t,s)\frac{\partial\Lambda}{\partial t}(t,s)$. This relation implies that

$$\frac{\partial\Lambda}{\partial t}(t,s) = \frac{\partial\vec{f}}{\partial s}(t,s) = i\left[\frac{\partial^2\Lambda}{\partial s^2}, \Lambda(t,s)\right] \tag{29}$$

Equation (29) can be also written in terms of the coordinates $\lambda(t,s)$ relative to the basis of Hermitian Pauli matrices as follows:

$$\frac{\partial\lambda}{\partial t}(t,s) = \lambda(t,s) \times \frac{\partial^2\lambda}{\partial s^2} \tag{30}$$

L.D.Faddeev and L.A. Takhtajan refer to the preceding equation as *the continuous isotropic Heisenberg feromagnetic model*([5], Part II.,Chapter 1).

4.2 The spherical case

The derivation of the corresponding Hamiltonian equations on the sphere is quite similar to the preceding case, so we will just sketch the main points. Let $\hat{\Gamma}(t,s)$ denote a family of curves in SU_2 that evolves according to the following equations

$$\frac{\partial\hat{\Gamma}}{\partial s}(t,s) = \hat{\Gamma}(t,s)U(t,s), \text{ and } \frac{\partial\hat{\Gamma}}{\partial t}(t,s) = \hat{\Gamma}V(t,s)$$

We assume that $\hat{\Gamma}(0,s) = \Gamma(s)$, and that $\frac{\partial\hat{\Gamma}}{\partial t}(0,s) = \Gamma(s)W(s)$, i.e., we assume that the curves define a tangent vector W at a point Γ in $\mathcal{L}_s^0(L)$. Then,

$$\frac{\partial U}{\partial t}(t,s) - \frac{\partial V}{\partial s}(t,s) + [U(t,s,V(t,s)] = 0$$

To be consistent with the previous notations we denote $C(s) = \frac{\partial U}{\partial t}(0,s)$, and we note that $C(s) - \frac{dW}{ds} + [\Gamma(s), W(s)] = 0$.Then,

$$df_\Gamma(W) = -\int_0^L \left\langle \frac{d^2\Lambda}{ds^2}, C\right\rangle ds$$

Let $\vec{f}$ denote the skew-symmetric matrix that corresponds to the Hamiltonian vector field generated by f, and let $\hat{C} = \frac{d\vec{f}}{ds} - [\Lambda, \vec{f}]$. It then follows from above that $[\Lambda, \hat{C}] = \frac{d^2\Lambda}{ds^2} + \left\langle \frac{d^2\Lambda}{ds^2}, \Lambda \right\rangle$. This further implies that

$$\hat{C} = [\frac{d^2\Lambda}{ds^2}, \Lambda] \text{ because } [[\Lambda, \hat{C}], \Lambda] = \hat{C}$$

The Hamiltonian flow is given by the following equations

$$\frac{\partial \Gamma}{\partial t}(t,s) = \Gamma(t,s)\vec{f}, \text{ and } \frac{\partial \Lambda}{\partial s} = \Gamma(t,s)\Lambda(t,s) \tag{31}$$

But then,

$$\frac{\partial \Lambda}{\partial t}(t,s) - \frac{\partial \vec{f}}{\partial s}(t,s) + [\Lambda(t,s), \vec{f}(t,s)] = 0$$

This last equation reduces to

$$\frac{\partial \Lambda}{\partial t}(t,s) = [\frac{\partial^2 \Lambda}{\partial s^2}, \Lambda(t,s)] \tag{32}$$

because $\frac{\partial \vec{f}}{\partial s}(t,s) = [\Lambda(t,s), \vec{f}(t,s)] + \hat{C}$ Equation (31) describes the flow of the Hamiltonian vector field in the spherical case. The reader should keep in mind however, that in the hyperbolic case Λ is Hermitian, while in the spherical case, Λ is skew-Hermitian: to pass from one to the other, one simply has to multiply $\Lambda(t,s)$ by i. Thus equations (28), (29), and (31) are representations of the same partial differential equation, Heisenberg's magnetic equation.

4.3 The non-linear Schroedinger equation

Our setting is confined to the smoothly periodic solutions Heisenberg's equation correspond to the smoothly periodic frames of arc length parametrized curves. Each such curve can be lifted to a curve in the orthonormal frame bundle of the underlying space , but the lifing is not unique. The Serret-Frenet frames are often cited in the literature, although they are not always the most convenient(demonstrated, for instance in [11]).

Let us now consider any choice of orthonormal frames $f_1(t,s)$, $f_2(t,s)$, $f_3(t,s)$ adapted to the curve $\gamma(t,s)$ that evolves according to (28). Let $\varphi(t,s)$ denote the curve in SU_2 such that $f_1(t,s) = \varphi B_1 \varphi^{-1}$, $f_2(t,s) = \varphi B_2 \varphi$ and $f_3 = \varphi B_3 \varphi^{-1}$, and such that $\phi(t,0) = I$ for all t. Recall that adapted means that $f_1(t,s) = \Lambda(t,s)$, which is equivalent to saying that $\phi(t,s) B_1 \phi^*(t,s) = \Lambda(t,s)$. Curves $\phi(t,s)$ then evolve according to the following differential equations:

$$\frac{\partial \phi}{\partial s}(t,s) = \phi(t,s)U(t,s), \text{ and } \frac{\partial \phi}{\partial t} = \phi(t,s)V(t,s)$$

Matrices $U(t,s)$ and $V(t,s)$ are not independent of each other since they conform to the following equation

$$\frac{\partial U}{\partial t}(t,s) - \frac{\partial V}{\partial t}(t,s) + [U(t,s), V(t,s)] = 0 \tag{33}$$

which follows from the properties of the covariant derivative on SU_2, as we have demonstrated earlier. Moreover, $V(t,0) = 0$, because our curves are anchored at $s = 0$, which then implies through the above equation that $\frac{dU}{dt}(t,0) = 0$.

We shall denote by $u_1(t,s)$, $u_2(t,s)$, $u_3(t,s)$ the functions such that $U(t,s)$ for $u_1A_1 + u_2A_2 + u_3A_3$. Written more explicitly,

$$\begin{aligned}\frac{\partial \varphi}{\partial s}(t,s) &= \varphi(t,s)(u_1(t,s)A_1 + u_2(t,s)A_2 + u_3(t,s)A_3) \\ &= \varphi(t,s)\frac{1}{2}\begin{pmatrix} iu_1(t,s) & u_2(t,s) + iu_3(t,s) \\ -u_2(t,s) + iu_3(t,s) & -iu_1(t,s) \end{pmatrix}\end{aligned}$$

It will be convenient to use $u(t,s)$ to denote the complex function $u_1(t,s) + iu_2(t,s)$. Recall that $|u(t,s)|^2 = k(t,s)^2$ with $k(t,s)$ the geodesic curvature of the base curve $\gamma(t,s)$

In what follows we will show that the quantity $\psi(t,s) = u(t,s)e^{i\int_0^s u_1(t,x)\,dx}$ satisfies the non-linear Schroedinger equation. To show this fact it will be nesessary to obtain the precise relation between matrices $U(t,s)$ and $V(t,s)$.

Since $\Lambda(t,s) = \varphi(t,s)B_1\varphi^*(t,s)$, it follows that

$$\begin{aligned}\frac{\partial \Lambda}{\partial t} &= \frac{\partial \varphi}{\partial t}B_1\varphi^* + \varphi B_1\frac{\partial \varphi^*}{\partial t} \\ &= \varphi(VB_1 - B_1V)\varphi^* \\ &= \varphi[B_1, V]\varphi^*\end{aligned}$$

Similarly,

$$\frac{\partial \Lambda}{\partial s} = \varphi[B_1, U]\varphi^*, \quad \text{and} \quad \frac{\partial^2 \Lambda}{\partial s^2} = \varphi\left([[B_1, U], U] + \left[B_1, \frac{\partial U}{\partial s}\right]\right)\varphi^*$$

The fact that $\Lambda(t,s)$ evolves according Heisenberg's magnetic equation implies that

$$[B_1, V] = i([[[B_1, U], U], B_1] + [[B_1, \frac{\partial U}{\partial s}], B_1]) \tag{34}$$

The above relations simplify through the Lie algebraic relations given in (23) and (24], which we now recall for the calculations that follow.

$$[A, [A, B]] = \langle A, B\rangle A - \langle A, A\rangle B$$

for any $A, B \in \mathfrak{h}$.

$$[[A,B],B] = \langle B,B\rangle A - \langle A,B\rangle B$$

for any $A \in su_2$ and $B \in \mathfrak{h}$. Then,

$$[B_1,U],U] = \langle U,B_1\rangle U - \langle U,U\rangle B_1 = -(u_2^2+u_3^2)B_1 + u_1u_2B_2 + B_3u_1u_3$$

and so, $[[[B_1,U],U],B_1] = u_1u_3A_2 - u_1u_2A_3$.

Similarly,

$$[B_1,\frac{\partial U}{\partial s}] = \frac{\partial u_3}{\partial s}B_2 - \frac{\partial u_2}{\partial s}B_3, \text{ and } [[B_1,\frac{\partial U}{\partial s}],B_1] = -\frac{\partial u_3}{\partial s}A_3 - \frac{\partial u_2}{\partial s}A_2$$

Then, equation (33) reduces to

$$[B_1,V] = i(u_1(u_3A_2 - u_2A_3) - \frac{\partial u_3}{\partial s}A_3 - \frac{\partial u_2}{\partial s}A_2)$$

which can also be written as

$$[B_1,V] = -u_1(u_3B_2 - u_2B_3) + \frac{\partial u_3}{\partial s}B_3 + \frac{\partial u_2}{\partial s}B_2$$

because $A_j = iB_j, j = 1,2,3$. Let $V(t,s) = v_1(t,s)A_1 + v_2(t,s)A_2 + v_3(t,s)A_3$. Then, $[B_1,V] = v_3B_2 - v_2B_3$, which together with the relations above yields

$$v_2 = -u_1u_2 - \frac{\partial u_3}{\partial s}, \text{ and } v_3 = -u_1u_3 + \frac{\partial u_2}{\partial s} \tag{35}$$

Equation (34) can be written more compactly in terms of complex functions as

$$v(t,s) = -u_1(t,s)u(t,s) + i\frac{\partial u}{\partial s}(t,s), \text{ where } v(t,s) = v_2(t,s) + iv_3(t,s) \tag{36}$$

The zero curvature equation $\frac{\partial \mathbb{U}}{\partial t} - \frac{\partial V}{\partial s} + [U,V] = 0$ implies that

$$\frac{\partial u_1}{\partial t} = \frac{\partial v_1}{\partial s} + \frac{1}{2}\frac{\partial}{\partial s}(u_2^2+u_3^2) \tag{37}$$

and that

$$\frac{\partial u}{\partial t} = i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - \frac{\partial u_1}{\partial s}u - i(v_1+u_1^2)u \tag{38}$$

Equation (36) implies that

$$\frac{\partial}{\partial t}\int^{s} u_1(t,x)dx = v_1 + \frac{1}{2}(u_2^2+u_3^2) + c,$$

for some function $c(t)$. However, this function must be equal to zero because our curves are anchored, i.e., both $v_1(t,0)=0$, and $u(t,0)=0$.

Upon substituting $v_1 = \frac{\partial}{\partial t}\int^s u_1(t,x)dx - \frac{1}{2}|u|^2$ in (37) one gets

$$\frac{\partial u}{\partial t} + iu\frac{\partial}{\partial t}\int^s u_1(t,x)\,dx = i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - u\frac{\partial u_1}{\partial s} - i\left(-\frac{1}{2}|u|^2 + u_1^2\right)u \qquad (39)$$

Now we shall multiply both sides of the above equation by $e^{i\int u_1\,dx}$ and then recognize the left-hand side of the above equation as $\frac{\partial}{\partial t}ue^{i\int_0^s u_1(t,x)\,dx}$ which then yields

$$\frac{\partial}{\partial t}ue^{i\int^s u_1(t,x)\,dx} = (i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - u\frac{\partial u_1}{\partial s} - i(u_1^2 - \frac{1}{2}|u|^2))ue^{i\int^s u_1\,dx}$$

If we introduce now the function $\psi(t,s) = u(t,s)e^{i\int^s u_1(t,x)\,dx}$,
then $\frac{\partial\psi}{\partial s} = \left(\frac{\partial u}{\partial s} + iuu_1\right)e^{i\int u_1\,dx}$, and $\frac{\partial^2\psi}{\partial s^2} = \left(\frac{\partial^2 u}{\partial s^2} + 2iu_1\frac{\partial u}{\partial s} + iu\frac{\partial u_1}{\partial s} - u_1^2 u\right)e^{i\int u_1\,dx}$.
It follows that

$$i\frac{\partial^2\psi}{\partial s^2} = \left(i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - u\frac{\partial u_1}{\partial s} - iu_1^2 u\right)e^{i\int u_1\,dx}.$$

Upon substituting the preceding expression in equation (38) one gets

$$\frac{\partial}{\partial t}\psi(t,s) = i\frac{\partial^2\psi}{\partial s^2} + i\frac{1}{2}|\psi|^2\psi \qquad (40)$$

and we have finally arrived at the non-linear Schroedinger's equation. Of course, we can now retrace the steps and express Schroedinger's equation in matrix form through the zero-curvature equation, as done in [5]. Simply let $\mathbb{U} = \frac{1}{2}\begin{pmatrix} 0 & \psi \\ -\overline{\psi} & 0 \end{pmatrix}$, and $V = \frac{1}{2}\begin{pmatrix} i\frac{\partial}{\partial s}|\psi|^2 & i\psi(|\psi|^2 + c) \\ i\overline{\psi}|\psi|^2 & -i\frac{\partial}{\partial s}|\psi|^2 \end{pmatrix}$. An easy computation shows that

$$\frac{\partial V}{\partial t} - \frac{\partial V}{\partial s} + [U,V] = 0 \qquad (41)$$

In the spherical case,

$$[A_1, V] = i([[[A_1, U], U], A_1] + [[A_1, \frac{\partial U}{\partial s}], BA_1])$$

is the analogue od equation (33). Since it can be made to coicide with (33) after multiplying both sides by $-i$, it becomes evident that the same calculations repeated from the hyperbolic case would lead to the same end result, the non-linear Schroedinger equation.

The preceding developement clarifies Hasimoto's first observation that $\psi = ke^{i\int \tau}\, dx$, where k and τ are the curvature and the torsion of a curve $\gamma(t,s)$ that satisfies the filament equation

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial s} \times \frac{\partial^2 \gamma}{\partial s^2}$$

is a solution of the non-linear Schroedinger equation.

If one assumes that the solution curve of the filament equation is already parametrized by arc-length, then it follows that its tangent vector $T(t,s) = \frac{\partial \gamma}{\partial s}(t,s)$ satisfies

$$\frac{\partial T}{\partial t} = T \times \frac{\partial^2 T}{\partial s^2}$$

If the tangent vector T in $\mathbb{R}^3$ is interpreted as the coordinate vector of Λ relative to the basis consisting of the Pauli matrices, as done by Faddeev and Takhtajan, then we get Heisenberg's magnetic equation (28). When φ corresponds to the Serret-Frenet frame than u_1 is the torsion $u_2 = 0$ and u_3 is the geodesic curvature. Hence, the complex function $\psi = ue^{i\int u_1\, ds}$ coincides with Hasimoto's function $ke^{i\int \tau\, ds}$.

However, both assumptions in the preceding paragraphs are geometrically unjustified. In the first place, the filament equation is not a priori independent of the choice of parametrization of γ. So one cannot assume that its solutions can always be parametrized by arc-length. Secondly, to interpret tangent vectors in R^3 as the coordinate vectors of Hermitian matrices relative to the basis of Pauli matrices fundamentally obscures the geometric origins of the subject matter: such interpretations confuse the hyperbolic curvature with the Euclidean curvature, and they also confuse hyperbolic tangents with Euclidean curves. To get the true solutions of our Hamiltonian system equations (28) and (29) need to be solved, a fact which is often neglected in the literature on infinite domensional Hamiltonian systems.

5 Elastic Hyperbolic Curves and Solitons

For mechanical systems the Hamiltonian function stands for the total energy of the system, and its critical points carry important qualitative information about the system. In this setting, the critical points for the Hamiltonian $f(\gamma) = \frac{1}{2}\int_0^L k^2\, ds$ correspond to the elastic curves, and therefore, not too surprisingly, also carry important information about solutions of the corresponding differential system. More precisely, we shall show that they generate soliton solutions for either Heisenberg's magnetic equation, or for the non-linear Schroedinger equation.

We remind the reader that the elastic curves are the projections of extremal curves corresponding to the variational problem of minimizing $\frac{1}{2}\int_0^L k^2\, ds$ over curves in a

Riemannian space that satisfy given boundary conditions at the initial and terminal points of the curve. Here k denotes the geodesic curvature of the curve.

This problem has several formulations depending on the type of frames used to describe the curves. The Serret- Frenet frames will do, but they are awkward for variational problems that involve only the curvature and not the torsion: Here, we shall proceed by relying on the formalism developed in [11], which in this context can be explained as follows.

We may begin with the most general frame $\hat{\phi}(s)$ adapted to the curve γ in SU_2 or $\mathbb{H}^3$. As we have explained earlier in the paper, $\hat{\phi}(s)$ is a curve in SU_2 that is a solution of $\frac{d\hat{\phi}}{ds}(s) = \hat{\phi}(s)U(s)$ with $U(s) = u_1(s)A_1 + u_2(s)A_2 + u_3(s)A_3$, where the geodesic curvature $k(s)$ is related to $U(s)$ through the formula

$$k^2(s) = u_2^2(s) + u_3^2(s)$$

We shall now show that there is a frame adapted to the curve for which $u_1(s) = 0$. Let $\psi(s)$ denote the solution of the following equation in SU_2: $\frac{d\psi}{ds}(s) = -\psi(s)(u_1(s)A_1)$. Then, $\phi(s) = \hat{\phi}(s)\psi(s)$ is a solution of

$$\frac{d\phi}{ds}(s) = \phi(s)(\psi^*(u_2(s)A_2 + u_3(s)A_3)\psi(s)$$

Since $[A_1, B_1] = 0$, and $[A_1, A_2] = -A_3, [A_1, A_3] = A_2$, it follows that

$$\psi(s)B_1\psi^*(s) = B_1, \text{ and } \psi(s)(u_2(s)A_2 + u_3(s)A_3)\psi^*(s) = v_2(s)A_2 + v(s)_3A_3$$

with $u_2^2(s) + u_3^2(s) = v_2^2(s) + v_3^2(s)$. The frame ϕ is the frame adapted to the curve γ for which $u_1(s) = 0$ (we now replace $v(s$ by $u(s))$. Such a frame can also be described in terms of covariant derivatives as follows:

$$\frac{D_\gamma}{ds}v_2(s) = -u_3(s)v_1(s), \text{ and } \frac{D_\gamma}{ds}v_3(s) = u_2(s)v_1(s)$$

We shall call frames defined above strongly adapted frames: They are unique up to a rotation around the tangential direction. The elastic problem which will be considered now is the following:

Minimize $\frac{1}{2}\int\limits_0^L (u_2^2 + u_3^2)\, ds$ over all solution curves $\frac{dg}{ds}(s) = g(s)(B_1 + u_2A_2 + u_3A_3)$ that satisfy the given boundary conditions $g(0) = g_0$ and $g(L) = g_1$.

The projections $\gamma(s)$of the extremal curves $g(s)$corresponding to the above variational problem will be called elastic. It can be shown that there exists an optimal solution for any pair of boundary points provided that the length L is suitably large([10]). In this case we shall be interested in periodic boundary conditions, and since the problem is left invariant, we may take the initial point to be equal to the group identity.

As usual, the Maximum Principle of optimal control leads to the appropriate Hamiltonian on the cotangent bundle of T^*G of G. To take advantage of the left-invariant symmetries of our variational problem we shall trivialize the cotangent bundle T^G by the left-translations and write $T^*G = G \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of $\mathfrak{g} = sl_2(C)$.

Let us use $h_1, h_2, h_3, H_1, H_2, H_3$ to denote the coordinates of l in $\mathfrak{g}^*$ relative to the dual basis $B_1^*, B_2^*, B_3^*, A_1^*, A_2^*, A_3^*$ defined by the Pauli matrices B_1, B_2, B_3 and $A_1 = iB_1, A_2 = iB_2, A_3 = iB_3$.

An easy application of the Maximum Principle shows that the regular extremal curves of our elastic problem are the integral curves of the Hamiltonian vector field

$$H = \frac{1}{2}(H_2^2 + H_3^3) + h_1$$

and it also shows that the optimal control functions are of the form $u_i(s) = H_i(s), 2 \leq i \leq 3$. There are also the abnormal extremal curves, but it can be shown ([10]) that the optimal solutions for this problem are the projections of regular extremal curves, and therefore the abnormal extremals can be ignored here. . Since the integral curves of a smooth vector field which are closed are necessarily smooth, it follows that the projections of closed solutions of our Hamiltonian system will be smoothly periodic

There are several ways to write the equations of the Hamiltonian flow corresponding to a given Hamiltonian function. The most direct way, although perhaps not the most illuminating, makes use of the Poisson tables involving the variables $h_1, h_2, h_3, H_1, H_2, H_3$. The Poisson table is isomorphic to the Lie bracket table (Table 1) from which it immediately follows that

$$\begin{aligned} \frac{dH_1}{ds} &= \{H_1, H\} = H_2\{H_1, H_2\} + H_3\{H_1, H_3\} + \{H_1, h_1\} = 0 \\ \frac{dH_2}{ds} &= \{H_2, H\} = H_3\{H_2, H_3\} + \{H_2, h_1\} = -H_3H_1 + h_3 \\ \frac{dH_3}{ds} &= \{H_3, H\} = H_2H_1 - h_2 \\ \frac{dh_1}{ds} &= \{h_1, H\} = H_3h_2 - H_2h_3 \\ \frac{dh_2}{ds} &= \{h_2, H\} = -H_3h_1 - H_3 \\ \frac{dh_3}{ds} &= \{h_3, H\} = H_2h_1 + H_2 \end{aligned} \tag{42}$$

Alternatively, one may write the extremal equations in, what is sometimes called the Lax-pair form, since SU_2 is semi-simple, as follows:

$$\frac{dL}{ds} = [DH(L), L] \tag{43}$$

where $L = \sum_{i=1}^{i=3}(H_iA_i + h_iB_i)$, and where $DH(L)$ denotes the differential of H at L. In this case, $DH(L) = H_2A_2 + H_3A_3 + B_1$.L

It will be convenient to use M and P to denote the Hermitian and the skew symmetric part of L respectively, i.e., $M = \sum_{i=1}^{i=3} H_iA_i$, and $P = \sum_{i=1}^{i=3} h_iB_i$, and similarly do the same for $D(H(L)$ and write $dH(L) = \Omega + B_1$. Then (42) becomes

$$\frac{dM}{ds} = [\Omega, M] + [B_1, P], \text{and } \frac{dP}{ds} = [\Omega, P] + [B_1, M]$$

It follows almost immediately from the preceding equations that $I_2 = \langle M, P\rangle$ and $I_1 = \langle M, M\rangle + \langle P, P\rangle$ are constants of motion, which together with H_1 and H produces four independent constants of motion. This observation implies that our Hamiltonian system is completely integrable.

To get the necessary conditions for the existence of soliton solutions, assume that $L(t)$ is a particular extremal curve that projects onto an elastic curve. The corresponding frame $\phi(s)$ is the solution of $\frac{d\phi}{ds}(s) = \phi(s)U(s)$ such that $\phi(0) = I$. The matrix $U(s)$ is of the form $U(s) = H_2(s)A_2 + u_3(s)A_3$. We would like to show that there exists a real number ξ such that $\psi(t,s) = u(s+\xi t)$, where $u(s) = H_2(s)+iH_3(s)$, is a solution of the non-linear Schroedinger's equation.

Let $w(t) = h_2(t) + ih_3(t)$, It follows from equations (41) that

$$\frac{du}{ds}(s) = iH_1u(s) - iw(s), \text{ and } \frac{dw}{ds} = i(h_1+1)u(s) \tag{44}$$

Then,

$$\frac{\partial\psi}{\partial t} = i\xi(H_1\psi - w), \text{ and } \frac{\partial^2\psi}{\partial s^2} = -H_1^2\psi + H_1w + (h_1+1)\psi$$

Since $H = \frac{1}{2}|\psi|^2 + h_1$,

$$\begin{aligned} -i\frac{\partial\psi}{\partial t} - (\frac{\partial^2\psi}{\partial s^2} + \frac{1}{2}|\psi|^2\psi) &= \xi(H_1\psi - w) - (H_1^2\psi + H_1w + (h_1+1)\psi\frac{1}{2}|\psi|^2\psi \\ &= (-\xi - H_1)w + (\xi H_1 + H_1^2 - 1 - H)\psi \end{aligned}$$

The preceding quantity is a solution of the non-linear Schroedinger's equation whenever

$$\xi = -H_1 \text{ and } H = -1$$

It remains to show that the closed solutions exist and that, moreover, they occur on the energy level $H = -1$.

For this we need to integrate the above equations, at least on the level of the Lie algebra. To begin with, note that

$$(H_2h_3 - H_3h_2)^2 + (H_2h_2 + H_3h_3)^2 = (H_2^2 + H_3^2)(h_2^2 + h_3^2)$$

Then,

$$\begin{aligned} (\tfrac{d}{ds}h_1)^2 &= (H_2h_3 - H_3h_2)^2 = (H_2^2 + H_3^2)(h_2^2 + h_3^2) - (H_2h_2 + H_3h_3)^2 \\ &= (H_2^2 + H_3^2)(I_1 + H_1^2 + H_2^2 + H_3^2 - h_1^2) - (I_2 - h_1H_1)^2 \\ &= 2(H - h_1)(I_1 + H_1^2 + 2(H - h_1) - h_1^2) - (I_2 - h_1H_1)^2 \\ &= 2h_1^3 + c_1h_1^2 + c_2h_1 + c_3 \end{aligned} \tag{45}$$

where c_1, c_2, c_3 are the constants of motion given by the following expressions

$c_1 = -(H_1^2 + 2H + 4), c_2 = (2I_2H_1 - 2H_1^2 - 8H - 2I_1), c_3 = 2H(I_1 + H_1^2 + 2H) - I_2^2$

.

Therefore, $h_1(s)$ is expressed in terms of elliptic functions, and since $k^2 = H_2^2 + H_3^2 = 2(H - h_1)$ the same can be said for the curvature of the projected elastic curve. The remaining variables u and w, in the notations of equations (43), can be integrated in terms of two angles θ and ϕ, which are analogous to the angles of nutation and precession in the literature on the rigid body, that are defined as follows.

First note that $I_1 = h_1^2 + |w|^2 - (H_1^2 + |u|^2)$, and since $|u|^2 = 2(H - h_1)$,

$$(h_1 + 1)^2 + |w|^2 = I_1 + H_1^2 + 2H + 1 = J^2$$

where J^2 denotes $I_1 + H_1^2 + 2H + 1$, which is constant along each extremal trajectory.Then, define θ and ϕ through the following formulas

$$(h_1(s) + 1) = J\cos\theta(s) \text{ and } w(s) = \sin\theta(s)e^{i\phi(s)} \tag{46}$$

It follows that

$$\frac{dh_1}{ds} = -J\sin\theta\frac{d\theta}{ds}, \text{ and } \frac{dw}{ds} = w(J\frac{cos\theta}{\sin\theta}\frac{d\theta}{ds} + i\frac{d\phi}{ds})$$

Now ,we note that

$$\begin{aligned}\frac{u}{w} = \frac{u\bar{w}}{|w|^2} &= \frac{H_2h_2 + H_3h_3 + i(H_3h_2 - H_2h_3)}{J^2 - (h_1 + 1)^2} \\ &= \frac{I_2 - h_1H_1 + i\frac{dh_1}{ds}}{J^2\sin^2\theta} \\ &= \frac{I_2 - h_1H_1}{J^2\sin^2\theta} - \frac{i}{J\sin\theta}\frac{d\theta}{ds}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{dw}{ds} = i(h_1 + 1)u &= iJw\cos\theta\frac{(I_2 - h_1H_1)}{J^2\sin^2\theta} + (\frac{\cos\theta}{\sin\theta})\frac{d\theta}{ds}w \\ &= w((\frac{\cos\theta}{\sin\theta})\frac{d\theta}{ds} + i\frac{d\phi}{ds})\end{aligned}$$

hence,

$$\frac{d\phi}{ds} = \frac{J\cos\theta(I_2 + H_1 - H_1J\cos\theta)}{J^2\sin^2\theta} \tag{47}$$

Equations (44) and (45) yield the following differential equation for θ

$$(\frac{d\theta}{ds})^2 = 2(H + 1 - J\cos\theta) - \frac{(I_2 - H_1(-1 + J\cos\theta))^2}{\sin^2\theta} \tag{48}$$

We now return to the question of existence of periodic trajectories. Since we have assumed that the extremal curve projects onto a closed optimal curve in $SL_2(C)$, it follows that the curvature $u(s)$ is closed. Since $u(s) = H_2(s) + iH_3(s)$, both H_2 and H_3 are closed, which implies that h_1 is closed, because the Hamiltonian H is constant on each extremal trajectory. It then follows from equation (45) that w is periodic of period L whenever $\phi(0) = \phi(L)$. Therefore, the extremal trajectory is closed, hence periodic, whenever

$$\int_0^L \frac{J\cos\theta(I_2 + H_1 - H_1 J\cos\theta)}{J^2\sin^2\theta}\, ds = 0 \tag{49}$$

For the existence of solitons it is required that $H = -1$, in which case equation (47) reduces to

$$(\frac{d\theta}{ds})^2 = -2J\cos\theta - \frac{(I_2 - H_1(-1 + J\cos\theta))^2}{\sin^2\theta} \tag{50}$$

It remains to show that there exists a solution of equation (49) such that (48) holds for suitable constants I_1, I_2 and H_1, which can be readily verified computationally.

6 Complete Integrability

There are further connections between elasic curves and the solutions of the non-linear Schroedinger equation that were first noticed by J.Langer and R. Perline in [13], namely that the integrals of motion for both Hamiltonian systems are related. We will illustrate this phenomenon by showing that the function $f(\gamma) = \int_0^L k^2\tau\, ds$ is an integral of motion for Heisenberg's magnetic equation, while at the same time the quantity $k^2(s)\tau(s)$ is a constant of motion for the elastic problem.

We shall first show that $k^2\tau$ is a constant of motion for the elastic curves. Since both the Serret-Frenet frame and the one used in this paper are adapted to the curve, it follows that they rotate around each other in the plane perpendicular to the tangent vector. If we denote by β the angle through which the frame used above rotates relative to the Serret-Frenet frame, then the rate $\frac{d\beta}{ds}(s)$ is equal to the torsion τ of the underlying curve γ. This relation can be derived easily, or one can consult

([12], p 461) for a demonstration. Moreover, $\tan\beta = -\frac{H_2}{H_3}$. Then,

$$\begin{aligned}
\sec^2\beta\frac{d\beta}{ds} &= \frac{(H_2\frac{dH_3}{ds} - H_3\frac{dH_2}{ds})}{H_3^2} \\
&= \frac{H_2(H_2H_1 - h_2) - H_3(-H_3H_1 + h_3)}{H_3^2} \\
&= \frac{H_1(H_2^2 + H_3^2) - H_2h_2 - H_3h_3}{H_3^2} \\
&= \frac{H_1(H_2^2 + H_3^2) - (I_2 + H_1h_1)}{H_3^2} \\
&= \frac{2H_1H - I_2}{H_3^2}
\end{aligned}$$

and therefore,

$$k^2\tau = 2H_1H - I_2$$

We shall now show that the functions $f[\gamma) = \frac{1}{2}\int_0^L k^2\,ds$ and $g(\gamma) = \int_0^L k\tau\,ds$ Poisson commute, where the Poisson bracket is given by the usual formula

$$\{f, g\}(\gamma) = \omega_\gamma(\vec{f}(\gamma), \vec{g}(\gamma)$$

with $\vec{f}$ and $\vec{g}$ denoting the Hamiltonian vector fields associated with the functions.

Suppose now that $\Lambda(s)$ denotes the Hermitian matrix that corresponds to the tangent vector of a curve γ. Then, the normal vector $N(s)$ to $\gamma(s)$ is given by $N = \frac{1}{k}\frac{d\Lambda}{ds}$. Hence the binormal vector $B(s)$ is given by $B = [\Lambda, \frac{d\Lambda}{ds}]$. According to the Serret-Frenet equations, $\frac{dN}{ds} = -k\Lambda + \tau B$, and hence

$$\tau = \left\langle \frac{dN}{ds}, B \right\rangle = \left\langle -\frac{1}{k^2}\frac{dk}{ds}\frac{d\Lambda}{ds} + \frac{1}{k}\frac{d^2\Lambda}{ds^2}, \frac{1}{k}[\Lambda, \frac{d\Lambda}{ds}] \right\rangle$$

It follows that

$$k^2\tau = \left\langle [\Lambda, \frac{d\Lambda}{ds}], \frac{d^2\lambda}{ds^2} \right\rangle$$

Let $W(s)$ be an arbitrary tangent vector at γ. Then the directional derivative of g at γ in the direction W is given by the following expression

$$\begin{aligned}
dg_\gamma(W) &= \int_0^L \left\langle \dddot{W}, [\Lambda, \dot{\Lambda}] \right\rangle + \left\langle \ddot{\Lambda}, [\dot{W}, \dot{\Lambda}] \right\rangle + \left\langle \ddot{\Lambda}, [\Lambda, \ddot{W}] \right\rangle ds \\
&= \int_0^L 2\left\langle [\ddot{\Lambda}, \Lambda], \ddot{W} \right\rangle - \left\langle [\ddot{\Lambda}, \dot{\Lambda}], \dot{W} \right\rangle ds \\
&= -2\int_0^L \left\langle (\frac{d}{ds}([\ddot{\Lambda}, \Lambda]) - [\ddot{\Lambda}, \dot{\Lambda}]), \dot{W} \right\rangle ds
\end{aligned}$$

where the dots indicate derivatives with respect to s. I then follows that

$$\begin{aligned}\frac{1}{i}\frac{d\vec{g}}{ds} &= [[\ddot{\Lambda},\dot{\Lambda}],\Lambda] + 2[\frac{d}{ds}[\ddot{\Lambda},\Lambda],\Lambda] \\ &= 2[[\dddot{\Lambda},\Lambda],\Lambda] + [[\ddot{\Lambda},\dot{\Lambda}],\Lambda] \\ &= 2\dddot{\Lambda} - 2\langle\dddot{\Lambda},\Lambda\rangle\Lambda - \left\langle\Lambda,\ddot{\Lambda}\right\rangle\dot{\Lambda}\end{aligned}$$

Recall now that $\frac{d\vec{f}}{ds} = i[\ddot{\Lambda},\Lambda]$. An easy calculation shows that

$$[\frac{d\vec{f}}{ds},\frac{d\vec{g}}{ds}] = (\left\langle\Lambda,\ddot{\Lambda}\right\rangle + 2\langle\dddot{\Lambda},\Lambda\rangle - 2\left\langle\ddot{\Lambda},\Lambda\right\rangle - \left\langle\dddot{\Lambda},\ddot{\Lambda}\right\rangle)\Lambda$$

Hence,

$$\{f,g\} = \frac{1}{i}\int_0^L(-2\left\langle\dddot{\Lambda},\ddot{\Lambda}\right\rangle + 2\langle\Lambda,\dddot{\Lambda}\rangle\left\langle\Lambda,\ddot{\Lambda}\right\rangle + \left\langle\Lambda,\ddot{\Lambda}\right\rangle\left\langle\dot{\Lambda},\ddot{\Lambda}\right\rangle)\,ds$$

The above integral is zero for the following reasons:

The integral of the first term is zero, because $2\left\langle\dddot{\Lambda},\ddot{\Lambda}\right\rangle = \frac{d}{ds}\left\langle\ddot{\Lambda},\ddot{\Lambda}\right\rangle$. Since $2\left\langle\Lambda,\ddot{\Lambda}\right\rangle\langle\Lambda,\dddot{\Lambda}\rangle = \frac{d}{ds}\left\langle\Lambda,\ddot{\Lambda}\right\rangle^2 - 2\left\langle\Lambda,\ddot{\Lambda}\right\rangle\left\langle\dot{\Lambda},\ddot{\Lambda}\right\rangle$, the above inegrand reduces to one term $-\left\langle\Lambda,\ddot{\Lambda}\right\rangle\left\langle\dot{\Lambda},\ddot{\Lambda}\right\rangle$. But then $-\frac{1}{2}\frac{d}{ds}\left\langle\dot{\Lambda},\dot{\Lambda}\right\rangle^2 = \left\langle\Lambda,\ddot{\Lambda}\right\rangle\left\langle\dot{\Lambda},\ddot{\Lambda}\right\rangle$, because $\left\langle\dot{\Lambda},\dot{\Lambda}\right\rangle = \left\langle\Lambda,\ddot{\Lambda}\right\rangle$, and our claim is proved.

It might be instructive at this point to translate our results in terms of the language of mathematical physics. To begin with, the vector $\int_0^L\Lambda(s)\,ds$ is called the total spin in([5]). Here we recognize it as the moment map discussed in the previous section. It is a conserved quantity, since the Hamiltonian is invariant under the action of SU_2. Of course, this fact can be verified directly as follows:

$$\frac{\partial}{\partial t}\int_0^L\Lambda(t,s)\,ds = \int_0^L\frac{\partial\Lambda}{\partial t}(t,s)\,ds = i\int_0^L[\frac{\partial^2\Lambda}{\partial ds^2},\Lambda]\,ds = i\int_0^L\frac{\partial}{\partial s}[\Lambda,\dot{\Lambda}]\,ds = 0$$

The function $\int_0^L k^2\tau\,ds = i\int_0^L\left\langle\Lambda,[\dot{\Lambda},\ddot{\Lambda}]\right\rangle\,ds$ corresponds to the momentum in [17] for the following reasons : Recall that the tangent vector Λ and the adapted frame ϕ are related by the formula $\Lambda = \phi B_1\phi^*$. When ϕ is strongly adapted to Λ then the frame deformation matrix U is of the form $U = \begin{pmatrix} 0 & u \\ -\bar{u} & o\end{pmatrix}$, and when Λ evolves according to Heisenberg's equation, u evolves according to the non-linear Schroedinger equation. The reader can readily verify that when ϕ is strongly adapted to Λ then

$$i\left\langle\Lambda,[\dot{\Lambda},\ddot{\Lambda}]\right\rangle = \left\langle[[B_1,U],[B_1,\dot{U}]],B_1\right\rangle = Im\bar{u}\dot{u}$$

which up to a constant factor appears as the constant C_2 in ([17]) where it is refered to as the momentum. Our Hamiltonian can be then expressed as $\frac{1}{2}\int_0^L u(a)u^*(s)\,ds$ which then corresponds to the probability distribution of the number of particles.

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