GEOMETRY OF THE CROSS-RATIO AND KP-HIERARCHY

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Geometry of the cross-ratio and KP-hierarchy*

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Generalization of the cross-ratio to polarizations of linear spaces (finite or infinite-dimensional) is given. This cross-ratio appears to be a cocycle of the canonical (tautalogical) bundle over the corresponding Grassmannian with coefficients in the sheaf of automorphisms of fibers. Applications to Sato's Grassmannian and KP-hierarchy is given.

Keywords: Cross-ratio, Riccati equations, infinite-dimensional completely integrable systems, Grassmannian Sato, cohomologies with coefficients in a sheaf.

MSC: 14M15, 34H05

1 Sato's Grassmannian and KP-hierarchy.

We shall use the following standard notations. For the basic ring $B = \boldsymbol{C}[[x]] = \sum_{i\geq 0} a_i x^i \mid a_i \in \boldsymbol{C}$ – (formal Taylor series) – let $E = \boldsymbol{C}[[x]]((\partial^{-1}))$ – be the ring of formal pseudodifferential operators with coefficients in B, equipped with the ∂^{-1}-adic topology, i.e. the norm of an element $f = \sum_{i\leq j} a_i \partial^i$, where $a_j \neq 0$, is defined as $\nu(f) = j$.

Consider the direct sum decomposition $E = E_+ \oplus E_-$, where $E_+ = B[\partial]$ is the ring of differential operators, and $E_- = B[[\partial]]\partial^{-1}$ is the ring of operators of the negative order (Volterra-type operators). So that for any $P \in E$ one has $P = P_+ + P_-$, where P_+ is the differential part and P_- is the Volterra part of P. A decomposition of a space into a direct sum of its two subspaces is called polarization.

Let E be a Hilbert space equipped with a polarization

$$E = E_+ \oplus E_-. \tag{1}$$

Define the subgroup $GL_{res}(E) \subset GL(E)$ as follows:

If $A \in GL(E)$ is represented as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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relative to the decomposition (1) then $A \in GL_{res}(E)$ if b and c are Hilbert-Shmidt operators. It follow that a and d are Fredholm operators.

Sato's Grassmannian $Gr(E)$ is the set of all closed subspaces $W \subset E$ such that

1. The orthogonal projection $pr_+ : W \to E_+$ is a Fredholm operator.
2. The orthogonal projection $pr_- : W \to E_-$ is a Hilbert-Shmidt operator.

The group $GL_{res}(E)$ acts on $Gr(E)$ by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot T = (c + dT)(a + bT)^{-1}.$$

The tangent space to $Gr(E)$ at a point W is $T_W = Hom(W, E/W)$.

Consider the affine subspace $E' = \partial + E_-$ and the multiplicative group $G = 1 + E_-$. Both of them have the tangent space E_-. Define the mapping $\xi : G \to E'$ for $S \in G$ by the formula $\xi : S \mapsto L = S\partial S^{-1}$, and the mapping $\eta : G \to Gr(E)$ by the formula $\eta : S \mapsto S^{-1}W_0$, where $W_0 \subset Gr(E)$ is a fix subspace. Differentials of these mappings at a point S act on $A \in E_-$ by formulae $d\xi : A \mapsto [AS^{-1}, L]$, and $d\eta : A \mapsto -S^{-1}A$ accordingly.

The main result (known as Sato's correspondence) [1],[2] is as follows. The operator of multiplication by z^{-n} transfers by $d\xi(d\eta)^{-1}$ into the commutator $[L_+^n, L]$, i.e. into the right hand side of KP-equation. Given n, the operator $z^{-n} : E \to E$ on $Gr(E)$ defines a vector field that does not depend on t, i.e. the right hand side of "constant coefficients" Riccati equation on Sato's Grassmsnnian. Analogues theory for noncommutative rings of coefficients was developed in [4], and for the multidimensional x in [5].

2 Operator cross-ratio

Consider first the finite-dimensional case and subspaces of half-dimension. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ be four n-dimensional subspaces of $\boldsymbol{R}^{2n}$ which correspond to four points of $Gr_n(\boldsymbol{R}^{2n})$ with matrix coordinates P_1, P_2, P_3, P_4. Suppose that $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3, \mathcal{P}_4$ define the polarizations of $\boldsymbol{R}^{2n}$, i.e. $\mathcal{P}_1 \oplus \mathcal{P}_2 = \mathcal{P}_3 \oplus \mathcal{P}_4 = \boldsymbol{R}^{2n}$. The polarization $\mathcal{P}_i, \mathcal{P}_j$ will be denoted by Π_{ij}. The class of matrices which are similar to the matrix $DV(\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\mathcal{P}_4) = (P_1 - P_2)^{-1}(P_2 - P_3)(P_3 - P_4)^{-1}(P_4 - P_1)$ (invers matrices are defined since Π_{12} and Π_{34} are polarizations) is an invariant of the ordered four points of Grassmannian relative to the Möbius transformations [3]. It is called matrix cross-ratio.

The projection parallel to a subspace $\mathcal{P}_i$ will be denoted by π_i or by the figure i above the arrow which gives the corresponding mapping. If $\mathcal{P}_i, \mathcal{P}_j$ define a polarization then the image of π_i in $\mathcal{P}_j$ is uniquely defined.

Theorem 1 *Let Π_{12} and Π_{34} be the polarizations. Then $DV(\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\mathcal{P}_4)$ is the matrix of the composition mapping $\mathcal{P}_1 \xrightarrow{4} \mathcal{P}_3 \xrightarrow{2} \mathcal{P}_1$ of the space $\mathcal{P}_1$ on itself.*

Proof. Let $(g, P_3 g)$ be the projection of an element $(f, P_1 f)$ of the space $\mathcal{P}_1$ on $\mathcal{P}_3$ parallel to $\mathcal{P}_4$. This means that $(f - g, P_1 f - P_3 g) \in \mathcal{P}_4$, i.e. $P_1 f - P_3 g =$

$P_4(f-g)$. Hence, $g = (P_4 - P_3)^{-1}(P_4 - P_1)f$. Similarly, the projection of an element $(g, P_3 g)$ on $\mathcal{P}_1$ parallel to $\mathcal{P}_2$ is $(h, P_1 h)$, where $h = (P_2 - P_1)^{-1}(P_2 - P_3)g = (P_1 - P_2)^{-1}(P_2 - P_3)(P_3 - P_4)^{-1}(P_4 - P_1)f$, as it was desired.

The analog of the theorem 1 was proved in [6] only for the determinant of τ-function. Our theorem allows to remove the restriction of half-dimension and to define cross-ratio for a pair of polarizations of the space $\boldsymbol{R}^m$ with similar dimensions, i.e. $dim\mathcal{P}_1 = dim\mathcal{P}_3 = k; \quad dim\mathcal{P}_2 = dim\mathcal{P}_4 = m-k$. Moreover, this gives the possibility to define an operator cross-ratio for infinite-dimensional cases. The invariance of the operator cross-ratio relative to the Möbius group is inherited by the construction itself due to the linearity of the projection operator. May be it would be adequate to call it the cross-mapping rather than the cross-ratio, but we remain the initial term as a historical reminiscence.

Note The τ-function of the theory of integrable systems [6] is only the determinant of an operator cross-ratio. Remaining invariants of the corresponding operators were not used before.

Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ be subspaces of the linear space E. The operator $DV(\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\mathcal{P}_4)$ defined in the theorem 1 will be denoted by $DV_{12;34}$.

Lemma 1 *Let projections parallel to $\mathcal{P}_2$ and parallel to $\mathcal{P}_4$ are isomorphisms of the spaces $\mathcal{P}_1$ and $\mathcal{P}_3$. Then $DV_{12;34} = DV_{34;12}$*

Proof. Consider the mappings

$$DV_{12;34} : \mathcal{P}_1 \xrightarrow{4} \mathcal{P}_3 \xrightarrow{2} \mathcal{P}_1; \qquad DV_{34;12} : \mathcal{P}_3 \xrightarrow{2} \mathcal{P}_1 \xrightarrow{4} \mathcal{P}_3.$$

If one identifies $\mathcal{P}_1$ and $\mathcal{P}_3$ by using projection π_4, then both maps coinside.

Thus the cross-ratio does not depends on the order of polarization pairs.

Lemma 2 *Let $\mathcal{P}_1 \oplus \mathcal{P}_2 = \mathcal{P}_3 \oplus \mathcal{P}_4 = \mathcal{P}_1 \oplus \mathcal{P}_4 = \mathcal{P}_3 \oplus \mathcal{P}_2 = E$. Then $DV_{12;34} = DV_{14;32}$.*

Proof. Consider the mappings

$$DV_{12;34} : \mathcal{P}_1 \xrightarrow{4} \mathcal{P}_3 \xrightarrow{2} \mathcal{P}_1; \qquad DV_{14;32} : \mathcal{P}_1 \xrightarrow{2} \mathcal{P}_3 \xrightarrow{4} \mathcal{P}_1.$$

We have

$$\mathcal{P}_1 \xrightarrow{4} \mathcal{P}_3 = (\mathcal{P}_3 \xrightarrow{4} \mathcal{P}_1)^{-1}; \qquad \mathcal{P}_1 \xrightarrow{2} \mathcal{P}_3 = (\mathcal{P}_3 \xrightarrow{2} \mathcal{P}_1)^{-1},$$

which follows the lemma.

Lemma 3 *Let $\mathcal{P}_1 \oplus \mathcal{P}_2 = \mathcal{P}_3 \oplus \mathcal{P}_4 = E$. Then $DV_{12;43} = I - DV_{12;34}$.*

Proof. Designe the image of $h_1 \in \mathcal{P}_1$ under the mapping $DV_{12;43} : \mathcal{P}_1 \xrightarrow{3} \mathcal{P}_4 \xrightarrow{2} \mathcal{P}_1$. We have $\pi_3 h_1 = h_1 + h_3$, where $h_3 \in \mathcal{P}_3$ and $(h_1 + h_3) \in \mathcal{P}_4$. Further on, $\pi_2(h_1 + h_3) = h_1 + h_3 + h_2$, where $h_2 \in \mathcal{P}_2$ and $(h_1 + h_3 + h_2 \in \mathcal{P}_1)$. Let us rewrite the image of $DV_{12;43}$ in the form $(h_1 + h_3 + h_3) = h_1 - (h_1 - (h_1 + h_3 + h_2))$.

Since $(h_1 + h_3) \in \mathcal{P}_4$ and $(h_1 - (h_1 + h_3)) \in \mathcal{P}_3$, then $(h_1 - (h_1 + h_3))$ is the image of h_1 under the projection π_4 on $\mathcal{P}_3$. The subtraction of h_2 gives the image of its π_2-projection on $\mathcal{P}_1$. Hence $(h_1 - (h_1 + h_3 + h_2)) = DV_{12;34}h_1$. Consequently, $DV_{12;43} = I - DV_{12;34}$.

Lemmas 1-3 defines generators of the representation of the group of permutations of four spaces in the group generated by identity and $D = DV_{12;34}$.

Let us show that operator cross-ratio defines a cocycle with values in operators.

Lemma 4 *Let two subspaces* $\mathcal{P}_i, \quad (i = 1, 2)$ *be equivalent relative to the group* GL_{res}, *and three subspaces* $\mathcal{Q}_j, \quad (j = 1, 2, 3)$ *complement them to the whole space, that is* $\mathcal{P}_i \oplus \mathcal{Q}_j = E, \quad (i = 1, 2;\ j = 1, 2, 3)$. *Then*

$$DV(\mathcal{P}_1\mathcal{Q}_1\mathcal{P}_2\mathcal{Q}_2) \cdot DV(\mathcal{P}_1\mathcal{Q}_3\mathcal{P}_2\mathcal{Q}_1) \cdot DV(\mathcal{P}_1\mathcal{Q}_2\mathcal{P}_2\mathcal{Q}_3) = I. \tag{2}$$

That means that the product of these three operators is the identity.

Proof. Consider the chain of mappings

$$\mathcal{P}_1 \xrightarrow{\mathcal{Q}_2} \mathcal{P}_2 \xrightarrow{\mathcal{Q}_1} \mathcal{P}_1 \xrightarrow{\mathcal{Q}_1} \mathcal{P}_2 \xrightarrow{\mathcal{Q}_3} \mathcal{P}_1 \xrightarrow{\mathcal{Q}_3} \mathcal{P}_2 \xrightarrow{\mathcal{Q}_2} \mathcal{P}_1$$

which defines the left hand side of the formula (2). The composition of the second and the third mappings, as well as the composition of the fourth and the fifth ones, are identities. After its reduction, the remaining composition gets identity too.

Let us clarify the geometrical sense of the lemma 4.

Consider the canonical (tautalogical) bundle γ over the manifold $Gr(E)$, i.e. the bundle whose fiber in a point $W \in Gr(E)$ is the linear space W itself. Introduce the following local trivialization of γ. Fix a point $W_+ \in Gr(E)$. The chart $\mathcal{U}_V$ on γ is defined by a subspace $V \subset E$ that complement W_+, i.e. $W_+ \oplus V = E$, and besides the projecting operator $\pi_V : E \to W_+$ parallel to the subspace V is asumed to be bounded. In other words, for the decomposition $h = w + v$, where $h \in E,\ w \in W_+,\ v \in V$, the following estimate with a constant C has to be valid: $\parallel w \parallel \leq C \parallel h \parallel$ (the space V does not have "infinitesimally small" angles with W_+).The coordinates of a point $(W, x) \in \gamma$, where $x \in \gamma$, in the chart $\mathcal{U}_V$ will be $(W, \pi_V x) \in Gr(E) \times W_+$. Let us calculate the transformation formula of coordinates from a chart $\mathcal{U}_{V_1}$ to a chart $\mathcal{U}_{V_2}$. Let (W, y) be the coordinates of a point $(W, x) \in \gamma$ in the chart $\mathcal{U}_{V_1}$. Here $y \in W_+$. Then $x = \pi^{-1}_{\mathcal{U}_{V_1}} y$, and the same point has in the chart $\mathcal{U}_{V_2}$ coordinates $(W, \pi_{\mathcal{U}_{V_2}} \cdot \pi^{-1}_{\mathcal{U}_{V_1}} y)$. But $\pi_V = \pi_V^{-1}$, since π_V is a projector. Hence, the transition function is defined by the formula

$$W_+ \xrightarrow{V_1} W \xrightarrow{V_2} W_+$$

which coincides with the cross-ratio $DV(W_+, V_2; W, V_1)$. By the lemma 4, the transition from $\mathcal{U}_{V_1}$ to $\mathcal{U}_{V_2}$, further on, to $\mathcal{U}_{V_3}$, and finaly, back to $\mathcal{U}_{V_1}$, gives the cocycle property (2). But charts $\mathcal{U}_V$-type do not cover all the Grassmann manifold. In contrast to finite-dimensional cases, in a Hilbert space two subspaces W_+ and W of equal dimensions do not have in general a common complementary subspace V. The exact necessary and sufficient conditions for two subspaces to have a common complementary subspace was found in [7]. To avoid this difficulty we exchange the transformation $DV(W_+, V_2; W, V_1)$ by $DV(W, V_1; W_+, V_2)$. By lemma 1, this gives the same cross-ratio but now it defines a transform of W, that allow us to change W_+. Now all the Grassmannian is covered by $\mathcal{U}_V$-type charts.

So, the transition from a chart $\mathcal{U}_{V_1}$ to a chart $\mathcal{U}_{V_2}$ is defined by the transform $DV(W, V_1; W_+, V_2)$ which acts on coordinates $x \in W$ as a linear fractional function from operator coordinates of a subspace W. These transforms are defined on intersections of charts $\mathcal{U}_V$-type. Hence, in view of (2), the cross-ratio defines a cocycle $\{DV\}$ on Grassmannian with coefficients in the sheaf of automorphisms of fibers W of canonical bundle γ, i.e.

$$\{DV\} \in H^1(Gr, Aut(W)).$$

3 Integrals of KP-hierarchy

While finding τ-function [1] it was obtained an infinite matrix that was upper-triangular except for a finite-dimensional block and with units on the remainig part of the main diagonal. That was the reason for existing its determinant. It may be checked that it is the matrix of the operator cross-ratio of some four linear subspaces. If we use the lemma 3 and change the order of these four subspaces we obtain the infinite matrix A that will be (except for a finite-dimensional block) "asymptotically nilpotent", i.e. upper-triangular with zeros on the main diagonal. We shall call such matrices almost asymptotically nilpotent. Almost asymptotically nilpotent matrices and all its powers have a trace which gives us the possibility to define ζ-function of a cross-ratio.

The cross-ratio relates to four points of Sato's Grassmannian $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ the class of operators that are similar to $DV(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4)$. Each subspace $\mathcal{P}_i$ defines, due to Sato's correspondence, a solution of KP-hierarchy therewith the image of each $\mathcal{P}_i$ is obtained by Möbius transform. Hence, the cross-ratio of four subspaces remains invariant in the flow of KP-hierarchy. It means that the corresponding matrices change isospectrally. Invariants of almost asymptotically nilpotent matrix relative to isospectral deformations are traces of its powers. It is natural to describe these invariants in terms of ζ-function.

Let we have three solutions $W_i(t)$, $(i = 1, 2, 3)$ of Riccati equation on Sato's Grassmannian. Using Sato's correspondence we obtain three solutions $L_i(t)$, $(i = 1, 2, 3)$ of KP-hierarchy on the space E'. Take any subspace W and designe the cross-ratio $DV(W, W_1(t), W_2(t), W_3(t))$. We obtain the operator, spector of which does not depend on t. Coefficients of its characteristic operators (or its characteristic numbers) give the set of integrals, and its ζ function remains invariant. It is convenient for calculations to take stationary solutions of the corresponding Riccati equation as some of W_i.

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