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COMPUTATION OF BOUNDED DEGREE NEVANLINNA-PICK INTERPOLANTS BY SOLVING NONLINEAR EQUATIONS*

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ABSTRACT. This paper provides a procedure for computing scalar rational Nevanlinna-Pick interpolants of a bounded degree. It turns out that the computation of such interpolants boils down to a system of nonlinear equations to be solved in a nonconvex region. The existence and the uniqueness of the solution are guaranteed by Georgiou's proof of bijectivity of a map between a class of nonnegative trigonometric polynomials and a class of numerator/denominator polynomial pairs of interpolants. Based on further analysis of this map, a numerical continuation method for solving the nonlinear equations is proposed. Numerical examples illustrate the reliability of the proposed procedure.

KEYWORDS. Nevanlinna-Pick interpolation, positive realness, rationality, system of nonlinear equations, continuation method.

AMS CLASSIFICATION: 30E05, 34A12, 42A15, 42A70.

1. INTRODUCTION

Nevanlinna-Pick interpolation theory has been recognized as a tool applicable to various areas in systems and control [8, 19]. Recent developments of the interpolation theory with complexity constraint have introduced new approaches in signal processing, circuit theory and robust control [3, 4, 5, 7, 12, 14]. To fully exploit the theory in such applications, it is significant to devise a numerically efficient algorithm to compute any rational interpolant of a bounded degree.

The theory in [3, 4, 5] completely characterizes all the *strictly* positive real Nevanlinna-Pick interpolants of a bounded degree. More precisely, it shows the diffeomorphism between a class of *positive* trigonometric polynomials and a class of rational *strictly* positive real interpolants, a stronger assertion of Georgiou's conjecture of bijectivity in [11]. The problem of computing each such interpolant amounts to solving an optimization problem. To solve the optimization problem, an algorithm has been developed based on a continuation method in [2, 10, 15].

The algorithm may work even to derive non-strictly positive real interpolants that correspond to some nonnegative trigonometric polynomials. However, the algorithm *cannot* find any positive real interpolant that has poles on the unit circle. This is because, in such cases, the algorithm requires spectral factorization with spectral zeros on the unit circle, which is numerically impossible. The computation of positive real functions with poles on the unit circle is important; for instance it corresponds to spectral lines in spectral estimation. Furthermore, a reliable algorithm ought to be numerically stable for spectral zeros in the vicinity of the unit circle. Therefore, we need to develop a procedure to be able to generate such interpolants.

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Recently, Georgiou proved the bijectivity between a class of *nonnegative* trigonometric polynomials and a class of (not necessarily strictly) positive real interpolants in [12]. However, the proof in [12] does not offer any means of actually constructing such interpolants from arbitrarily specified nonnegative trigonometric polynomials. The objective of this paper is to develop a computational algorithm that is able to compute *any positive real* interpolant, especially the ones that have poles on the unit circle.

In this paper, to compute rational Nevanlinna-Pick interpolants, we will propose a rather different approach from optimization-based approach taken in [3, 4, 5, 10, 15]. The key theorem that motivated this work is Theorem 2 in [12], which states the bijectivity mentioned above. If we rewrite Theorem 2 in [12] in terms of coefficient vectors of rational interpolants, we naturally come across a system of nonlinear equations to be solved in a nonconvex region. A powerful method to solve such system numerically is a *continuation method* [1]. In the continuation method, we implicitly define a trajectory with the help of a homotopy. As a homotopy, we choose a linear combination between the system corresponding to the so-called *central solution* and the system of our interest. As it will turn out later, the solution trajectory generated by this homotopy has nice properties (such as smoothness and no bifurcation). Therefore, the continuation method using the predictor-corrector steps will be an appropriate method for tracing the trajectory numerically. We expect that the proposed procedure is numerically more reliable than the previous one in [2, 10, 15], especially when we would like to determine positive real interpolants with poles that are very close to, or exactly on the unit circle.

The paper is organized as follows. In Section 2, we will review the Nevanlinna-Pick interpolation problem with degree constraint. Section 3 is devoted to an exposition of properties of a map, which are important in solving the system of nonlinear equations. In Section 4, we will derive a system of nonlinear equation to be solved for the computation of any positive real interpolant of a bounded degree. Section 5 explains a continuation method that we apply to the system of nonlinear equations. Two examples are given in Section 6 to illustrate the efficiency and reliability of the procedure. The case when derivative constraints are involved is discussed in the Appendix.

2. Nevanlinna-Pick interpolation with degree constraint

In this section, we will formulate a Nevanlinna-Pick interpolation problem with degree constraint. For the problem, we will review Theorem 2 in [12] concerning the bijectivity between the class of interpolants and that of nonnegative trigonometric polynomials, which is pertinent to the main result in this paper.

2.1. Nevanlinna-Pick interpolation problem with degree constraint.

The core problem considered in this paper is the *Nevanlinna-Pick interpolation problem with degree constraint*, formulated as follows.

Problem 2.1. Given a set of self-conjugate complex number pairs $\{(z_j, w_j)\}_{j=0}^n$ with z_0=0 and distinct $\{z_j\}_{j=0}^n \subset \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, determine any function f which satisfies the following conditions:

- f is positive real, i.e., f is holomorphic in $\mathbb{D}$ and maps $\mathbb{D}$ into the closed right half-plane $\mathbb{C}_+$,
- f satisfies the interpolation conditions:

$$f(z_j) = w_j, \quad j = 0, 1, \ldots, n, \tag{2.1}$$

- f is rational of degree at most n.

Remark 2.1. The results in this paper are still valid even in the case where the interpolation constraints include derivative constraints at each point z_j as

$$f^{(k)}(z_j) = w_{jk}, \quad k = 1, \dots, m.$$

The only difference from the results presented in this paper appears in the matrix K that will be defined in (4.2) (see Appendix A). Here, for simplicity of notation, we just deal with the case without derivative constraints.

The problem setting is the same as the one in [12], but is different from the series of work in [3, 4, 15] based on convex optimization, in that we are interested in *all the positive real interpolants* of a bounded degree, that is, not only strictly positive real functions but also non-strictly positive real functions. The necessary and sufficient condition for the existence of rational interpolants of degree at most n is well-known to be expressed as the nonnegativity of the *Pick matrix* (see [12, p. 632]), which is the same as the classical result without degree constraint (see e.g. [18]). If the Pick matrix is nonnegative definite but singular, there is a unique solution to this problem. Hereafter, we assume that the Pick matrix is positive definite, leading to the case where there are infinitely many rational interpolants of degree at most n. This assumption will be essential in a proof in Section 3.

2.2. **Complete characterization of all the solutions.** In [11], for the covariance extension problem (which is essentially the same problem as the Nevanlinna-Pick interpolation problem), Georgiou conjectured that *the class of all the interpolants of degree at most n is completely characterized in terms of the class of nonnegative trigonometric polynomials of degree at most n.* This conjecture was proven to be true in [12, Theorem 2]. Even though the theorem in [12] deals with both complex and real rational interpolants, here, we restrict ourselves to real rational interpolants which are relevant to applications. In addition, we remove the normalization that was taken into account in [12].

Theorem 2.1. [12] *For each element in the class of real nonnegative trigonometric polynomials of degree at most n:*

$$\mathcal{D} := \left\{ \begin{array}{r} d(z, z^{-1}) := \displaystyle\sum_{j=0}^{n} d_j (z^j + z^{-j}) \geq 0, \forall\, |z| = 1 \\ d_j \in \mathbb{R},\ j = 0, 1, \dots, n \end{array} \right\}, \tag{2.2}$$

there is a unique pair of real polynomials (α, β) *satisfying the following four conditions:*

C1: $\deg \alpha \leq n$, $\deg \beta \leq n$ *and* $\alpha(0) > 0$,

C2: *the function* $f := \beta/\alpha$ *satisfies the interpolation conditions (2.1),*

C3: α *and* β *satisfies*

$$\alpha(z)\beta^*(z) + \beta(z)\alpha^*(z) = d(z, z^{-1}), \tag{2.3}$$

where $\alpha^*(z) := \alpha(z^{-1})$,

C4: $\alpha + \beta$ *has all its roots in* $\mathbb{D}^c := \{z \in \mathbb{C} :\ |z| \geq 1\}$,

Note that the combination of two conditions **C3** and **C4** is equivalent to the condition that $f := \beta/\alpha$ is positive real (see [12]), and therefore, that the conditions **C1**–**C4** are the same as three conditions in Problem 2.1. Conversely, it is obvious that any Nevanlinna-Pick interpolant $f = \beta/\alpha$ of degree at most n corresponds to some nonnegative trigonometric polynomial $d(z, z^{-1})$ with (2.3). Consequently, the class $\mathcal{D}$ completely characterizes the class of all the solutions to the Nevanlinna-Pick interpolation problem with degree constraint.

Now, define the inner-product of the real functions a and b in L^2 space by

$$\langle a, b\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} a^*(e^{i\theta})b(e^{i\theta})d\theta. \tag{2.4}$$

Since, as will be shown later, the conditions **C1** and **C2** give rise to a linear relation between α and β, denoted by $\beta = \kappa(\alpha)$, the above theorem can be restated as follows.

Theorem 2.2. *The map $G : \mathcal{A} \to \mathcal{D}$ defined by*

$$G(\alpha) := \alpha^* \left[\kappa(\alpha)\right] + \left[\kappa(\alpha)\right]^* \alpha \tag{2.5}$$

is bijective. Here, the domain $\mathcal{A}$ is the class of all polynomials α which generate the Nevanlinna-Pick interpolants with degree constraint, defined as

$$\mathcal{A} := \left\{\alpha : \ \deg\alpha \le n,\ \alpha(0) > 0,\ \kappa(\alpha)/\alpha \text{ is positive real}\right\}. \tag{2.6}$$

In fact, the map G has stronger properties than the bijectivity. This point will be discussed in the next section.

3. Properties of the map G

Now, we will discuss important properties of the map G in (2.5). As was stated in Theorem 2.2, the map is a bijection. However, we can actually show that *the map G is a homeomorphism*. In the region of *strictly* positive real interpolants, *it is even a diffeomorphism*, as shown by Byrnes and Lindquist in [6]. Both of these properties will turn out to be vital in justifying a numerical continuation method. First we have the following theorem.

Theorem 3.1. *The map $G : \mathcal{A} \to \mathcal{D}$ in (2.5) is a homeomorphism.*

To prove this theorem, we show a normalized version of the theorem.

Lemma 3.1. *The map $\tilde{G} : \tilde{\mathcal{A}} \to \tilde{\mathcal{D}}$ defined by*

$$\tilde{G}(\alpha) := \alpha^* \left[\kappa(\alpha)\right] + \left[\kappa(\alpha)\right]^* \alpha \tag{3.1}$$

is bijective. Here, the domain $\tilde{\mathcal{A}}$ and the range $\tilde{\mathcal{D}}$ are normalized sets of $\mathcal{A}$ and $\mathcal{D}$ respectively, defined as

$$\tilde{\mathcal{A}} := \left\{\alpha \in \mathcal{A} : \ \langle \alpha, \kappa(\alpha)\rangle = 1\right\}, \tag{3.2}$$

$$\tilde{\mathcal{D}} := \left\{d \in \mathcal{D} : \ d_0 = 1\right\}. \tag{3.3}$$

Note that the normalization $\langle \alpha, \kappa(\alpha)\rangle = 1$ is equivalent to

$$\left\langle \tilde{G}(\alpha), 1\right\rangle = 2. \tag{3.4}$$

Proof. Since $\tilde{G}$ is known to be a bijection due to Theorem 2.2, it suffices to show that it is continuous and that the domain $\tilde{\mathcal{A}}$ is closed and bounded.

First we address the boundedness of $\tilde{\mathcal{A}}$. Let us take $\alpha \in \tilde{\mathcal{A}}$ and form the corresponding positive real function

$$f(z) := \frac{\kappa(\alpha(z))}{\alpha(z)}.$$

Note that the polynomials α and $\kappa(\alpha)$ can be written respectively as $\alpha_0\tilde{\alpha}$ and $\beta_0\tilde{\beta}$ using polynomials $\tilde{\alpha}$ and $\tilde{\beta}$ with constant terms equal one. Also note that we have a relation $\beta_0 = w_0\alpha_0$, since $f(z_0) = w_0$ with the assumption $z_0 = 0$. Hence, $f = w_0\tilde{\beta}/\tilde{\alpha}$.

Since f is positive real, the roots of $\tilde{\alpha}$ are in the closed unit disc. Thus, and since α_0 is taken positive, it suffices to show that α_0 is bounded from above. In order to prove the boundedness of α_0, we need to utilize the positivity of the Pick matrix. To this end, along the same line as in [13, 3], we introduce the filter bank

$$H(z) := \begin{bmatrix} H_0(z) \\ \vdots \\ H_n(z) \end{bmatrix},$$

where $H_j(z) := 1/(1 - z_j z)$. Then the Pick matrix can be represented as

$$\Sigma = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(f+f^*)H^* d\theta = \frac{w_0}{2\pi}\int_{-\pi}^{\pi} H\left(\frac{\tilde{\alpha}}{\tau}\right)^{-1}\frac{\tilde{\alpha}\tilde{\beta}^* + \tilde{\beta}\tilde{\alpha}^*}{\tau\tau^*}\left(\frac{\tilde{\alpha}}{\tau}\right)^{-*} H^* d\theta.$$

where we have suppressed the evaluation at $e^{i\theta}$, and τ is defined by $\tau(z) := \prod_{j=0}^{n}(1 - z_j z)$. Since $\tilde{\alpha}/\tau \in Span\{H_j\}$, there is a vector $k \in \mathbb{C}^{n+1}$ such that $\tilde{\alpha}/\tau = k^H H$. Therefore, and since the Pick matrix is positive definite, there is an $\varepsilon > 0$ such that

$$\alpha_0^2\varepsilon \le \alpha_0^2 k^H \Sigma k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\tilde{G}(\alpha)(e^{i\theta})}{|\tau(e^{i\theta})|^2}d\theta \le \frac{M}{2\pi}\int_{-\pi}^{\pi}\tilde{G}(\alpha)(e^{i\theta})d\theta = 2M,$$

for some finite number M satisfying $M \ge \max_{\theta\in[-\pi,\pi]} 1/\left|\tau(e^{i\theta})\right|^2$. Here, the finiteness of M follows from the assumption that all the interpolation point are in $\mathbb{D}$, and the last equality follows from (3.4). Since ε and M are independent of α_0, we have shown that α_0 is bounded from above, and thus $\tilde{\mathcal{A}}$ is bounded.

Secondly, we deal with the continuity of $\tilde{G}$. For arbitrary two elements α_1 and α_2 in $\tilde{\mathcal{A}}$, we have

$$\begin{aligned} \tilde{G}(\alpha_1) - \tilde{G}(\alpha_2) &= \alpha_1^*\left[\kappa(\alpha_1)\right] + \left[\kappa(\alpha_1)\right]^*\alpha_1 - \left\{\alpha_2^*\left[\kappa(\alpha_2)\right] + \left[\kappa(\alpha_2)\right]^*\alpha_2\right\}, \\ &= (\alpha_1-\alpha_2)^*\kappa(\alpha_1) + \alpha_2^*\kappa(\alpha_1-\alpha_2) \\ &\quad + \left[\kappa(\alpha_1)\right]^*(\alpha_1-\alpha_2) + \left[\kappa(\alpha_1-\alpha_2)\right]^*\alpha_2. \end{aligned}$$

Therefore, since the linear map κ is bounded and the set $\tilde{\mathcal{A}}$ is bounded, it is easy to show that there exists a constant $C > 0$ satisfying

$$\|\tilde{G}(\alpha_1) - \tilde{G}(\alpha_2)\| \le C\|\alpha_1 - \alpha_2\|.$$

Since α_1 and α_2 are arbitrary in $\tilde{\mathcal{A}}$, we have proven the continuity of $\tilde{G}$ in $\tilde{\mathcal{A}}$.

Finally the closedness of $\tilde{\mathcal{A}}$ follows immediately by using the property of continuity and bijectivity of $\tilde{G}$ and the closedness of $\tilde{\mathcal{D}}$ [16, page xx]. This completes the proof. ☐

Proof of Theorem 3.1. The map G can be written as a composite of three maps:

$$G = M \circ \tilde{G}_{ex} \circ N,$$

where

$$\begin{aligned} &N : \mathcal{A} \mapsto \tilde{\mathcal{A}} \times \mathbb{R}^+ : && N(\alpha) := \begin{bmatrix} \dfrac{\alpha}{(\langle\alpha,\kappa(\alpha)\rangle)^{1/2}} \\ \langle\alpha,\kappa(\alpha)\rangle \end{bmatrix}, \\ &\tilde{G}_{ex} : \tilde{\mathcal{A}} \times \mathbb{R}^+ \mapsto \tilde{\mathcal{D}} \times \mathbb{R}^+ : && \tilde{G}(\alpha, r) := \begin{bmatrix} \tilde{G}(\alpha) \\ r \end{bmatrix}, \\ &M : \tilde{\mathcal{D}} \times \mathbb{R}^+ \mapsto \mathcal{D} : && M(d, r) := rd. \end{aligned}$$

The maps N and M play roles of normalization and inverse scaling, respectively. Due to Lemma 3.1, the map $\tilde{G}_{ex}$ is homeomorphic. Since the maps N and M are also homeomorphic, so is the composite map G. $\square$

Now we phrase a smoothness result from the theory developed for strictly positive real interpolants. Here we consider the subsets of $\mathcal{A}$ and $\mathcal{D}$ as:

$$\begin{aligned} \mathcal{A}^+ &:= \{\alpha \in \mathcal{A} : \kappa(\alpha)/\alpha \text{ is strictly positive real}\}, \\ \mathcal{D}^+ &:= \{d \in \mathcal{D} : d(z, z^{-1}) > 0 \ \forall |z| = 1\}. \end{aligned}$$

The following theorem was proven by Byrnes and Lindquist:

Theorem 3.2. [6] *The map $G^+ : \mathcal{A}^+ \to \mathcal{D}^+$, defined by $G^+(\alpha) := G(\alpha)$ for all $\alpha \in \mathcal{A}^+$, is a diffeomorphism.*

4. Derivation of a system of nonlinear equations

Theorem 2.1 was proved in [12], but the proof does not offer any means to construct interpolants. In engineering applications, we are interested in actually computing the polynomials α and β in the theorem, by specifying $d(z, z^{-1})$ in $\mathcal{D}$. This generates a set of *design parameters*; in robust control they can be tuning parameters to meet certain specifications and in spectral estimation they can incorporate *a priori* knowledge about the process. In the case when we specify $d(z, z^{-1})$ which is strictly positive on the unit circle, the corresponding pair (α, β) yields a *strictly* positive real interpolant. An algorithm to find such interpolant has been developed based on convex optimization in [3, 4, 5] and modified in [2, 9, 15] with a continuation method to avoid numerical problems.

Although the modified algorithm successfully works to compute any *strictly* positive real function with a degree bound, it is still important to devise a method to generate any *positive* real function. There are two reasons for this. One reason is that there are situations where we need to compute such functions. The other reason is that we can expect a more reliable procedure than the previous one, in the sense that the accuracy of the solution does not decrease even the solution has poles close to the unit circle. To devise a method for computing any positive real interpolant, we will next translate the above theorem into a form using vectors.

Here, we will rewrite the conditions **C1**–**C4** in terms of coefficient vectors of polynomials α and β.

First, due to the condition **C1**, we can parameterize the polynomials α and β as

$$\alpha(z) := \boldsymbol{\alpha}^T \boldsymbol{z}, \quad \beta(z) := \boldsymbol{\beta}^T \boldsymbol{z},$$

where the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{z}$ are defined by

$$\begin{aligned} \boldsymbol{\alpha} &:= \begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_n \end{bmatrix}^T, \quad \alpha_0 > 0, \\ \boldsymbol{\beta} &:= \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_n \end{bmatrix}^T, \\ \boldsymbol{z} &:= \begin{bmatrix} 1 & z & \cdots & z^n \end{bmatrix}^T. \end{aligned}$$

Next, due to the condition **C2**, we have a linear relation between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as

$$\boldsymbol{\beta} = K \boldsymbol{\alpha}, \tag{4.1}$$

where

$$K := \Gamma^{-1} W \Gamma \tag{4.2}$$

with

$$\Gamma := \begin{bmatrix} 1 & z_0 & \cdots & z_0^n \\ 1 & z_1 & \cdots & z_1^n \\ \vdots & \vdots & & \vdots \\ 1 & z_n & \cdots & z_n^n \end{bmatrix}, \quad W = \begin{bmatrix} w_0 & & & \\ & w_1 & & \\ & & \ddots & \\ & & & w_n \end{bmatrix}. \tag{4.3}$$

Note that the Vandermonde matrix Γ is nonsingular because of the distinct assumption of $\{z_j\}$. To express the condition **C3** in terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we introduce a Hankel + Toeplitz operator defined for a vector.

Definition 4.1. For a vector $\boldsymbol{v} := \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix}^T$, a linear continuous operator $S : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is defined by

$$S(\boldsymbol{v}) := \begin{bmatrix} v_1 & \cdots & v_m \\ \vdots & \iddots & \\ v_m & & \end{bmatrix} + \begin{bmatrix} v_1 & \cdots & v_m \\ & \ddots & \vdots \\ & & v_1 \end{bmatrix}.$$

Remark 4.1. The continuity of the linear operator is due to [17, Lemma 1.20].

With this operator, the relation (2.3) can be written as

$$S(\boldsymbol{\alpha})\boldsymbol{\beta} = \boldsymbol{d},$$

where $\boldsymbol{d} := \begin{bmatrix} 2 & d_1 & \cdots & d_n \end{bmatrix}^T$. The condition **C4** is described by means of the parameter vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as

$$\boldsymbol{\alpha} + \boldsymbol{\beta} \in \mathcal{S},$$

where the Schur stability region $\mathcal{S}$ in $\mathbb{R}^{n+1}$ is defined by

$$\mathcal{S} := \left\{ \boldsymbol{v} \in \mathbb{R}^{n+1} : \ \boldsymbol{v}^T \boldsymbol{z} \neq 0, \forall z \in \mathbb{D} \right\}$$

From the vector expressions of the conditions **C1**–**C4**, Theorem 2.1 can be rephrased as follows.

Corollary 4.1. *For each vector* $\boldsymbol{d}$ *in the region*

$$\hat{\mathcal{D}} := \left\{ \boldsymbol{d} \in \mathbb{R}^{n+1} : \sum_{j=0}^{n} d_j (z^j + z^{-j}) \in \mathcal{D} \right\}, \tag{4.4}$$

the system of nonlinear equations

$$\boldsymbol{g}(\boldsymbol{\alpha}) := S(\boldsymbol{\alpha}) K \boldsymbol{\alpha} - \boldsymbol{d} = 0 \tag{4.5}$$

has a unique solution in the region

$$\hat{\mathcal{A}} := \left\{ \boldsymbol{\alpha} \in \mathbb{R}^{n+1} : \ \alpha_0 > 0, \ (I + K)\boldsymbol{\alpha} \in \mathcal{S} \right\}. \tag{4.6}$$

The remaining issue is, for each specified vector $\boldsymbol{d}$ in $\hat{\mathcal{D}}$, to find the unique solution to the system of nonlinear equations (4.5) in $\hat{\mathcal{A}}$. However, because of the nonlinearity of $\boldsymbol{g}$, this is not a trivial issue, and we need in general to rely on some numerical methods. In order to find the unique solution, we shall use a continuation method, which is one of the standard numerical tools for solving a system of nonlinear equations. The performance of the continuation method crucially depends on properties of the map G in (2.5), which will be discussed in the next section.

5. An algorithm based on a continuation method

A continuation method is a standard tool to solve a system of nonlinear equations [1]. The basic idea is to gradually deform the system of equations from an easy one to the one of interest. In the following, we will introduce a homotopy that defines a trajectory to be traced numerically, discuss the properties of the trajectory crucial for the use of the continuation method, and explain the numerical method of tracing the trajectory, that is, the predictor-corrector steps.

5.1. **Construction of a homotopy.** To construct a homotopy, we notice that if $\boldsymbol{d}$ is chosen as $\boldsymbol{\tau} := [2\tau_0, \tau_1, \cdots, \tau_n]^T \in \hat{\mathcal{D}}$ which consists of coefficients of the trigonometric polynomial

$$\sum_{j=0}^{n} \tau_j (z^j + z^{-j}) := \frac{1}{M} \prod_{j=0}^{n} (1 - z_j z)(1 - z_j z)^*,$$

where M is a scaling to make $\tau_0 = 1$, then the corresponding system of nonlinear equation

$$\boldsymbol{g}_\tau(\boldsymbol{\alpha}) := S(\boldsymbol{\alpha}) K \boldsymbol{\alpha} - \boldsymbol{\tau} = 0 \tag{5.1}$$

is easy to solve. In fact, the system becomes linear in this case, and the solution is generally called the *central solution* (see [3, Appendix A]).

We design a convex homotopy $\boldsymbol{h} : \mathbb{R}^{n+1} \times [0,1] \to \mathbb{R}^{n+1}$ as

$$\begin{aligned} \boldsymbol{h}(\boldsymbol{\alpha}, \nu) &:= (1-\nu)\boldsymbol{g}_\tau(\boldsymbol{\alpha}) + \nu \boldsymbol{g}(\boldsymbol{\alpha}), \\ &= S(\boldsymbol{\alpha}) K \boldsymbol{\alpha} - \boldsymbol{\tau} + \nu (\boldsymbol{\tau} - \boldsymbol{d}), \quad \nu \in [0,1]. \end{aligned}$$

Note that $\boldsymbol{h}(\boldsymbol{\alpha}, 0) = \boldsymbol{g}_\tau(\boldsymbol{\alpha})$ and $\boldsymbol{h}(\boldsymbol{\alpha}, 1) = \boldsymbol{g}(\boldsymbol{\alpha})$. Therefore, $\boldsymbol{h}(\boldsymbol{\alpha}, 0) = 0$ is easy to solve, while $\boldsymbol{h}(\boldsymbol{\alpha}, 1) = 0$ is our problem at hand.

For each $\nu \in [0,1]$, the system $\boldsymbol{h}(\boldsymbol{\alpha}, \nu) = 0$ has a unique solution in $\hat{\mathcal{A}}$, due to Corollary 4.1 and the convexity of $\hat{\mathcal{D}}$. Let us denote the unique solution of the system $\boldsymbol{h}(\boldsymbol{\alpha}, \nu) = 0$ as $\hat{\boldsymbol{\alpha}}(\nu)$, and we call the class $\{\hat{\boldsymbol{\alpha}}(\nu)\}_{\nu=0}^{1}$ the *trajectory.* Our objective is to trace this implicitly defined trajectory numerically from $\nu = 0$ to $\nu = 1$, and to obtain $\hat{\boldsymbol{\alpha}}(1)$. For this purpose, we use a numerical continuation method with predictor-corrector steps. Before proceeding the exposition of the predictor-corrector steps, we shall analyze the properties of the trajectory.

5.2. **Properties of the trajectory.** To apply a continuation method to a trajectory tracing, the trajectory should enjoy some favorable properties, since otherwise, the method is likely to break down or end up with a wrong solution. For example, the trajectory should have sufficient smoothness (such as continuity and differentiability), but neither a bifurcation nor a turning point (see [1]). We shall next study these properties for the trajectory $\{\hat{\boldsymbol{\alpha}}(\nu)\}_{\nu=0}^{1}$.

First, due to Lemma 3.1 and Theorem 3.2, the property of $\{\hat{\boldsymbol{\alpha}}(\nu)\}_{\nu=0}^{1}$ concerning the smoothness follows immediately.

Proposition 5.1. *The trajectory* $\{\hat{\boldsymbol{\alpha}}(\nu)\}_{\nu=0}^{1}$ *is continuously differentiable in the interval* $[0,1)$. *In addition, it is continuous at* $\nu = 1$.

A direct consequence of Proposition 5.1 is the following.

Corollary 5.1. *The trajectory* $\{\hat{\boldsymbol{\alpha}}(\nu)\}_{\nu=0}^{1}$ *does not have any turning point in* $[0,1)$.

Because of the uniqueness of $\hat{\boldsymbol{\alpha}}(\nu)$ for each $\nu \in [0,1]$, we also have that

Corollary 5.2. *The trajectory* $\{\hat{\boldsymbol{\alpha}}(\nu)\}_{\nu=0}^{1}$ *does not have any bifurcation.*

The three properties above justify the use of a continuation method to trace the trajectory numerically.

Owing to the continuous differentiability of the trajectory, we can express the trajectory as a solution of an ordinary differential equation with an initial value. When we take a derivative of $\boldsymbol{h}(\hat{\boldsymbol{\alpha}}(\nu),\nu)=0$ with respect to ν, we have

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{\alpha}}(\hat{\boldsymbol{\alpha}}(\nu),\nu)\cdot\frac{d\hat{\boldsymbol{\alpha}}}{d\nu}(\nu)+\frac{\partial \boldsymbol{h}}{\partial \nu}(\hat{\boldsymbol{\alpha}}(\nu),\nu)=0,$$

or equivalently,

$$\frac{d\hat{\boldsymbol{\alpha}}}{d\nu}(\nu)=\boldsymbol{v}(\hat{\boldsymbol{\alpha}}(\nu)),\quad \boldsymbol{v}(\hat{\boldsymbol{\alpha}}(\nu)):=-\left[\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{\alpha}}(\boldsymbol{\alpha},\nu)\right]^{-1}\frac{\partial \boldsymbol{h}}{\partial \nu}(\boldsymbol{\alpha},\nu)\Bigg|_{\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}(\nu)}.$$

The invertibility of the Jacobian $\partial\boldsymbol{h}/\partial\boldsymbol{\alpha}$ is guaranteed on the trajectory in the interval $[0,1)$ because of the differentiability of $\hat{\boldsymbol{\alpha}}$. Since we can easily compute $\hat{\boldsymbol{\alpha}}(0)$, the problem to solve is the ordinary differential equation with an initial value:

$$\left\{\begin{array}{lcl}\dfrac{d\hat{\boldsymbol{\alpha}}}{d\nu}(\nu) & = & \boldsymbol{v}(\hat{\boldsymbol{\alpha}}(\nu)),\\ \hat{\boldsymbol{\alpha}}(0) & : & \text{given.}\end{array}\right.$$

To solve this initial value problem numerically, we use predictor-corrector steps.

5.3. **The predictor step.** In the predictor step, given a point $\hat{\boldsymbol{\alpha}}(\nu)$ on the trajectory, we move the point to a new point as

$$\boldsymbol{\alpha}(\nu+\delta\nu):=\hat{\boldsymbol{\alpha}}(\nu)+\delta\nu\cdot\boldsymbol{v}(\hat{\boldsymbol{\alpha}}(\nu)).$$

This point may not be on the trajectory, and hence we use the notation $\boldsymbol{\alpha}(\nu+\delta\nu)$ instead of $\hat{\boldsymbol{\alpha}}(\nu+\delta\nu)$. To determine the new point $\boldsymbol{\alpha}(\nu+\delta\nu)$, we need to compute $\boldsymbol{v}(\hat{\boldsymbol{\alpha}}(\nu))$ and to provide $\delta\nu$.

The directional vector $\boldsymbol{v}(\hat{\boldsymbol{\alpha}}(\nu))$ consists of two factors: one is the inverse of $\partial\boldsymbol{h}/\partial\boldsymbol{\alpha}$, which is the Jacobian of $\boldsymbol{g}$ and can be written explicitly as

$$\nabla\boldsymbol{g}(\boldsymbol{\alpha})=S(\boldsymbol{\alpha})K+\sum_{j=0}^{n}\alpha_j S(\boldsymbol{k}_j), \tag{5.2}$$

where $\boldsymbol{k}_j$ is the $(j+1)$-th column of the matrix K in (4.1). Note that, due to the differentiability of $\hat{\boldsymbol{\alpha}}$, the Jacobian is nonsingular for all ν in the interval $[0,1)$. The other factor is

$$\frac{\partial \boldsymbol{h}}{\partial \nu}=\boldsymbol{\tau}-\boldsymbol{d},$$

which is independent of both $\boldsymbol{\alpha}$ and ν, and can thus be computed off-line.

The step size must be chosen in a careful way. This is because a small step size will cause a long time to arrive at $\nu=1$ and to obtain $\hat{\boldsymbol{\alpha}}(1)$, while a large step size might ruin the convergence rate of corrector step that follows the predictor step. Next, we propose a reasonable way to make this trade-off.

On the trajectory, we have $\boldsymbol{h}(\hat{\boldsymbol{\alpha}}(\nu),\nu)=0$, and in particular,

$$e_1^T\boldsymbol{h}(\hat{\boldsymbol{\alpha}}(\nu),\nu)=e_1^TS(\hat{\boldsymbol{\alpha}}(\nu))K\hat{\boldsymbol{\alpha}}(\nu)-e_1^T\boldsymbol{\tau}=2\hat{\boldsymbol{\alpha}}(\nu)^TK\hat{\boldsymbol{\alpha}}(\nu)-2=0.$$

One of the criterion for not deviating too far from the trajectory is to require

$$1-\mu\le\boldsymbol{\alpha}(\nu+\delta\nu)^TK\boldsymbol{\alpha}(\nu+\delta\nu)\le 1+\mu,$$

for some small number $\mu > 0$. For a given direction $\boldsymbol{v}(\hat{\boldsymbol{\alpha}}(\nu))$, we can compute the maximal step length $\delta\nu_*$ as

$$\begin{aligned}
(\boldsymbol{\alpha} + \delta\nu_* \boldsymbol{v}(\hat{\boldsymbol{\alpha}}))^T K (\boldsymbol{\alpha} + \delta\nu_* \boldsymbol{v}(\hat{\boldsymbol{\alpha}})) &= 1 \pm \mu, \\
\delta\nu_*^2 \boldsymbol{v}(\hat{\boldsymbol{\alpha}})^T K \boldsymbol{v}(\hat{\boldsymbol{\alpha}}) + \delta\nu_* (\boldsymbol{v}(\hat{\boldsymbol{\alpha}})^T K \boldsymbol{\alpha} + \boldsymbol{\alpha}^T K \boldsymbol{v}(\hat{\boldsymbol{\alpha}})) \pm \mu &= 0, \\
\delta\nu_*^2 + \delta\nu_* \underbrace{\frac{\boldsymbol{v}(\hat{\boldsymbol{\alpha}})^T K \boldsymbol{\alpha} + \boldsymbol{\alpha}^T K \boldsymbol{v}(\hat{\boldsymbol{\alpha}})}{\delta\boldsymbol{\alpha}^T K \boldsymbol{v}(\hat{\boldsymbol{\alpha}})}}_{=:p} \pm \underbrace{\frac{\mu}{\delta\boldsymbol{\alpha}^T K \boldsymbol{v}(\hat{\boldsymbol{\alpha}})}}_{=:q} &= 0.
\end{aligned}$$

We pick the smallest positive solution:

$$h = \begin{cases} -\frac{p}{2} - \sqrt{\frac{p^2}{4} - |q|} & \text{if } p < 0 \ \& \ \frac{p^2}{4} > |q|, \\ -\frac{p}{2} + \sqrt{\frac{p^2}{4} + |q|} & \text{otherwise.} \end{cases}$$

5.4. **The corrector step.** In the corrector step, given a point $\boldsymbol{\alpha}(\nu+\delta\nu)$ which is obtained in the predictor step, we pull the point back to the trajectory by fixing ν-value, and obtain $\hat{\boldsymbol{\alpha}}(\nu + \delta\nu)$. This is equivalent to solving the system of nonlinear equations

$$\tilde{\boldsymbol{h}}(\boldsymbol{\alpha}) := \boldsymbol{h}(\boldsymbol{\alpha}, \nu + \delta\nu) = 0.$$

We use Newton's method with an initial point $\boldsymbol{\alpha}(\nu+\delta\nu)$ to solve this system numerically. The Newton's method uses iterations

$$\boldsymbol{\alpha}_{k+1} = \boldsymbol{\alpha}_k - \nabla\tilde{\boldsymbol{h}}(\boldsymbol{\alpha}_k)^{-1} \cdot \tilde{\boldsymbol{h}}(\boldsymbol{\alpha}_k), \quad k = 0, 1, 2, \ldots.$$

The Jacobian $\nabla\tilde{\boldsymbol{h}}(\boldsymbol{\alpha}_k)$ is nonsingular if $\boldsymbol{\alpha}_k$ is close enough to the trajectory, due to the continuity of the Jacobian.

6. Illustrating Examples

In this section, we will consider two examples which indicate the potential of the new procedure for computing rational Nevanlinna-Pick interpolants. Both examples are of a quite academic character. The first example illustrates the ability of the procedure to compute interpolants with spectral zeros on the unit circle as well as the continuity of the map G in (2.5). The second example illustrates that, for appropriately chosen trigonometric polynomial, the solution is robust against errors in the interpolation data. Such robustness is an important feature in applications.

Example 6.1. Consider the positive real function

$$f(z) = 1 + \frac{1 + e^{i\theta_0} z}{1 - e^{i\theta_0} z} + \frac{1 + e^{-i\theta_0} z}{1 - e^{-i\theta_0} z}.$$

The associated spectral density $\Phi = 2\,\mathrm{Re}\, f$ corresponds to a sinusoid in (Gaussian) white noise. Let us take $\theta_0 = 0.5$ and interpolation points at 0 and $0.5e^{\pm i\theta_0}$. We can then compute the interpolation data $\{w_k\}$. Now, for various choices of trigonometric polynomial $d(z, z^{-1})$, we will recover corresponding positive real functions $f(z)$. Associated with each interpolant is the spectral factor $w(z)w^*(z) := f(z) + f^*(z)$. The zeros of w will also be the zeros of d. Now we choose a family of these zeros converging to the true one at $e^{i\theta_0}$. In Figure 1, the poles and the (chosen) zeros of the spectral factor are plotted. We see that as the spectral zeros approach the unit circle, the poles approach the same points. When the zero is chosen to be exactly $e^{i\theta_0}$, we get a cancellation in the spectral factor and we have recovered the true function f. In Figure 2, the corresponding spectral densities Φ

are plotted. We note that densities gradually tend to the singular density corresponding to the true f.

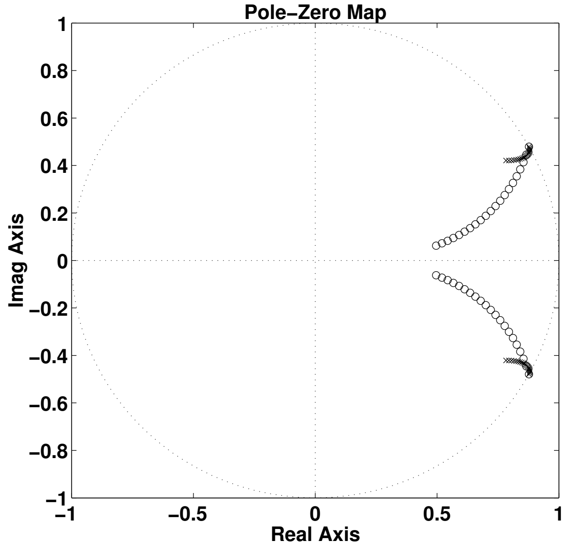

FIGURE 1. The given spectral zeros (o) and corresponding poles (x) of the positive real functions consistent with the interpolation data in Example 6.1

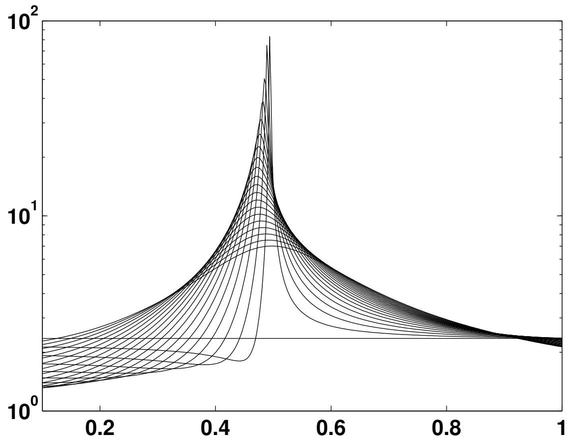

FIGURE 2. The spectral densities associated with the interpolants in Example 6.1

Example 6.2. In this example, we try to identify the positive real function of degree four:

$$f(z) = 1 + \frac{1+e^{i\theta_0}z}{1-e^{i\theta_0}z} + \frac{1+e^{-i\theta_0}z}{1-e^{-i\theta_0}z} + \frac{1+e^{i\theta_1}z}{1-e^{i\theta_1}z} + \frac{1+e^{-i\theta_1}z}{1-e^{-i\theta_1}z}.$$

Compared to Example 6.1, we have added another sinusoid. Here we are interested in studying the effect of error in the interpolation data. Therefore, we fixed the trigonometric polynomial d to be the true one, that is with zeros on the boundary. Let us take $\theta_0 = 0.5$ and $\theta_1 = 0.8$, and the interpolation points at 0, $0.5e^{\pm i\theta_0}$ and $0.5e^{\pm i\theta_1}$. Then, we compute the interpolation data and afterwards we perturb the interpolation points to $0, 0.5e^{\pm i\theta_0} \pm i\varepsilon, 0.5e^{\pm i\theta_1} + \varepsilon$. Now, we sweep ε from -0.005 to 0.005. Within this range, the Pick matrix turns out to be positive definite, and therefore, we can solve interpolation problems. In Figure 3, the spectral densities associated with the interpolants are plotted for a family of errors $\{\varepsilon\}$. The densities again seem to approach to the true one with pole/zero cancellation as the errors go to zero.

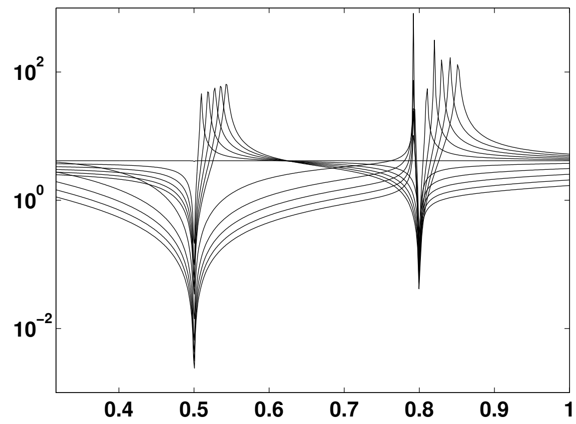

FIGURE 3. The spectral densities associated with the interpolants in Example 6.2

7. Conclusions

In this paper, we have developed a procedure to compute any real rational, positive real Nevanlinna-Pick interpolant with a degree bound. We have shown that the problem of finding such an interpolant from each specified nonnegative trigonometric polynomial boils down to solving a system of nonlinear equations. Since the existence and the uniqueness of the solution to the system in a nonconvex region were proved by Georgiou, the remaining problem was to devise a method to find the unique solution to the system. We have adopted one of the standard methods, namely a continuation method, to solve the system numerically. The reliability of the proposed method was demonstrated through two numerical examples. We emphasize that the procedure proposed in this paper is not just an extension of the previously developed solver based on convex optimization approach.

Even though we can see experimentally that the developed procedure converges well, a theoretical proof of the convergence is missing. Besides, real-world engineering problems, for instance the spectral estimation problem, should be examined by utilizing the proposed procedure. Furthermore, it will be important to extend this procedure to the multivariable cases, that is, to both matrix-valued and tangential Nevanlinna-Pick interpolation problems. These will be future research subjects. As far as the matrix-valued interpolation problem is concerned, we have the following conjecture.

Conjecture: The map G: $\mathcal{A} \mapsto \mathcal{D}$, defined by

$$G([A]) := [\kappa(A)]^* A + A^* [\kappa(A)], \tag{7.1}$$

is homeomorphism, where

$$
\begin{aligned}
\mathcal{A} \;&:=\; \{[A]\}\,, \\
&[A] := \left\{ \tilde{A} \in \mathcal{A}_{all} : \begin{array}{l} \tilde{A}(z) = A(z)U \text{ for some orthogonal matrix } U \in \mathbb{R}^{\ell\times\ell} \\ \left([\kappa(A)]^* A + A^* [\kappa(A)]\right) U = U \left([\kappa(A)]^* A + A^* [\kappa(A)]\right) \end{array} \right\} \\
&\mathcal{A}_{all} := \left\{ A : \begin{array}{l} A(z) = \sum_{j=0}^n A_j z^j, \; A_j \in \mathbb{R}^{\ell\times\ell} \\ \kappa(A)A^{-1} \text{ is positive real} \end{array} \right\} \\
\mathcal{D} \;&:=\; \left\{ D : \begin{array}{l} D(z) = \sum_{j=0}^n (D_j z^j + D_j^T z^{-j}) \geq 0, \; \forall |z| = 1 \\ D_j \in \mathbb{R}^{\ell\times\ell}, \; D_0 + D_0^T > 0 \end{array} \right\}.
\end{aligned}
$$

Note that the image of G does not depend on a particular choice of the element in the equivalent class $[A]$.

Appendix A. Derivative interpolations

Even in the case where interpolation constraints include derivative constraints, all the results stated in this paper still hold. The only difference appears in the construction of K, which is explained here.

Suppose that the interpolation constraints are

$$f^{(k)}(z_{jk}) = w_{jk}, \quad j = 0, 1, \ldots, n, \quad k = 0, 1, \ldots, m_j - 1, \tag{A.1}$$

where $f^{(k)}$ means the k-th derivative of f, and m_j is called the *multiplicity of* z_j. Let the total number of interpolation condition be $N+1$, that is,

$$N + 1 := \sum_{j=0}^{n} m_j.$$

Then, the following holds.

Proposition A.1. *A rational function f of degree at most N satisfies the interpolation conditions (A.1) if and only if the coefficient vectors of the numerator polynomial and the denominator polynomial, denoted respectively by $\boldsymbol{\beta} := [\beta_0, \beta_1, \cdots, \beta_N]^T$ and $\boldsymbol{\alpha} := [\alpha_0, \alpha_1, \cdots, \alpha_N]^T$, has a linear relation:*

$$\boldsymbol{\beta} = \Gamma^{-1} W \Gamma \boldsymbol{\alpha}.$$

Here, the matrix Γ is a controllability matrix

$$\Gamma := \begin{bmatrix} b & Ab & \cdots & A^N b \end{bmatrix}, \tag{A.2}$$

with the controllable pair (A, b). The block-diagonal matrix $A :=$ blockdiag $\begin{bmatrix} A_0 & A_1 & \cdots & A_n \end{bmatrix}$ has each block of size $m_j \times m_j$

$$A_j := \begin{bmatrix} z_j & & & \\ 1 & z_j & & \\ & \ddots & \ddots & \\ & & 1 & z_j \end{bmatrix}.$$

The column vector b is given as

$$b = \begin{bmatrix} e_0^T & e_1^T & \cdots & e_n^T \end{bmatrix}^T,$$

where the vector $e_j := \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ is of size $m_j \times 1$. The block-diagonal matrix

$$W := \text{blockdiag} \begin{bmatrix} W_0 & W_1 & \cdots & W_n \end{bmatrix}$$

has each block of size $m_j \times m_j$ as

$$W_j := \begin{bmatrix} w_{j0} & & & \\ w_{j1} & w_{j0} & & \\ \vdots & \ddots & \ddots & \\ w_{jm_j} & \cdots & w_{j1} & w_{j0} \end{bmatrix}, \quad j = 0, 1, \ldots, n.$$

The proof is straightforward, and hence omitted here. It is easy to verify that the matrices Γ and W in (4.3) are special cases of Γ in (A.2) and W in (A.3).

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