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LQG BALANCING FOR DISCRETE-TIME INFINITE-DIMENSIONAL LINEAR SYSTEMS

MARK R. OPMEER* AND RUTH F. CURTAIN †

Abstract. In this paper, we study the existence of linear quadratic gaussian (LQG) -balanced realizations for discrete-time infinite dimensional systems. LQG-balanced realizations are those for which the smallest nonnegative selfadjoint solutions of the control and filter Riccati equations are equal. We show that the control (filter) Riccati equation has a nonnegative selfadjoint solution if and only if the system is output (input) stabilizable. Our main result is that the transfer function of a discrete-time linear system has an approximately controllable and observable LQG-balanced realization iff it has an input and output stabilizable realization. The corresponding control and filter Riccati equations have unique nonnegative selfadjoint solutions. Moreover, approximately controllable and observable LQG-balanced realizations are unique up to a unitary state-space transformation. Finally, we show that the spectrum of the product of the smallest nonnegative selfadjoint solutions of the control and filter Riccati equations is independent of the particular realization.

Key words. balanced realization, discrete-time system, infinite-dimensional system, LQG-balanced realization, normalized factorization, Riccati equations

AMS subject classifications. 47A48, 47N70, 93B28, 93C55

1. Introduction. Simple models are normally preferred over complex ones in control systems design. Sometimes it is obvious how to construct a simple model for a physical system, but sometimes it is not obvious what the characteristics essential to the controller design of a physical system are. One way of obtaining a simple model in this last case is to first obtain a sophisticated model that takes every aspect that could be of interest into account and then perform model reduction on this sophisticated model. A simple model reduction procedure was introduced by Moore [10] and is now a textbook subject (see e.g. Zhou and Doyle [22], Chapter 7). The method proposed by Moore consists of truncating a balanced realization. A balanced realization (also called Lyapunov- or internally balanced) is a realization for which the controllability and observability gramians are equal and diagonal. This procedure is only applicable to stable systems. Alternatively for unstable systems one can use truncations of a LQG-balanced realization, which for rational transfer functions always exists. A LQG-balanced realization is a realization for which the smallest nonnegative selfadjoint solutions of the standard LQG control and filter Riccati equations are equal and diagonal. This method was proposed by Verriest [17], [18] and further developed by Jonckheere and Silverman [7]. For an alternative treatment see Mustafa and Glover [11]. The discrete-time case was considered in Hoffmann et al. [6].

In the case that the system is infinite-dimensional, the model/controller approximation becomes essential. One would like to use the methods of balanced truncation and LQG-balanced truncation in this case too. The existence of Lyapunov-balanced and LQG-balanced realizations for irrational transfer functions is however nontrivial. Sufficient conditions for the existence of Lyapunov-balanced realizations were proven by Young [20], [21]. The purpose of this article is to give necessary and sufficient conditions for the existence of LQG-balanced realizations for discrete-time infinite-dimensional systems.

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The proof is based on the correspondence between the Riccati equations of the plant and the Lyapunov equations of a certain closed-loop system and the result of Young on the existence of Lyapunov-balanced realizations. Although discrete-time systems have bounded operators, there are a number of features that make the infinite-dimensional case more complicated than the finite-dimensional case. One is that input and output stabilizability, the natural infinite-dimensional generalizations of stabilizability and detectability, are not sufficient to obtain unique solutions of the control Riccati equation. Another is that the natural factorization generated by the closed-loop system need not be coprime. These uniqueness and coprime properties were key features in the finite-dimensional proofs. Consequently, we have been forced to develop different proofs, predominantly algebraic in nature, to get around these complications. We exploit the factorization idea previously used by Meyer and Franklin [9] (see also Ober and McFarlane [12]) for the finite-dimensional continuous-time case.

This article is organized as follows. We begin by reviewing the known finite-dimensional theory on discrete-time LQG-balancing in Section 2. In Section 3 we review the relevant theory of discrete-time infinite-dimensional systems, and in particular the linear quadratic regulator problem for this class of systems. Some of our results appear to be new. The previous standard results (e.g. Halanay and Ionescu [5]) assume a type of “exponential” stabilizability that is too strong for our purposes. In Section 4 we review the relevant theory on Lyapunov-balanced realizations for discrete-time infinite-dimensional systems. The key result on the connection between normalized factorizations and the linear quadratic regulator theory is developed in Section 5. In Section 6 we define the LQG-characteristic values and show that they are system invariants. In Section 7 we derive many algebraic relations between the solutions of the control and filter Riccati equations and the Lyapunov equations of the closed-loop system. The often tedious algebraic proofs are relegated to the Appendix. Finally, all of the results from the previous sections are linked up in Section 8 to prove our main result: an input and output stabilizable discrete-time system possesses an approximately controllable and observable LQG-balanced realization. These realizations are unique up to a unitary state-space transformation. This represents an elegant generalization of the finite-dimensional theory under minimal assumptions.

2. LQG-balanced realizations: the finite-dimensional case. In this section we review some of the results on finite-dimensional LQG-balanced realizations. We consider systems of the form

$$(2.1) \quad x_{n+1} = Ax_n + Bu_n \quad x(0) = x_0 \quad y_n = Cx_n + Du_n,$$

where A, B, C, D are matrices of compatible dimensions. For simplicity we consider the linear quadratic regulator (LQR) problem for the cost functional

$$J(x_0, u) := \sum_{n=0}^{\infty} \|u_n\|^2 + \|y_n\|^2,$$

where y is given in terms of x_0 and u by (2.1). The LQR problem consists of finding for a given x_0 that u for which $J(x_0, u)$ is minimal. As is well-known, this problem has a unique solution when (A, B, C, D) is minimal: the optimal input u^{\min} is given by the state feedback $u_n^{\min} = -(I + D^*D + B^*QB)^{-1}(D^*C + B^*QA)x_n$, where Q is the unique nonnegative solution of the Riccati equation

$$A^*QA - Q + C^*C = (C^*D + A^*QB)(I + D^*D + B^*QB)^{-1}(D^*C + B^*QA),$$

and the optimal cost is given by $J(x_0, u^{\min}) = \langle x_0, Qx_0 \rangle$. By duality the ‘optimal filter cost’ is given by $\langle x_0, Px_0 \rangle$, where P is the unique nonnegative solution of the Riccati equation

$$APA^* - P + BB^* = (BD^* + APC^*)(I + DD^* + CPC^*)^{-1}(DB^* + CPA^*).$$

The quantity $\langle x_0, Px_0 \rangle$ can be interpreted as a measure of the difficulty of reconstructing the initial state x_0 from noisy measurements. The eigenvalues of the product PQ are similarity invariants. It was shown by Fuhrmann and Ober [3] that the square-roots of the eigenvalues of PQ (called the *LQG-characteristic values*) are the singular values of a certain Hankel operator associated with the system. These invariants can be interpreted as a measure of how important the subspace generated by the eigenvector is for the compensator design. This can be seen from the LQG-balanced realization. A LQG-balanced realization is a realization of the plant such that $P = Q = \Lambda$, where Λ is the diagonal matrix containing the LQG-characteristic values. Let λ_i be the square root of an eigenvalue of PQ with eigenvector x_i of length one. Then, in the LQG-balanced realization, the optimal cost with initial condition x_i is λ_i and the difficulty of reconstructing this initial state from noisy measurements is also λ_i . The idea behind LQG-balanced truncation is to restrict the system to the subspace generated by the eigenvectors corresponding to the largest eigenvalues. Since this subspace is most important for compensator design, the system obtained by LQG-balanced truncation seems to be a reasonable approximation. There is a bound on the distance between a plant and a LQG-balanced truncation of the plant in terms of the discarded LQG-characteristic values, see Mustafa and Glover [11, Section 8.4.5].

The existence of LQG-balanced realizations in the finite-dimensional case is easily proven as follows.

1. Start with a minimal realization (A, B, C, D) and compute the solutions Q and P of the Riccati equations.
2. Write $P^{1/2}QP^{1/2} = U\Lambda^2U^*$, with Λ diagonal.
3. Let $T := Q^{1/2}U\Lambda^{-1/2}$.

Then it is easily seen that $(TAT^{-1}, TB, CT^{-1}, D)$ is a LQG-balanced realization.

In the infinite-dimensional case this proof no longer works. The main problem is that the singular value decomposition performed at step two cannot always be made in the infinite-dimensional case (one has to assume a compactness condition). Even if it can, then usually the singular values form a sequence with zero as limit point and the operator $\Lambda^{-1/2}$ mentioned at the third step is unbounded. Because of this unboundedness, it is unclear whether the expressions TAT^{-1} and TB make sense. To avoid the problems mentioned above we take a different approach. This approach was already mentioned in the introduction: we use the known result for the existence of Lyapunov-balanced realizations. In this article we only consider the *existence* of LQG-balanced realizations. The study of the properties of truncated LQG-balanced realizations will be done elsewhere. In slight contrast with the definition above we will call a realization *LQG-balanced* if $P = Q$. We do not require that they are ‘diagonal’, since this is not always possible in infinite dimensions.

3. Discrete-time infinite-dimensional systems. In this section we review that part of the theory of discrete-time infinite-dimensional systems that we need in this article. Discrete-time systems for infinite-dimensional systems have been treated in a number of texts (e.g. [1], [5], [16]). However, the standard treatments of the linear quadratic theory assume the strong concept of power stabilizability, i.e. the existence of an F such that $\|(A + BF)^n\| \leq M\lambda^n$ for some constants $M > 0, 0 < \lambda < 1$ and

all positive integers n . Unfortunately, this concept is not suitable for a nice theory of LQG-balanced realizations. So in this section we re-examine the basic concepts under weaker stabilizability assumptions.

A *discrete-time infinite-dimensional system* or simply a *system* is a quadruple of bounded operators $(A, B, C, D) \in \mathcal{L}(X) \times \mathcal{L}(U, X) \times \mathcal{L}(X, Y) \times \mathcal{L}(U, Y)$, where X, U, Y are separable Hilbert spaces. For an input u and initial condition x_0 the state x and output y of the system are defined by

$$(3.1) \quad x_{n+1} = Ax_n + Bu_n \quad x(0) = x_0 \quad y_n = Cx_n + Du_n.$$

The *observability map* \mathcal{C} of a discrete-time system (A, B, C, D) is defined by

$$(3.2) \quad (\mathcal{C}x)_i := CA^i x \quad i \in \mathbb{N}.$$

The discrete-time system (A, B, C, D) is said to be *approximately observable* if $\ker \mathcal{C} = \{0\}$ (see Curtain and Zwart [1]). There are many generalizations of the finite-dimensional concept of observability to an infinite dimensional setting and the concept of approximate observability is one of them. Our main use of approximate observability is that realizations may, without loss of generality, be assumed to be approximately controllable and observable. This may not always be the case for other generalizations of ‘observability’. Here approximately controllable and observable plays the role that ‘minimal’ plays in finite dimensions.

The discrete-time system (A, B, C, D) is said to be *output stable* if the image of \mathcal{C} is contained in $l^2(\mathbb{N}; Y)$. The *observability gramian* L_C of an output stable system is defined as $L_C := \mathcal{C}^* \mathcal{C}$. We now give an alternative characterization of output stability in terms of solutions of a certain Lyapunov equation.

LEMMA 3.1. *Let (A, B, C, D) be a discrete-time system. The following are equivalent statements.*

1. *The system is output stable,*
2. *the observation Lyapunov equation*

$$(3.3) \quad A^*LA - L + C^*C = 0$$

has a nonnegative selfadjoint solution.

If one (and hence both) of the above holds, then the observability gramian is the smallest nonnegative selfadjoint solution of the observation Lyapunov equation.

Proof. We first show the implication 1. implies 2.. It is easily seen that the observability gramian is given by the formula $L_C = \sum_{i=0}^{\infty} A^{*i} C^* C A^i$ and substituting this into the observation Lyapunov equation shows that it is a solution.

The implication 2. implies 1. is proven as follows. Suppose that the observation Lyapunov equation has a nonnegative selfadjoint solution L . Then multiplying (3.3) from the left with A^{*n} and from the right with A^n and summing from $n = 0$ to N gives

$$\begin{aligned} \sum_{n=0}^N A^{*n} C^* C A^n &= \sum_{n=0}^N A^{*n} L A^n - \sum_{n=0}^N A^{*n+1} L A^{n+1} \\ &= L - A^{*N+1} L A^{N+1} \leq L. \end{aligned}$$

Letting $N \rightarrow \infty$ shows that L_C is a bounded map and so \mathcal{C} is bounded. That L_C is smaller than any other nonnegative selfadjoint solution of the observation Lyapunov equation is obvious from the above inequality. \square

The following result shows that strong stability implies the uniqueness of solutions of Lyapunov equations. We remind the reader that an operator A is called strongly (or asymptotically) stable if for all $x \in X$ we have $A^n x \rightarrow 0$ as $n \rightarrow \infty$.

LEMMA 3.2. *Let (A, B, C, D) be output stable and let A be strongly stable. Then the observability gramian is the unique nonnegative selfadjoint solution of the observation Lyapunov equation (3.3).*

Proof. According to Lemma 3.1 the observability gramian is a nonnegative selfadjoint solution of the observation Lyapunov equation, so we only have to show that it is the unique nonnegative selfadjoint solution. Let L be a nonnegative selfadjoint solution of the observation Lyapunov equation. Then, as in the proof of Lemma 3.1 we have for all $N \in \mathbb{N}$

$$\sum_{n=0}^N A^{*n} C^* C A^n = \sum_{n=0}^N A^{*n} L A^n - \sum_{n=0}^N A^{*(n+1)} L A^{n+1} = L - A^{*(N+1)} L A^{N+1}.$$

We then have for all $x, y \in X$

$$\left\langle \sum_{n=0}^N A^{*n} C^* C A^n x, y \right\rangle = \langle Lx, y \rangle - \langle L A^{N+1} x, A^{N+1} y \rangle.$$

Letting $N \rightarrow \infty$ and using that A is strongly stable we have for all $x, y \in X$

$$\langle L_C x, y \rangle = \langle Lx, y \rangle.$$

This implies that $L = L_C$. Since L was an arbitrary nonnegative selfadjoint solution this implies that L_C is the unique nonnegative selfadjoint solution of the observation Lyapunov equation. \square

A discrete-time system (A, B, C, D) is called *output stabilizable* if there exists an $F \in \mathcal{L}(X, U)$ such that $(A + BF, 0, [F; C + DF], 0)$ is output stable. Output stabilizability is a necessary and sufficient condition for the solvability of the LQR problem. To show this we first review some well-known results on the LQR problem in infinite dimensions. For a system (A, B, C, D) with input, state and output related by (3.1) we consider the cost functional

$$J(x_0, u) := \sum_{n=0}^{\infty} \|u_n\|^2 + \|y_n\|^2.$$

The well-known *linear quadratic regulator problem* is: find a sequence u^{\min} such that $J(x_0, u^{\min}) \leq J(x_0, u)$ for all sequences u . A system is said to satisfy the *finite cost condition* if for every initial state x_0 there exists a u such that $J(x_0, u) < \infty$. Just as in the finite-dimensional case one can prove that if the finite cost condition is satisfied, then there exists a unique optimal control u^{\min} , we actually have $J(x_0, u^{\min}) < J(x_0, u)$ for all other sequences u and $J(x_0, u^{\min}) = \langle x_0, Qx_0 \rangle$, where Q is the smallest nonnegative selfadjoint solution of the Control Algebraic Riccati Equation (CARE) associated with the system (A, B, C, D)

$$(3.4) \quad A^*QA - Q + C^*C = (C^*D + A^*QB)(S + B^*QB)^{-1}(D^*C + B^*QA),$$

where $S := I + D^*D$. Moreover, u^{\min} can be given by a state feedback. All of this can be found for example in Curtain and Zwart [1, Exercise 6.34]. The operator Q

is called the *optimal cost operator*. The following lemma gives conditions that are equivalent to the finite cost condition.

LEMMA 3.3. *The following statements about a discrete-time system (A, B, C, D) are equivalent.*

1. *The discrete-time system is output stabilizable.*
2. *The discrete-time system satisfies the finite cost condition.*
3. *The CARE (3.4) of the discrete-time system has a nonnegative selfadjoint solution.*

Proof. Suppose the discrete-time system is output stabilizable. Then there exists an F such that $(A + BF, 0, [F; C + DF], 0)$ is output stable, denote the observability map of this system by \mathcal{C}_F . Add the equation $u := Fx$ to the equations (3.1). Call the solution (u, x, y) of this set of equations $\bar{u}, \bar{x}, \bar{y}$. Then $[\bar{u}, \bar{y}] = \mathcal{C}_F x_0$ and since \mathcal{C}_F is bounded we see that $[\bar{u}, \bar{y}]$ has finite l^2 norm. That is: $J(x_0, \bar{u}) < \infty$ and the system satisfies the finite cost condition.

An outline of the proof that 2. implies 3. can be found in Curtain and Zwart [1, Exercise 6.34].

For the implication 3. implies 1. we will show that the feedback $F := -(B^*QB + I + D^*D)^{-1}(D^*C + B^*QA)$, where Q is a solution of the CARE (3.4) is output stabilizing. We will do this by showing that Q is a solution of the observation Lyapunov equation of the system $(A + BF, 0, [F; C + DF], 0)$. So we want to show that Q satisfies

$$(A + BF)^*Q(A + BF) - Q + [F^*, C^* + F^*D^*] \begin{bmatrix} F \\ C + DF \end{bmatrix} = 0.$$

This is equivalent to

$$(3.5) \quad \begin{aligned} A^*QA - Q + C^*C + F^*(B^*QB + I + D^*D)F \\ + F^*(B^*QA + D^*C) + (A^*QB + C^*D)F = 0. \end{aligned}$$

Substituting for F in (3.5) we obtain

$$A^*QA - Q + C^*C = (C^*D + A^*QB)(S + B^*QB)^{-1}(D^*C + B^*QA),$$

which is precisely the CARE (3.4). \square

The *controllability map* \mathcal{B} of a discrete-time system (A, B, C, D) is defined for finitely nonzero U -valued sequences u by

$$(3.6) \quad \mathcal{B}u := \sum_{i=0}^{\infty} A^i B u_{-i-1}.$$

The discrete-time system (A, B, C, D) is said to be *approximately controllable* if $\ker \mathcal{B}^* = \{0\}$. The discrete-time system (A, B, C, D) is said to be *input stable* if \mathcal{B} extends to a bounded map from $l^2(\mathbb{Z}^-; U)$ to X . The *controllability gramian* L_B of an input stable system is defined as $L_B := \mathcal{B}\mathcal{B}^*$. A discrete-time system (A, B, C, D) is called *input stabilizable* if there exists an $L \in \mathcal{L}(Y, X)$ such that $(A + LC, [L, B + LD], 0, 0)$ is input stable. The following dual results of the results proven earlier hold.

LEMMA 3.4. *Let (A, B, C, D) be a discrete-time system. The following are equivalent statements.*

1. *The system is input stable,*

2. the control Lyapunov equation

$$(3.7) \quad ALA^* - L + BB^* = 0$$

has a nonnegative selfadjoint solution.

If one (and hence both) of the above holds, then the controllability gramian is the smallest nonnegative selfadjoint solution of the control Lyapunov equation.

LEMMA 3.5. *Let (A, B, C, D) be input stable and let A^* be strongly stable. Then the controllability gramian is the unique nonnegative selfadjoint solution of the control Lyapunov equation (3.7).*

LEMMA 3.6. *The following statements about a discrete-time system (A, B, C, D) are equivalent.*

1. *The discrete-time system is input stabilizable.*
2. *The dual system (A^*, C^*, B^*, D^*) satisfies the finite cost condition.*
3. *The Filter Algebraic Riccati Equation (FARE)*

$$(3.8) \quad APA^* - P + BB^* = (BD^* + APC^*)(R + CPC^*)^{-1}(DB^* + CPA^*),$$

where $R := I + DD^*$, of the discrete-time system has a nonnegative selfadjoint solution.

We now give a condition under which the CARE (3.4) has a unique nonnegative selfadjoint solution.

LEMMA 3.7. *Let (A, B, C, D) be an input and output stabilizable discrete-time system. Let Q be a nonnegative selfadjoint solution of the CARE (3.4) and assume that $A_Q := A - B(S + B^*QB)^{-1}(D^*C + B^*QA)$ is strongly stable. Then Q is the unique nonnegative selfadjoint solution of the CARE (3.4).*

Proof. For the proof we need the following algebraic relations which are proven in the Appendix (Lemmas 10.3 and 10.4). Suppose Q_1 and P_1 are nonnegative selfadjoint solutions of the CARE and FARE respectively and define A_{Q_1} similarly to A_Q above and $A_{P_1} := A - (BD^* + AP_1C^*)(R + CP_1C^*)^{-1}C$, then the following relation holds

$$(3.9) \quad (I + P_1Q_1)A_{Q_1} = A_{P_1}(I + P_1Q_1).$$

The following algebraic relation is also proven in the Appendix (Lemma 10.4). If Q_1 and Q_2 are nonnegative selfadjoint solutions of the CARE and A_{Q_1} and A_{Q_2} are defined similarly as A_Q above, then

$$(3.10) \quad Q_1 - Q_2 = A_{Q_2}^*(Q_1 - Q_2)A_{Q_1}.$$

With induction it follows that for all $n \in \mathbb{N}$ we have

$$(3.11) \quad Q_1 - Q_2 = A_{Q_2}^{*n}(Q_1 - Q_2)A_{Q_1}^n.$$

Using these facts we now prove the statement. Since (A, B, C, D) is input stabilizable there exists a nonnegative selfadjoint solution P of the FARE (3.8). Since A_Q is assumed to be strongly stable and equation (3.9) shows that A_P is similar to A_Q we have that A_P is strongly stable. Now let \tilde{Q} be an arbitrary nonnegative selfadjoint solution of the CARE. According to equation (3.9) $A_{\tilde{Q}}$ is similar to the strongly stable operator A_P and hence is strongly stable. Since $A_{\tilde{Q}}$ is strongly stable there exists for every $x \in X$ a real number c_x such that for every $n \in \mathbb{N}$ we have $\|A_{\tilde{Q}}^n x\| \leq c_x$. By the uniform boundedness theorem this implies that there exists a real number c such

that for every $n \in \mathbb{N}$ we have $\|A_Q^n\| \leq c$. Using equation (3.11) with $Q_1 = Q$ and $Q_2 = \tilde{Q}$ we have for all $x \in X$ and $n \in \mathbb{N}$

$$\|(Q - \tilde{Q})x\| = \|A_Q^{*n}(Q - \tilde{Q})A_Q^n x\| \leq \|A_Q^{*n}\| \|Q - \tilde{Q}\| \|A_Q^n x\| \leq c \|Q - \tilde{Q}\| \|A_Q^n x\|.$$

Since A_Q is strongly stable the right-hand side converges to zero as $n \rightarrow \infty$. This implies that the left-hand side is zero and so $\tilde{Q} = Q$. \square

The *input-output map* \mathcal{D} of a discrete-time system (A, B, C, D) is defined for finitely nonzero U -valued sequences u by

$$(3.12) \quad (\mathcal{D}u)_k := \sum_{i=0}^{\infty} CA^i Bu_{k-i-1} + Du_k.$$

The discrete-time system (A, B, C, D) is said to be *input-output stable* if \mathcal{D} extends to a bounded map from $l^2(\mathbb{Z}; U)$ to $l^2(\mathbb{Z}; Y)$.

We define the transfer function G of a system (A, B, C, D) by

$$(3.13) \quad G(z) = D + \sum_{i=0}^{\infty} CA^i Bz^{i+1}$$

for those z for which the sum converges absolutely. Note that it converges absolutely for $|z| < 1/r(A)$ ($r(A)$ denotes the spectral radius of A) and it is equal to $D + Cz(I - zA)^{-1}B$ for those z . It is obvious that the transfer function of a system can be constructed from the input-output map and vice versa, in this sense transfer functions and input-output maps are equivalent notions. Given a transfer function G we call any system (A, B, C, D) such that (3.13) holds a *realization* of the transfer function. We note that the functions that appear as transfer functions of discrete-time infinite-dimensional systems are exactly the operator-valued functions that are analytic on some disc centered at the origin.

The (time-domain) *Hankel operator* Γ of a system is defined as $\Gamma := \mathcal{C}\mathcal{B}$ where \mathcal{C} and \mathcal{B} are the observability and controllability maps of the system, respectively. It is easily seen that the Hankel operator does not depend on the particular realization, but only on the input-output map.

4. Lyapunov-balanced realizations. In this section we review some results from Young [20], [21] and Ober and Wu [13] and translate them in terms more suitable for our purposes. The following result on the existence of Lyapunov-balanced realizations was proven by Young [20], [21]. We recall that an input and output stable system is called Lyapunov-balanced if its controllability and observability gramians are equal (again: we do not require them to be ‘diagonal’).

LEMMA 4.1. *Every transfer function which has a bounded Hankel operator has an approximately controllable and observable Lyapunov-balanced realization. Moreover, approximately controllable and observable Lyapunov-balanced realizations are unique up to a unitary transformation.*

The next corollary gives an alternative condition for the existence of Lyapunov-balanced realizations.

COROLLARY 4.2. *A transfer function has a Lyapunov-balanced realization if and only if it has a realization that is both input and output stable.*

Proof. Since a Lyapunov-balanced realization is input and output stable one implication is immediate. If the transfer function has a realization such that both its

controllability map \mathcal{B} and observability map \mathcal{C} are bounded then its Hankel operator $\Gamma = \mathcal{C}\mathcal{B}$ is bounded and Lemma 4.1 shows that it has a Lyapunov-balanced realization. \square

The following result was proven by Ober and Wu [13].

LEMMA 4.3. *Let (A, B, C, D) be Lyapunov-balanced and approximately controllable and approximately observable. Then A and A^* are strongly stable.*

Combing Lemmas 4.3, 3.2 and 3.5 we have the following corollary.

COROLLARY 4.4. *The gramian of an approximately controllable and approximately observable Lyapunov-balanced realization is the unique nonnegative selfadjoint solution of both the control and the observation Lyapunov equation.*

5. Normalized factorizations. In this section we generalize a result of Meyer and Franklin [9] on the connection between normalized factorizations and linear quadratic regulator theory to the infinite-dimensional case. This result will allow us to relate LQG-balanced realizations to Lyapunov-balanced realizations of a normalized factorization of the given transfer function.

Given an output stabilizable discrete-time system (A, B, C, D) with optimal cost operator Q we form the *optimal closed-loop system*

$$(5.1) \quad \check{A} := A + BF, \quad \check{B} := BW^{-1/2}, \quad \check{C} := [F; C + DF], \quad \check{D} := [I; D]W^{-1/2}$$

where

$$W := S + B^*QB, \quad S := I + D^*D, \quad F := -W^{-1}(D^*C + B^*QA).$$

(Note that we use the notation $[-; -]$ for a block column vector and $[-, -]$ for a block row vector.) We first remark that the F above is the optimal state feedback operator for the LQR problem. We obtain the optimal closed-loop system from the system (A, B, C, D) by choosing $u = Fx + W^{-1/2}\check{u}$ and considering \check{u} as the input of this new system and $[u; y]$ as the output. This amounts to closing the loop by the optimal state feedback operator, considering the input and output of the plant as the new output and prefiltering the new input.

Our first result in this section states that that the optimal cost operator of the plant equals the observability gramian of the optimal closed-loop system.

LEMMA 5.1. *Let (A, B, C, D) be an output stabilizable discrete-time system. Denote its optimal cost operator by Q and define its optimal closed-loop system by (5.1). Denote the observability gramian of the optimal closed-loop system by L_C . Then $Q = L_C$.*

Proof. From the discussion above it is obvious that if $\check{u} = 0$, then the output of the optimal closed-loop system is $[u^{\min}; y^{\min}]$, the optimal input and output of the plant. From this it follows that

$$\langle L_C x_0, x_0 \rangle = \langle \mathcal{C}x_0, \mathcal{C}x_0 \rangle = \left\| \begin{bmatrix} u^{\min} \\ y^{\min} \end{bmatrix} \right\|^2 = J(x_0, u^{\min}) = \langle Qx_0, x_0 \rangle.$$

Since this holds for all x_0 in the state space we have $Q = L_C$. \square

The next lemma shows that the observability gramian of the optimal closed-loop system satisfies two additional equations.

LEMMA 5.2. *Let (A, B, C, D) be an output stabilizable discrete-time system. Denote its optimal cost operator by Q and define its optimal closed-loop system $(\check{A}, \check{B}, \check{C}, \check{D})$*

by (5.1). Denote the observability gramian of the optimal closed-loop system by L_C . Then

$$(5.2) \quad \check{B}^* L_C \check{B} + \check{D}^* \check{D} = I,$$

$$(5.3) \quad \check{B}^* L_C \check{A} + \check{D}^* \check{C} = 0.$$

Proof. The equations (5.2) and (5.3) are readily verified using (5.1) and the fact that $Q = L_C$ from Lemma 5.1. \square

An input-output map \mathcal{D} is called *inner* if it maps $l^2(\mathbb{Z}; U)$ into $l^2(\mathbb{Z}; Y)$ and satisfies $\mathcal{D}^* \mathcal{D} = I$. In the next lemma we give necessary and sufficient conditions on a realization for the input-output map to be inner.

LEMMA 5.3. *Let (A, B, C, D) be an output stable realization of the input-output map \mathcal{D} . Denote the observability gramian of this system by L_C . If*

$$B^* L_C B + D^* D = I,$$

and

$$B^* L_C A + D^* C = 0,$$

then \mathcal{D} is inner. If the system (A, B, C, D) is approximately controllable then these conditions are also necessary.

Proof. We take u_k equal to u at the k -th position and zero elsewhere and compute

$$(\mathcal{D}u_k)_m = \begin{cases} 0 & m < k \\ Du & m = k \\ CA^{m-k-1}Bu & m > k \end{cases}.$$

We define v_i similarly to u_k above and compute for $k > i$

$$\langle (\mathcal{D}u_k)_n, (\mathcal{D}v_i)_n \rangle = \begin{cases} 0 & n < k \\ \langle Du, CA^{k-i-1}Bv \rangle & n = k \\ \langle CA^{n-k-1}Bu, CA^{n-i-1}Bv \rangle & n > k \end{cases}.$$

We then compute

$$\begin{aligned} \langle \mathcal{D}u_k, \mathcal{D}v_i \rangle &= \sum_{n=-\infty}^{\infty} \langle (\mathcal{D}u_k)_n, (\mathcal{D}v_i)_n \rangle \\ &= \langle Du, CA^{k-i-1}Bv \rangle + \sum_{n=k+1}^{\infty} \langle CA^{n-k-1}Bu, CA^{n-i-1}Bv \rangle \\ &= \langle u, D^* CA^{k-i-1}Bv \rangle + \sum_{n=k+1}^{\infty} \langle u, B^* A^{n-k-1} C^* CA^{n-k-1} AA^{k-i-1} Bv \rangle \\ &= \langle u, (D^* C + B^* L_C A) A^{k-i-1} Bv \rangle. \end{aligned}$$

By the assumptions we thus have $\langle \mathcal{D}u_k, \mathcal{D}v_i \rangle = 0$ for $k > i$. Then obviously $\langle \mathcal{D}u_k, \mathcal{D}v_i \rangle = 0$ for $k \neq i$.

For $k = i$ we have

$$\begin{aligned} \langle \mathcal{D}u_k, \mathcal{D}v_i \rangle &= \sum_{n=-\infty}^{\infty} \langle (\mathcal{D}u_k)_n, (\mathcal{D}v_i)_n \rangle \\ &= \langle Du, Dv \rangle + \sum_{n=k+1}^{\infty} \langle CA^{n-k-1}Bu, CA^{n-k-1}Bv \rangle \\ &= \langle D^*Du, v \rangle + \sum_{j=0}^{\infty} \langle B^*A^{*j}C^*CA^jBu, v \rangle = \langle (D^*D + B^*L_C B)u, v \rangle. \end{aligned}$$

By the assumptions we thus have $\langle \mathcal{D}u_k, \mathcal{D}v_i \rangle = \langle u, v \rangle$ for $k = i$.

Let u and v be finitely nonzero sequences. Then

$$u = \sum_{k=-n}^n u^k e_k, \quad v = \sum_{i=-n}^n v^i e_i,$$

where $u^k, v^i \in U$ and e_j is the element of $l^2(\mathbb{Z}; U)$ with a one at the j -th position and zeros elsewhere. Then

$$\begin{aligned} \langle \mathcal{D}u, \mathcal{D}v \rangle &= \sum_{k=-n}^n \sum_{i=-n}^n \langle \mathcal{D}(u^k e_k), \mathcal{D}(v^i e_i) \rangle = \sum_{j=-n}^n \langle \mathcal{D}(u^j e_j), \mathcal{D}(v^j e_j) \rangle \\ &= \sum_{j=-n}^n \langle u^j, v^j \rangle = \langle u, v \rangle. \end{aligned}$$

From the above we have for every finitely nonzero sequence u that $\|\mathcal{D}u\| = \|u\|$. Since the set of finitely nonzero sequences is dense in $l^2(\mathbb{Z}; U)$ this implies that \mathcal{D} has a continuous extension to a map from $l^2(\mathbb{Z}; U)$ to $l^2(\mathbb{Z}; Y)$, so \mathcal{D} is stable. Further, since this extension satisfies $\langle \mathcal{D}u, \mathcal{D}v \rangle = \langle u, v \rangle$, we must have $\mathcal{D}^*\mathcal{D} = I$, so \mathcal{D} is inner.

Suppose that \mathcal{D} is inner and the realization is approximately controllable. Since \mathcal{D} is inner we have for all $i < 0$ that $\langle \mathcal{D}u_0, \mathcal{D}v_i \rangle = \langle u_0, v_i \rangle = 0$ where u_0 and v_i are defined as above. From the above we see that this implies that $\langle u, (D^*C + B^*L_C A)A^{-i-1}Bv \rangle = 0$. Since this holds for all $u \in U$ we must have $(D^*C + B^*L_C A)A^{-i-1}Bv = 0$ for all $v \in U$ and $i < 0$. This implies that for finitely nonzero U -valued sequences z we have $(D^*C + B^*L_C A)\mathcal{B}z = 0$. Since the system is approximately controllable the set of elements of the form $\mathcal{B}z$ is dense in the state space so $D^*C + B^*L_C A = 0$ on a dense set and by continuity on the whole state space.

Since \mathcal{D} is inner we have $\langle \mathcal{D}u_0, \mathcal{D}v_0 \rangle = \langle u_0, v_0 \rangle$, where u_0 and v_0 are as above. This implies that $\langle (D^*D + B^*L_C B)u, v \rangle = \langle u, v \rangle$ for all $u, v \in U$ and hence $D^*D + B^*L_C B = I$. \square

Combining Lemmas 5.2 and 5.3 we have the following.

COROLLARY 5.4. *Let (A, B, C, D) be an output stabilizable discrete-time system. Then the input-output map of its optimal closed-loop system is inner.*

We can recover the system (A, B, C, D) from its optimal closed-loop system as follows.

LEMMA 5.5. *Let (A, B, C, D) be an output stabilizable discrete-time system. Let $(\check{A}, \check{B}, \check{C}, \check{D})$ be its optimal closed-loop system defined by (5.1). Partition \check{C} and \check{D} in the obvious way as $\check{C} = [\check{C}_1; \check{C}_2]$ and $\check{D} = [\check{D}_1; \check{D}_2]$. Then \check{D}_1 is boundedly invertible and*

$$(5.4) \quad A := \check{A} - \check{B}\check{D}_1^{-1}\check{C}_1, \quad B := \check{B}\check{D}_1^{-1}, \quad C := \check{C}_2 - \check{D}_2\check{D}_1^{-1}\check{C}_1, \quad D := \check{D}_2\check{D}_1^{-1}.$$

Proof. That \check{D}_1 is boundedly invertible is obvious from its definition. The identities (5.4) follow from simple algebraic manipulations. \square

To relate the input-output maps of the plant and its optimal closed-loop system we first study the series interconnection of two systems.

Consider two systems (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) such that the output space of the first system and the input space of the second system are equal. Define the series interconnection of these two systems as the system we obtain by choosing the input of the second system equal to the output of the first system. Obviously, the input-output map of the series interconnection is D_2D_1 , the composition of the input-output maps of the first and second system. A realization of this series interconnection is the following

$$A = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix}, \quad C = [D_2C_1, C_2], \quad D = D_2D_1.$$

If we apply the invertible state space transformation

$$\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$$

to this realization we obtain another realization of the series interconnection, namely

$$(5.5) \quad A_s = \begin{bmatrix} A_1 & 0 \\ A_2 + B_2C_1 - A_1 & A_2 \end{bmatrix}, \quad B_s = \begin{bmatrix} B_1 \\ B_2D_1 - B_1 \end{bmatrix},$$

$$C_s = [D_2C_1 + C_2, C_2], \quad D_s = D_2D_1.$$

We first use the series interconnection to obtain a result about the invertibility of input-output maps.

LEMMA 5.6. *Let (A, B, C, D) be a system with input-output map \mathcal{D} . If D is boundedly invertible, then the input-output map $\tilde{\mathcal{D}}$ of the system $(A - BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1})$ satisfies $\tilde{\mathcal{D}}\mathcal{D} = I = \mathcal{D}\tilde{\mathcal{D}}$. Thus in this case the input-output map \mathcal{D} has an inverse.*

Proof. Using equations (5.5) we see that the series interconnection of the two given systems has a realization

$$A_s = \begin{bmatrix} A & 0 \\ 0 & A - BD^{-1}C \end{bmatrix}, \quad B_s = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C_s = [0, -D^{-1}C], \quad D_s = I.$$

From this we see that $C_s A_s^k B_s = 0$ for all $k \geq 0$ and so the input-output map of the series interconnection is the identity. This implies that $\tilde{\mathcal{D}}\mathcal{D} = I$. The other equality mentioned follows from interconnecting the systems in the opposite order. \square

We now state the relation between the input-output map of a plant and its optimal closed-loop system.

LEMMA 5.7. *Let $(\check{A}, \check{B}, [\check{C}_1; \check{C}_2], [\check{D}_1; \check{D}_2])$ be a system such that \check{D}_1 is boundedly invertible. Denote its input-output map by $[\mathcal{M}; \mathcal{N}]$. Define the system (A, B, C, D) by (5.4) and denote its input-output map by \mathcal{D} . Then $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}$.*

Proof. The realization $(\check{A}, \check{B}, [\check{C}_1; \check{C}_2], [\check{D}_1; \check{D}_2])$ of $[\mathcal{M}; \mathcal{N}]$ gives us (by Lemma 5.6) the following realization (A_1, B_1, C_1, D_1) of \mathcal{M}^{-1} :

$$A_1 = \check{A} - \check{B}\check{D}_1^{-1}\check{C}_1, \quad B_1 = \check{B}\check{D}_1^{-1}, \quad C_1 = -\check{D}_1^{-1}\check{C}_1, \quad D_1 = \check{D}_1^{-1}.$$

It also gives us the realization (A_2, B_2, C_2, D_2) of \mathcal{N} :

$$A_2 = \check{A}, \quad B_2 = \check{B}, \quad C_2 = \check{C}_2, \quad D_2 = \check{D}_2.$$

Using (5.5) we obtain the following realization of the series interconnection which has input-output map $\mathcal{N}\mathcal{M}^{-1}$:

$$A_s = \begin{bmatrix} \check{A} - \check{B}\check{D}_1^{-1}\check{C}_1 & 0 \\ 0 & \check{A} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & \check{A} \end{bmatrix}, \quad B_s = \begin{bmatrix} \check{B}\check{D}_1^{-1} \\ 0 \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix},$$

$$C_s = [\check{C}_2 - \check{D}_2\check{D}_1^{-1}\check{C}_1, \check{C}_2] = [C, \check{C}_2], \quad D_s = \check{D}_2\check{D}_1^{-1} = D,$$

where we have used (5.4). It follows that $D_s = D$ and $C_s A_s^k B_s = C A^k B$ for all $k \geq 0$. This implies that $\mathcal{N}\mathcal{M}^{-1} = \mathcal{D}$. \square

We call an input-output map $[\mathcal{M}; \mathcal{N}]$ a *right factor* of the input-output map \mathcal{D} if \mathcal{M}^{-1} is the input-output map of a system, $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}$ and \mathcal{M} and \mathcal{N} are stable. We call the factor *normalized* if $[\mathcal{M}; \mathcal{N}]$ is inner. From Lemmas 5.5 and 5.7 and Corollary 5.4 we have the following.

COROLLARY 5.8. *Let (A, B, C, D) be an output stabilizable discrete-time system. Then the input-output map of its optimal closed-loop system is a normalized right factor of the input-output map of the plant.*

We next state a result about the uniqueness of a normalized right factor. We remark that we can interpret an operator V in $\mathcal{L}(U)$ as a map \mathcal{V} from $l^2(\mathbb{Z}; U)$ into itself by $(\mathcal{V}u)_k = V u_k$.

LEMMA 5.9. *If the input-output map of a system has a normalized right factor $[\mathcal{M}; \mathcal{N}]$, then all normalized right factors of this input-output map are $\{[\mathcal{M}V; \mathcal{N}V] : V \in \mathcal{L}(U) \text{ unitary}\}$.*

Proof. Let $[\mathcal{M}; \mathcal{N}]$ be an arbitrary normalized right factor of \mathcal{D} . Since $[\mathcal{M}; \mathcal{N}]$ is normalized we have

$$\mathcal{M}^* \mathcal{M} + \mathcal{N}^* \mathcal{N} = I.$$

Multiplying this equality with \mathcal{M}^{-*} from the left and \mathcal{M}^{-1} from the right we obtain

$$I + \mathcal{D}^* \mathcal{D} = \mathcal{M}^{-*} \mathcal{M}^{-1}.$$

Since the left-hand side of this equation does not depend on the particular factor we have for two normalized factors $[\mathcal{M}_1; \mathcal{N}_1]$ and $[\mathcal{M}_2; \mathcal{N}_2]$ of \mathcal{D} that $\mathcal{M}_1^{-*} \mathcal{M}_1^{-1} = \mathcal{M}_2^{-*} \mathcal{M}_2^{-1}$, which implies

$$(5.6) \quad \mathcal{M}_2^* \mathcal{M}_1^{-*} = \mathcal{M}_2^{-1} \mathcal{M}_1.$$

The operator on the right-hand side of (5.6) is the input-output map of some system (namely, the series interconnection of the systems corresponding to \mathcal{M}_1 and \mathcal{M}_2^{-1}). Define u_k to be the sequence equal to u at the k -th position and zero elsewhere. Then $\mathcal{M}_2^{-1}\mathcal{M}_1u_k$ is equal to zero at the positions i with $i < k$. The operator on the left-hand side of (5.6) is the adjoint of the input-output map of some system (namely, the series interconnection of the systems corresponding to \mathcal{M}_2 and \mathcal{M}_1^{-1}). This implies that $\mathcal{M}_2^*\mathcal{M}_1^{-*}u_k$ is zero at the positions i with $i > k$.

Since $\mathcal{M}_2^*\mathcal{M}_1^{-*} = \mathcal{M}_2^{-1}\mathcal{M}_1$ we must have that $\mathcal{M}_2^{-1}\mathcal{M}_1u_k$ is equal to zero at all positions except possibly the k -th one. So $\mathcal{M}_2^{-1}\mathcal{M}_1$ is a constant operator V . So $\mathcal{M}_1 = \mathcal{M}_2V$.

We have $\mathcal{N}_1 = \mathcal{D}\mathcal{M}_1 = \mathcal{D}\mathcal{M}_2V = \mathcal{N}_2V$.

It remains to be proven that V is unitary. We have that $V = \mathcal{M}_2^{-1}\mathcal{M}_1$ and so $V^* = \mathcal{M}_1^*\mathcal{M}_2^{-*} = \mathcal{M}_1^{-1}\mathcal{M}_2 = V^{-1}$ by (5.6). This proves that V is unitary. \square

In the finite-dimensional case the transfer function of the optimal closed-loop system is known to be a normalized *coprime* factorization. In the infinite-dimensional case this is still an open problem.

6. Some algebraic relations. In Section 5 we proved that the optimal cost operator equals the observability gramian of the optimal closed-loop system. This can also be interpreted as follows: the smallest nonnegative selfadjoint solution of the control algebraic Riccati equation equals the smallest nonnegative selfadjoint solution of the observation Lyapunov equation of a certain closed-loop system. In this section we show that a similar result holds for *all* nonnegative selfadjoint solutions of the control algebraic Riccati equation. We also study the relation between nonnegative selfadjoint solutions of the filter algebraic Riccati equation of the plant and nonnegative selfadjoint solutions of the control Lyapunov equation of the closed-loop system.

The CARE closed-loop system $(\check{A}, \check{B}, \check{C}, \check{D})$ associated with an output stabilizable discrete-time system (A, B, C, D) and a nonnegative selfadjoint solution Q of the CARE (3.4) is defined by (5.1). For the special case that Q is the smallest nonnegative selfadjoint solution of the CARE (3.4) the CARE closed-loop system is equal to the optimal closed-loop system defined earlier.

Lemma 5.5 holds in this more general case, which is obvious from its proof.

LEMMA 6.1. *Let (A, B, C, D) be a discrete-time system such that its CARE (3.4) has a nonnegative selfadjoint solution Q . Let $(\check{A}, \check{B}, \check{C}, \check{D})$ be its CARE closed-loop system defined by (5.1). Partition \check{C} and \check{D} in the obvious way as $\check{C} = [\check{C}_1; \check{C}_2]$ and $\check{D} = [\check{D}_1; \check{D}_2]$. Then \check{D}_1 is boundedly invertible and (5.4) holds.*

This lemma implies that the input-output map of the CARE closed-loop system is a factor of the input-output map of the plant by Lemma 5.7. If Q is not the smallest nonnegative selfadjoint solution of the CARE the factorization need not be normalized (see e.g. [15, Example 3.1.2]).

We now show that if a system is approximately controllable or observable, then its CARE closed-loop system is too.

LEMMA 6.2. *Let (A, B, C, D) be an output stabilizable discrete-time system and let Q be a nonnegative selfadjoint solution of its CARE (3.4). If (A, B, C, D) is approximately controllable, then its CARE closed-loop system is too. If (A, B, C, D) is approximately observable, then its CARE closed-loop system is too.*

Proof. As is well-known a system $(\check{A}, \check{B}, \check{C}, \check{D})$ is approximately controllable (observable) iff $(\check{A} + \check{B}K\check{C}, \check{B}, \check{C}, \check{D})$ is approximately controllable (observable). With $K = -W^{-1/2}[I, 0]$ we see that the CARE closed-loop system is approximately control-

lable (observable) iff $(A, \check{B}, \check{C}, \check{D})$ is approximately controllable (observable). Obviously $(A, \check{B}) = (A, BW^{-1/2})$ is approximately controllable iff (A, B) is approximately controllable and $(A, \check{C}) = (A, [F; C + DF])$ is approximately observable if (A, C) is (but not only if). \square

In the next four lemmas we prove a correspondence between the Riccati equations of a system and the Lyapunov equations of its CARE closed-loop system. Because this requires some extensive algebraic manipulations we have relegated some of the proofs to the Appendix.

LEMMA 6.3. *Let (A, B, C, D) be an output stabilizable discrete-time system and let Q be a nonnegative selfadjoint solution of its CARE (3.4). Let $(\check{A}, \check{B}, \check{C}, \check{D})$ be its CARE closed-loop system defined by (5.1). Then*

$$(6.1) \quad \check{B}^* Q \check{B} + \check{D}^* \check{D} = I,$$

$$(6.2) \quad \check{B}^* Q \check{A} + \check{D}^* \check{C} = 0,$$

and Q is a solution of the observation Lyapunov equation of $(\check{A}, \check{B}, \check{C}, \check{D})$:

$$(6.3) \quad \check{A}^* Q \check{A} - Q + \check{C}^* \check{C} = 0.$$

Proof. The equations (6.1) and (6.2) are readily verified using (5.1). Equation (6.3) is more complicated and the proof is given in the Appendix. \square

LEMMA 6.4. *Let $(\check{A}, \check{B}, [\check{C}_1; \check{C}_2], [\check{D}_1, \check{D}_2])$ be a discrete-time system such that \check{D}_1 is boundedly invertible and define the discrete-time system (A, B, C, D) by (5.4). If the nonnegative selfadjoint operator Q satisfies the two equations (6.2) and (6.3) then Q is a solution of the CARE (3.4) of (A, B, C, D) .*

Proof. See the Appendix. \square

LEMMA 6.5. *Let (A, B, C, D) be an input and output stabilizable discrete-time system and let Q be a nonnegative selfadjoint solution of its CARE (3.4) and P be a nonnegative selfadjoint solution of its FARE (3.8). Let $(\check{A}, \check{B}, \check{C}, \check{D})$ be its CARE closed-loop system defined by (5.1). Define $L := (I + PQ)^{-1}P$. Then L is a solution of the control Lyapunov equation of $(\check{A}, \check{B}, \check{C}, \check{D})$:*

$$(6.4) \quad \check{A} L \check{A}^* - L + \check{B} \check{B}^* = 0.$$

Proof. See the Appendix. \square

LEMMA 6.6. *Let $(\check{A}, \check{B}, [\check{C}_1; \check{C}_2], [\check{D}_1, \check{D}_2])$ be a discrete-time system such that \check{D}_1 is boundedly invertible and define the discrete-time system (A, B, C, D) by (5.4). If the nonnegative selfadjoint operators Q and L satisfy the four equations (6.1), (6.2), (6.3) and (6.4) and $I - QL$ is boundedly invertible then $L(I - QL)^{-1}$ is a solution of the FARE (3.8) of (A, B, C, D) .*

Proof. See the Appendix. \square

If Q and P are the smallest nonnegative selfadjoint solutions of their respective Riccati equations then the operator $L := (I + PQ)^{-1}P$ defined in Lemma 6.5 is actually the smallest nonnegative selfadjoint solution of the Lyapunov equation (6.4). To prove this we first prove the following lemma. We note that the Hankel norm of an input-output map is the norm of the associated Hankel operator.

LEMMA 6.7. *Let (A, B, C, D) be an input and output stabilizable discrete-time system. Then the Hankel norm of the input-output map of the optimal closed-loop system (5.1) is strictly smaller than one.*

Proof. Denote the optimal cost operators by P and Q , the gramians of the optimal closed-loop system by L_B and L_C and the Hankel operator of the optimal closed-loop system by Γ . From Lemma 5.1 it follows that $Q = L_C$. From Lemma 6.5 we know that $(I + PQ)^{-1}P$ is a solution of the control Lyapunov equation and hence by Lemma 3.4 we have $L_B \leq (I + PQ)^{-1}P$. This implies that $L_C^{1/2}L_B L_C^{1/2} \leq Q^{1/2}(I + PQ)^{-1}PQ^{1/2} = (I + Q^{1/2}PQ^{1/2})^{-1}Q^{1/2}PQ^{1/2}$. Now $\|\Gamma\| = \|\Gamma\Gamma^*\| = r(\Gamma\Gamma^*) = r(C\mathcal{B}\mathcal{B}^*C^*) = r(C^*C\mathcal{B}\mathcal{B}^*) = r(L_C L_B) = r(L_C^{1/2}L_B L_C^{1/2}) \leq r((I + Q^{1/2}PQ^{1/2})^{-1}Q^{1/2}PQ^{1/2})$. Next we show that if X is a nonnegative self-adjoint operator, then $(I + X)^{-1}X < I$. The inequality $(I + X)^{-1}X < I$ is equivalent to $X(I + X) < (I + X)^2$, which is equivalent to $0 < I + X$, which is true since X is nonnegative. Finally, we apply this last result with $X := Q^{1/2}PQ^{1/2}$ to obtain $\|\Gamma\| \leq r((I + X)^{-1}X) < 1$. \square

Lemma 6.7 has the following corollary.

COROLLARY 6.8. *Let (A, B, C, D) be an input and output stabilizable discrete-time system. Let L_B and L_C denote the gramians of an input and output stable realization of the input-output map of the optimal closed-loop system (5.1). Then $r(L_B L_C) < 1$.*

We now show for an approximately observable system that if Q and P are the smallest nonnegative selfadjoint solutions of their respective Riccati equations then the operator $L := (I + PQ)^{-1}P$ defined in Lemma 6.5 is actually the smallest nonnegative selfadjoint solution of the Lyapunov equation (6.4).

LEMMA 6.9. *Let (A, B, C, D) be an approximately observable input and output stabilizable discrete-time system. Let Q and P denote the optimal cost operators of the system and its dual, respectively. Define $L := (I + PQ)^{-1}P$. Then L is the controllability gramian of the optimal closed-loop system (5.1).*

Proof. We have by Lemma 5.1 that $Q = L_C$, the observability gramian of the optimal closed-loop system and by Lemma 6.5 that $L = (I + PQ)^{-1}P$ is a solution of the control Lyapunov equation of the optimal closed-loop system. Since the control Lyapunov equation of the optimal closed-loop system has a nonnegative selfadjoint solution the optimal closed-loop system is input stable and has a controllability gramian L_B which satisfies $L_B \leq L$. From Lemma 6.8 we know that $I - L_C L_B$ is boundedly invertible and hence by Lemma 6.6 we have that $\tilde{P} := L_B(I - L_C L_B)^{-1}$ is a solution of the FARE (3.8) of the system (A, B, C, D) . We thus have $P \leq \tilde{P}$. We now use Lemma 10.9 from the Appendix with $X_1 := P$, $X_2 := \tilde{P}$ and $Y := L_C = Q$ to conclude that $L = (I + PQ)^{-1}P \leq (I + \tilde{P}Q)^{-1}\tilde{P} = L_B$. Since we already had $L_B \leq L$ we must have $L = L_B$. Note that approximate observability ensures that $Q = Y > 0$. \square

7. LQG-characteristic values. In this section we show that the spectrum of the product of the optimal cost operators of a system and its dual does not depend on the realization, but only on the input-output map. This generalizes the result from finite-dimensional theory that the eigenvalues of PQ are similarity invariants. We define the *LQG-characteristic values* of an input and output stabilizable discrete-time system to be the square roots of the spectral values of the product of the optimal cost operators P and Q . We first prove the following lemma on the spectrum of PQ .

LEMMA 7.1. *Let (A, B, C, D) be an approximately observable input and output stabilizable discrete-time system. Let Q and P denote the optimal cost operators of the system and its dual, respectively, and let L_B and L_C denote the gramians of the optimal closed-loop system (5.1). Then $\lambda \in \sigma(PQ)$ iff $\lambda/(1 + \lambda) \in \sigma(L_B L_C)$.*

Proof. From Lemmas 5.1 and 6.9 we have that $L_B L_C = (I + PQ)^{-1} PQ$, from which it follows that $PQ = (I - L_B L_C)^{-1} L_B L_C$. Let $\lambda \in \mathbb{C} - \{-1\}$ and define $\mu := \lambda/(1 + \lambda)$, then $\lambda = \mu/(1 - \mu)$. We have

$$\begin{aligned} \lambda I - PQ &= \frac{\mu}{1 - \mu} I - L_B L_C (I - L_B L_C)^{-1} \\ &= \frac{1}{1 - \mu} [\mu I - (1 - \mu) L_B L_C (I - L_B L_C)^{-1}] \\ &= \frac{1}{1 - \mu} [\mu(I - L_B L_C) - (1 - \mu) L_B L_C] (I - L_B L_C)^{-1} \\ &= \frac{1}{1 - \mu} (\mu I - L_B L_C) (I - L_B L_C)^{-1}. \end{aligned}$$

This shows that $\lambda \in \sigma(PQ)$ iff $\mu = \lambda/(1 + \lambda) \in \sigma(L_B L_C)$. \square

The following lemma gives the desired result.

LEMMA 7.2. *Let (A_i, B_i, C_i, D) with $i = 1, 2$ be two approximately observable input and output stabilizable discrete-time systems. Let Q_i and P_i denote the optimal cost operators of the system and its dual, respectively. If the two systems have the same input-output map, then the spectra of $P_1 Q_1$ and $P_2 Q_2$ are equal, with the possible exception of zero.*

Proof. Denote the gramians of the optimal closed-loop system of (A_i, B_i, C_i, D) by L_{B_i} and L_{C_i} . Then according to Lemma 7.1 the Lemma would be proven if the nonzero elements in the spectrum of $L_{B_1} L_{C_1}$ equal the nonzero elements in the spectrum of $L_{B_2} L_{C_2}$. Since the input-output map of both optimal closed-loop systems is a normalized factor of the input-output map of the plant by Lemma 5.8 there exists a unitary V such that $[M_2; \mathcal{N}_2] = [M_1; \mathcal{N}_1]V$ by Lemma 5.9. For the Hankel operators of the optimal closed-loop systems this implies $\Gamma_2 = \Gamma_1 V$, which implies that $\Gamma_2 \Gamma_2^* = \Gamma_1 \Gamma_1^*$. Since the nonzero elements in the spectrum of $L_B L_C = \mathcal{B} \mathcal{B}^* \mathcal{C}^* \mathcal{C}$ equal the nonzero elements in the spectrum of $\Gamma \Gamma^* = \mathcal{C} \mathcal{B} \mathcal{B}^* \mathcal{C}^*$ by Lemma 10.10 from the Appendix this shows that the nonzero elements in the spectrum of $L_{B_1} L_{C_1}$ equal the nonzero elements in the spectrum of $L_{B_2} L_{C_2}$. \square

8. LQG-balanced realizations. In this section we prove the existence and some properties of LQG-balanced realizations. We first show that the input-output map of the optimal closed-loop system has a Lyapunov-balanced realization.

LEMMA 8.1. *Let (A, B, C, D) be an output stabilizable discrete-time system. Then the input-output map of the optimal closed-loop system has an approximately controllable and approximately observable Lyapunov-balanced realization.*

Proof. Lemma 5.8 shows that the optimal closed-loop system is input-output stable. This implies that the Hankel operator of the optimal closed-loop system is bounded and Lemma 4.1 then shows that the input-output map of the optimal closed-loop system has a Lyapunov-balanced realization. \square

From this Lyapunov-balanced realization we can construct a LQG-balanced realization of the plant.

THEOREM 8.2. *Let (A, B, C, D) be an input and output stabilizable discrete-time system. Then the input-output map of the system (A, B, C, D) has a LQG-balanced realization.*

Proof. Denote the approximately controllable and approximately observable Lyapunov-balanced realization of the input-output map of the optimal closed-loop system by $(\check{A}, \check{B}, \check{C}, \check{D})$. Denote the equal controllability and observability gramians of this realization by L . Define

$$A_s := \check{A} - \check{B}\check{D}_1^{-1}\check{C}_1, \quad B_s := \check{B}\check{D}_1^{-1}, \quad C_s := \check{C}_2 - \check{D}_2\check{D}_1^{-1}\check{C}_1, \quad D_s := \check{D}_2\check{D}_1^{-1}.$$

Then (A_s, B_s, C_s, D_s) is a realization of the input-output map of (A, B, C, D) according to Lemma 5.7. Since the input-output map of the optimal closed-loop system is inner (Lemma 5.8) and $(\check{A}, \check{B}, \check{C}, \check{D})$ is approximately controllable we know from Lemma 5.3 that $\check{B}^*L\check{B} + \check{D}^*\check{D} = I$ and $\check{B}^*L\check{A} + \check{D}^*\check{C} = 0$. From Lemma 6.4 we see that L is a solution of the CARE of the system (A_s, B_s, C_s, D_s) . From Corollary 6.8 we know that $I - L^2$ is boundedly invertible and Lemma 6.6 now tells us that $L(I - L^2)^{-1}$ is a solution of the FARE of the system (A_s, B_s, C_s, D_s) . From Lemma 4.3 we know that \check{A} is strongly stable. Using Lemma 3.7 we see that this implies that L is the unique solution of the CARE of the system (A_s, B_s, C_s, D_s) . Assume that the FARE of the system (A_s, B_s, C_s, D_s) has two nonnegative selfadjoint solutions P_1 and P_2 . From Lemma 6.5 we see that $(I + P_1L)^{-1}P_1$ and $(I + P_2L)^{-1}P_2$ are solutions of the control Lyapunov equation of the optimal closed-loop system of (A_s, B_s, C_s, D_s) , which is the balanced realization $(\check{A}, \check{B}, \check{C}, \check{D})$. From Lemma 4.4 we see that $(I + P_1L)^{-1}P_1 = (I + P_2L)^{-1}P_2$ which implies that $P_1 = P_2$. We recall that if a system (A_1, B_1, C_1, D_1) has a solution Q to its CARE and P to its FARE and S is a boundedly invertible operator then the system $(SAS^{-1}, SB, CS^{-1}, D)$ has a solution $S^{-1}QS^{-1}$ to its CARE and SPS^* to its FARE. It is easily seen that $I - L^2$ is a nonnegative operator (for example using that the Hankel operator has norm smaller than one), from which it follows that $S := (I - L^2)^{-1/4}$ is well-defined. Define $(A_l, B_l, C_l, D_l) = (SA_sS^{-1}, SB_s, C_sS^{-1}, D_s)$. Then the system (A_l, B_l, C_l, D_l) has $L(I - L^2)^{-1/2}$ as the unique solution to both its CARE and its FARE. \square

We now prove that LQG-balanced realizations are essentially unique.

LEMMA 8.3. *Let (A, B, C, D) be an approximately controllable and approximately observable LQG-balanced realization. Then all approximately controllable and approximately observable LQG-balanced realizations of the same input-output map are given by $\{(TAT^{-1}, TB, CT^{-1}, D) : T \in \mathcal{L}(X) \text{ unitary}\}$.*

Proof. It is obvious that the given realizations are indeed LQG-balanced, it remains to be proven that these are all approximately controllable and approximately observable LQG-balanced realizations. Let (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) be two approximately controllable and approximately observable LQG-balanced realizations of the same input-output map. Let Q_1 and Q_2 be the optimal cost operators of the respective systems. Define $S_i := (I + Q_i^2)^{1/4}$ and $(A_i^b, B_i^b, C_i^b, D_i^b) = (S_i A_i S_i^{-1}, S_i B_i, C_i S_i^{-1}, D_i)$ for $i = 1, 2$. Then $(A_i^b, B_i^b, C_i^b, D_i^b)$ has $Q_i^b := Q_i(I + Q_i^2)^{-1/2}$ as its optimal cost operator and $P_i^b := Q_i(I + Q_i^2)^{1/2}$ as the optimal cost operator of its dual system. Let $(\check{A}_i^b, \check{B}_i^b, \check{C}_i^b, \check{D}_i^b)$ be the optimal closed-loop system of $(A_i^b, B_i^b, C_i^b, D_i^b)$. Using Lemmas 5.1 and 6.9 we see that they are Lyapunov-balanced realizations with gramians $L_i := Q_i(I + Q_i^2)^{-1/2}$. According to Lemma 5.8 the input-output maps of $(\check{A}_1^b, \check{B}_1^b, \check{C}_1^b, \check{D}_1^b)$ and $(\check{A}_2^b, \check{B}_2^b, \check{C}_2^b, \check{D}_2^b)$ are normalized factors of the input-output map of (A_1, B_1, C_1, D_1) (which equals the input-output map of (A_2, B_2, C_2, D_2)). By Lemma 5.9 there exists an operator V such that $[\mathcal{M}_2; \mathcal{N}_2] = [\mathcal{M}_1; \mathcal{N}_1]V$. It is easily seen that $(\check{A}_1^b, \check{B}_1^b V, \check{C}_1^b, \check{D}_1^b V)$ is a Lyapunov-balanced realization of $[\mathcal{M}_1; \mathcal{N}_1]V = [\mathcal{M}_2; \mathcal{N}_2]$. From Lemma 6.2 it follows that both $(\check{A}_1^b, \check{B}_1^b V, \check{C}_1^b, \check{D}_1^b V)$ and $(\check{A}_2^b, \check{B}_2^b, \check{C}_2^b, \check{D}_2^b)$ are approximately controllable and approxi-

mately observable Lyapunov-balanced realizations of the input-output map $[\mathcal{M}_2; \mathcal{N}_2]$. From Lemma 4.1 it follows that there exists a unitary T such that

$$(T\check{A}_1^b T^{-1}, T\check{B}_1^b V, \check{C}_1^b T^{-1}, \check{D}_1^b V) = (\check{A}_2^b, \check{B}_2^b, \check{C}_2^b, \check{D}_2^b).$$

Using equations (5.4) for $i = 1, 2$ we see that

$$(TA_1^b T^{-1}, TB_1^b, C_1^b T^{-1}, D_1^b) = (A_2^b, B_2^b, C_2^b, D_2^b).$$

It now follows that

$$(S_2^{-1} T S_1 A_1 S_1^{-1} T^{-1} S_2, S_2^{-1} T S_1 B_1, C_1 S_1^{-1} T^{-1} S_2, D_1) = (A_2, B_2, C_2, D_2).$$

To complete the proof that (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) are unitarily equivalent we prove that $S_2^{-1} T S_1 = T$ or equivalently $T S_1 T^{-1} = S_2$. Since $(I + Q_i^2)^{-1} = I - [Q_i(I + Q_i^2)^{-1/2}]^2 = I - L_i^2$ we have $S_i = (I + Q_i^2)^{-1/4} = (I - L_i^2)^{1/4}$ and since $L_2 = T L_1 T^{-1}$ we then have

$$S_2 = (I - L_2^2)^{-1/4} = T(I - L_1^2)^{-1/4} T^{-1} = T S_1 T^{-1}.$$

This proves that all approximately controllable and approximately observable LQG-balanced realizations of the same input-output map are unitarily equivalent. \square

The following lemma states what the proof of Theorem 8.2 already indicated, namely that the filter and control Riccati equations of an approximately controllable and approximately observable LQG-balanced realization have unique nonnegative selfadjoint solutions.

LEMMA 8.4. *Let (A, B, C, D) be an approximately controllable and approximately observable LQG-balanced realization. Then both the CARE (3.4) and the FARE (3.8) of (A, B, C, D) have a unique nonnegative selfadjoint solution.*

Proof. As in the proof of Lemma 8.3 we construct the approximately controllable and approximately observable realization (A^b, B^b, C^b, D^b) and the approximately controllable and approximately observable Lyapunov-balanced realization $(\check{A}^b, \check{B}^b, \check{C}^b, \check{D}^b)$. From Lemma 4.3 we know that \check{A}^b is strongly stable. Lemma 3.7 shows that the CARE of (A^b, B^b, C^b, D^b) has a unique nonnegative selfadjoint solution. Obviously this implies that the CARE of (A, B, C, D) has a unique nonnegative selfadjoint solution. From Lemma 4.4 we know that the control Lyapunov equation of $(\check{A}^b, \check{B}^b, \check{C}^b, \check{D}^b)$ has a unique nonnegative selfadjoint solution. As in the proof of Theorem 8.2 it follows that the FARE of (A^b, B^b, C^b, D^b) and hence the FARE of (A, B, C, D) has a unique nonnegative selfadjoint solution. \square

9. Conclusions. We have proven the existence of LQG-balanced realizations for the class of transfer functions that are analytic on some disc centered at the origin and have a (infinite-dimensional) realization that is both input and output stabilizable. We have also proven that approximately controllable and approximately observable LQG-balanced realizations are essentially unique and that the Riccati equations of approximately controllable and approximately observable LQG-balanced realizations have unique nonnegative selfadjoint solutions. Analogous continuous-time results and error-bounds for truncations of LQG-balanced realizations will be given elsewhere.

10. Appendix.

10.1. Miscellaneous results on Riccati operators.

LEMMA 10.1. *Let P and Q be nonnegative selfadjoint operators. Define*

$$(10.1) \quad A_P := A - (BD^* + APC^*)(R + CPC^*)^{-1}C$$

$$(10.2) \quad A_Q := A - B(S + B^*QB)^{-1}(D^*C + B^*QA)$$

$$(10.3) \quad \underline{A} := A - BS^{-1}D^*C.$$

where $S := I + D^*D$ and $R := I + DD^*$. Then

$$(10.4) \quad A_P(I + PC^*R^{-1}C) = \underline{A} = (I + BS^{-1}B^*Q)A_Q$$

$$(10.5) \quad A_Q = (I + BS^{-1}B^*Q)^{-1}A_P(I + PC^*R^{-1}C)$$

$$(10.6) \quad A_P = (I + BS^{-1}B^*Q)A_Q(I + PC^*R^{-1}C)^{-1}.$$

Proof. We prove that $A_P(I + PC^*R^{-1}C) = \underline{A}$. The equality $\underline{A} = (I + BS^{-1}B^*Q)A_Q$ is proven similarly. By writing out A_P in full we have

$$\begin{aligned} & A_P(I + PC^*R^{-1}C) \\ &= A(I + PC^*R^{-1}C) - (BD^* + APC^*)(R + CPC^*)^{-1}C(I + PC^*R^{-1}C), \\ &= A(I + PC^*R^{-1}C) - (BD^* + APC^*)(R + CPC^*)^{-1}(R + CPC^*)R^{-1}C, \\ &= A + APC^*R^{-1}C - (BD^* + APC^*)R^{-1}C = A - BD^*R^{-1}C \\ &= A - BS^{-1}D^*C, \end{aligned}$$

since $D^*R^{-1} = S^{-1}D^*$. This completes the proof of (10.4). Equations (10.5) and (10.6) easily follow from (10.4). \square

Note that in the above lemma we have not assumed that P and Q are solutions of the Riccati equations.

We now prove that the Riccati equations can be written in several different but equivalent versions.

LEMMA 10.2.

1. P is a nonnegative selfadjoint solution of

$$(10.7) \quad A_P P(I + C^*R^{-1}CP)A_P^* - P + BS^{-1}B^* = 0,$$

where A_P is defined by (10.1), iff it is a nonnegative selfadjoint solution of

$$(10.8) \quad \underline{A}P(I + C^*R^{-1}CP)^{-1}\underline{A}^* - P + BS^{-1}B^* = 0,$$

where \underline{A} is defined by (10.3).

2. P is a nonnegative selfadjoint solution of (10.7) iff it is a nonnegative selfadjoint solution of the FARE (3.8).
3. Q is a nonnegative selfadjoint solution of

$$(10.9) \quad A_Q^*(I + QBS^{-1}B^*)QA_Q - Q + C^*R^{-1}C = 0,$$

where A_Q is defined by (10.2), iff it is a nonnegative selfadjoint solution of

$$(10.10) \quad \underline{A}^*Q(I + BS^{-1}B^*Q)^{-1}\underline{A} - Q + C^*R^{-1}C = 0,$$

where \underline{A} is defined by (10.3).

4. Q is a nonnegative selfadjoint solution of (10.9) iff it is a nonnegative selfadjoint solution of the CARE (3.4).

Proof. We shall prove the equivalence of the filter equations; the equivalence of the control equations is similar.

1. The equations (10.7) and (10.8) are equivalent iff the following holds

$$(10.11) \quad \underline{A}P(I + C^*R^{-1}CP)^{-1}\underline{A}^* = A_P P(I + C^*R^{-1}CP)A_P^*.$$

We use Lemma 10.1 (which tells us that $\underline{A} = A_P(I + PC^*R^{-1}C)$) to write the left-hand side of equation (10.11) as

$$A_P(I + PC^*R^{-1}C)P(I + C^*R^{-1}CP)^{-1}(I + C^*R^{-1}CP)A_P^*,$$

which is indeed equal to the right-hand side of equation (10.11).

2. To prove the equivalence of (10.7) and (3.8) we substitute in (10.7) for A_P from (10.1) and for $(I + C^*R^{-1}CP)A_P^*$ we substitute \underline{A}^* (using (10.4)) and then substitute (10.3) for \underline{A} . We then get

$$(A - (BD^* + APC^*)(R + CPC^*)^{-1}C)P(A^* - C^*DS^{-1}B^*) - P + BS^{-1}B^* = 0.$$

Rewriting this gives

$$\begin{aligned} APA^* - P + BB^* &= (BD^* + APC^*)(R + CPC^*)^{-1}CPA^* \\ &- (BD^* + APC^*)(R + CPC^*)^{-1}CPC^*DS^{-1}B^* + APC^*DS^{-1}B^* \\ &- BS^{-1}B^* + BB^*. \end{aligned}$$

We now focus on the last two lines of this last equation. We note that $I - S^{-1} = D^*DS^{-1}$ and we can thus rewrite these last two lines as

$$-(BD^* + APC^*)(R + CPC^*)^{-1}CPC^*DS^{-1}B^* + APC^*DS^{-1}B^* + BD^*DS^{-1}B^*$$

and this can be rewritten as

$$(BD^* + APC^*)(R + CPC^*)^{-1}[-CPC^* + R + CPC^*]DS^{-1}B^*.$$

Noting that $RDS^{-1} = D$ we see that this is equal to

$$(BD^* + APC^*)(R + CPC^*)^{-1}DB^*.$$

This completes the proof of the equivalence of (10.7) and (3.8). \square

We now show that the known relationship between A_Q and A_P extends to the infinite-dimensional case.

LEMMA 10.3. *Let (A, B, C, D) be an input and output stabilizable discrete-time system and let Q be a nonnegative selfadjoint solution of its CARE (3.4) and let P be a nonnegative selfadjoint solution of its FARE (3.8). Define A_P and A_Q by (10.1) and (10.2), respectively. Then*

$$(I + PQ)A_Q = A_P(I + PQ).$$

Proof. We use the FARE (10.7) to write

$$P = A_P P(I + C^* R^{-1} C P) A_P^* + B S^{-1} B^*,$$

which leads to

$$I + PQ = I + A_P P(I + C^* R^{-1} C P) A_P^* Q + B S^{-1} B^* Q$$

and so

$$(I + PQ)A_Q = (I + B S^{-1} B^* Q)A_Q + A_P P(I + C^* R^{-1} C P) A_P^* Q A_Q.$$

We use (10.5) to write the right-hand side as

$$A_P(I + P C^* R^{-1} C) + A_P P(I + C^* R^{-1} C P) A_P^* Q A_Q.$$

Rearranging gives

$$A_P + A_P P[C^* R^{-1} C + (I + C^* R^{-1} C P) A_P^* Q A_Q]$$

and using (10.5) again we obtain

$$A_P + A_P P[C^* R^{-1} C + A_Q^*(I + Q B S^{-1} B^*) Q A_Q].$$

According to the CARE (10.9) the term in square brackets equals Q . So the above is equal to $A_P(I + PQ)$. \square

We now prove a relation concerning the difference of two solutions of a Riccati equation.

LEMMA 10.4. *Let (A, B, C, D) be an output stabilizable discrete-time system and let Q_1 and Q_2 be nonnegative selfadjoint solutions of its CARE (3.4). Define A_{Q_1} and A_{Q_2} similarly to (10.2). Then*

$$Q_1 - Q_2 = A_{Q_2}^*(Q_1 - Q_2)A_{Q_1}.$$

Proof. Subtract the form (10.9) of the CARE for Q_1 and Q_2 to obtain

$$(10.12) \quad Q_1 - Q_2 = A_{Q_1}^*(I + Q_1 B S^{-1} B^*)Q_1 A_{Q_1} - A_{Q_2}^*(I + Q_2 B S^{-1} B^*)Q_2 A_{Q_2}.$$

According to Lemma 10.1 (say with $P = I$) we have

$$(10.13) \quad \begin{aligned} A_{Q_2} &= (I + B S^{-1} B^* Q_2)^{-1} A_P (I + P C^* R^{-1} C) \\ &= (I + B S^{-1} B^* Q_2)^{-1} (I + B S^{-1} B^* Q_1) A_{Q_1}. \end{aligned}$$

Combining equations (10.12) and (10.13) we obtain

$$\begin{aligned} Q_1 - Q_2 &= A_{Q_2}^*(I + Q_2 B S^{-1} B^*)Q_1 A_{Q_1} - A_{Q_2}^* Q_2 (I + B S^{-1} B^* Q_1) A_{Q_1} \\ &= A_{Q_2}^*(Q_1 - Q_2)A_{Q_1}. \end{aligned}$$

\square

10.2. Proofs for Section 6. In this section we prove the relationships between the CARE and FARE of a system and the control and observation Lyapunov equations of its CARE closed-loop system (Lemmas 6.3, 6.4, 6.5 and 6.6).

LEMMA 10.5. *Suppose that the system $(\check{A}, \check{B}, [\check{C}_1; \check{C}_2], [\check{D}_1; \check{D}_2])$ is such that \check{D}_1 is boundedly invertible and that there exists a nonnegative selfadjoint operator V such that*

$$(10.14) \quad \check{B}^* V \check{A} + \check{D}^* \check{C} = 0.$$

Define the system (A, B, C, D) by (5.4) and $S := I + D^* D$ and $R := I + DD^*$. Then

1. $\check{A} = A - B(S + B^* V B)^{-1}(D^* C + B^* V A)$,
2. $\check{A}^* V \check{A} - V + \check{C}^* \check{C} = \check{A}^*(I + V B S^{-1} B^*) V \check{A} - V + C^* R^{-1} C$.

Proof. We first prove the equality

$$(10.15) \quad S \check{C}_1 = -(B^* V \check{A} + D^* C).$$

From (10.14) and (5.4) we obtain

$$\check{D}_1^* B^* V \check{A} + \check{D}_1^* \check{C}_1 + \check{D}_2^* \check{C}_2 = 0.$$

So

$$\check{C}_1 = -B^* V \check{A} - D^* \check{C}_2 = -B^* V \check{A} - D^*(C + D \check{C}_1)$$

and this yields (10.15):

$$S \check{C}_1 = (I + D^* D) \check{C}_1 = -B^* V \check{A} - D^* C.$$

We now prove the first equality stated in the lemma. We take the just proven equality (10.15) and substitute $\check{A} = A + B \check{C}_1$ to obtain

$$S \check{C}_1 = -(B^* V (A + B \check{C}_1) + D^* C).$$

So

$$(S + B^* V B) \check{C}_1 = -(B^* V A + D^* C).$$

We now solve for \check{C}_1 and substitute to obtain

$$\check{A} = A + B \check{C}_1 = A - B(S + B^* V B)^{-1}(B^* V A + D^* C).$$

We now prove the equality

$$(10.16) \quad \check{C}^* \check{C} = \check{A}^* V B S^{-1} B^* V \check{A} + C^* R^{-1} C.$$

We have

$$\check{C}^* \check{C} = \check{C}_1^* \check{C}_1 + \check{C}_2^* \check{C}_2$$

and substituting for \check{C}_2 from (5.4) gives

$$\check{C}^* \check{C} = \check{C}_1^* \check{C}_1 + (C + D \check{C}_1)^*(C + D \check{C}_1).$$

Finally, substituting for \check{C}_1 from (10.15) and simplifying gives the result. The second equality stated in the lemma follows easily from (10.16). \square

Proof of Lemma 6.3 We only have to prove (6.3). Noting (6.2), we apply Lemma 10.5 with $V := Q$ to the CARE closed-loop system. Since \check{A} defined by (5.1) equals A_Q defined by (10.2) we have

$$\check{A}^*Q\check{A} - Q + \check{C}^*\check{C} = A_Q^*(I + QB_S^{-1}B^*)QA_Q - Q + C^*R^{-1}C.$$

Since Q is a solution of the CARE (3.4) of (A, B, C, D) the right-hand side of this equation is zero by Lemma 10.2 and we have shown (6.3). \square

Proof of Lemma 6.4 By assumption there exists a nonnegative selfadjoint Q such that $\check{B}^*Q\check{A} + \check{D}^*\check{C} = 0$. We apply Lemma 10.5 with $V := Q$ to obtain $\check{A} = A_Q$ given by (10.2) and

$$\check{A}^*Q\check{A} - Q + \check{C}^*\check{C} = A_Q^*(I + QB_S^{-1}B^*)QA_Q - Q + C^*R^{-1}C.$$

The left-hand side of this equation is zero since by assumption Q satisfies (6.3). This proves that the right-hand side is zero and Lemma 10.2 now shows that Q is a solution of the CARE of the system (A, B, C, D) . \square

LEMMA 10.6. *Let a system $(\check{A}, \check{B}, [\check{C}_1; \check{C}_2], [\check{D}_1; \check{D}_2])$ with \check{D}_1 boundedly invertible be given and assume that a nonnegative selfadjoint operator V exists such that*

$$\check{B}^*V\check{B} + \check{D}^*\check{D} = I.$$

Define the system (A, B, C, D) by (5.4). Then we have

1. $B^*VB + S = \check{D}_1^{-*}\check{D}_1^{-1}$,
2. $\check{B}\check{B}^*(I + VBS^{-1}B^*) = BS^{-1}B^*$.

Proof. 1. The given equation for V translates to

$$\check{D}_1^*B^*VB\check{D}_1 + \check{D}_1^*\check{D}_1 + \check{D}_1^*D^*D\check{D}_1 = I$$

and multiplying from the left with \check{D}_1^{-*} and from the right with \check{D}_1^{-1} gives the result.

2. The first equality implies that $(S + B^*VB)^{-1} = \check{D}_1\check{D}_1^*$ and so $B(S + B^*VB)^{-1}B^* = \check{B}\check{B}^*$. Hence

$$\begin{aligned} \check{B}\check{B}^*(I + VBS^{-1}B^*) &= B(S + B^*VB)^{-1}B^*(I + VBS^{-1}B^*) \\ &= B(S + B^*VB)^{-1}(S + B^*VB)S^{-1}B^* = BS^{-1}B^* \end{aligned}$$

which proves the second equality. \square

We now prove Lemma 6.5.

Proof of Lemma 6.5 We remark that $L = (I + PQ)^{-1}P = P(I + QP)^{-1}$ and so we have to show that

$$(10.17) \quad \check{A}P(I + QP)^{-1}\check{A}^* - P(I + QP)^{-1} + \check{B}\check{B}^* = 0.$$

Recalling that $\check{A} = A_Q$ from (5.1) and (10.2) and using Lemmas 10.1 (10.5) and 10.3 we can substitute $\check{A} = (I + BS^{-1}B^*Q)^{-1}A_P(I + PC^*R^{-1}C)$ and $(I + QP)^{-1}\check{A}^* = A_P^*(I + QP)^{-1}$ to obtain for the left-hand side of (10.17)

$$\begin{aligned} &(I + BS^{-1}B^*Q)^{-1}A_P(I + PC^*R^{-1}C)PA_P^*(I + QP)^{-1} \\ &\quad - P(I + QP)^{-1} + \check{B}\check{B}^* \end{aligned}$$

$$\begin{aligned}
&= (I + BS^{-1}B^*Q)^{-1}[A_P(I + PC^*R^{-1}C)PA_P^* - (I + BS^{-1}B^*Q)P \\
&\quad + (I + BS^{-1}B^*Q)^{-1}\check{B}\check{B}^*(I + QP)](I + QP)^{-1}.
\end{aligned}$$

We now apply Lemma 10.6 with $V := Q$ to obtain $(I + BS^{-1}B^*Q)^{-1}\check{B}\check{B}^* = BS^{-1}B^*$ and so we obtain for the left-hand side of (10.17)

$$\begin{aligned}
&(I + BS^{-1}B^*Q)^{-1}[A_P(I + PC^*R^{-1}C)PA_P^* - (I + BS^{-1}B^*Q)P \\
&\quad + BS^{-1}B^*(I + QP)](I + QP)^{-1}.
\end{aligned}$$

The term in square brackets is zero according to the FARE (10.7) and we have proven (10.17). \square

Proof of Lemma 6.6 We first recall from the proof of Lemma 6.4 that $\check{A} = A_Q$ where A_Q is defined by (10.2) and that equation (10.6) holds with $A_Q = \check{A}$. Define $P := L(I - QL)^{-1}$ and define A_P by (10.1). The second step in the proof is establishing the identity

$$(10.18) \quad (I - LQ)A_P = \check{A}(I - LQ)$$

or by (10.6) the equivalent identity

$$(10.19) \quad (I - LQ)(I + BS^{-1}B^*Q)\check{A} = \check{A}(I - LQ)(I + PC^*R^{-1}C).$$

Since $P = L(I - QL)^{-1} = (I - LQ)^{-1}L$ the right-hand side of equation (10.19) is equal to

$$\check{A} - \check{A}LQ + \check{A}LC^*R^{-1}C.$$

We substitute $Q - C^*R^{-1}C = \check{A}^*(I + QBS^{-1}B^*)Q\check{A}$ (this identity holds because Q is a solution of the CARE, see (10.9)) to obtain for the right-hand side of (10.19)

$$\check{A} - \check{A}L\check{A}^*(I + QBS^{-1}B^*)Q\check{A}.$$

The control Lyapunov equation tells us that $\check{A}L\check{A}^* = L - \check{B}\check{B}^*$ and so the right-hand side of (10.19) is equal to

$$\check{A} - L(I + QBS^{-1}B^*)Q\check{A} + \check{B}\check{B}^*(I + QBS^{-1}B^*)Q\check{A}.$$

Substituting $\check{B}\check{B}^*(I + QBS^{-1}B^*) = BS^{-1}B^*$ from Lemma 10.6 with $V = Q$ we obtain for the right-hand side of (10.19)

$$\check{A} - L(I + QBS^{-1}B^*)Q\check{A} + BS^{-1}B^*Q\check{A},$$

which is equal to the left-hand side of (10.19). This proves (10.18). We now make the third and last step of the proof that P is a solution of the FARE. We start with the control Lyapunov equation

$$\check{A}L\check{A}^* - L + \check{B}\check{B}^* = 0$$

and substitute $\check{A} = (I - LQ)A_P(I - LQ)^{-1}$ from (10.18) and $\check{A}^* = (I + C^*R^{-1}CP)A_P^*(I + QBS^{-1}B^*)^{-1}$ from (10.6) to obtain

$$(I - LQ)A_P(I - LQ)^{-1}L(I + C^*R^{-1}CP)A_P^*(I + QBS^{-1}B^*)^{-1}$$

$$-L + \check{B}\check{B}^* = 0.$$

We multiply by $(I - LQ)^{-1}$ from the left and by $(I + QBS^{-1}B^*)$ from the right to obtain

$$\begin{aligned} & A_P P(I + C^*R^{-1}CP)A_P^* - P(I + QBS^{-1}B^*) \\ & + (I - LQ)^{-1}\check{B}\check{B}^*(I + QBS^{-1}B^*) = 0. \end{aligned}$$

We again use the fact that $\check{B}\check{B}^*(I + QBS^{-1}B^*) = BS^{-1}B^*$ to obtain

$$\begin{aligned} & A_P P(I + C^*R^{-1}CP)A_P^* - P - PQBS^{-1}B^* \\ (10.20) \quad & + (I - LQ)^{-1}BS^{-1}B^* = 0. \end{aligned}$$

Using that $P = (I - LQ)^{-1}L$ we see that the sum of the two last terms of the left-hand side of (10.20) equals $BS^{-1}B^*$. This proves that P is a solution of the equivalent version (10.7) of the FARE. \square

The following lemmas are needed in the proof of Lemma 6.9.

LEMMA 10.7. *Let X be a boundedly invertible nonnegative selfadjoint operator on a Hilbert space \mathcal{H} . Then $X \leq I$ iff $X^{-1} \geq I$.*

Proof. Assume that $X \leq I$. Then for all $x \in \mathcal{H}$ we have $\langle Xx, x \rangle \leq \langle x, x \rangle$. Define $y := X^{1/2}x$. Then $\langle y, y \rangle \leq \langle X^{-1}y, y \rangle$. Since X is boundedly invertible all elements in \mathcal{H} are of the form $X^{1/2}x$ for some $x \in \mathcal{H}$ and we have proven that $X^{-1} \geq I$. The ‘if’ part of the statement is proven similarly. \square

LEMMA 10.8. *Let S and T be nonnegative selfadjoint operators on a Hilbert space \mathcal{H} . Then $(I + T)^{-1} \geq (I + S)^{-1}$ iff $T \leq S$.*

Proof. It is easy to see that $(I + T)^{-1} \geq (I + S)^{-1}$ iff $I \geq (I + T)^{1/2}(I + S)^{-1}(I + T)^{1/2}$. By Lemma 10.7 this is true iff $I \leq (I + T)^{-1/2}(I + S)(I + T)^{-1/2}$ and this is true iff $I + T \leq I + S$. \square

LEMMA 10.9. *Let X_1 and X_2 be nonnegative selfadjoint operators and Y a positive selfadjoint operator on a Hilbert space \mathcal{H} . Then $X_2 \geq X_1$ iff $(I + X_2Y)^{-1}X_2 \geq (I + X_1Y)^{-1}X_1$.*

Proof. Since Y is positive and thus has dense range we have $X_2 \geq X_1$ iff $Y^{1/2}X_2Y^{1/2} \geq Y^{1/2}X_1Y^{1/2}$. Using Lemma 10.8 this is true iff

$$\begin{aligned} & I - Y^{1/2}(I + X_2Y)^{-1}X_2Y^{1/2} = (I + Y^{1/2}X_2Y^{1/2})^{-1} \\ & \leq (I + Y^{1/2}X_1Y^{1/2})^{-1} = I - Y^{1/2}(I + X_1Y)^{-1}X_1Y^{1/2}. \end{aligned}$$

This is true iff $Y^{1/2}(I + X_2Y)^{-1}X_2Y^{1/2} \geq Y^{1/2}(I + X_1Y)^{-1}X_1Y^{1/2}$ and again using that Y has dense range this is true iff $(I + X_2Y)^{-1}X_2 \geq (I + X_1Y)^{-1}X_1$. \square

10.3. Proofs for Section 7. In this subsection we prove a lemma used in the proof of Lemma 7.2.

LEMMA 10.10. *Let H_1, H_2 be separable Hilbert spaces, $Z \in \mathcal{L}(H_1, H_2)$, $T \in \mathcal{L}(H_2, H_1)$ and $\lambda \neq 0$. Then $\lambda \in \sigma(ZT)$ iff $\lambda \in \sigma(TZ)$.*

Proof. Suppose that $\lambda \in \rho(ZT)$ and $\lambda \neq 0$. Then we have

$$\frac{1}{\lambda}(I + T(\lambda I - ZT)^{-1}Z)(\lambda I - TZ) = I$$

$$= (\lambda I - TZ) \frac{1}{\lambda} (I + T(\lambda I - TZ)^{-1} Z)$$

and this implies that $\lambda I - TZ$ has a bounded inverse; i.e., $\lambda \in \rho(TZ)$. \square

That $\lambda \neq 0$ is really needed follows from the example of the left and right shift on $l^2(\mathbb{N})$. If we compose them in one way we get the identity, but the other possible composition is not invertible.

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