SHIFT REALIZATIONS AND THEIR ALGEBRAIC RICCATI EQUATIONS

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Shift realizations and their algebraic Riccati equations *

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Abstract

We study the solutions of the discrete time algebraic Riccati equation for stable linear systems. We give sharpened existence results for such solutions (in comparison to our earlier work [4]) under an extra assumption that makes the state space isomorphism techniques applicable. This extends some of the known results on the parameterization of stable spectral factors for realizations with infinite dimensional state spaces.

Keywords: Algebraic Riccati equation, spectral factorization, minimality, state space isomorphism.

AMS Classification: 47A15, 47A68, 47N70, 93B28.

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Notations

The set of complex numbers and real numbers are denoted by $\mathbb{C}$ and $\mathbb{R}$, respectively. The right and the left half plane are denoted by $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \Re s > 0\}$ and $\mathbb{C}_- := \{s \in \mathbb{C} \mid \Re s < 0\}$. Positive and negative real numbers are written by $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$ and $\mathbb{R}_- := \{x \in \mathbb{R} \mid x < 0\}$. Imaginary axis is $i\mathbb{R}$. The open unit disc and the unit circle are $\mathbb{D}$ and $\mathbb{T}$, respectively. Natural numbers, integers, nonnegative integers and negative integers are denoted by $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{Z}_- := \mathbb{Z} \backslash \mathbb{Z}_+$.

The letters U, V, Y and H denote (possibly infinite-dimensional) separable Hilbert spaces. For any such U, its inner product is denoted by $\langle \cdot, \cdot \rangle_U$, its norm by $||\cdot||_U$, and its identity operator by I_U. The closure and the orthogonal complement of any set $S \subset U$ are denoted by $\overline{S}$ and $S^\perp$, respectively. Sometimes we write also $S^\perp = U \ominus S$, to emphasize that the orthogonal complement is to be taken in U.

The bounded operators from U to Y are denoted by $\mathcal{L}(U;Y)$, and if $U = Y$ we write $\mathcal{L}(U;U) = \mathcal{L}(U)$. The $\mathcal{L}(U;Y)$-valued bounded analytic functions in $\mathbb{D}$ are denoted by $H^\infty(\mathbb{D}; \mathcal{L}(U;Y))$.

Given a Hilbert space U, we define the sequence spaces

$$
\begin{aligned}
&Seq\,(U) := \big\{ \{z_i\}_{i\in\mathbb{Z}} \mid z_i \in U \quad \text{and} \quad \exists I \in \mathbb{Z} \quad \forall i \le I : z_i = 0 \big\},\\
&Seq_+\,(U) := \big\{ \{z_i\}_{i\in\mathbb{Z}} \mid z_i \in U \quad \text{and} \qquad \forall i < 0 : z_i = 0 \big\},\\
&Seq_-\,(U) := \big\{ \{z_i\}_{i\in\mathbb{Z}} \in Seq\,(U) \mid z_i \in U \quad \text{and} \qquad \forall i \ge 0 : z_i = 0 \big\},\\
&\ell^2(\mathbb{Z}; U) := \big\{ \{z_i\}_{i\in\mathbb{Z}} \subset U \mid \sum_{i\in\mathbb{Z}} ||z_i||_U^2 < \infty \big\} \quad \text{for} \quad 1 \le p < \infty,\\
&\ell^2(\mathbb{Z}_+; U) := \big\{ \{z_i\}_{i\in\mathbb{Z}_+} \subset U \mid \sum_{i\in\mathbb{Z}_+} ||z_i||_U^2 < \infty \big\} \quad \text{for} \quad 1 \le p < \infty.\\
&\ell^2(\mathbb{Z}_-; U) := \big\{ \{z_i\}_{i\in\mathbb{Z}_-} \subset U \mid \sum_{i\in\mathbb{Z}_-} ||z_i||_U^2 < \infty \big\} \quad \text{for} \quad 1 \le p < \infty.
\end{aligned}
$$

The following linear operators are defined for $\tilde{z} \in Seq\,(U)$:

- the projections for $j, k \in \mathbb{Z} \cup \{\pm\infty\}$

$$
\begin{aligned}
&\pi_{[j,k]}\tilde{z} := \{w_j\}; \quad w_i = z_i \quad \text{for} \quad j \le i \le k, \quad w_i = 0 \quad \text{otherwise},\\
&\pi_j := \pi_{[j,j]}, \quad \pi_+ := \pi_{[1,\infty]}, \quad \pi_- := \pi_{[-\infty,-1]},\\
&\bar{\pi}_+ := \pi_0 + \pi_+, \quad \bar{\pi}_- := \pi_0 + \pi_-,
\end{aligned}
$$

- the bilateral forward time shift τ and its inverse, the backward time shift τ^*

$$\tau\tilde{u} := \{w_j\} \quad \text{where} \quad w_j = u_{j-1},$$
$$\tau^*\tilde{u} := \{w_j\} \quad \text{where} \quad w_j = u_{j+1}.$$

The spaces $\ell^2(\mathbb{Z}_-;U)$ and $\ell^2(\mathbb{Z}_+;U)$ are regarded naturally as closed subspaces of $\ell^2(\mathbb{Z};U)$ as well as ranges of π_- and $\bar{\pi}_+$, respectively.

1 Introduction

Let $\mathcal{D} : \ell^2(\mathbb{Z};U) \to \ell^2(\mathbb{Z};Y)$ be a bounded linear operator which is causal and shift (or translation) invariant in the sense that

$$\pi_-\mathcal{D}\bar{\pi}_+ = 0 \quad \text{and} \quad \tau\mathcal{D} = \mathcal{D}\tau. \tag{1}$$

As it is well-known, such operators appear as I/O maps of discrete time linear systems, and they are equivalent to the multiplication by the corresponding transfer function $\widehat{\mathcal{D}} \in H^\infty(\mathbb{D};\mathcal{L}(U;Y))$ [1].

It has been known from the 1960's that factorizations of the (causal) Hankel operator

$$\Gamma_{\mathcal{D}} := \bar{\pi}_+\mathcal{D}\pi_- \in \mathcal{L}(\ell^2(\mathbb{Z}_-;U);\ell^2(\mathbb{Z}_+;U))$$

induce certain state space realizations of $\mathcal{D}$ that are referred to as being "canonical". First such results were given by probably by Kalman for rational transfer functions in terms of (algebraic) polynomial models; the state space of the constructed realization is a certain factorial monoid. For a lucid exposition, see [1] and all the classical references therein.

Analogous canonical state space realizations can be (and, indeed, have been) given for pretty close any thinkable class of shift invariant operators (both in continuous and discrete time), by using as the state space e.g. the closure of range$(\Gamma_{\mathcal{D}})$ (in the discrete time case case taken in $\ell^2(\mathbb{Z}_+;Y)$. For this reason, some of such realizations are often called *Hankel range realizations*. However, the state space can be chosen in a number of other ways — indeed, other variants are given in Definitions 3.1 and 3.2 of this paper. We remark that for continuous time *well-posed linear systems*, corresponding realization results are given in [5]. The analogous discrete time case does not entail as severe technical complications as does the "well-posed" case.

[1] In this introduction, we use freely the definitions and notations from Section 2. See also [4, Chapter 1] for further details and proofs.

Let us recall some notation from discrete time linear systems (shortly, DLSs). Any state space realization of $\mathcal{D}$ is the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$, described in the usual manner by the system of difference equations on Hilbert spaces

$$\begin{cases} x_{j+1} &= Ax_j + Bu_j, \\ y_j &= Cx_j + Du_j, \quad j \in \mathbb{Z}; \end{cases}$$

and it is related to $\mathcal{D}$ by the transfer function identity

$$\widehat{\mathcal{D}}(z) = D + zC\,(I - zA)^{-1}\,B \quad \text{for all} \quad z \in \mathbb{D}.$$

Any DLS ϕ together with a self-adjoint $J \in \mathcal{L}(Y)$ defines the familiar *discrete time algebraic Riccati equation (shortly, DARE)*

$$\text{(2)} \qquad \begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P, \\ \Lambda_P = D^*JD + B^*PB, \\ \Lambda_P K_P = -D^*JC - B^*PA, \end{cases}$$

whose certain self-adjoint solutions $P \in ric_0(\phi, J)$ — the *regular H^∞-solutions* in the language of [4] — are of particular operator theoretic interest.

The purpose of this paper is to study the regular H^∞-solutions and associated factorization results in the special case when ϕ (defining DARE (2) is any of the "canonical" realization of its I/O map $\mathcal{D}$. We shall give full proofs of even all purely technical results, in case they have not already been treated in [4].

Considering such special cases can be defended from the following points of view:

- An assumption can be removed from many factorization results involving $\mathcal{D}$ or $\mathcal{D}^*J\mathcal{D}$ in [4], if the special structure of the canonical realizations ϕ of $\mathcal{D}$ can be used. See the discussion following Lemma 4.1;
- Such sharpened results can be carried over to arbitrary *minimal* realizations ϕ if $\mathcal{D}$, in case when range $(\Gamma_{\mathcal{D}})$ is closed; see the main result of this paper, Theorem 7.1. Namely, such ϕ's are then equivalent to *exactly observable and controllable* canonical realization whose main operator is a part of a shift. In particular, we see that large families of regular H^∞-solutions $P \in ric_0(\phi, J)$ exists for such operator DAREs. All these results necessarily depend on the State Space Isomorphism Theorem 6.1 that does not hold in infinite dimensions without extra assumptions.

- Finally, one should never fail to check “abstract infinite-dimensional” system theory results in terms of some concrete realizations whose structure is known and amenable to computations. The following is typical for the canonical (Hankel range) realizations: the state space variants of various problems give back the “state space -free” version of the same problem, when applied to canonical realizations[2]. This is seen for discrete time algebraic Riccati equations in the results of this paper.

Only for the fun of it, we give in Theorem 8.1 rather general conditions (in terms of inner-outer factorizations) for the closedness of the range of Hankel operators, allowing operator-valued functions as their symbols.

Most of this paper was written during my somewhat interesting post-doc episode in summer 2000 at Imperial College, London. The proof reading with some additions and corrections was partly carried out in spring 2003 at the Mittag–Leffler Institute, Stockholm. Also that was an interesting episode but in a different way.

2 Background on linear systems

In this preparatory section we recall some notions and basic facts from the theory of discrete time linear systems and their algebraic Riccati equations.

Let U, H and Y be separable Hilbert spaces, and let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(U;H)$, $C \in \mathcal{L}(H;Y)$ and $D \in \mathcal{L}(U;Y)$. Then a *discrete time linear system* (DLS) on U, H and Y is the quadruple $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, defining the system of

[2]It is then very much in doubt, if such “infinite-dimensional” state space results should be regarded as solutions of the original problem at all. As a rule, a general state space solution “R” that claim to be a contribution to any given “state space -free” problem “P” is in fact quite useless, unless it restricts the mathematical form of the realization according to the practical application that one is interested in.

Namely, if result “R” can be applied on minimal canonical realizations under the restrictive conditions that ensure the state space isomorphism to hold, it only changes (in this particular case) the “state-space free” problem to the same problem but merely written in different symbols — a problem posed in “abstract” operators in an “abstract” state space. For more general realizations (not having e.g. the state space isomorphism property), the original problem “P” is transformed by the solution “R” to a more difficult problem (of which we know less) than the original one. This is quite disencouraging when it comes to such “abstract theories” that do not assume anything on their realizations that does **not** hold for canonical realizations.

This might to cast some light on what is “reasonable” regarding purely functional analytic “infinite-dimensional state space” generalizations, nowadays appearing in system theory.

difference equations

$$
(3) \qquad \begin{cases} x_{j+1} & = Ax_j + Bu_j \\ y_j & = Cx_j + Du_j, \quad j \in \mathbb{Z}, \end{cases}
$$

where the sequences $\tilde{u} := \{u_j\}_{j\in\mathbb{Z}} \subset U$, $\{x_j\}_{j\in\mathbb{Z}} \subset X$, $\tilde{y} := \{y_j\}_{j\in Z} \subset Y$. We call U the *input space*, Y the *output space* and H the *state space* of ϕ. For the solvability of (3) we assume that the input sequence $\tilde{u}$ has only finitely many nonzero elements u_j for $j < 0$, and the initial state is set by $x_N = 0$ for some N negative enough; in symbols $\tilde{u} \in Seq_-(U)$.

The operator A is called the *main operator* (or sometimes *semigroup generator*) of ϕ. As usual, the *controllability* and *observability* maps of DLS ϕ are defined by

$$
(4) \qquad \begin{aligned} \mathcal{B}_\phi \tilde{u} &:= \sum_{j>1} A^j B u_{-j} \in X, \\ \mathcal{C}_\phi x &:= \{CA^j x\}_{j\geq 0} \subset Y, \quad x \in X, \end{aligned}
$$

where $\tilde{u} := \{u_j\}_{j<0} \subset U$ again has only finitely many nonzero elements, in order to have the sum well-defined. Roughly speaking, $\mathcal{B}_\phi : Seq_-(U) \to H$ maps past inputs into present states, and $\mathcal{C}_\phi : H \to Seq_+(Y)$ maps present states into future outputs.

Finally the mapping $\mathcal{D}_\phi : Seq(u) \ni \tilde{u} \mapsto \tilde{y} \in Seq(Y)$ described by (3) is called *input-output mapping* of ϕ. Conversely, the DLS ϕ is called a *realization* of it I/O map $\mathcal{D}_\phi$.

When the mappings

$$
\begin{aligned} &\mathcal{D}_\phi : \ell^2(\mathbb{Z}_+; U) \subset Seq_-(U) \to \ell^2(\mathbb{Z}_+; Y) \subset Seq_-(Y), \\ &\mathcal{B}_\phi : Seq_-(U) \subset \ell^2(\mathbb{Z}_-; U) \to H \quad \text{and} \\ &\mathcal{C}_\phi : H \to \ell^2(\mathbb{Z}_+; Y) \subset Seq_+(Y) \end{aligned}
$$

map boundedly between (the dense subsets of) the indicated Hilbert spaces, we say that the corresponding ϕ is *I/O stable*, *input stable* and *output stable*, respectively. In this case, we obtain linear bounded extensions by density and shift-invariance, denoted by $\mathcal{D}_\phi : \ell^2(\mathbb{Z}; U) \to \ell^2(\mathbb{Z}; Y)$ and $\overline{\mathcal{B}_\phi} : \ell^2(\mathbb{Z}_-; U) \to H$[3]. An input, output and I/O stable ϕ with a power-bounded main operator is called *stable*.

The controllability, observability and minimality notions are so essential in this report that we reserve them a formal definition:

[3]There will be need (for clarity) to write an overline on the extended version of the controllability map, but similar danger of confusion does not occur in the case of I/O maps

Definition 2.1. *We say that ϕ is* approximately (infinite-time) controllable *if*

$$\text{range}\,(\mathcal{B}_\phi) := \mathcal{B}_\phi\, Seq_-\,(U) \tag{5}$$

is dense in H, and approximately (infinite-time) observable *if* $\ker(\mathcal{C}_\phi) = \{0\}$. *An approximately observable and controllable DLS is called* minimal*.*

Two minimal, stable realizations of a same I/O stable I/O map can be extremely different, even if they are weakly isomorphic in the sense of Lemma 6.1. An example of this can be found in e.g. [1].

An input stable DLS ϕ is *exactly (infinite-time) controllable* if range $\left(\overline{\mathcal{B}_\phi}\right) = H$[4] and *exactly (infinite-time) observable* if $\mathcal{C}_\phi$ is coercive from H into $\ell^2(\mathbb{Z}_+;Y)$. It is well known that any bounded, causal and shift invariant operator $\mathcal{D} : \ell^2(\mathbb{Z};U) \to \ell^2(\mathbb{Z};Y)$ (for definition, see (1)) is an I/O map of a minimal stable DLS which than can even be chosen to satisfy an energy dissipation inequality if $\mathcal{D}$ is contractive. Hence, we call such operators $\mathcal{D}$ simply "I/O -maps of I/O stable DLS's" for the rest of this paper.

The *transfer function* of ϕ is defined by

$$\widehat{\mathcal{D}}_\phi(z) := D + zC(I - zA)^{-1}B \quad \text{for all} \quad z^{-1} \notin \sigma(A).$$

As is well known, the I/O map $\mathcal{D}_\phi$ (for z-transformable input sequences $\tilde{u}$) can be represented by a multiplication by $\widehat{\mathcal{D}}_\phi$. Also ϕ is I/O stable if and only if $\widehat{\mathcal{D}}_\phi \in H^\infty(\mathbb{D};\mathcal{L}(U;Y))$.

The *(causal) Hankel operator*[5] of the I/O stable I/O map $\mathcal{D}_\phi$ is denoted by $\bar{\pi}_+\mathcal{D}_\phi\pi_-$, and it maps the past input space $\ell^2(\mathbb{Z}_-;U)$ into the future output space $\ell^2(\mathbb{Z}_+;Y)$. The Hankel operator is connected to $\mathcal{B}_\phi$ and $\mathcal{C}_\phi$ by the following crucial factorizing relation, the Hankel condition

$$\bar{\pi}_+\mathcal{D}_\phi\pi_- = \mathcal{C}_\phi\mathcal{B}_\phi \quad \text{on all of} \quad \ell^2(\mathbb{Z}_-;U). \tag{6}$$

In fact, any DLS ϕ can be fully characterized by its main operator A, three operators $\mathcal{B}_\phi$, $\mathcal{C}_\phi$ and $\mathcal{D}_\phi$ satisfying (6) and the intertwining equations

$$\tau\mathcal{D}_\phi = \mathcal{D}_\phi\tau, \quad \bar{\pi}_+\tau^*\mathcal{C}_\phi = \mathcal{C}_\phi A, \quad \text{and} \quad \mathcal{B}_\phi\tau^*\pi_+ = A\mathcal{B}_\phi, \tag{7}$$

(and some self-evident boundedness assumptions) where the second equation is posed on H and the two other equations on $\ell^2(\mathbb{Z}_-;U)$. In fact it is sometimes more practical to use operators A, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ (satisfying (6) and (7)

[4] Note that we write range $\left(\overline{\mathcal{B}_\phi}\right) := \overline{\mathcal{B}_\phi}\ell^2(\mathbb{Z}_-;U)$ instead of range$\,(\mathcal{B}_\phi)$, by which we mean what is defined in (5)

[5] In Kalmans use of language the Hankel operator is known as the "restricted I/O maps"

but written without subindex ϕ) without explicit reference to operators C, B and D. We say that the DLS is given in *I/O form*, and it is then denoted by the quadruple $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$; for all the details that remain obscure, see [4, Chapter 1] that hopefully helps.

We proceed to discuss the *(discrete time) algebraic Riccati equations (DARE)*. Such equations arise in e.g. (inner-outer; spectral) factorization and optimal control problems involving DLSs; for a comprehensive operator-theoretic treatment see [4, Chapters 2, 3 and 4]. We only recall some notations and basic facts in the following.

Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS and $J \in \mathcal{L}(Y)$ be a self-adjoint, possibly non-definite *cost operator*. Then the shift-invariant operator $\mathcal{D}_\phi^* J \mathcal{D}_\phi$ is called the *Popov operator*, and its frequency domain counterpart is just the common spectral function. (Sometimes the Toepliz operator $\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+$ is called Popov operator instead.) The *discrete time algebraic Riccati equation (DARE)* is given by

$$
\left\{
\begin{array}{l}
A^*PA - P + C^*JC = K_P^* \Lambda_P K_P, \\
\Lambda_P = D^*JD + B^*PB, \\
\Lambda_P K_P = -D^*JC - B^*PA,
\end{array}
\right. \tag{8}
$$

where $P = P^* \in \mathcal{L}(H)$ is the operator to be solved. We always require from the solution P that its *indicator operator* Λ_P satisfies $\Lambda_P, \Lambda_P^{-1} \in \mathcal{L}(U)$ and that its *feedback operator* satisfies $K_P \in \mathcal{L}(H; U)$. In this case we write $P \in Ric(\phi, J)$, but the symbol $Ric(\phi, J)$ refers also DARE (8) defined by ϕ and J.

It the defining ϕ is both output stable and I/O stable, then equation (8) is called H^∞DARE, and it is then referred to by the symbol $ric(\phi, J)$. Moreover, we say that $P \in Ric(\phi, J)$ is a *regular H^∞-solution* of H^∞DARE (8), if the associated *spectral DLS* $\phi_P := \left(\begin{smallmatrix} A & B \\ -K_P & I \end{smallmatrix}\right)$ is both output stable and I/O stable, and the *residual cost operator*

$$L_{A,P} := \operatorname*{slim}_{j \to \infty} A^{*j} P A^j$$

exists and vanishes[6]. In this case we write shortly $P \in ric_0(\phi, J)$.

It is shown in [4, Chapters 3 and 4][7] under very mild technical assumptions that for a stable ϕ, the solutions $P \in ric_0(\phi, J)$ correspond injectively to the causal I/O stable spectral factors $\mathcal{D}_{\phi_P}$ of the Popov operator $\mathcal{D}_\phi^* J \mathcal{D}_\phi$. Moreover, in the case of *positive* cost operator J, this correspondence

$$ric_0(\phi) \ni P \mapsto \operatorname{range}(\mathcal{D}_{\phi_P} \bar{\pi}_+) \subset \ell^2(\mathbb{Z}_+; U)$$

[6]The symbol slim denotes the limit taken in strong operator topology.

[7]See also [2, 3, **?**] for another, earlier approach in a different setting.

is order-preserving if the solution set $ric_0(\phi, J)$ is partially ordered as self-adjoint operators, and the ranges of the Toeplitz operators are partially ordered by the subspace inclusion.

3 Canonical realizations

In this section we define and describe some canonical realizations, whose main operator A is a part of either the bilateral or the unilateral backward shift, denoted by τ^* and $S^* := \bar{\pi}_+ \tau^*$, respectively.

3.1 Realization on the bilateral shift

Any factorization $\mathcal{D} = \mathcal{T}_1 \mathcal{T}_2$ into shift-invariant factors $\mathcal{T}_1$ and $\mathcal{T}_2$ is associated to a realization of $\mathcal{D}$ in the following lemma. That no causality properties for factors are assumed, amounts to the fact that a part of a *bilateral* shift must be used as a main operator.

Lemma 3.1. *Let $\mathcal{D} : \ell^2(\mathbb{Z}; U) \to \ell^2(\mathbb{Z}; Y)$ be an I/O map of an I/O stable DLS. Let $\mathcal{T}_1 : \ell^2(\mathbb{Z}; V) \to \ell^2(\mathbb{Z}; Y)$ and $\mathcal{T}_2 : \ell^2(\mathbb{Z}; U) \to \ell^2(\mathbb{Z}; V)$ be bounded, shift-invariant (but possibly noncausal) operators, such that $\mathcal{D} = \mathcal{T}_1 \mathcal{T}_2$. Then the quadruple of linear operators*

$$\Phi_b(\mathcal{T}_1, \mathcal{T}_2) := \begin{bmatrix} (\tau^*|H)^j & \mathcal{T}_2 \pi_- \tau^{*j} \\ \bar{\pi}_+ \mathcal{T}_1 | H & \mathcal{D} \end{bmatrix} \text{ with } H := \overline{\mathrm{range}\,(\mathcal{T}_2 \pi_-)} \tag{9}$$

is an approximately controllable, input stable, output stable and I/O stable DLS, whose state space H is closed in and equipped with the inner product of $\ell^2(\mathbb{Z}; V)$.

Proof. To shorten the notations, define $A := \tau^*|H$, $\mathcal{B} := \mathcal{T}_2 \pi_- | \, Seq_-(U)$ and $\mathcal{C} = \bar{\pi}_+ \mathcal{T}_1 | H$. We proceed to show that the quadruple $\Phi := \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix} \right]$ is a DLS in I/O form.

We first show that A is a contraction in $\mathcal{L}(H)$, and valid as a main operator of a DLS. For any $x \in \mathrm{range}\,(\mathcal{B})$ there is a $\tilde{u} \in \ell^2(\mathbb{Z}_-; U)$ such that $x = \mathcal{T}_2 \pi_- \tilde{u}$. Now

$$\tau^* x = \tau^* \mathcal{T}_2 \pi_- \tilde{u} = \mathcal{T}_2 \tau^* \pi_- \tilde{u} = \mathcal{T}_2 \pi_- \cdot (\pi_{[-\infty,-2]} \tau^* \tilde{u}) \in \mathrm{range}\,(\mathcal{B})\,.$$

By the contractivity of τ^*, we have $\tau^* H \subset H$. Hence $A = \tau^*|H$ defines an element of $\mathcal{L}(H)$ which was to be shown.

To deal with the controllability operator candidate, we have to show that $\mathcal{B}\tau^* = A\mathcal{B} + B\pi_0$ where $B := \mathcal{B}\tau^*\pi_0 = \mathcal{B}\pi_{-1}\tau^* \in \mathcal{L}(U;H)$, with the obvious identification of spaces U and range (π_0). We have

$$\begin{aligned}\mathcal{B}\tau^* &= \mathcal{T}_2\pi_-\tau^* = \mathcal{T}_2(\pi_{-1} + \pi_{[-\infty,-2]})\tau^* \\ &= \mathcal{T}_2\tau^*\pi_0 + \mathcal{T}_2\tau^*\pi_- = B\pi_0 + \tau^*\mathcal{T}_2\pi_- = B\pi_0 + A\mathcal{B}.\end{aligned}$$

The observability part $\bar{\pi}_+\tau^*\mathcal{C} = \mathcal{C}A$ is even more trivial. By the boundedness of $\mathcal{B}$, $C = \pi_0\mathcal{C} \in \mathcal{L}(H;Y)$ with the obvious identification of spaces Y and range (π_0). By a direct calculation

$$\begin{aligned}\bar{\pi}_+\tau^*\mathcal{C} &= \bar{\pi}_+\tau^* \cdot \bar{\pi}_+\mathcal{T}_1|H = \bar{\pi}_+\tau^*\pi_+\mathcal{T}_1|H \\ &= \bar{\pi}_+\tau^*\mathcal{T}_1|H = \bar{\pi}_+\mathcal{T}_1 \cdot \tau^*|H = \bar{\pi}_+\mathcal{T}_1|H \cdot \tau^*|H = \mathcal{C}A,\end{aligned}$$

where the second to the last equality follows from the already proved fact that H is τ^*-invariant. Noting that the Hankel condition $\mathcal{C}\mathcal{B} = \bar{\pi}_+\mathcal{D}\pi_-$ is immediate from the assumption that $\mathcal{D} = \mathcal{T}_1\mathcal{T}_2$, we conclude that $\Phi_b(\mathcal{T}_1, \mathcal{T}_2)$ is an I/O stable DLS. The rest of the claims are trivial. □

It is now time to give things names:

Definition 3.1. *The DLS $\Phi_b(\mathcal{T}_1, \mathcal{T}_2)$ as defined in Lemma 3.1 is called the* bilateral backward shift realization (shortly, BBSR) *of the I/O map $\mathcal{D}$, associated to the factorization $\mathcal{D} = \mathcal{T}_1\mathcal{T}_2$.*

Recall that exact (infinite time) controllability of input stable DLS $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right]$ means that the (extended) controllability map satisfies range $\left(\overline{\mathcal{B}}\right) = H$. For general exactly controllable realizations, it is possible that ker $\left(\overline{\mathcal{B}}\right) \neq \{0\}$. The exact controllability of BBSR is dealt in the following:

Proposition 3.1. *Let $\mathcal{D} : \ell^2(\mathbb{Z};U) \to \ell^2(\mathbb{Z};Y)$ be an I/O map of an I/O stable DLS and let the operator $\mathcal{T}_1$ and $\mathcal{T}_2$ be as in Lemma 3.1. Assume, in addition, that $\mathcal{T}_2$ has a bounded inverse in $\mathcal{L}(\ell^2(\mathbb{Z};V);\ell^2(\mathbb{Z};U))$.*

Then the BBSR $\Phi_b(\mathcal{T}_1, \mathcal{T}_2)$ is exactly (infinite time) controllable, and its (extended) controllability map $\overline{\mathcal{B}}$ is a bounded bijection from $\ell^2(\mathbb{Z}_-;U)$ onto H.

Proof. Clearly the (extended) controllability map

$$\mathcal{T}_2\pi_- : \ell^2(\mathbb{Z}_-;U) \to \overline{\text{range}\,(\mathcal{T}_2\pi_-)} \subset \ell^2(\mathbb{Z};V)$$

has a bounded left inverse $\pi_-\mathcal{T}_2^{-1}|\overline{\text{range}\,(\mathcal{T}_2\pi_-)}$ which implies the coercivity of $\mathcal{T}_2\pi_-$. The exact controllability of $\Phi_b(\mathcal{T}_1, \mathcal{T}_2)$ follows because coercive operators have closed ranges, and $\Phi_b(\mathcal{T}_1, \mathcal{T}_2)$ is approximately controllable by Lemma 3.1. □

It is close to a standing assumption in [4] that $\mathcal{D}$ has a (J, S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$; here $J \in \mathcal{L}(Y)$ and $S \in \mathcal{L}(U)$ are self-adjoint cost operators, and the outer factor $\mathcal{X}$ has a bounded inverse in $\ell^2(\mathbb{Z}; U)$. Then the backward shift realization $\Phi_b(\mathcal{N}, \mathcal{X}) = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ satisfies the conditions of 3.1. Moreover, it is well known that if the Toeplitz (Popov) operator $\bar{\pi}_+\mathcal{D}^*\mathcal{D}\bar{\pi}_+$ is coercive, then an $(I; S)$-inner-outer factorization exists. In particular, any such a $\mathcal{D}$ has an input stable and output stable realization, whose main operator $A \in \mathcal{L}(H)$ is a restriction of a bilateral shift and $\overline{\mathcal{B}} : \ell^2(\mathbb{Z}_-; U) \to H$ is a bounded bijection between the indicated spaces. Hence, the space $\ell^2(\mathbb{Z}_-; U)$ could be regarded as the state space of such a realization as well, but we omit these considerations because they are inessential for the purpose of this paper.

3.2 Realization on the unilateral shift

We now proceed to give another type of canonical realization for $\mathcal{D}$, whose main operator is now a part of a *unilateral* shift S^*. We start from a factorization $\mathcal{D} = \mathcal{T}_1\mathcal{T}_2$ where the first factor $\mathcal{T}_1$ is now an *anticausal*, bounded and shift-invariant operator. Such operators appear trivially as the adjoints of I/O maps for I/O stable DLSs.

Lemma 3.2. *Let $\mathcal{D} : \ell^2(\mathbb{Z}; U) \to \ell^2(\mathbb{Z}; Y)$ be an I/O map of an I/O stable DLS. Let $\mathcal{T}_1 \in \ell^2(\mathbb{Z}; V) \to \ell^2(\mathbb{Z}; Y)$ and $\mathcal{T}_2 : \ell^2(\mathbb{Z}; U) \to \ell^2(\mathbb{Z}; V)$ be bounded and shift-invariant operators. Assume that $\mathcal{D} = \mathcal{T}_1\mathcal{T}_2$ and that $\mathcal{T}_1$ is anticausal; i.e. $\bar{\pi}_+\mathcal{T}_1\pi_- = 0$. Then the quadruple of linear operators*

$$\Phi_u(\mathcal{T}_1, \mathcal{T}_2) := \begin{bmatrix} (S^*|H)^j & \bar{\pi}_+\mathcal{T}_2\pi_-\tau^{*j} \\ \bar{\pi}_+\mathcal{T}_1|H & \mathcal{D} \end{bmatrix} \text{ with } H := \overline{\mathrm{range}\,(\bar{\pi}_+\mathcal{T}_2\pi_-)} \tag{10}$$

defines an approximately controllable, input stable, output stable and I/O stable DLS, whose state space H is closed in and equipped with the inner product of $\ell^2(\mathbb{Z}_+; V)$. Moreover, the main operator of $\Phi_u(\mathcal{T}_1, \mathcal{T}_2)$ is a strongly stable contraction.

Proof. For brevity, we define again $A := S^*|H$, $\mathcal{B} := \bar{\pi}_+\mathcal{T}_2\pi_-|\, Seq_-(U)$ and $\mathcal{C} = \bar{\pi}_+\mathcal{T}_1|H$. We proceed to show that the quadruple $\Phi := \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ is a DLS in I/O form, which amounts to doing some cleaning work.

We first show that the state space H is invariant under A, and $A \in \mathcal{L}(H)$. (This involves only showing that the range of a Hankel operator is S^*-invariant, but we do it the hard "system theory way"). For any $x \in \mathrm{range}\,(\mathcal{B})$ there is a $\tilde{u} \in Seq_-(U)$ such that $x = \bar{\pi}_+\mathcal{T}_2\pi_-\tilde{u}$. Thus for any such x

$$\begin{aligned} S^*x &= \bar{\pi}_+\tau^* \cdot \bar{\pi}_+\mathcal{T}_2\pi_-\tilde{u} = \bar{\pi}_+\tau^*\mathcal{T}_2\pi_-\tilde{u} \\ &= \bar{\pi}_+\mathcal{T}_2\tau^*\pi_-\tilde{u} = \bar{\pi}_+\mathcal{T}_2\pi_- \cdot (\pi_{[-\infty,-2]}\tau^*\tilde{u}) \in \mathrm{range}\,(\mathcal{B})\,. \end{aligned}$$

Because S^* is a bounded operator $\ell^2(\mathbb{Z}_+;V)$, also the closure of range $(\mathcal{B})$ in $\ell^2(\mathbb{Z}_+;V)$ (equalling H by definition) is invariant under S^*. Because S^* is a strongly stable contraction on $\ell^2(\mathbb{Z}_+;V)$, we conclude that A generates a strongly stable discrete semigroup in $\mathcal{L}(H)$.

We proceed to show that $\mathcal{B}$ is a valid controllability operator for a DLS whose main operator is $A = S^*|H$. As in Lemma 3.1, we show that that $\mathcal{B}\tau^* = A\mathcal{B} + B\pi_0$ where $B := \mathcal{B}\tau^*\pi_0 = \mathcal{B}\pi_{-1}\tau^* \in \mathcal{L}(U;H)$, with the obvious identification of spaces U and range (π_0). We have on $Seq_-(U)$

$$\begin{aligned}\mathcal{B}\tau^* &= \bar{\pi}_+\mathcal{T}_2\pi_-\tau^* = \bar{\pi}_+\mathcal{T}_2(\pi_{-1} + \pi_{[-\infty,-2]})\tau^* \\ &= \bar{\pi}_+\mathcal{T}_2\pi_- \cdot \tau^*\pi_0 + \bar{\pi}_+\mathcal{T}_2\tau^*\pi_- = B\pi_0 + \bar{\pi}_+\tau^* \cdot \bar{\pi}_+\mathcal{T}_2\pi_- \\ &= B\pi_0 + S^*|H \cdot \bar{\pi}_+\mathcal{T}_2\pi_- = B\pi_0 + A\mathcal{B}.\end{aligned}$$

For the observability intertwining $\bar{\pi}_+\tau^*\mathcal{C} = \mathcal{C}A$, the anticausality assumption is used. By a direct calculation, we have on H

$$\begin{aligned}\mathcal{C}A &= \bar{\pi}_+\mathcal{T}_1|H \cdot S^*|H = \bar{\pi}_+\mathcal{T}_1\bar{\pi}_+\tau^*|H \\ &= \bar{\pi}_+\mathcal{T}_1\tau^*\pi_+|H = \bar{\pi}_+\tau^* \cdot \pi_+\mathcal{T}_1\pi_+|H,\end{aligned}$$

where the second equality follows from the already proved fact that H is $S^* = \bar{\pi}_-\tau^*$-invariant. Because $\mathcal{T}_1$ is anticausal and shift-invariant, $\pi_+\mathcal{T}_1\pi_+ = \pi_+\mathcal{T}_1\bar{\pi}_+$. But then we may continue the previous calculation by

$$\bar{\pi}_+\tau^* \cdot \pi_+\mathcal{T}_1\pi_+|H = \bar{\pi}_+\tau^* \cdot \pi_+\mathcal{T}_1\bar{\pi}_+|H = \bar{\pi}_+\tau^* \cdot \bar{\pi}_+\mathcal{T}_1\bar{\pi}_+|H = \bar{\pi}_+\tau^*\mathcal{C}.$$

It remains to verify the Hankel condition $\bar{\pi}_+\mathcal{D}\pi_- = \mathcal{C}\mathcal{B}$. We have on $Seq_-(U)$

$$\begin{aligned}\mathcal{C}\mathcal{B} &= \bar{\pi}_+\mathcal{T}_1|H \cdot \bar{\pi}_+\mathcal{T}_2\pi_- = \bar{\pi}_+\mathcal{T}_1\bar{\pi}_+\mathcal{T}_2\pi_- \\ &= \bar{\pi}_+\mathcal{T}_1\mathcal{T}_2\pi_- - \bar{\pi}_+\mathcal{T}_1\pi_-\mathcal{T}_2\pi_- = \bar{\pi}_+\mathcal{D}\pi_- - \bar{\pi}_+\mathcal{T}_1\pi_- \cdot \mathcal{T}_2\pi_-\end{aligned}$$

because $\mathcal{D} = \mathcal{T}_1\mathcal{T}_2$. Now the required Hankel condition follows from the anticausality $\bar{\pi}_+\mathcal{T}_1\pi_- = 0$ of $\mathcal{T}_1$. All the remaining claims are trivial. □

Note that much less than full anticausality of $\mathcal{T}_1$ was needed to verify the identity $\bar{\pi}_+\tau^*\mathcal{C} = \mathcal{C}A$. But that $\bar{\pi}_+\mathcal{D}\pi_- = \mathcal{C}\mathcal{B}$ holds, depends quite crucially on the anticausality assumption in its strongest form.

Definition 3.2. *The DLS $\Phi_u(\mathcal{T}_1, \mathcal{T}_2)$ as defined in Lemma 3.2 is called the* unilateral backward shift realization (shortly, UBSR) *of the I/O map $\mathcal{D}$, associated to the factorization $\mathcal{D} = \mathcal{T}_1\mathcal{T}_2$.*

Clearly, using the factorization $\mathcal{D} = \mathcal{T}_1\mathcal{T}_2$ with $\mathcal{T}_1 = \mathcal{I}$ and $\mathcal{T}_2 = \mathcal{D}$, we obtain what is commonly known as the *Hankel range realization* as the UBSR $\Phi_u(\mathcal{I}, \mathcal{D})$. The following statement involving the exact controllability of $\Phi_u(\mathcal{T}_1, \mathcal{T}_2)$ is trivial, but it involves us with a fundamental assumption on the closed Hankel range.

Proposition 3.2. *Under the assumptions of Lemma 3.2, the following are equivalent:*

(i) The unilateral backward shift realization $\Phi_u(\mathcal{T}_1, \mathcal{T}_2)$ is exactly (infinite-time) controllable; and

(ii) the factor $\mathcal{T}_2$ is such that its causal Hankel operator

$$\bar{\pi}_+ \mathcal{T}_2 \pi_- : \ell^2(\mathbb{Z}_-; U) \to \ell^2(\mathbb{Z}_+; V)$$

has closed range.

After these algebraic manipulations begins the real fun: in the next section we leave the realization theory behind ourselves, and proceed to apply the factorization results in [4, Chapter 3 and 4] to $\Phi_b(\mathcal{T}_1, \mathcal{T}_2)$ and $\Phi_u(\mathcal{T}_1, \mathcal{T}_2)$.

4 Riccati equations for canonical realizations

The following lemma is only slightly less general than [4, Theorem 142]:

Lemma 4.1. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an approximately controllable, I/O stable and output stable DLS. Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator.*

(i) To each solution $P \in ric_0(\phi, J)$, the spectral factorization

$$\mathcal{D}_\phi^* J \mathcal{D}_\phi = \mathcal{D}_{\phi_P}^* \Lambda_P \mathcal{D}_{\phi_P} \tag{11}$$

of the Popov operator is associated.

(ii) Assume that the Popov operator has the factorization of the form

$$\mathcal{D}_\phi^* J \mathcal{D}_\phi = \mathcal{D}_{\phi'}^* \Lambda \mathcal{D}_{\phi'}$$

where

$$\phi' = \begin{pmatrix} A & B \\ -K & I \end{pmatrix}, \quad K \in \mathcal{L}(H; U), \quad \Lambda = \Lambda^*, \quad \Lambda^{-1} \in \mathcal{L}(U),$$

is an I/O stable and output stable DLS. Then $\phi' = \phi_P$ and $\Lambda = \Lambda_P$ for some $P \in ric_0(\phi, J)$.

Claim (ii) of previous lemma seems somewhat weaker than one might expect. An *a priori* assumption has been made on the structure of the realization of the stable spectral factor $\mathcal{D}_{\phi'}$: it must be possible to realize it with the *same* main operator A and input operator B that appear in the original DLS ϕ. Whenever this is the case, a regular H^∞-solution of H^∞DARE (8) can be found, such that its indicator Λ_P and spectral DLS ϕ_P give the spectral factorization in question through (11). Without such an assumption, one does not get very far in proving the converse direction in claim (ii) of Lemma 4.1 — the formulas just seem to have "too many letters" in them[8].

On the other hand, one would certainly expect that if the realization ϕ is "complicated enough" (in some sense) for its I(O-map $\mathcal{D}_\phi$, then certainly it is complicated enough to parameterize at least some of its spectral factors appearing as $\mathcal{D}_{\phi_P}$ through (11). However, this need not be the case, as the following example shows: let

$$\phi = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad U = Y = \mathbb{C}, \quad \text{and} \quad J = 1. \tag{12}$$

Then $\mathcal{D}_\phi = \mathcal{I}$, $\mathcal{D}_\phi^* J \mathcal{D}_\phi = \mathcal{I}$, and the H^∞DARE (8) takes the simple form $-P = 0$. Indeed, only one spectral factor (equalling $\mathcal{I}$) is covered by the single solution $P = 0 \in ric_0(\phi, 1)$, even though any inner function is a stable spectral factor of the trivial spectral function $\widehat{\mathcal{D}}_\phi(e^{i\theta})^* J \widehat{\mathcal{D}}_\phi(e^{i\theta}) = I$. However, all these other spectral factors cannot be realized in the required form because such $A = 0$ and $B = 0$ lack the necessary "complexity" to accommodate their "zero-pole" structure.

After such a discouragingly trivial counterexample, one might be left wondering whether solutions of DARE are any good in parameterizing the spectral factors. In other words, is it typical or not for a *reasonable* H^∞DAREs to have many solutions $P \in ric_0(\phi, J)$? Fortunately, a large number of stable spectral factorizations can be parameterized for special DLSs ϕ and their DAREs. The unilateral and bilateral backward shift realizations are examples of such DLSs, as the following Lemmas 4.2 and 4.3 will show.

Lemma 4.2. *Make the same assumptions as in Lemma 3.1, but assume in addition that $\mathcal{T}_2$ has a bounded, shift-invariant (but possibly noncausal) inverse on $\ell^2(\mathbb{Z}; U)$. Assume that $\mathcal{D}' : \ell^2(\mathbb{Z}; U) \to \ell^2(\mathbb{Z}; U)$ is an I/O map of an I/O stable DLS, such that*

(i) the feed-through part of $\mathcal{D}'$ satisfies $\pi_0 \mathcal{D}' \pi_0 = I$; and

[8] Indeed, such "letter sickness" is maybe the most serious ailment in contemporary infinite-dimensional system theory, and some rather extreme examples can be given to point out the gravity of the matter. One is simply trying to say too much, yet willing to assume only too little or too inappropriate — without understanding the impossibility of the whole mission.

(ii) the Popov operator has the factorization of the form

$$\mathcal{D}^* J \mathcal{D} = \mathcal{D}'^* \Lambda \mathcal{D}'$$

for some boundedly invertible, self-adjoint $\Lambda \in \mathcal{L}(U)$.

Then the H^∞*DARE associated to the BBSR* (9) *has a regular* H^∞*-solution* $P \in ric_0(\Phi_b(\mathcal{T}_1, \mathcal{T}_2), J)$*, such that*

$$\mathcal{D}' = \mathcal{D}_{(\Phi_b(\mathcal{T}_1,\mathcal{T}_2))_P} \quad \textit{and} \quad \Lambda = \Lambda_P. \tag{13}$$

In fact, the corresponding spectral DLS $(\Phi_b(\mathcal{T}_1, \mathcal{T}_2))_P$ *is given by the BBSR*

$$(\Phi_b(\mathcal{T}_1, \mathcal{T}_2))_P = \begin{bmatrix} (\tau^*|H)^j & \mathcal{T}_2 \pi_- \tau^{*j} \\ \bar{\pi}_+ \mathcal{D}' \mathcal{T}_2^{-1}|H & \mathcal{D}' \end{bmatrix}$$

with state space $H = \overline{\text{range}\,(\mathcal{T}_2 \pi_-)}$.

Proof. We show that $\mathcal{D}'$ can be realized in form $\mathcal{D}' = \mathcal{D}_{\phi'}$ for some I/O stable and output stable DLS $\phi' = \left(\begin{smallmatrix} A & B \\ -K & I \end{smallmatrix} \right)$ whose state space, together with the operators A and B, coincide with those of $\Phi_b(\mathcal{T}_1, \mathcal{T}_2)$. First note that by assumption (i), the feed-through operator of *any* realization of $\mathcal{D}'$ equals $I \in \mathcal{L}(U)$. Because the BBSR $\Phi_b(\mathcal{T}_1, \mathcal{T}_2)$ is easiest given in I/O form, it is most convenient to show that A and the controllability map $\mathcal{B}_{\phi'}$ coincide with those of $\Phi_b(\mathcal{T}_1, \mathcal{T}_2)$, if the realization ϕ' is chosen as described next.

Define the bounded operator $\mathcal{T}' : \ell^2(\mathbb{Z}; V) \to \ell^2(\mathbb{Z}; U)$ by $\mathcal{T}' := \mathcal{D}' \mathcal{T}_2^{-1}$. This is possible because $\mathcal{T}_2$ was assumed to be boundedly invertible from $\ell^2(\mathbb{Z}; U)$ onto $\ell^2(\mathbb{Z}; V)$, and clearly such a $\mathcal{T}'$ is always shift-invariant but not necessarily causal. Now the operators $\mathcal{T}'$ and $\mathcal{T}_2$ define the BBSR

$$\Phi_b(\mathcal{T}', \mathcal{T}_2) = \begin{bmatrix} (\tau^*|H')^j & \mathcal{T}_2 \pi_- \tau^{*j} \\ \bar{\pi}_+ \mathcal{T}'|H' & \mathcal{D}' \end{bmatrix}$$

whose state space $H' := \overline{\text{range}\,(\mathcal{T}_2 \pi_-)}$, main operator $\tau^*|H'$ and controllability map $\mathcal{T}_2 \pi_-$ are the same as those of $\Phi_b(\mathcal{T}_1, \mathcal{T}_2)$. So we use $\phi' := \Phi_b(\mathcal{T}_1, \mathcal{T}_2)$ to realize $\mathcal{D}'$, and we note that this realization is approximately controllable, by Lemma 3.1.

By claim (ii) of Lemma 4.1, there exists $P \in ric_0(\Phi_b(\mathcal{T}_1, \mathcal{T}_2), J)$ such that (13) holds, which completes the proof. □

Let us return for a moment to the trivial spectral factorization of the Popov operator $\mathcal{D}^* J \mathcal{D} = \mathcal{I} \in \mathcal{L}(\ell^2(\mathbb{Z}; \mathbb{C}))$ where $\mathcal{D} = \mathcal{I}$ and $J = 1$. In contrast to (12), we shall this time use the BBSR $\Phi_b(\mathcal{I},\mathcal{I})$ as the fundamental realization; namely

$$(14) \qquad \Phi_b(\mathcal{I},\mathcal{I}) = \begin{bmatrix} (\tau^*|H)^j & \pi_-\tau^{*j} \\ 0 & \mathcal{I} \end{bmatrix} \text{ with } H = \overline{\text{range}\,(\pi_-)} = \ell^2(\mathbb{Z}_-;\mathbb{C}),$$

by noting that the observability map vanishes as $\bar{\pi}_+\ell^2(\mathbb{Z}_-;\mathbb{C}) = \{0\}$. Let $\mathcal{N} \in \mathcal{L}(\ell^2(\mathbb{Z};\mathbb{C}))$ be now any inner I/O map (i.e. $\mathcal{N}^*\mathcal{N} = \mathcal{I}$) whose feed-through part $N := \pi_0\mathcal{N}\pi_0$ is nonzero. By normalizing $\mathcal{N}^\circ := N^{-1}\mathcal{N}$, we obtain an I/O map whose feed-through operator is identity, and it satisfies the spectral factorization identity

$$\mathcal{D}^*J\mathcal{D} = (\mathcal{N}^\circ)^*\,\Lambda\mathcal{N}^\circ$$

of the form appearing in Lemma 4.1, where $\Lambda := N^*N$ is positive and real.

Now we are in a situation where Lemma 4.2 applies. Indeed, there exists a solution $P \in ric_0(\Phi_b(\mathcal{I},\mathcal{I}), J)$, such that $\Lambda = \Lambda_P$ and

$$(\Phi_b(\mathcal{N}^\circ,\mathcal{I}))_P = \begin{bmatrix} (\tau^*|H)^j & \pi_-\tau^{*j} \\ \bar{\pi}_+\mathcal{N}^\circ|H & \mathcal{N}^\circ \end{bmatrix} \quad \text{with} \quad H = \ell^2(\mathbb{Z}_-;U).$$

This is strongly in contrast to the trivial realization (12) of the same I/O map. There only the trivial spectral factor $\mathcal{I}$ was covered by solutions of DARE, whereas all stable inner spectral factors are parameterized in the context of realization (14). Note that $\Phi_b(\mathcal{I},\mathcal{I})$ is exactly controllable (in infinite time), its state space $\ell^2(\mathbb{Z}_-;U)$, in a sense, is "very large", and the main operator is an unilateral forward shift.

We proceed to the case of H^∞DAREs associated to UBSRs. In the context of the Popov operator $\mathcal{D}^*J\mathcal{D} = \mathcal{I}$, this situation appears to be an intermediate cases between the two extremes, involving the realizations (12) and (14).

Lemma 4.3. *Make the same assumptions as in Lemma 3.2. Assume, moreover, that there exists a bounded, shift-invariant and causal operator* $\mathcal{T}' : \ell^2(\mathbb{Z};V) \to \ell^2(\mathbb{Z};U)$*, such that*

- *(i) the bounded, shift-invariant operator* $\mathcal{D}' := \mathcal{T}'^*\mathcal{T}_2 : \ell^2(\mathbb{Z};U) \to \ell^2(\mathbb{Z};U)$ *is an I/O map of an I/O stable DLS;*
- *(ii) the feed-through part of* $\mathcal{D}'$ *satisfies* $\pi_0\mathcal{D}'\pi_0 = I$*; and*
- *(iii) the Popov operator has the factorization of the form*

$$\mathcal{D}^*J\mathcal{D} = \mathcal{D}'^*\Lambda\mathcal{D}'$$

for some boundedly invertible, self-adjoint $\Lambda \in \mathcal{L}(U)$*.*

Then the H^∞DARE associated to the UBSR (10) *has a regular H^∞-solution $P \in ric_0(\Phi_u(\mathcal{T}_1, \mathcal{T}_2), J)$, such that*

$$\mathcal{D}' = \mathcal{D}_{(\Phi_u(\mathcal{T}_1,\mathcal{T}_2))_P} \quad \textit{and} \quad \Lambda = \Lambda_P. \tag{15}$$

In fact, the corresponding spectral DLS $(\Phi_u(\mathcal{T}_1, \mathcal{T}_2))_P$ is given by the UBSR

$$(\Phi_u(\mathcal{T}_1, \mathcal{T}_2))_P = \begin{bmatrix} (\tau^*|H)^j & \bar{\pi}_+\mathcal{T}_2\pi_-\tau^{*j} \\ \bar{\pi}_+T'^*|H & \mathcal{D}' \end{bmatrix}, \tag{16}$$

with state space $H = \overline{\text{range}\,(\bar{\pi}_+\mathcal{T}_2\pi_-)}$.

Proof. The claim essentially follows by noting that output stable and I/O stable USBR $\Phi_u(\mathcal{T}'^*, \mathcal{T}_2)$ appearing on the right hand side of equation (16), indeed, has the same main operator and controllability map as the USBR $\Phi_u(\mathcal{T}_1, \mathcal{T}_2)$ defining the H^∞DARE. Because $\Phi_u(\mathcal{T}_1^*, \mathcal{T}_2)$ is approximately controllable, an application of claim (ii) of Lemma 4.1 completes the proof. For the missing details, follow the proof of Lemma 4.2. □

We complete this section with a few comments on the previous lemma. Firstly, it is an abstract form of a "pole-placement" condition that the spectral factor $\mathcal{D}'$ is assumed in (i) to be causal and I/O stable. In the rational case, all the spectral factors parameterized by DARE in Lemma 4.3 have all poles outside $\mathbb{D}$.

Let us consider the Hankel range realization $\Phi_u(\mathcal{I}, \mathcal{D})$ with $\mathcal{T}_1 = \mathcal{I}$ and $\mathcal{T}_2 = \mathcal{D}$. Let as assume, in addition, that we have the factorization $\mathcal{D} = \mathcal{N}_1\mathcal{N}_2\mathcal{X}$, where $\mathcal{N}_1$ is (J, S_1)-inner, $\mathcal{N}_2$ is (S_1, S)-inner, and $\mathcal{X}$ is outer with a bounded inverse. (Clearly $\mathcal{D} = \mathcal{N}\mathcal{X}$ is a (J, S)-inner-outer factorization, if we set $\mathcal{N} := \mathcal{N}_1\mathcal{N}_2$.) We assume that all the self-adjoint operators J, S_1 and S are boundedly invertible. Defining

$$\mathcal{T}' := J\mathcal{N}_1S_1^{-1} \quad \text{and} \quad \mathcal{D}' := \mathcal{T}'^*\mathcal{T}_2 = \mathcal{N}_2\mathcal{X}$$

we note that $\mathcal{D}'$ satisfies condition (i) of Lemma 4.3. Moreover, because

$$\mathcal{D}'^*S_1\mathcal{D}' = \mathcal{X}^*\mathcal{N}_2^*S_1\mathcal{N}_2\mathcal{X} = \mathcal{X}^*S\mathcal{X} = \mathcal{D}^*J\mathcal{D},$$

we see that condition (iii) of Lemma 4.3 is satisfied with $\Lambda = S_1$. After a trivial normalization of $\mathcal{D}'$ (under the additional restrictive assumption that the feed-through of the spectral factor $\mathcal{D}'$ is boundedly invertible), also the normalization in condition (ii) can be made to hold (but after the normalization, a different $\Lambda \neq S_1$ must be used in condition (iii)). We conclude that by Lemma 4.3, most of the stable spectral factors of $\mathcal{D}$ are parameterized by the solutions of the H^∞DARE, in case we use the Hankel range realization $\Phi_u(\mathcal{I}, \mathcal{D})$ for $\mathcal{D}$.

5 On the state space isomorphism

Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS whose state space is H. Given another Hilbert space H' and a bounded bijection $T \in \mathcal{L}(H, H')$, a mapping between DLSs can be defined through

$$\eta_T \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{pmatrix}. \tag{17}$$

Clearly, $\eta_T(\phi)$ has the same input space U and the output space Y as the original ϕ. Also the I/O maps of ϕ and $\eta_T(\phi)$ are the same, but the state space of $\eta_T(\phi)$ is H'. It is easy to see that the group of boundedly invertible operators in $T \in \mathcal{L}(H)$ acts on the set of DLSs whose state space is H, through the mapping $T \mapsto \eta_T$. The inverse of η_T is $\eta_{T^{-1}}$ in the sense that $\eta_{T^{-1}}(\eta_T(\phi)) = \eta_T(\eta_{T^{-1}}(\phi)) = \phi$. The following Propositions 5.1 and 5.2 are immediate.

Proposition 5.1. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS whose state space is H. Let H' be another Hilbert space, and $T \in \mathcal{L}(H, H')$ be a bounded bijection.*

(i) Then $A^j_{\eta_T(\phi)} = TA^jT^{-1}$, $\mathcal{B}_{\eta_T(\phi)} = T\mathcal{B}_\phi$ and $\mathcal{C}_{\eta_T(\phi)} = \mathcal{C}_\phi T^{-1}$.

(ii) Conversely, if $\phi' = \left(\begin{smallmatrix} A' & B' \\ C' & D \end{smallmatrix}\right)$ is a DLS such that $A' = TAT^{-1}$, $\mathcal{B}_{\phi'} = T\mathcal{B}_\phi$ and $\mathcal{C}_{\phi'} = \mathcal{C}_\phi T^{-1}$, then $\phi' = \eta_T(\phi)$.

Proof. As an example, let us verify $\mathcal{B}_{\eta_T(\phi)} = T\mathcal{B}_\phi$. For any $\tilde{u} \in Seq_-(U)$ we have by the finiteness of all the sums $\mathcal{B}_{\eta_T(\phi)} = \sum_{j\geq 0} (TAT^{-1})^j \cdot TBu_{-j-1} = \sum_{j\geq 0} TA^jT^{-1}TBu_{-j-1} = T\sum_{j\geq 0} A^jBu_{-j-1} = T\mathcal{B}_\phi\tilde{u}$, and we are done. □

Proposition 5.2. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS whose state space is H. Let H' be Hilbert space, and $T \in \mathcal{L}(H, H')$ be a bounded bijection. Then*

(i) $\eta_T(\phi)$ is input stable if and only if ϕ is input stable,

(ii) $\eta_T(\phi)$ is output stable if and only if ϕ is output stable,

(iii) $\eta_T(\phi)$ is approximately observable if and only if ϕ is approximately observable,

(iv) $\eta_T(\phi)$ is approximately controllable if and only if ϕ is approximately controllable.

(v) Assume, in addition, that ϕ is input stable. Then $\eta_T(\phi)$ is (infinite-time) exactly controllable if and only if ϕ is (infinite-time) exactly controllable.

Proof. Claims (i), (ii), and (iii) follow immediately from Proposition 5.1. Let us prove claim (iv). Because $\mathcal{B}_{\eta_T(\phi)} = T\mathcal{B}_\phi$ on $Seq_-(U)$, we conclude that range $(\mathcal{B}_{\eta_T\phi}) = \mathcal{B}_{\eta_T(\phi)}\ Seq_-(U) = T\mathcal{B}_\phi\ Seq_-(U) = T\,\mathrm{range}\,(\mathcal{B}_\phi)$. Because a bounded bijection maps dense sets onto dense sets, the approximate controllabilities of $\eta_T(\phi)$ and ϕ are equivalent. The final claim (v) follows by considering the range of linear extensions $\overline{\mathcal{B}_\phi}$ and $\overline{\mathcal{B}_{\eta_T(\phi)}}$ that clearly satisfy $\overline{\mathcal{B}_{\eta_T(\phi)}} = T\overline{\mathcal{B}_\phi}$. □

Proposition 5.3. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS with state space H, and let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let H' be a Hilbert space, and $T \in \mathcal{L}(H, H')$ be a bounded bijection. Then*

$$Q \in Ric(\eta_T(\phi), J) \Leftrightarrow P := T^*QT \in Ric(\phi, J).$$

Moreover, both the solutions P and Q are simultaneously nonnegative or positive.

Proof. Assume that $Q \in Ric(\eta_T(\phi), J)$. By writing the appropriate DARE and regrouping terms, we obtain

$$\begin{cases} T^{-*}A^* \cdot T^*QT \cdot AT^{-1} - Q + T^{-*}C^*JCT^{-1} = K_Q'^*\Lambda_Q' K_Q', \\ \Lambda_Q' K_Q' = (-D^*JC - B^* \cdot T^*QT \cdot A)T^{-1}, \\ \Lambda_Q' = D^*JD + B^*T^*QTB, \end{cases}$$

where $\Lambda\prime_Q$ and K_Q' are the indicator and the feedback operator, associated to solution Q of DARE $Ric(\eta_T(\phi), J)$. Writing $P := T^*QT$ and using the definitions of the indicator and feedback operators $\Lambda_P := D^*JD + B^*PB$ and $K_P := \Lambda_P^{-1}(-D^*JC - B^*PA)$ of DARE $Ric(\phi, J)$, we get

$$\begin{cases} A^*PA - P + C^*JC = \left(K_Q'T\right)^* \cdot \Lambda_Q' \cdot \left(K_Q'T\right), \\ \Lambda_Q' \cdot K_Q'T = -D^*JC - B^*PA, \\ \Lambda_Q' = \Lambda_P, \end{cases}$$

where the first equation has been multiplied by T^* from the left and by T from the right. We conclude from the second and third equation that $K_Q'T = K_P$. Inserting this to the first two of the three equations above gives

$$\begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_Q K_P, \\ \Lambda_P K_P = -D^*JC - B^*PA, \\ \Lambda_P = D^*JD + B^*PB. \end{cases}$$

Hence $P \in Ric(\phi, J)$ because both Q and P are self-adjoint whenever one of them is. We have now shown the first implication

$$Q \in Ric(\eta_T(\phi), J) \Rightarrow T^*QT \in Ric(\phi, J).$$

The converse implication can be reduced to this, by using $\eta_{T^{-1}}(\eta_T(\phi)) = \phi$. Indeed,

$$P \in Ric(\phi, J) \Leftrightarrow P \in Ric(\eta_{T^{-1}}(\eta_T(\phi)), J) \Rightarrow T^{-*}PT^{-1} \in Ric(\eta_T(\phi), J).$$

Writing $P = T^*QT$, we get from the previous

$$T^*QT \in Ric(\phi, J) \Rightarrow T^{-*}T^*QTT^{-1} = Q \in Ric(\eta_T(\phi), J),$$

which completes the proof. □

It remains to treat the correspondence of the spectral DLSs ϕ_P and $(\eta_T(\phi))_Q$ where $P = T^*QT$ as in Proposition 5.3. After that, the (regular) H^∞-solutions of a $H^\infty DARE$ are easily dealt with.

Lemma 5.1. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS. Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $T \in \mathcal{L}(H)$ is a bounded bijection, and the operators P and Q be related by $P = T^*QT$. Then*

$$P \in ric(\phi, J) \Leftrightarrow Q \in ric(\eta_T(\phi), J) \tag{18}$$

and

$$P \in ric_0(\phi, J) \Leftrightarrow Q \in ric_0(\eta_T(\phi), J). \tag{19}$$

Proof. In the notation of the previous proof, we saw that $K'_Q T = K_P$ and $\Lambda'_Q = \Lambda_P$. By a direct computation we get

$$\begin{aligned}(\eta_T(\phi))_Q &= \begin{pmatrix} TAT^{-1} & TB \\ -K'_Q & I \end{pmatrix} = \begin{pmatrix} TAT^{-1} & TB \\ -K_P T^{-1} & I \end{pmatrix} \\ &= \eta_T \begin{pmatrix} A & B \\ -K_P & I \end{pmatrix} = \eta_T(\phi_P).\end{aligned} \tag{20}$$

Hence the spectral DLSs are connected by $(\eta_T(\phi))_Q = \eta_T(\phi_P)$, and equivalence (18) follows trivially from Propositions 5.2, as the H^∞-solutions are, by definition, those whose spectral DLSs are both output stable and I/O stable. So as to the *regular* H^∞-solutions, assume that $L_{A,P} := \operatorname{slim}_{j\to\infty} A^{*j}PA^j = 0$. Then for any $z_0 \in H'$ we have

$$\begin{aligned}&\lim_{j\to\infty} \left(TAT^{-1}\right)^{*j} Q \left(TAT^{-1}\right)^j z_0 \\ &= T^{-*} \lim_{j\to\infty} A^{*j}PA^j T^{-1} z_0 = T^{-*} L_{A,P} T^{-1} z_0 = 0,\end{aligned}$$

where the operations with limits are legal because all the operators are continuous. Because $z_0 \in H'$ was arbitrary, we conclude $L_{TAT^{-1},Q} = 0$. This proves the first direction of equivalence (19), and the converse direction is similar. $\square$

6 Exact controllability and observability of minimal realizations

Our intention is to prove two state space isomorphism results, namely Lemma 6.1 (weak similarity) and Theorem 6.1 (strong similarity of the state space). To achieve this, we must first prove two auxiliary propositions. We remind that $\overline{\mathcal{B}}$ denotes the bounded extension of an input stable controllability map $\mathcal{B}$ to all of $\ell^2(\mathbb{Z}_-;U)$; minimality of the DLS $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ means that $\overline{\text{range}\,(\mathcal{B})} = H$ and $\ker(\mathcal{C}) = \{0\}$.

Proposition 6.1. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an input stable, output stable, I/O stable and minimal DLS, with state space* H*. Then*

(i) $\ker(\bar{\pi}_+ \mathcal{D} \pi_-) = \ker\left(\overline{\mathcal{B}}\right)$*, regarded as subsets of* $\ell^2(\mathbb{Z}_-;U)$*, and*

(ii) $\mathcal{C}\,\text{range}\left(\overline{\mathcal{B}}\right) = \text{range}\,(\bar{\pi}_+ \mathcal{D} \pi_-) \subset \text{range}\,(\mathcal{C})$ *and* $\text{range}\,(\mathcal{C}) \subset \overline{\text{range}\,(\bar{\pi}_+ \mathcal{D} \pi_-)}$*, regarded as subsets of* $\ell^2(\mathbb{Z}_+;Y)$*.*

Proof. We show that $\bar{\pi}_+ \mathcal{D} \pi_- = \mathcal{C}\overline{\mathcal{B}}$ on all of $\ell^2(\mathbb{Z}_-;U)$. In the dense set $Seq_-(U) \subset \ell^2(\mathbb{Z}_-;U)$ we already know this identity as a basic axiom for DLSs in I/O form. Let $\tilde{u} \in \ell^2(\mathbb{Z}_-;U)$ be arbitrary, and choose a sequence $Seq_-(U) \ni \tilde{u}_j \to \tilde{u}$ as $j \to \infty$. Then

$$\bar{\pi}_+ \mathcal{D} \pi_- \tilde{u}_j = \mathcal{C}\mathcal{B}\tilde{u}_j = \mathcal{C}\overline{\mathcal{B}}\tilde{u}_j \quad \text{for all} \quad j. \tag{21}$$

The left hand side of this equation converges to $\bar{\pi}_+ \mathcal{D} \pi_- \tilde{u}$ by the boundedness of $\mathcal{D}$. On the right hand side, $\overline{\mathcal{B}}\tilde{u}_j \to \overline{\mathcal{B}}\tilde{u}$ in the norm of H, by the definition of the extension $\overline{\mathcal{B}}$. By the output stability, $\mathcal{C}\left(\overline{\mathcal{B}}\tilde{u}_j\right) \to \mathcal{C}\overline{\mathcal{B}}\tilde{u}$, giving the limit on the right hand side of (21). By the uniqueness of the limit $\bar{\pi}_+ \mathcal{D} \pi_- = \mathcal{C}\overline{\mathcal{B}}$ on all of $\ell^2(\mathbb{Z}_-;U)$.

Now we are prepared to prove claim (i). It now follows that

$$\ker(\bar{\pi}_+ \mathcal{D} \pi_-) = \ker\left(\mathcal{C}\overline{\mathcal{B}}\right) = \ker\left(\overline{\mathcal{B}}\right) \cup \{\tilde{u} \in \ell^2(\mathbb{Z}_-) \mid \overline{\mathcal{B}}\tilde{u} \in \ker(\mathcal{C})\}.$$

But we have assumed $\ker(\mathcal{C}) = \{0\}$, and thus claim (i) follows.

The first part in claim (ii) is a consequence of the identity $\bar{\pi}_+ \mathcal{D} \pi_- = \mathcal{C}\overline{\mathcal{B}}$. The continuity of $\mathcal{C}$ and the approximate controllability $\overline{\operatorname{range}(\mathcal{B})} \subset \overline{\operatorname{range}\left(\overline{\mathcal{B}}\right)} = H$ implies

$$\begin{aligned} \operatorname{range}(\mathcal{C}) &= \mathcal{C}\, \overline{\operatorname{range}\left(\overline{\mathcal{B}}\right)} \subset \overline{\mathcal{C}\operatorname{range}\left(\overline{\mathcal{B}}\right)} \\ &= \overline{\operatorname{range}\left(\mathcal{C}\overline{\mathcal{B}}\right)} = \overline{\operatorname{range}(\bar{\pi}_+ \mathcal{D} \pi_-)}. \end{aligned}$$

This proves the rest of the inclusions in claim (ii). □

Proposition 6.2. *Let $S_1, S_2 \in \mathcal{L}(H, H')$ be injective. Then $T := S_2^{-1} S_1 : \operatorname{dom}(T) \to \operatorname{range}(T)$ is a bijective linear mapping between the spaces*

$$\begin{aligned} &\operatorname{dom}(T) := S_1^{-1}(\operatorname{range}(S_1) \cap \operatorname{range}(S_2)) \qquad \textit{and} \\ &\operatorname{range}(T) := S_2^{-1}(\operatorname{range}(S_1) \cap \operatorname{range}(S_2)). \end{aligned}$$

Furthermore, it is closed operator on H with this domain.

Proof. Clearly T is well-defined, linear and injective on the vector space $\operatorname{dom}(T)$, though it might happen that $\operatorname{range}(S_1) \cap \operatorname{range}(S_2) = \{0\}$ in which case $\operatorname{dom}(T)$ is trivial by the injectivity of S_1. Also the vector space $\operatorname{range}(T)$ is, indeed, the range of T with this domain.

Choose an arbitrary sequence $\{x_j\}_{j \geq 0} \subset \operatorname{dom}(T)$ such that $x_j \to x$ and $Tx_j \to y$ for some $x, y \in H$. We show that $x \in \operatorname{dom}(T)$ and $Tx = y$. Because S_1 is bounded, $S_1 x_j \to S_1 x$. Because $Tx_j \to y$ and S_2 is bounded, $S_1 x_j = S_2 T x_j \to S_2 y$. By the uniqueness of the limit, $S_1 x = S_2 y \in \operatorname{range}(S_1) \cap \operatorname{range}(S_2)$, which directly implies that $x \in \operatorname{dom}(T)$ and $y = Tx$. Thus T is closed. □

Lemma 6.1. *Let $\phi_1 := \left(\begin{smallmatrix} A_1 & B_1 \\ C_1 & D \end{smallmatrix}\right)$ and $\phi_2 := \left(\begin{smallmatrix} A_2 & B_2 \\ C_2 & D \end{smallmatrix}\right)$ be be two I/O stable, input stable, output stable, minimal realizations of the same I/O map $\mathcal{D}$. Denote the state spaces of ϕ_1 and ϕ_2 by H_1 and H_2, respectively.*

(i) Then there exists a closed, bijective linear operator

$$T : H_1 \supset \operatorname{dom}(T) \to \operatorname{range}(T) \subset H_2,$$

given by $T := \mathcal{C}_{\phi_2}^{-1} \mathcal{C}_{\phi_1}$, where

$$\begin{aligned} &\operatorname{dom}(T) := \mathcal{C}_{\phi_1}^{-1}(\operatorname{range}(\mathcal{C}_{\phi_1}) \cap \operatorname{range}(\mathcal{C}_{\phi_2})) \qquad \textit{and} \\ &\operatorname{range}(T) := \mathcal{C}_{\phi_2}^{-1}(\operatorname{range}(\mathcal{C}_{\phi_1}) \cap \operatorname{range}(\mathcal{C}_{\phi_2})). \end{aligned}$$

The inclusions $\operatorname{range}\left(\overline{\mathcal{B}_{\phi_1}}\right) \subset \operatorname{dom}(T)$ and $\operatorname{range}\left(\overline{\mathcal{B}_{\phi_2}}\right) \subset \operatorname{range}(T)$ hold. Moreover, T is densely defined, injective, and it has dense range. For all $x_0 \in \operatorname{range}(T)$, $A_2 x_0 = T A_1 T^{-1} x_0$ and $C_2 x_0 = C_1 T^{-1} x_0$ hold. For all $u \in U$, $B_2 u = T B_1 u$ holds.

(ii) The operator

$$\tilde{T} : \text{range}\left(\overline{\mathcal{B}_{\phi_1}}\right) \to \text{range}\left(\overline{\mathcal{B}_{\phi_2}}\right)$$

given by

$$\tilde{T} := \left(\overline{\mathcal{B}_{\phi_2}}|\ker\left(\bar{\pi}_+\mathcal{D}\pi_-\right)^{\perp}\right)\left(\overline{\mathcal{B}_{\phi_1}}|\ker\left(\bar{\pi}_+\mathcal{D}\pi_-\right)^{\perp}\right)^{-1}$$

is a bijection between $\text{range}\left(\overline{\mathcal{B}_{\phi_1}}\right)$ *and* $\text{range}\left(\overline{\mathcal{B}_{\phi_2}}\right)$*, and it satisfies* $\tilde{T}x = Tx$ *for all* $x \in \text{range}\left(\overline{\mathcal{B}_{\phi_1}}\right)$*.*

(iii) If both ϕ_1 *and* ϕ_2 *are exactly (infinite time) controllable, then* $T : H_1 \to H_2$ *is a bounded bijection,* $T = \tilde{T}$ *and* $\phi_2 = \eta_T(\phi_1)$*.*

Proof. By Proposition 6.2 and output stabilities of ϕ_1 and ϕ_2, the operator $T := \mathcal{C}_{\phi_2}^{-1}\mathcal{C}_{\phi_1}$ is closed and bijective between its natural domain and range. We proved in the proof of Proposition 6.1 that

$$\mathcal{C}_{\phi_1}\overline{\mathcal{B}_{\phi_1}} = \bar{\pi}_+\mathcal{D}\pi_- = \mathcal{C}_{\phi_2}\overline{\mathcal{B}_{\phi_2}} \quad \text{on } \ell^2(\mathbb{Z}_-;U),$$

and hencerange $\left(\overline{\mathcal{B}_{\phi_1}}\right) \subset \text{dom}(T)$. Because $T^{-1} = \mathcal{C}_{\phi_1}^{-1}\mathcal{C}_{\phi_2}$ with $\text{dom}(T^{-1}) = \text{range}(T)$, we conclude that $\text{range}\left(\overline{\mathcal{B}_{\phi_2}}\right) \subset \text{range}(T)$ because the assumptions for ϕ_1 and ϕ_2 are the same in this proposition. By the minimality assumptions, the operator T is densely defined and has dense range.

To show that $\text{dom}(T)$ is A_1-invariant, let $x_0 \in \text{dom}(T)$ be arbitrary. Then $\mathcal{C}_{\phi_1}x_0 \in \text{range}(\mathcal{C}_{\phi_1}) \cap \text{range}(\mathcal{C}_{\phi_2})$ and $\bar{\pi}_+\tau^*\mathcal{C}_{\phi_1}x_0 = \mathcal{C}_{\phi_1}A_1x_0 \in \text{range}(\mathcal{C}_{\phi_1})$. Similarly $\bar{\pi}_+\tau^*\mathcal{C}_{\phi_1}x_0 \in \text{range}(\mathcal{C}_{\phi_2})$, and thus

$$\mathcal{C}_{\phi_1}A_1x_0 = \bar{\pi}_+\tau^*\mathcal{C}_{\phi_1}x_0 \in \text{range}(\mathcal{C}_{\phi_1}) \cap \text{range}(\mathcal{C}_{\phi_2}).$$

We conclude that $A_1x_0 \in \text{dom}(T)$. By interchanging ϕ_1 and ϕ_2, we see by a similar argument that $\text{range}(T)$ is A_2-invariant.

Now, the linear mapping $x_0 \mapsto (TA_1 - A_2T)x_0$ is well defined for all $x_0 \in \text{dom}(T)$, and it maps into $\text{range}(T)$. We show that it vanishes on $\text{dom}(T)$. We have

$$\begin{aligned}\mathcal{C}_{\phi_2}(TA_1 - A_2T)x_0 &= \left(\mathcal{C}_{\phi_1}A_1 - \mathcal{C}_{\phi_2}A_2 \cdot \mathcal{C}_{\phi_2}^{-1}\mathcal{C}_{\phi_1}\right)x_0 \\ &= \left(\mathcal{C}_{\phi_1}A_1 - \bar{\pi}_+\tau^*\mathcal{C}_{\phi_2} \cdot \mathcal{C}_{\phi_2}^{-1}\mathcal{C}_{\phi_1}\right)x_0 = \left(\mathcal{C}_{\phi_1}A_1 - \bar{\pi}_+\tau^*\mathcal{C}_{\phi_1}\right)x_0 = 0.\end{aligned}$$

Because $\ker(\mathcal{C}) = \{0\}$, we conclude that $TA_1T^{-1} = A_2$ in $\text{range}(T)$. Because $\mathcal{C}_{\phi_1} = \mathcal{C}_{\phi_2}T$ in $\text{dom}(T)$, the identity $C_1 = C_2T$ holds in $\text{dom}(T)$. This is equivalent to $C_1T^{-1} = C_2$ in $\text{range}(T)$, because T is a bijection between its domain and range.

Let $\tilde{u} \in \ell^2(\mathbb{Z}_-; U)$ be arbitrary. Denote $x_0 := \overline{\mathcal{B}_{\phi_1}}\tilde{u}$ and $x_1 := \overline{\mathcal{B}_{\phi_2}}\tilde{u}$. Then

$$\mathcal{C}_{\phi_1} x_0 = \bar{\pi}_+ \mathcal{D} \pi_- \tilde{u} = \mathcal{C}_{\phi_2} x_1;$$

hence $x_0 \in \operatorname{dom}(T)$ and $T x_0 = x_1$. But now $T\overline{\mathcal{B}_{\phi_1}} = \overline{\mathcal{B}_{\phi_2}}$ in all of $\ell^2(\mathbb{Z}_-; U)$, and in particular $TB_1 = B_2$ in U. Claim (i) now follows.

Because the previous argument proved $T\overline{\mathcal{B}_{\phi_1}} = \overline{\mathcal{B}_{\phi_2}}$ in all of $\ell^2(\mathbb{Z}_-; U)$, we conclude that $\tilde{T} = T$ in range $\left(\overline{\mathcal{B}_{\phi_1}}\right)$ from the fact that $\overline{\mathcal{B}_{\phi_1}}|\ker(\bar{\pi}_+ \mathcal{D} \pi_-)^{\perp}$ is an injection onto range $\left(\overline{\mathcal{B}_{\phi_1}}\right)$, by claim (i) of Proposition 6.1. Now claim (ii) follows. The final claim (iii) follows from the Closed Graph Theorem and the inclusion range $\left(\overline{\mathcal{B}_{\phi_1}}\right) \subset \operatorname{dom}(T)$. □

Lemma 6.2. *Let $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an input stable, output stable, I/O stable and minimal DLS, such that the range of the Hankel operator $\bar{\pi}_+ \mathcal{D} \pi_- : \ell^2(\mathbb{Z}_-; U) \to \ell^2(\mathbb{Z}_+; Y)$ is closed.*

Then Φ is exactly (infinite time) observable and exactly (infinite-time) controllable.

Proof. Consider the restriction

$$M := \bar{\pi}_+ \mathcal{D} \pi_- |\ker(\bar{\pi}_+ \mathcal{D} \pi_-)^{\perp} = \mathcal{C}\overline{\mathcal{B}}|\ker(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)^{\perp}.$$

Because the Hankel operator is assumed to have closed range,

$$M : \ker(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)^{\perp} \to \overline{\operatorname{range}(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)}$$

is a bounded bijection with a bounded inverse. By Proposition 6.1, $\ker\left(\overline{\mathcal{B}_\phi}\right) = \ker(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)$. Suppose that ϕ is not exactly controllable. Then there is a sequence $\{\tilde{u}_j\}_{j \geq 0} \subset \ker\left(\overline{\mathcal{B}}\right)^{\perp} = \ker(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)^{\perp}$ such that $||\tilde{u}_j||_{\ell^2(\mathbb{Z}_-;U)} = 1$ but $\overline{\mathcal{B}}\tilde{u}_j \to 0$ as $j \to \infty$. But then, because $\mathcal{C}$ is bounded,

$$M\tilde{u}_j = \mathcal{C}\left(\overline{\mathcal{B}}\tilde{u}_j\right) \to 0$$

which implies that M is not coercive. This contradiction proves that ϕ is exactly controllable.

Suppose that ϕ is not exactly observable, meaning that $\mathcal{C}$ is not coercive. Because $\ker(\mathcal{C}) = \{0\}$ by the minimality assumption, then there is a sequence $\{x_j\}_{j \geq 0} \subset H$ such that $||x_j||_H = 1$ for all j but $\mathcal{C}x_j \to 0$. By the already proved exact controllability, $x_j = \overline{\mathcal{B}}\tilde{u}_j$ for some sequence $\{\tilde{u}_j\}_{j \geq 0} \subset \ker(\bar{\pi}_+ \mathcal{D} \pi_-)^{\perp}$. Because $\overline{\mathcal{B}} : \ker(\bar{\pi}_+ \mathcal{D} \pi_-)^{\perp} \to H$ is a bounded bijection with a bounded inverse, the sequence $\{\tilde{u}_j\}_{j \geq 0}$ is bounded away from zero: $||\tilde{u}_j||_{\ell^2(\mathbb{Z}_-;U)} \geq \epsilon > 0$ for some ϵ and all j. But now

$$\bar{\pi}_+ \mathcal{D} \pi_- \tilde{u}_j = \mathcal{C}\left(\overline{\mathcal{B}}\tilde{u}_j\right) = \mathcal{C}x_j \to 0.$$

But this is a contradiction against the coercivity of $\bar{\pi}_+ \mathcal{D} \pi_- |\ker(\bar{\pi}_+ \mathcal{D} \pi_-)^{\perp}$, which is equivalent to closedness of $\operatorname{range}(\bar{\pi}_+ \mathcal{D} \pi_-)$. □

It is well known that if the Hankel operator $\bar{\pi}_+\mathcal{D}\pi_-$ has a closed range, then there are essentially only one kind of minimal realizations for $\mathcal{D}$. This is the famous state space isomorphism theorem, found e.g in [1]. It states that an effective coordinatization of the state space is possible for any minimal realization satisfying the conditions of the following Theorem 6.1:

Theorem 6.1. *Let $\mathcal{D}$ be an I/O map of an I/O stable DLS, such that* $\operatorname{range}(\bar{\pi}_+\mathcal{D}\pi_-)$ *is closed. Then all minimal input stable and output stable realizations of $\mathcal{D}$ are state space isomorphic to each other by a bounded bijection. Moreover, all such realizations are exactly (infinite time) observable and exactly (infinite-time) controllable.*

Proof. Just combine claim (iii) of Lemma 6.1 with Lemma 6.2. □

7 H^∞ Riccati equations of minimal realizations

In this section, the main result of this paper is given. It should be compared to [4, Theorem 142], appearing as Lemma 4.1 in this paper. In the proof, the State Space Isomorphism Theorem 6.1 used; therefore we need make the restrictive assumption that the range of the Hankel operator $\bar{\pi}_+\mathcal{D}_\phi\pi_-$ is closed.

Theorem 7.1. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_{\mathcal{T}^{*j}} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an input stable, output stable, I/O stable and minimal DLS, such that the Hankel operator $\bar{\pi}_+\mathcal{D}\pi_- : \ell^2(\mathbb{Z}_-;U) \to \ell^2(\mathbb{Z}_+;Y)$ has a closed range. Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator.*

(i) Each solution $P \in ric_0(\Phi, J)$ gives rise to the (stable) spectral factorization

$$\mathcal{D}^*J\mathcal{D} = \mathcal{D}^*_{\phi_P}\Lambda_P\mathcal{D}_{\phi_P} \tag{22}$$

of the Popov operator.

(ii) Conversely, let $\mathcal{D}' : \ell^2(\mathbb{Z};U) \to \ell^2(\mathbb{Z};U)$ be an I/O map of an I/O stable DLS, such that $\pi_0\mathcal{D}'\pi_0 = I$. Assume that $\mathcal{N}\mathcal{D}' = \mathcal{D}$ for some I/O stable and (J,Λ)-inner I/O map $\mathcal{N} : \ell^2(\mathbb{Z};U) \to \ell^2(\mathbb{Z};U)$, where $\Lambda \in \mathcal{L}(U)$ is boundedly invertible and self-adjoint.

Then the spectral factorization

$$\mathcal{D}^*J\mathcal{D} = \mathcal{D}'^*\Lambda\mathcal{D}'$$

of the Popov operator is of the form of equation (22), with $\mathcal{D}' = \mathcal{D}_{\phi_P}$ and $\Lambda = \Lambda_P$ for some $P \in ric_0(\Phi, J)$.

Proof. The first claim (i) is a special case of claim (i) of Lemma 4.1. To prove claim (ii), we first write down the USBR $\Phi_u(\mathcal{I}, \mathcal{D})$ for $\mathcal{D}$. It is given by

$$\Phi_u(\mathcal{I}, \mathcal{D}) = \begin{bmatrix} (S^*|H')^j & \bar{\pi}_+ \mathcal{D} \pi_- \tau^{*j} \\ \bar{\pi}_+|H' & \mathcal{D} \end{bmatrix},$$

with the state space $H' = \operatorname{range}(\bar{\pi}_+ \mathcal{D} \pi_-)$, because the Hankel operator $\bar{\pi}_+ \mathcal{D} \pi_- : \ell^2(\mathbb{Z}_-; U) \to \ell^2(\mathbb{Z}_+; Y)$ is assumed to have a closed range. The UBSR $\Phi_u(\mathcal{I}, \mathcal{D})$ is input stable, output stable and (infinite-time) exactly controllable, again by the assumption on the range of the Hankel operator. As the realization $\Phi_u(\mathcal{I}, \mathcal{D})$ is trivially (even exactly) observable, we conclude that it is certainly minimal.

We are now in the situation described by Lemma 4.3, with $\mathcal{T}_1 = \mathcal{I}$, $\mathcal{T}_2 = \mathcal{D}$ and $\mathcal{T}' = J \mathcal{N} \Lambda^{-1}$; see the example following Lemma 4.3. We conclude that a regular H^∞-solution $P' \in ric_0(\Phi_u(\mathcal{I}, \mathcal{D}), J)$ exists, such that

$$\mathcal{D}' = \mathcal{D}_{(\Phi_u(\mathcal{I}, \mathcal{D}))_{P'}} \quad \text{and} \quad \Lambda = \Lambda'_{P'}. \tag{23}$$

It is time to connect this to our data, i.e. realization Φ. As both $\Phi_u(\mathcal{I}, \mathcal{D})$ and Φ satisfy the conditions of State Space Isomorphism Theorem 6.1, there exists a boundedly invertible $T \in \mathcal{L}(H; H')$ such that $\eta_T(\Phi) = \Phi_u(\mathcal{I}, \mathcal{D})$. By equivalence (19) of Lemma 5.1, we conclude that $P := T^* P' T \in ric_0(\Phi, J)$. Moreover, for the spectral DLSs of solutions P and P' we have

$$\mathcal{D}_{\phi_P} = \mathcal{D}_{(\Phi_u(\mathcal{I}, \mathcal{D}))_{P'}} = \mathcal{D}',$$

where the first equality is by (20) and the second one by (23). As $\Lambda = \Lambda'_{P'}$ by (23), we have $\Lambda'_{P'} = \Lambda_P$ by the discussion following Proposition 5.3. This completes the proof. □

We remark that the solution P appearing in claim (ii) of Theorem 7.1 is unique in the set $ric_0(\Phi, J)$, by [4, claim (ii) of Proposition 111] and the state space isomorphism property of the realizations involved.

8 Closed range Hankel operators

In this final section, we give general conditions for an I/O map $\mathcal{D}$ to have a closed range Hankel operator. It is rather easy to see that the Hankel operator $\bar{\pi}_+ \mathcal{N} \pi_-$ is a partial isometry, if $\mathcal{N}$ is inner from both sides; i.e. $\mathcal{N}^* \mathcal{N} = \mathcal{N} \mathcal{N}^* = \mathcal{I}$. Neither is it too difficult to give a counter example of a compact infinite-rank Hankel operator whose symbol is inner from the left and continuous on $\mathbb{T}$. By the classical theorem of Kronecker, it is well-known (at least in the scalar case) that Hankel operators with rational symbols are finite-rank; hence they have closed ranges. Theorem 8.1 is an infinite-dimensional generalization that (in a sense) comprises both of these instances. As a preparation, we need the following proposition:

Proposition 8.1. *Let H, H_1 and H_2 be Hilbert spaces and $T_1 \in \mathcal{L}(H; H_1)$ and $T_2 \in \mathcal{L}(H; H_2)$ such that* $\ker(T_1) \cap \ker(T_2) = \{0\}$ *and* $\operatorname{range}(T_1)$ *is closed. Define* $T := \left[\begin{smallmatrix} T_1 \\ T_2 \end{smallmatrix}\right] \in \mathcal{L}(H; \begin{smallmatrix} H_1 \\ \oplus \\ H_2 \end{smallmatrix})$.

(i) If $\dim(\ker(T_1)) < \infty$*, then* $\operatorname{range}(T)$ *is closed.*

(ii) Conversely, if T_2 is compact and $\operatorname{range}(T)$ *is closed, then* $\dim(\ker(T_1)) < \infty$.

Proof. Using the orthogonal splitting $H = \begin{smallmatrix} \ker(T_1)^\perp \\ \oplus \\ \ker(T_1) \end{smallmatrix}$, we obtain the block matrix representation

$$T := \begin{bmatrix} T_1|\ker(T_1)^\perp & 0 \\ T_2|\ker(T_1)^\perp & T_2|\ker(T_1) \end{bmatrix} : \begin{smallmatrix} \ker(T_1)^\perp \\ \oplus \\ \ker(T_1) \end{smallmatrix} \to \begin{smallmatrix} H_1 \\ \oplus \\ H_2 \end{smallmatrix}. \tag{24}$$

For contradiction, assume that we have a sequence $\{u_j\}_{j\geq 0} \subset H$ such that $||u_j||_H = 1$ for all j but $Tu_j \to 0$ as $f \to \infty$. Decomposing orthogonally, we write $u_j = v_j \oplus w_j$ where $v_j \in \ker(T_1)^\perp$ and $w_j \in \ker(T_1)$ for all j. As $\dim(\ker(T_1)) < \infty$, we may assume without loss of generality (by taking a subsequence if necessary) that $w_j \to w \in \ker(T_1))$ as $j \to \infty$ in the norm topology. As $Tu_j \to 0$, also $T_1|\ker(T_1)^\perp v_j = T_1 u_j \to 0$ as $j \to \infty$. Because T_1 has closed range, it follows trivially that $T_1|\ker(T_1)^\perp : \ker(T_1)^\perp \to H_1$ is coercive; hence $v_j \to 0$ as $j \to \infty$ and also $T_2|\ker(T_1)^\perp v_j \to 0$ as $j \to \infty$. But now

$$T_2|\ker(T_1)\, w_j = T_2 u_j - T_2|\ker(T_1)^\perp v_j \to 0 \quad \text{as} \quad j \to \infty,$$

because $T_2 u_j \to 0$ by the counter assumption $Tu_j \to 0$. Because $\ker(T_1) \cap \ker(T_2) = \{0\}$, we conclude that $w = \lim_{j\to\infty} w_j = 0$. We have now shown that both $v_j \to 0$ and $w_j \to 0$ in the norm of H; hence $u_j \to 0$ as $j \to \infty$. This is a contradiction against the assumption that $||u_j||_H = 1$ for all j. This completes the proof of claim (i).

In order to prove claim (ii), we use again decomposition (24). For contradiction, assume that $\dim(\ker(T_1)) = \infty$. As $\ker(T_1) \cap \ker(T_2) = \{0\}$, it follows that $T_2|\ker(T_1)$ is a compact injective operator with an infinite-dimensional range. Hence there exists a sequence $\{v_j\}_{j\geq 0} \subset \ker(T_1)$ such that $||v_j||_H = 1$ but $T_2|\ker(T_1)\, v_j \to 0$. But then $Tv_j \to 0$, and as an injective operator, it cannot have a closed range. ☐

Theorem 8.1. *Assume that an I/O stable I/O map has the inner–outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, such that $\mathcal{N} : \ell^2(\mathbb{Z}; U) \to \ell^2(\mathbb{Z}; U)$ is inner from both sides and $\mathcal{X} : \ell^2(\mathbb{Z}; U) \to \ell^2(\mathbb{Z}; U)$ is outer with a bounded inverse. Then the following holds:*

(i) range $(\bar{\pi}_+ \mathcal{D} \pi_-)$ *is closed if* $n(\mathcal{D}) < \infty$ *where*

$$(25) \quad n(\mathcal{D}) := \dim \left(\ker (\bar{\pi}_+ \mathcal{N} \pi_-) / \ker (\bar{\pi}_+ \mathcal{N} \pi_-) \cap \ker \left(\bar{\pi}_+ \mathcal{X}^{-1} \pi_- \right) \right).$$

(ii) Conversely, if $\bar{\pi}_+ \mathcal{X} \pi_-$ *is compact and* range $(\bar{\pi}_+ \mathcal{D} \pi_-)$ *is closed, then* $n(\mathcal{D}) < \infty$.

Proof. As $\mathcal{X}^{-1} : \ell^2(\mathbb{Z}; U) \to \ell^2(\mathbb{Z}; U)$ is a causal bounded bijection, it is enough to show that the operator

$$(26) \qquad \bar{\pi}_+ \mathcal{D} \pi_- \mathcal{X}^{-1} \pi_- = \bar{\pi}_+ \mathcal{N} \pi_- - \mathcal{D} \bar{\pi}_+ \mathcal{X}^{-1} \pi_-$$

has closed range in order to prove claim (i). Because

$$(\bar{\pi}_+ \mathcal{N} \pi_-)^* \mathcal{D} \bar{\pi}_+ = \pi_- \mathcal{N}^* \bar{\pi}_+ \cdot \mathcal{N} \mathcal{X} \bar{\pi}_+ = \pi_- \mathcal{N}^* \mathcal{N} \mathcal{X} \bar{\pi}_+ = \pi_- \mathcal{X} \bar{\pi}_+ = 0,$$

we conclude that range $(\bar{\pi}_+ \mathcal{N} \pi_-) \perp$ range $(\mathcal{D} \bar{\pi}_+ \mathcal{X}^{-1} \pi_-)$ in decomposition (26). Hence, for any $\pi_- \tilde{u} \in \ell^2(\mathbb{Z}_-; U)$ we have

$$(27) \qquad \begin{aligned} & ||\bar{\pi}_+ \mathcal{D} \pi_- \mathcal{X}^{-1} \pi_- \tilde{u}||^2_{\ell^2(\mathbb{Z}_+; Y)} \\ & = ||\bar{\pi}_+ \mathcal{N} \pi_- \tilde{u}||^2_{\ell^2(\mathbb{Z}_+; Y)} + ||\mathcal{D} \bar{\pi}_+ \mathcal{X}^{-1} \pi_- \tilde{u}||^2_{\ell^2(\mathbb{Z}_+; Y)}. \end{aligned}$$

Moreover, because $\bar{\pi}_+ \mathcal{X}^{-1} \mathcal{N}^* \bar{\pi}_+ \cdot \mathcal{D} \bar{\pi}_+ = \bar{\pi}_+ \mathcal{X}^{-1} \mathcal{N}^* \mathcal{N} \mathcal{X} \bar{\pi}_+ = \pi_+$, we conclude that there exists a constant $c_1 > 0$ such that

$$||\mathcal{D} \bar{\pi}_+ \tilde{v}||_{\ell^2(\mathbb{Z}_+; Y)} \geq c_1 ||\pi_+ \tilde{v}||_{\ell^2(\mathbb{Z}_+; U)} \quad \text{for any} \quad \tilde{v} \in \ell^2(\mathbb{Z}_+; U).$$

This together with (27) gives the first inequality in

$$(28) \qquad \begin{aligned} c ||\bar{\pi}_+ \begin{bmatrix} \mathcal{N} \\ \mathcal{X}^{-1} \end{bmatrix} \pi_- \tilde{u}||_{\ell^2\left(\mathbb{Z}_+; \begin{smallmatrix} Y \\ \oplus \\ U \end{smallmatrix}\right)} & \leq ||\bar{\pi}_+ \mathcal{D} \pi_- \mathcal{X}^{-1} \pi_- \tilde{u}||^2_{\ell^2(\mathbb{Z}_+; Y)} \\ & \leq C ||\bar{\pi}_+ \begin{bmatrix} \mathcal{N} \\ \mathcal{X}^{-1} \end{bmatrix} \pi_- \tilde{u}||_{\ell^2\left(\mathbb{Z}_+; \begin{smallmatrix} Y \\ \oplus \\ U \end{smallmatrix}\right)} \end{aligned}$$

for all $\pi_- \tilde{u} \in \ell^2(\mathbb{Z}_-; U)$ for some constants $0 < c, C < \infty$; the latter inequality in (28) being a trivial consequence of (26). In particular

$$\begin{aligned} & \ker \left(\bar{\pi}_+ \mathcal{D} \pi_- \mathcal{X}^{-1} \pi_- \right) = \ker (\bar{\pi}_+ \mathcal{N} \pi_-) \cap \ker \left(\bar{\pi}_+ \mathcal{X}^{-1} \pi_- \right) \\ & = \ker \left(\bar{\pi}_+ \begin{bmatrix} \mathcal{N} \\ \mathcal{X}^{-1} \end{bmatrix} \pi_- \right), \end{aligned}$$

where all the Hankel operators are defined on $\ell^2(\mathbb{Z}_-;U)$. Hence, by dividing the common null space away from (28), it is enough to prove that the Hankel operator

$$\bar{\pi}_+ \begin{bmatrix} \mathcal{N} \\ \mathcal{X}^{-1} \end{bmatrix} \pi_- : \ell^2(\mathbb{Z}_-;U) \to \ell^2 \left(\mathbb{Z}_+; \begin{smallmatrix} Y \\ \oplus \\ U \end{smallmatrix} \right)$$

has closed range. To this end we use claim (i) of Proposition 8.1, and we define first the spaces

$$H := \ell^2(\mathbb{Z}_-;U) \ominus \left(\ker(\bar{\pi}_+ \mathcal{N} \pi_-) \cap \ker\left(\bar{\pi}_+ \mathcal{X}^{-1} \pi_-\right)\right),$$

$H_1 := \ell^2(\mathbb{Z}_+;Y)$, $H_2 := \ell^2(\mathbb{Z}_+;U)$, and denote the operators

$$T_1 := \bar{\pi}_+ \mathcal{N} \pi_- | H \quad \text{and } T_2 := \bar{\pi}_+ \mathcal{X}^{-1} \pi_- | H.$$

Then $T_1 \in \mathcal{L}(H;H_1)$, $T_2 \in \mathcal{L}(H;H_2)$ and $\ker(T_1) \cap \ker(T_2) = \{0\}$ by construction. Because $n(\mathcal{D}) < \infty$ for $n(\mathcal{D})$ given by (25), it follows that $\dim(\ker(T_1)) < \infty$. As $\mathcal{N}$ is inner from both sides, it follows by a quite straightforward computation that $\bar{\pi}_+ \mathcal{N} \pi_-$ is a partial isometry and hence $\mathrm{range}\,(T_1)$ is closed. Hence, all the conditions of Proposition 8.1 are satisfied, and it follows that $\mathrm{range}\left(\left[\begin{smallmatrix} T_1 \\ T_2 \end{smallmatrix}\right]\right) = \mathrm{range}\left(\bar{\pi}_+ \left[\begin{smallmatrix} \mathcal{N} \\ \mathcal{X}^{-1} \end{smallmatrix}\right] \pi_-\right)$ is closed. Claim (i) is now proved.

The latter claim (ii) follows immediatly from claim (ii) of Proposition 8.1, noting only that $\mathrm{range}\,(\bar{\pi}_+ \mathcal{X}^{-1} \pi_-)$ is closed if and only if $\mathrm{range}\,(\bar{\pi}_+ \mathcal{X} \pi_-)$ is closed. This equivalence follows because first by causality

$$\bar{\pi}_+ \mathcal{X} \bar{\pi}_+ \cdot \bar{\pi}_+ \mathcal{X}^{-1} \pi_- = -\bar{\pi}_+ \mathcal{X} \pi_- \cdot \pi_- \mathcal{X}^{-1} \pi_-,$$

where both the Toepliz operator $\bar{\pi}_+ \mathcal{X} \bar{\pi}_+ \in \mathcal{L}(\ell^2(\mathbb{Z}_+;U))$ and $\pi_- \mathcal{X}^{-1} \pi_- \in \mathcal{L}(\ell^2(\mathbb{Z}_-;U))$ are bounded bijections as

$$\begin{aligned} &\bar{\pi}_+ \mathcal{X} \bar{\pi}_+ \cdot \bar{\pi}_+ \mathcal{X}^{-1} \bar{\pi}_+ = \bar{\pi}_+ \mathcal{X} \mathcal{X}^{-1} \bar{\pi}_+ = \bar{\pi}_+ \quad \text{and} \\ &\pi_- \mathcal{X} \pi_- \cdot \pi_- \mathcal{X}^{-1} \pi_- = \pi_- \mathcal{X} \mathcal{X}^{-1} \pi_- - \pi_- \mathcal{X} \bar{\pi}_+ \cdot \bar{\pi}_+ \mathcal{X}^{-1} \pi_- = \pi_-, \end{aligned}$$

because both $\mathcal{X}$ and $\mathcal{X}^{-1}$ are causal. □

We make some remarks involving the condition $n(\mathcal{D}) < \infty$ where $n(\mathcal{D})$ is given by (25). In the special case $\mathcal{N} = \mathcal{I}$ we see that $n(\mathcal{D}) = n(\mathcal{X})$ is exactly the codimension of $\ker(\bar{\pi}_+ \mathcal{X}^{-1} \pi_-)$. But this is equal to the rank of the same Hankel operator, surely having closed range whenever this rank is finite. On the other hand, assume that the inner factor $\mathcal{N}$ is nontrivial but $\mathcal{X} = \mathcal{I}$. Then certainly the codimension of $\ker(\bar{\pi}_+ \mathcal{X} \pi_-) = \ell^2(\mathbb{Z}_-;U)$ is infinite but now

$$\ker(\bar{\pi}_+ \mathcal{N} \pi_-) = \ker(\bar{\pi}_+ \mathcal{N} \pi_-) \cap \ker\left(\bar{\pi}_+ \mathcal{X}^{-1} \pi_-\right),$$

implying $n(\mathcal{D}) = n(\mathcal{N}) = 0$.

Dedication

This report is dedicated to my opponent, prof. Anders Lindquist, who asked for some minimality in my defence, on March 17th., 2000.

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