FREQUENCY-DOMAIN ANALYSIS OF LINEAR TIME-PERIODIC SYSTEMS

H. SANDBERG

REPORT No. 29, 2002/2003, spring

ISSN 1103-467X
ISRN IML-R- -29-02/03- -SE+spring

INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

Frequency-Domain Analysis of Linear Time-Periodic Systems

Henrik Sandberg
Department of Automatic Control, Lund Institute of Technology

August 15, 2003

Abstract. In this report we study how a time-varying system with a time-periodic integral kernel (impulse response), $g(t, \tau) = g(t + T, \tau + T)$, can be expanded into a sum of essentially time-invariant systems. This allows us to define a linear frequency response operator for periodic systems, called the Harmonic Transfer Function (HTF). The HTF is a direct analog of the transfer function for time-invariant systems, but it captures the frequency coupling of a time-periodic system. It can, for example, be used to compute the induced L_2−norm of periodic systems. The report also includes analysis of convergence of truncated HTFs, which is essential for practical computations as the HTF is an infinite-dimensional operator.

Keywords. time-periodic impulse response, frequency response operator, convergence properties, norm computation

2000 Mathematics Subject Classification. 47N70

Notation. Signals defined in continuous time on an interval I will belong to the spaces $L_1(I)$ or $L_2(I)$. The standard norms on these spaces will be denoted by $||\cdot||_{L_1(I)}$ and $||\cdot||_{L_2(I)}$. When the interval I is of no importance, it will be left out. We will denote square-summable sequences by ℓ_2, and the norm by $||\cdot||_{\ell_2}$. $\boldsymbol{R}$ denotes the real axis, $\boldsymbol{R}_+$ the non-negative real axis, and $\boldsymbol{Z}$ the set of all integers. j is the imaginary unit, $j\boldsymbol{R}$ the imaginary axis, and * denotes the Hilbert-adjoint of an operator.

1 Introduction

Consider a linear operator G defined on signals u in L_2:

$$y = Gu.$$

In this paper we will restrict ourselves to a particular set of bounded (stable) operators G. The set of bounded operators, not necessarily causal, will be denoted by L_∞ and has a finite L_2-induced norm:

$$||G||_{L_\infty} = \sup_{||u||_{L_2}\leq 1} ||Gu||_{L_2}. \quad (1)$$

As is well known the set of operators in L_∞ with the norm (1) forms a Banach algebra, see Feintuch [5]. Thus the set is closed under addition and multiplication of operators (addition and multiplication defined in the natural sense), and every Cauchy sequence of operators will converge to an operator in L_∞. How to compute the norm (1) depends on how G is represented. If, for example, there exists a finite-dimensional state-space realization of G, solutions to certain Riccati equations can be used. In this paper we will pursue a frequency domain method.

We will have a special interest for operators G that have a representation in the time domain with a *causal* integral kernel:

$$y(t) = \int_{-\infty}^{t} g(t,\tau)u(\tau)d\tau, \quad (2)$$

where $u(t)$ and $y(t)$ belong to $L_2(-\infty,\infty) = L_2(\boldsymbol{R})$. $g(t,\tau)$ defined for $t \geq \tau$ will be called the impulse response of G. Conditions for when G can be represented as an integral equation (2) is given in, for example, Sandberg [10, 9].

If there is a real positive number T such that

$$g(t+T,\tau+T) = g(t,\tau), \quad \text{for all} \quad t \geq \tau, \quad (3)$$

then the operator (or the system it represents) is said to be *periodic* with period T. Periodic systems will be the topic of this paper. We will obtain a frequency-domain representation of periodic systems G, originally represented in time (2). The search for a frequency-domain representation is motivated by the fact that frequency domain methods are very successful in the study of time-invariant systems, i.e. systems whose impulse response satisfies $g(t,\tau) = g(t-\tau)$, for all $t \geq \tau$.

1.1 Background

The study of periodic systems has a long history in applied mathematics and control. Classical results in this field are those by Floquet and Hill. One reason for the many studies of periodic systems is that natural and man-made systems often obey the periodicity property (3). Some examples are: planets and satellites in orbit, rotors of wind mills and helicopters, sampled-data systems, and AC power systems. A good article that reviews many of the results for periodic systems is that of Bittanti and Colaneri [3].

The problem of frequency-domain analysis of linear time-periodic systems in continuous time has been studied by several authors in the past. Recently the following studies have been made: Wereley [11] where the so-called *harmonic transfer function* was defined and studied for systems with state-space representations. It is a direct generalization of the transfer function for time-invariant systems. Furthermore, an analog to the Nyquist criterion was presented. As some existence and convergence issues were left untreated by Wereley; Zhou et al. have written a series of papers, including [13, 14], where they have made many of the results rigorous. Möllerstedt and Bernhardsson [7, 8] took a different starting point, and the harmonic transfer function was defined on the basis of the impulse response of the system. We will take that same approach in this study, and we will try to make that approach more rigorous. Möllerstedt et al. successfully applied the frequency-domain analysis of periodic systems for modeling of power systems [8].

The subject has of course been studied by several other authors for different applications. In particular, sampled-data systems have received a lot of attention as this is important in control, see for example Jury [6], Araki et al. [1], Yamamoto et al. [12]. A more general study of frequency domain representations of time-varying linear systems has also been presented, see Ball et al. [2].

2 Properness and Roll-off

As our final goal is a frequency domain description of G, we will many times represent the input signal $u(t)$ and output signal $y(t)$ in $L_2(\boldsymbol{R})$, by their Fourier transforms $\hat{u}(j\omega)$ and $\hat{y}(j\omega)$. ω is the angular frequency. This presents no problems as $L_2(\boldsymbol{R})$ is isomorphic under the Fourier transform with $L_2(j\boldsymbol{R})$, see Dym and McKean [4]. We define the norms as follows

$$||u||_{L_2} = ||u(\cdot)||_{L_2(R)} = \left(\int_{-\infty}^{\infty} |u(t)|^2 dt \right)^{1/2} \tag{4}$$

$$= ||\hat{u}(\cdot)||_{L_2(jR)} = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} |\hat{u}(j\omega)|^2 d\omega \right)^{1/2}, \tag{5}$$

and the equality of (4) and (5) follows from Plancherel's theorem.

For implementational and analysis reasons we will truncate the representations of signals and systems, and therefore it will be interesting to study how the systems treat high-frequency signals. Hence a projection operator which we call P_Ω is defined. Its representation in the frequency domain is given by

$$\hat{y}_\Omega(j\omega) = \widehat{(P_\Omega y)}(j\omega) = \begin{cases} \hat{y}(j\omega), & |\omega| \leq \Omega, \\ 0, & |\omega| > \Omega. \end{cases} \tag{6}$$

Notice that P_Ω is not causal in the time domain. It is also convenient to define $Q_\Omega = I - P_\Omega$. In order for a truncated systems to be a good approximation we need some sort of roll-off, corresponding to strictly properness for linear time-invariant systems, see Zhou [15]. For physical systems this is a generic property. We call a system strictly proper if

$$||G - P_{\Omega_1} G P_{\Omega_2}||_{L_\infty} \to 0 \quad \text{as} \quad \Omega_1, \Omega_2 \to \infty.$$

To give sufficient conditions for properness we first notice that we can decompose the problem into two separate problems:

$$||G - P_{\Omega_1} G P_{\Omega_2}||_{L_\infty} \leq ||(I - P_{\Omega_1})G||_{L_\infty} + ||G(I - P_{\Omega_2})||_{L_\infty}.$$

Definition 1 (Möllerstedt [7]) *If for a system $G \in L_\infty$ there are positive constants C_1 and k_1 such that*

$$||(I - P_\Omega)G||_{L_\infty} \le C_1 \cdot \Omega^{-k_1},$$

then G is said to have output roll-off k_1*, and if there are positive constants C_2 and k_2 such that*

$$||G(I - P_\Omega)||_{L_\infty} \le C_2 \cdot \Omega^{-k_2},$$

then G is said to have input roll-off k_2*.*

For systems with output roll-off k_1 and input roll-off k_2 we have strictly properness and the following rate of convergence for truncated operators $P_{\Omega_1} G P_{\Omega_2}$ when Ω_1 and Ω_2 is increased:

$$||G - P_{\Omega_1} G P_{\Omega_2}||_{L_\infty} \le C_1 \cdot \Omega_1^{-k_1} + C_2 \cdot \Omega_2^{-k_2}. \tag{7}$$

Next we give sufficient conditions on the impulse response $g(t, \tau)$ of a stable periodic system that ensures roll-off. Similar theorems are given by Möllerstedt in [7].

Theorem 1 *Assume that $\partial^i g(t, \tau)/\partial t^i$ are smooth causal impulse responses that generate operators in L_∞ for $i = 0 \dots k_1$ and that*

$$\frac{\partial^i g}{\partial t^i}(t, t) = 0, \quad i = 0, \dots, k_1 - 2$$

$$\operatorname*{ess\ sup}_{0 \le t \le T} \left| \frac{\partial^{(k_1-1)} g}{\partial t^{(k_1-1)}}(t, t) \right| < \infty$$

then G has output roll-off k_1.

Proof. Differentiate the output (2) repeatedly using the Leibniz rule and the assumptions of the theorem

$$y(t) = \int_{-\infty}^{t} g(t, \tau) u(\tau) d\tau$$

$$y'(t) = g(t, t)u(t) + \int_{-\infty}^{t} \frac{\partial g}{\partial t}(t, \tau) u(\tau) d\tau \quad \dots$$

$$y^{(k_1)}(t) = \frac{\partial^{(k_1-1)} g}{\partial t^{(k_1-1)}}(t, t)u(t) + \int_{-\infty}^{t} \frac{\partial^{k_1} g}{\partial t^{k_1}}(t, \tau) u(\tau) d\tau.$$

If we by $G_t^{k_1}$ mean the operator generated by the impulse response $\partial^{k_1} g/\partial t^{k_1}$, we obtain the bound

$$||y^{(k_1)}||_{L_2} \le \underbrace{\left(\operatorname*{ess\ sup}_{0 \le t \le T} \left| \frac{\partial^{(k_1-1)} g}{\partial t^{(k_1-1)}}(t, t) \right| + ||G_t^{k_1}||_{L_\infty} \right)}_{C_1} ||u||_{L_2}.$$

Now the Fourier transform of $y^{(k_1)}(t)$ is $(j\omega)^{k_1} \hat{y}(j\omega)$, and by Plancherel's theorem we have the bounds

$$\Omega^{k_1} ||Q_\Omega y||_{L_2} \le ||\hat{Q}_\Omega(\cdot)(\cdot)^{k_1} \hat{y}(\cdot)||_{L_2(jR)} \le ||(\cdot)^{k_1} \hat{y}(\cdot)||_{L_2(jR)} = ||y^{(k_1)}||_{L_2},$$

and the result follows. ∎

Theorem 2 *Assume that* $\partial^i g(t,\tau)/\partial\tau^i$ *are smooth causal impulse responses that generate operators in* L_∞ *for* $i = 0 \dots k_2$ *and that*

$$\frac{\partial^i g}{\partial \tau^i}(t,t) = 0, \quad i = 0, \dots, k_2 - 2$$
$$\operatorname*{ess\,sup}_{0 \le t \le T} \left| \frac{\partial^{(k_2-1)} g}{\partial \tau^{(k_2-1)}}(t,t) \right| < \infty$$

then G *has input roll-off* k_2.

Proof. Similar to the proof of Theorem 1, but use partial integration instead. For details see Möllerstedt [7]. The constant C_2 is given by

$$C_2 = \operatorname*{ess\,sup}_{0 \le t \le T} \left| \frac{\partial^{(k_2-1)} g}{\partial \tau^{(k_2-1)}}(t,t) \right| + ||G_\tau^{k_2}||_{L_\infty}.$$

■

Example 1 (Finite-Dimensional State-Space Models) *A* $T-$*periodic system with a finite-dimensional state-space realization can be written as*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t),$$

with $A(t) = A(t+T)$, $B(t) = B(t+T)$, *etc. We assume all matrices are bounded and as differentiable as is required. The impulse response of the system is given by* $g(t,\tau) = C(t)\Phi_A(t,\tau)B(\tau) + D(t)\delta(t-\tau)$, *where* Φ_A *is the transition matrix for* $\dot{x}(t) = A(t)x(t)$. *For this system sufficient roll-off conditions become*

1. $D(t) \equiv 0$ *gives input and output roll-off 1.*
2. 1.+$[C(t)B(t)] \equiv 0$ *gives input and output roll-off 2.*
3. (a) 2.+ $[(C'(t) + C(t)A(t))B(t)] \equiv 0$ *gives output roll-off 3.*
   (b) 2.+ $[C(t)(B'(t) - A(t)B(t))] \equiv 0$ *gives input roll-off 3.*
4. (a) 3.*a*+ $[C''(t)B(t) + 2C'(t)A(t)B(t) + C(t)A'(t)B(t) + C(t)A^2(t)B(t)] \equiv 0$ *gives output roll-off 4.*
   (b) 3.*b*+ $[C(t)B''(t) - 2C(t)A(t)B'(t) - C(t)A'(t)B(t) + C(t)A^2(t)B(t)] \equiv 0$ *gives input roll-off 4.*

Notice how these conditions reduce to well-known Markov-parameter conditions for roll-off of time-invariant systems. Also it is seen that no direct term implies strictly properness.

In the following proposition some simple properties for calculations with systems with input-output roll-off are stated:

Proposition 1 (Roll-off) *The following basic rules apply for systems with roll-off:*

(a) If H has output roll-off k_1 and G is bounded, then HG has an output roll-off of at least k_1.

(b) If G has input roll-off k_2 and H is bounded, then HG has an input roll-off of at least k_2.

(c) If G has a time-invariant impulse response, that is $g(t,\tau) = g(t-\tau)$ for all $t \geq \tau$, then if it has output roll-off k_1, it has input-roll off k_1, and vice versa. Moreover $|\hat{g}(j\omega)| \leq C_1 \cdot |\omega|^{-k_1}$.

(d) If H is a time-invariant system with output roll-off k_1 and G has output roll-off k_2, then HG has output roll-off $k_1 + k_2$.

(e) If H is a time-invariant system with input roll-off k_1 and G has input roll-off k_2, then GH has input roll-off $k_1 + k_2$.

Proof. Follows from Definition 1, the property $||GH||_{L_\infty} \leq ||G||_{L_\infty}||H||_{L_\infty}$, and $Q_\Omega H = Q_\Omega H Q_\Omega$ for time-invariant H. ■

2.1 The Truncated Operator $P_{\Omega_1} G P_{\Omega_2}$ and Some Approximation Errors

When working with a periodic system numerically, some sort of approximate system needs to be stored. In this paper operators that are truncated in the frequency domain are frequently used. It is neccessary to know how large the errors are and how they propagate. In this section some results of this sort are given.

If we want to compute the norm (1) of a system with roll-off, the following result holds:

Lemma 1 *The $L_\infty-$norm of $P_{\Omega_1} G P_{\Omega_2}$ converges to the $L_\infty-$norm of G as Ω_1, Ω_2 tend to infinity, and the error is bounded:*

$$0 \leq ||G||_{L_\infty} - ||P_{\Omega_1} G P_{\Omega_2}||_{L_\infty} \leq C_1 \cdot \Omega_1^{-k_1} + C_2 \cdot \Omega_2^{-k_2}.$$

Proof. Follows directly from (7). ■

If we multiply two truncated systems with roll-off, the error is also bounded:

Lemma 2 *Assume F and G are strictly proper, and that F has output roll-off k_1 and G has input roll-off k_2. Then*

$$||FG - (P_{\Omega_1} F P_{\Omega_3})(P_{\Omega_3} G P_{\Omega_2})||_{L_\infty} \leq ||G||_{L_\infty} \cdot C_1 \cdot \Omega_1^{-k_1} + ||F||_{L_\infty} \cdot C_2 \cdot \Omega_2^{-k_2},$$

holds asymptotically as $\Omega_1, \Omega_2, \Omega_3$ tend to infinity.

Proof. We make an orthogonal decomposition of the Hilbert space L_2, so that $L_2 = P_\Omega L_2 \oplus Q_\Omega L_2$, in this basis F and G take the operator-matrix forms

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \quad G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

where $F_{11}: P_{\Omega_3}L_2 \to P_{\Omega_1}L_2$, $F_{12}: Q_{\Omega_3}L_2 \to P_{\Omega_1}L_2$ etc. Now we obtain

$$FG - (P_{\Omega_1}FP_{\Omega_3})(P_{\Omega_3}GP_{\Omega_2}) = \begin{bmatrix} 0 & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} 0 & G_{12} \\ G_{21} & G_{22} \end{bmatrix} + \begin{bmatrix} 0 & F_{11}G_{12} \\ F_{21}G_{11} & 0 \end{bmatrix}.$$

And hence,

$$\begin{aligned} ||FG - (P_{\Omega_1}FP_{\Omega_3})(P_{\Omega_3}GP_{\Omega_2})||_{L_\infty} &\le (C_1\Omega_1^{-k_1} + C_3\Omega_3^{-k_3})(C_2\Omega_2^{-k_2} + C_4\Omega_3^{-k_4}) \\ &\quad + ||F_{11}||_{L_\infty}C_2\Omega_2^{-k_2} + ||G_{11}||_{L_\infty}C_1\Omega_1^{-k_1}. \end{aligned}$$

As F and G are strictly proper we have introduced the input roll-off k_3 for F, and the output roll-off k_4 for G. For large $\Omega_1, \Omega_2, \Omega_3$, the first term decays to zero faster than the two other terms, and the asymptotic bound follows. ■

Lemma 3 *If we approximate $(I+G)^{-1}$ with $(I+P_{\Omega_1}GP_{\Omega_2})^{-1}$ (assuming they both exist), and G has output roll-off k_1 and input roll-off k_2, then the $L_\infty-$norm error is bounded:*

$$\begin{aligned} &||(I+G)^{-1} - (I+P_{\Omega_1}GP_{\Omega_2})^{-1}||_{L_\infty} \\ &\qquad \le ||(I+G)^{-1}||_{L_\infty} \cdot ||(I+P_{\Omega_1}GP_{\Omega_2})^{-1}||_{L_\infty} \cdot (C_1 \cdot \Omega_1^{-k_1} + C_2 \cdot \Omega_2^{-k_2}). \end{aligned}$$

Proof. Notice first that

$$||(I+G)^{-1} - (I+P_{\Omega_1}GP_{\Omega_2})^{-1}||_{L_\infty} \le ||(I+G)^{-1}||_{L_\infty}||I - (I+G)(I+P_{\Omega_1}GP_{\Omega_2})^{-1}||_{L_\infty}.$$

Using the same basis as in the proof of Lemma 2, we notice that the second factor is

$$I - \begin{bmatrix} I+G_{11} & G_{12} \\ G_{21} & I+G_{22} \end{bmatrix} \begin{bmatrix} (I+G_{11})^{-1} & 0 \\ 0 & I \end{bmatrix} = -\begin{bmatrix} 0 & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} (I+G_{11})^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

The norm of the first factor is bounded by $C_1\Omega_1^{-k_1} + C_2\Omega_2^{-k_2}$, and the second factor is bounded by $||(I+P_{\Omega_1}GP_{\Omega_2})^{-1}||_{L_\infty}$, and the result follows. ■

3 Fourier Expansions of Periodic Systems

Until now we have only represented a system G with a convolution integral in the time-domain. For time-invariant systems it is well-known that convolution becomes simple multiplication if we represent signals and systems in the frequency domain. We will now look for an analog frequency-domain representation of linear time-periodic systems. For reasons that will become clearer later it simplifies to consider systems in a subset $B \subset L_\infty$. The subset B is defined as

$$B = \{G : ||G||_B < \infty \text{ and } g(t,\tau) \text{ fulfills (3)}\} \tag{8}$$

where

$$||G||_B = \sum_{k=0}^{\infty} \left(\int_{r=kT}^{(k+1)T} \int_{t=0}^{T} |g(t,t-r)|^2 dt\, dr \right)^{\frac{1}{2}}. \tag{9}$$

The norm $||\cdot||_B$ is a combination of a Hilbert-Schmidt norm and an ℓ_1-norm. We now give some basic properties of the operators that belong to the subset B.

Proposition 2 $||\cdot||_B$ *is a norm, and all operators in* B *are bounded on* L_2*:*

$$||G||_{L_\infty} = \sup_{||u||_{L_2}\leq 1} ||Gu||_{L_2} \leq 2||G||_B.$$

Proof. See Appendix A1. ■

Notice that $||G||_B$ might be a very poor estimate of $||G||_{L_\infty}$, so the norm is not used for explicit calculations. The reason for introducing the set B is that it simplifies the Fourier analysis, as will be seen. The subset B is not empty:

Example 2 (Exponentially stable systems are in B**)** *All periodic systems that are uniformly exponentially stable, i.e. there are positive constants* K, κ *such that*

$$|g(t,\tau)| \leq K \cdot e^{-\kappa(t-\tau)}, \quad t \geq \tau$$

are in B*.*

If G has a periodic impulse response then for every given $r \in \boldsymbol{R}_+$, the impulse response $g(t, t-r)$ is T-periodic in t. For $G \in B$, $g(\cdot, \cdot - r)$ belongs to $L_2[0,T]$ for almost all $r \in \boldsymbol{R}_+$. This follows from Fubini's theorem (see for example Dym and McKean [4] and consider iterated integrals). Hence, for almost all r we can expand the impulse response in a Fourier series with convergence in $L_2[0,T]$:

$$g(t, t-r) = \sum_{l=-\infty}^{\infty} g_l(r) e^{jl\omega_0 t}, \tag{10}$$

where

$$g_l(r) = \frac{1}{T}\int_0^T e^{-jl\omega_0 t} g(t, t-r)dt \tag{11}$$

and $\omega_0 = 2\pi/T$. If the impulse response is real $g_{-l}(r) = \overline{g_l(r)}$. We will assume the impulse response is real from now on. The Fourier coefficients are relatively nice functions of r:

Proposition 3 *For the Fourier coefficients of* $G \in B$ *it holds that*

$$g_l(\cdot) \in L_1(\boldsymbol{R}_+) \cap L_2(\boldsymbol{R}_+)$$

for all $l \in \boldsymbol{Z}$*.*

Proof. See Appendix A2. ■

Next we see how truncated Fourier expansions of systems in B behave. The following lemma proves convergence:

Lemma 4 (Truncated Fourier representations) *A truncated Fourier expansion of* $G \in B$ *with* N *frequencies is defined as*

$$G_N : \quad g_N(t,\tau) = \sum_{l=-N}^{N} g_l(t-\tau) e^{jl\omega_0 t},$$

where $\tau = t - r$*. It has the following properties:*

(i) G_N *belongs to* B.

(ii) $||G - G_N||_{L_\infty} \leq 2||G - G_N||_B \to 0$ *as* $N \to \infty$.

Proof. (i): We have $G_N = \sum_{l=-N}^{N} G^l$, where each G^l has an impulse response $g^l(t, t-r) = g_l(r)e^{jl\omega_0 t}$. Now,

$$||G^l||_B = \sum_{k=0}^{\infty} \left(\int_{kT}^{(k+1)T} \int_0^T \left| g_l(r)e^{jl\omega_0 t} \right|^2 dt\, dr \right)^{1/2} = \sum_{k=0}^{\infty} \left(\int_{kT}^{(k+1)T} T|g_l(r)|^2 dr \right)^{1/2}$$
$$\leq \sqrt{T}||G||_B.$$

The last inequality is due to the Cauchy-Schwarz inequality applied to (11). The result follows from the triangular inequality with the $B-$norm.

(ii): Study first the error-function

$$\gamma_N(r) = \int_0^T |g(t, t-r) - \sum_{l=-N}^{N} g_l(r)e^{jl\omega_0 t}|^2 dt$$

which is finite for almost all r. By the convergence of the Fourier series in $L_2[0, T]$ we get that

$$\lim_{N\to\infty} \gamma_N(r) = 0,$$

almost everywhere, and $\{\gamma_N(r)\}_{N=0}^{\infty}$ is then a decreasing sequence.

Next we see that the limit in (ii) indeed exists. This can be seen as $\{||G - G_N||_B\}_{N=1}^{\infty}$ is a bounded decreasing sequence. By definition and interchanging the order of limits we then have

$$\begin{aligned} \lim_{N\to\infty} ||G - G_N||_B &= \lim_{N\to\infty} \sum_{k=0}^{\infty} \left(\int_{kT}^{(k+1)T} \gamma_N(r)dr \right)^{1/2} \\ &= \sum_{k=0}^{\infty} \lim_{N\to\infty} \left(\int_{kT}^{(k+1)T} \gamma_N(r)dr \right)^{1/2} \\ &= \sum_{k=0}^{\infty} \left(\int_{kT}^{(k+1)T} \lim_{N\to\infty} \gamma_N(r)dr \right)^{1/2} = 0. \end{aligned}$$

To justify the interchange of the order of the limit and the summation, we notice that the rest term of the series:

$$\sum_{k=n}^{\infty} \left(\int_{kT}^{(k+1)T} \gamma_N(r)dr \right)^{1/2} \leq \sum_{k=n}^{\infty} \left(\int_{kT}^{(k+1)T} \gamma_0(r)dr \right)^{1/2} \to 0, \quad n \to \infty,$$

tends to zeros uniformly in N. Hence the interchange is justified.

The interchange of the order of the limit and integration is a property of the Lebesgue integral of decreasing sequences of functions (a.e.), see for example Exercise 6 at page 10 of Dym and McKean [4]. The final result follows with the help of Proposition 1. ■

3.1 Frequency Coupling and Steady-State Response

To see the difference between a time-invariant and a time-periodic system it is instructive to study the steady-state response to an harmonic input signal, $u(t) = e^{j\omega t}$ with frequency ω. For time-invariant systems it is well known that the output also is an harmonic of the same frequency. This is, however, not the case in the time-periodic case. If we for simplicity study a finite Fourier expansion of G, we obtain

$$\begin{aligned} y(t) = G_N e^{j\omega t} &= \int_{-\infty}^{t} \left(\sum_{l=-N}^{N} g_l(t-\tau) e^{jl\omega_0 t} \right) e^{j\omega\tau} d\tau \\ &= \sum_{l=-N}^{N} e^{jl\omega_0 t} \int_{-\infty}^{\infty} g_l(t+\tau) e^{-j\omega\tau} d\tau \\ &= \sum_{l=-N}^{N} \hat{g}_l(j\omega) e^{j(l\omega_0+\omega)t} = \left(\sum_{l=-N}^{N} \hat{g}_l(j\omega) e^{jl\omega_0 t} \right) e^{j\omega t}. \end{aligned} \tag{12}$$

This shows that the response includes a whole range of frequencies, with a difference of ω_0. This is a well-known property of linear periodic systems, see for example Wereley [11] or Zhou et al. [13]. Hence these systems have frequency coupling. It also shows that a frequency-domain approach could be successful, as there is still a fairly simple relation between frequencies in input and output.

4 The Harmonic Transfer Function

By including a sufficient amount of frequencies in the Fourier expansion G_N of G, we can come arbitrarily close to G itself in $L_\infty-$sense. Let $y_N = G_N u$ and $y = Gu$, then from (2) and Lemma 4 we have

$$y_N(t) = \int_{-\infty}^{t} \left(\sum_{l=-N}^{N} g_l(t-\tau) e^{jl\omega_0 t} \right) u(\tau) d\tau = \sum_{l=-N}^{N} [g_l(\cdot) e^{jl\omega_0 \cdot} * u(\cdot) e^{jl\omega_0 \cdot}](t) \tag{13}$$

where $*$ is the standard convolution product. By applying the Fourier transform on (13) we get

$$\hat{y}_N(j\omega) = \sum_{l=-N}^{N} \hat{g}_l(j\omega - jl\omega_0) \hat{u}(j\omega - jl\omega_0). \tag{14}$$

The Fourier transform of $g_l(t)$, denoted by $\hat{g}_l(j\omega)$, is well defined by Proposition 2, and even bounded and continuous for all ω as $g_l \in L_1$, see [4]. By Lemma 4(ii) and Plancherel's theorem, we know that $\hat{y}_N(j\omega)$ converges to $\hat{y}(j\omega)$ in $L_2(j\boldsymbol{R})$ as $N \to \infty$. Therefore we can put $N = \infty$ in (14) if we mean convergence in L_2-sense, and not point wise convergence.

In Araki et al. [1] the Sample-Data(SD)-Fourier transform was defined, and it will also be useful here. The SD-transform is an isometric isomorphism between $L_2(j\boldsymbol{R})$ and a Hilbert space we will denote by $L_2^Z(jI_0)$. It maps the Fourier transform into an infinite-dimensional column-vector-valued function. The SD-transform of $\hat{u}(j\omega)$ is denoted by $\hat{U}(j\omega)$ and is defined as

$$\hat{U}(j\omega) = \begin{bmatrix} \dots & \hat{u}(j\omega + 2j\omega_0) & \hat{u}(j\omega + j\omega_0) & \hat{u}(j\omega) & \hat{u}(j\omega - j\omega_0) & \hat{u}(j\omega - 2j\omega_0) & \dots \end{bmatrix}^T.$$

As the vector contains repeated versions of $\hat{u}(j\omega)$, it is enough to consider $\omega \in (-\omega_0/2, \omega_0/2] = I_0$ to be able to take the inverse SD-transform. We define the norm in $L_2^Z(jI_0)$ as

$$||\hat{U}(\cdot)||_{L_2^Z(jI_0)} = \frac{1}{\sqrt{2\pi}} \left(\int_{I_0} ||\hat{U}(j\omega)||_{\ell_2}^2 d\omega \right)^{1/2} = \frac{1}{\sqrt{2\pi}} \left(\int_{I_0} \sum_{k=-\infty}^{\infty} |\hat{u}(j\omega + jk\omega_0)|^2 d\omega \right)^{1/2}. \tag{15}$$

For signals $u \in L_2$, we now have three representations: $u(t)$, $\hat{u}(j\omega)$, and $\hat{U}(j\omega)$. We will see in the proof of Theorem 3 that the following extended Plancherel's theorem is true:

$$||u||_{L_2} = ||u(\cdot)||_{L_2(R)} = ||\hat{u}(\cdot)||_{L_2(jR)} = ||\hat{U}(\cdot)||_{L_2^Z(jI_0)}. \tag{16}$$

If u has finite L_2-norm, then $\hat{U}(j\omega)$ is in ℓ_2 (its elements are square summable) for almost all $\omega \in I_0$, that is $||\hat{U}(j\omega)||_{\ell_2} < \infty$ a.e.

If we write (14) when $N = \infty$ in the matrix-vector form

$$\begin{bmatrix} \vdots \\ \hat{y}(j\omega + j\omega_0) \\ \hat{y}(j\omega) \\ \hat{y}(j\omega - j\omega_0) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & \hat{g}_0(j\omega + j\omega_0) & \hat{g}_1(j\omega) & \hat{g}_2(j\omega - j\omega_0) & \\ \ddots & \hat{g}_{-1}(j\omega + j\omega_0) & \hat{g}_0(j\omega) & \hat{g}_1(j\omega - \omega_0) & \ddots \\ & \hat{g}_{-2}(j\omega + j\omega_0) & \hat{g}_{-1}(j\omega) & \hat{g}_0(j\omega - j\omega_0) & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \hat{u}(j\omega + j\omega_0) \\ \hat{u}(j\omega) \\ \hat{u}(j\omega - j\omega_0) \\ \vdots \end{bmatrix}$$

we see that we can write it in a compact form using the SD-transform as

$$\hat{Y}(j\omega) = \hat{G}(j\omega)\hat{U}(j\omega), \quad \omega \in I_0. \tag{17}$$

We call $\hat{G}(j\omega)$ the *Harmonic Transfer Function* (HTF) of G, in analogy with Wereley [11] and Möllerstedt [8]. A similar object was called the FR operator in [1] in the special case of sampled-data systems.

We may now state the counterpart of Theorem 4 and 5 in Araki et al. [1]:

Theorem 3 *For linear periodic systems G in B, we can define the HTF $\hat{G}(j\omega)$ as above, and for any input signal $u \in L_2$ it holds that*

$$||y||_{L_2}^2 = \frac{1}{2\pi} \int_{I_0} ||\hat{Y}(j\omega)||_{\ell_2}^2 d\omega = \frac{1}{2\pi} \int_{I_0} ||\hat{G}(j\omega)\hat{U}(j\omega)||_{\ell_2}^2 d\omega. \tag{18}$$

$\hat{G}(j\omega)$ is a bounded operator on ℓ_2 for almost all ω in I_0 and

$$||G||_{L_\infty} = \operatorname*{ess\,sup}_{\omega \in I_0} ||\hat{G}(j\omega)||_\infty, \tag{19}$$

where $||\cdot||_\infty$ represents the induced ℓ_2-norm.

Proof. The first part follows from Plancherel's theorem and (17):

$$||y||_{L_2}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{y}(j\omega)|^2 d\omega = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{I_0}|\hat{y}(j\omega + jk\omega_0)|^2 d\omega$$
$$= \frac{1}{2\pi}\int_{I_0}\sum_{k=-\infty}^{\infty}|\hat{y}(j\omega + jk\omega_0)|^2 d\omega = \frac{1}{2\pi}\int_{I_0}||\hat{Y}(j\omega)||_{\ell_2}^2 d\omega.$$

The interchange of integration and summation is a property of the Lebesgue integral for a sum with positive terms, see for example [4]. By inserting (17) the result follows.

That $||G||_{L_\infty} \leq \operatorname{ess\,sup}_{\omega \in I_0} ||\hat{G}(j\omega)||_\infty$ follows directly from (18). The reverse inequality, $||G||_{L_\infty} \geq \operatorname{ess\,sup}_{\omega\in I_0} ||\hat{G}(j\omega)||_\infty$, follows from a proof similar to the proof of Theorem 5 in Araki et al. [1]. ■

4.1 Computation and Truncation of HTFs

Theorem 3 gives us a way to numerically compute the norm of G. To see this let us return to the truncated systems of Section 2 and see how it carries over into the formalism of this section:

Corollary 1 *For $G \in B$, if we choose $\Omega_1 = (N_1 + 1/2)\omega_0$ and $\Omega_2 = (N_2 + 1/2)\omega_0$ then the HTF of $P_{\Omega_1} G P_{\Omega_2}$ is given by the $(2N_1 + 1) \times (2N_2 + 1)-$matrix:*

$$\hat{G}_{(N_1,N_2)}(j\omega) = \begin{bmatrix} \hat{g}_{N_1-N_2}(j\omega + jN_2\omega_0) & \dots & \hat{g}_{N_1+N_2}(j\omega - jN_2\omega_0) \\ \vdots & & \vdots \\ \hat{g}_{-N_2}(j\omega + jN_2\omega_0) & \dots & \hat{g}_{N_2}(j\omega - jN_2\omega_0) \\ \vdots & & \vdots \\ \hat{g}_{-N_1-N_2}(j\omega + jN_2\omega_0) & \dots & \hat{g}_{-N_1+N_2}(j\omega - jN_2\omega_0) \end{bmatrix}, \tag{20}$$

and

$$\hat{P}_{\Omega_1}(j\omega)\hat{G}(j\omega)\hat{P}_{\Omega_2}(j\omega) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \hat{G}_{(N_1,N_2)}(j\omega) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The truncated HTF converges to $\hat{G}(j\omega)$ in the $L_\infty-$norm according to (7).

Hence we can represent a linear periodic system in B arbitrarily well with finite-dimensional matrices. By griding the frequency interval I_0 and computing the maximum singular value of the finite-dimensional matrix (20) we get an estimate of $||G||_{L_\infty}$, where the rate of convergence depends upon the roll-off of G according to Lemma 1. By using Lemma 2-3 we also see how much error we make when we approximate products and inverses of periodic systems with truncated frequency representations.

4.2 Roll-off of $\hat{g}_l(j\omega)$

In principle we have here described a linear time-periodic system as a (generally infinite) sum of modulated linear *time-invariant* systems, each one of them represented by a transfer function $\hat{g}_l(j\omega)$. One can ask what are the properties of these individual transfer functions. Some simple observations can be made by using Corollary 1, (19), and the definitions of input and output roll-off:

- $\hat{g}_l(j\omega)$ is continuous in ω, bounded (at least by $||G||_{L_\infty}$), and $\lim_{|\omega|\to\infty} \hat{g}_l(j\omega) = 0$ for each l. Also $\hat{g}_l(j\omega)$ is in $L_2(j\boldsymbol{R})$.
- If G has input roll-off k_2, then for all l

$$|\hat{g}_l(j\omega)| \leq \begin{cases} \sup_{\omega} |\hat{g}_l(j\omega)|, & |\omega| \leq \omega_{b,l} \\ C_2 \cdot |\omega|^{-k_2}, & |\omega| > \omega_{b,l} \end{cases} \tag{21}$$

  where $\omega_{b,l} = (C_2/ \sup |\hat{g}_l(j\omega)|)^{1/k_2}$. That is, all the transfer functions $\hat{g}_l(j\omega)$ have at least roll-off k_2 in the classical sense.

- If G has output roll-off k_1 and input roll-off k_2, then we have

$$\sup_{\omega} |\hat{g}_l(j\omega)| \leq \min\{C_1 \cdot |l\omega_0|^{-k_1} + C_2 \cdot |l\omega_0|^{-k_2}, ||G||_{L_\infty}\}. \tag{22}$$

5 Conclusion

In this report we have studied linear time-periodic systems from a frequency domain point of view. Previous studies in this field are often based on a state-space approach, see [11, 13], whereas we have here taken an input-output operator approach. We have identified a set of periodic impulse responses, denoted by B, that allow us to expand the corresponding systems in a series of modulated time-invariant systems, with convergence in an induced L_2-norm sense. The main result of this report is that we can construct a linear frequency response operator for these systems, the HTF. The HTF is an analog of the transfer function for linear time-invariant systems. By computing the maximum of the norm of the HTF on the imaginary axis we obtain the induced L_2-norm of the periodic system. This can be compared with computing the L_∞-norm of transfer functions.

We have also put some effort into the problem of how truncated HTFs converge. This problem has been approached by introducing the concepts of input and output roll-off. The roll-off of a system tells how high-frequency signals are damped. In the report conditions on the impulse response are given that ensures roll-off. The HTF of a system with roll-off can be truncated into a finite-dimensional matrix without introducing large error in induced L_2-norm sense. When the convergence rate of truncated HTFs is known we can compute approximations of series or feedback connections of periodic systems. This is believed to be important as robustness of connections often is better studied in the frequency domain. In the future robustness studies with the HTF should be considered.

Acknowledgments

The author is thankful to Dr. Jarmo Malinen, Prof. Anders Rantzer, Prof. Bo Bernhardsson, and Dr. Erik Möllerstedt for many fruitful discussions and inspiration. The author is also grateful to the Mittag-Leffler institute and its personel for five inspiring months at the institute.

References

[1] M. Araki, Y. Ito, and T. Hagiwara. "Frequency response of sampled-data systems." Automatica, **32:4**, pp. 483–497, 1996.

[2] J. Ball, I. Gohberg, and M. Kaashoek. "A frequency response function for linear time-varying systems." Math. Control Signals Systems, **8:4**, pp. 334–351, 1995.

[3] S. Bittanti and P. Colaneri. "Periodic control." In Webster, Ed., Encyclopedia of Electrical and Electronics Engineering, vol. 16, pp. 59–73. John Wiley and Sons, 1999.

[4] H. Dym and H. McKean. Fourier series and integrals. Academic Press, 1972.

[5] A. Feintuch. Robust control theory in Hilbert space. Applied Mathematical Sciences 130. Springer-Verlag, 1998.

[6] E. Jury. Sampled-data control systems. Wiley, New York, 1958.

[7] E. Möllerstedt. Dynamic Analysis of Harmonics in Electrical Systems. PhD thesis, Dept. of Automtic Control, Lund Institute of Technology, Sweden, 2000.

[8] E. Möllerstedt and B. Bernhardsson. "Out of control because of harmonics - an analysis of the harmonic response of an inverter locomotive." IEEE Control Systems Magazine, **20:4**, pp. 70–81, August 2000.

[9] I. W. Sandberg. "Integral representations for linear maps." IEEE Transactions on Circuits and Systems, **35:5**, pp. 536–544, May 1988.

[10] I. W. Sandberg. "Linear maps and impulse responses." IEEE Transactions on Circuits and Systems, **35:2**, pp. 201–206, February 1988.

[11] N. Wereley. Analysis and Control of Linear Periodically Time Varying Systems. PhD thesis, Dept. of Aeronautics and Astronautics, MIT, 1991.

[12] Y. Yamamoto and P. P. Khargonekar. "Frequency response of sampled-data systems." IEEE Transactions on Automatic Control, **41:2**, pp. 166–176, February 1996.

[13] J. Zhou and T. Hagiwara. "Existence conditions and properties of the frequency response operators of continuous-time periodic systems." SIAM Journal of Control and Optimization, **40:6**, pp. 1867–1887, 2002.

[14] J. Zhou and T. Hagiwara. "H_2 and H_∞ norm computations of linear continuous-time periodic systems via the skew analysis of frequency response operators." Automatica, **38**, pp. 1381–1387, 2002.

[15] K. Zhou and J. Doyle. Essentials of Robust Control. Prentice Hall, Upper Saddle River, New Jersey, 1998.

A Proofs

A.1 Proof of Proposition 1

It is fairly obvious that $||\cdot||_B$ fulfills the requirements of a norm, as it is a combination of the Hilbert-Schmidt- and the ℓ_1-norm.

In [7] it was shown that

$$||G||_{L_\infty} \leq \sum_{k=0}^{\infty} \left(\int_{t=kT}^{(k+1)T} \int_{\tau=0}^{T} |g(t,\tau)|^2 d\tau dt \right)^{\frac{1}{2}}.$$

But by periodicity of $g(t,\tau)$ (3) and the triangular inequality we see that

$$\frac{1}{2}||G||_B \leq \sum_{k=0}^{\infty} \left(\int_{t=kT}^{(k+1)T} \int_{\tau=0}^{T} |g(t,\tau)|^2 d\tau dt \right)^{\frac{1}{2}} \leq 2||G||_B,$$

hence the norms are equivalent and the result follows.

A.2 Proof of Proposition 2

We begin to prove that g_l belongs to L_2. By applying the Cauchy-Schwarz inequality to (11) we have for almost all r

$$|g_l(r)|^2 \leq \frac{1}{T} \int_0^T |g(t,t-r)|^2 dt.$$

If we now integrate and sum in the r-direction we obtain

$$\sum_{k=0}^{\infty} \left(\int_{kT}^{(k+1)T} |g_l(r)|^2 dr \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{T}} ||G||_B \tag{23}$$

which is finite for $G \in B$. We also have the inequality

$$\left(\int_0^\infty |g_l(r)|^2 dr \right)^{\frac{1}{2}} \leq \sum_{k=0}^{\infty} \left(\int_{kT}^{(k+1)T} |g_l(r)|^2 dr \right)^{\frac{1}{2}}$$

by the triangular inequality on L_2, which proves $g_l(\cdot) \in L_2[0,\infty)$.

Next we prove that g_l belongs to L_1. We have

$$\int_{kT}^{(k+1)T} |g_l(r)| dr \leq \sqrt{T} \left(\int_{kT}^{(k+1)T} |g_l(r)|^2 dr \right)^{\frac{1}{2}},$$

again by the Cauchy-Schwarz inequality. If we sum over all k the right hand side converges by (23), and the result follows.