

**ON THE NEVANLINNA-PICK  
INTERPOLATION PROBLEM: ANALYSIS  
OF THE MCMILLAN DEGREE OF  
THE SOLUTIONS**

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# On the Nevanlinna-Pick interpolation problem: analysis of the McMillan degree of the solutions

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## **Abstract**

In this paper we investigate some aspect of the Nevanlinna-Pick and Schur interpolation problem formulated for Schur-functions considered on the right half plane of  $\mathbb{C}$ . We consider the well established parametrization of the solution  $Q = T_{\Theta}(S)$  (see e.g. [2],[6]), where  $\Theta$  is completely determined by the interpolation data and  $S$  is an arbitrary Schur function. We then compare the relations between the realizations of  $Q$  and  $S$  induced by  $\Theta$ . We show in particular that  $S$  generates a solution with a low McMillan degree if and only if  $S$  satisfies some interpolation conditions formulated on the left-half plane of  $\mathbb{C}$ . This analysis can be considered to be partially complementary to the results of A. Lindquist, C. Byrnes et al. on Carathéodory functions, [3], [5], [4].

**Keywords:** Schur parameters, Interpolation, Schur functions, Positive Real Functions.

**MSC2000 Numbers:** 30E05, 93B29, 93B30

# 1 Introduction

We consider here the problem of parametrizing the interpolating Schur functions  $Q$  for a given set of Schur data by means of a linear fractional transformation with a bound on the degree. The general problem (i.e. without bound) has been studied extensively (see [2], [6] and references therein). The technique which most widespread in the literature (and which will also use) starts from the interpolating conditions to build a linear fractional transformation (LFT), and the whole set of solutions is parametrized by means of the LFT of another Schur function  $S$ , which is chosen freely in the unit ball of  $H^\infty$ . Generically, the degree of  $Q$  is the sum of the number of interpolating conditions and the degree of the parameter function  $S$ ; that is, if  $n$  is the number of interpolating conditions, we have  $\deg(Q) = n + \deg(S)$ . Our aim is to characterize those Schur functions  $S$  for which there is degree drop in the corresponding  $Q$ . This is an important problem which has been studied with different methods (see [3], [4], [5], [8], [10] [11]). Here we use state space realizations (Section 3) for the LFT of  $S$ , denoted by  $T_\Theta(S)$ , and we analyse the controllability and observability subspace of this realization. It turns out that, while the realization is always controllable, it is not necessarily observable. A study of the unobservability subspace is thus what is developed in Section 4. In Section 5 these results are used to characterize the Schur functions  $S$  which yield a lower degree interpolant  $Q$  proving that a drop in the degree can be achieved only by imposing interpolation conditions on  $S$  in the left-half plane.

# 2 Preliminaries and notation

Let  $F$  be a rational  $p \times m$  matrix of McMillan degree  $N$ , whose entries lie in the Hardy space of the right half-plane. We shall denote by  $\mathbb{C}^+$  the right half-plane, and by  $\mathcal{H}_+^2$  the corresponding Hardy space of vector or matrix valued functions (the proper dimension will be understood from the context). The space  $\mathcal{H}_+^2$  is naturally endowed with the scalar product,

$$\langle F, G \rangle = \frac{1}{2\pi} \text{Tr} \int_{-\infty}^{\infty} F(iy)G(iy)^* dy, \quad (2.1)$$

and we shall denote by  $\| \cdot \|_2$  the associated norm. Note that if  $M$  is a complex matrix,  $\text{Tr}$  stands for its trace,  $M^T$  for its transpose and  $M^*$  for its transpose conjugate. Similarly, we define  $\mathcal{H}_+^\infty$  to be Hardy space of essentially bounded functions analytic on the right half plane.

Suppose  $F$  is right invertible in  $H_+^\infty$ . The observable pair  $(C, A)$  is said to be a *right null pair* for  $F$  (see [2]) if  $FC(sI - A)^{-1}$  is analytic in the spectrum of  $A$ .

We assume that we are given a set of interpolation points  $s_1, \dots, s_n$  in the right half-plane  $\mathbb{C}^+$  and interpolating conditions

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad (2.2)$$

with  $u_i, v_i$  row vectors in  $\mathbb{C}^p$ ,  $\|u_i\| = 1$  and  $\|v_i\| < 1$  for  $i = 1, \dots, n$  and we want to find the solutions  $Q$  to the problem

$$u_i Q(s_i)^* = v_i \quad i = 1, \dots, n \quad (2.3)$$

which are Schur-functions.

It is well-known (see eg. [2]) that all solutions of this problem can be given using a rational fractional representation defined by a J-inner function. This representation is even valid for a more general form of the interpolation problem (2.3) which allows for multiplicities of the interpolation nodes. This can be defined in the following way.

**Problem 2.1 (Schur, Nevanlinna-Pick).** *Given the matrix  $\mathcal{A}$  of size  $n \times n$  with eigenvalues in the left half plane  $\mathbb{C}^-$ , and matrices  $U, V$  of size  $n \times p$  and a constant matrix  $D$  of size  $p \times p$  parameterize all Schur-functions  $Q$  satisfying*

$$\begin{cases} (Q(s)U^* - V^*)(sI + \mathcal{A}^*)^{-1} & \text{is analytic on } \mathbb{C}^+ \\ Q(\infty) = D. \end{cases} \quad (2.4)$$

In this case the eigenvalues of  $-\mathcal{A}^*$  determine the interpolation nodes.

To avoid pathological cases, we assume that the functions  $Q(s)u_i^*$  are non constant. Notice that this assumption is always satisfied if for all  $i = 1, \dots, n$  we have  $u_i D^* \neq v_i$ .

Among the solutions of the Problem 2.1 we would like to consider solutions with "low complexity". In the scalar case – i.e. when  $p = 1$  – this can be formulated as solutions with McMillan-degree no greater than  $n$  or in other words as solutions in the form of ratio of two polynomials with degree  $n$ . This type of formulation can be applied in the multivariate case, as well.

First let us denote by  $\Xi$  the inner function determined by the interpolation nodes and directions, i.e.

$$\Xi(s) = \left( \begin{array}{c|c} \mathcal{A} & U \\ \hline -U^* \mathcal{R}^{-1} & I \end{array} \right),$$

where  $\mathcal{R}$  is the solution of the Lyapunov-equation

$$\mathcal{A}\mathcal{R} + \mathcal{R}\mathcal{A}^* + UU^* = 0.$$

**Problem 2.2.** *Given the matrix  $\mathcal{A}$  of size  $n \times n$  with eigenvalues in the left half plane  $\mathbb{C}^-$ , and matrices  $U, V$  of size  $n \times p$  and a constant strictly contractive matrix  $D$  of size  $p \times p$ , such that  $u_i D^* \neq v_i$ ,  $i = 1, \dots, n$ , parameterize all functions  $Q$ , for which*

(i)  $Q$  is a Schur-function;

(ii)

$$\begin{cases} (Q(s)U^* - V^*)(sI + \mathcal{A}^*)^{-1} & \text{is analytic on } \mathbb{C}^+ \\ Q(\infty) = D. \end{cases} \quad (2.5)$$

(iii)  $Q$  can be written in the form

$$Q = G^* (F^*)^{-1}, \quad (2.6)$$

where  $G, F$  are stable rational functions such that  $\Xi F^*$  is stable,  $F(\infty) = I$ ,

In Section 5 we show among others that equation (2.6) implies that the function  $Q$  has McMillan-degree no greater than  $n$ .

The following lemma shows that without loss of generality we can assume that  $D = 0$ .

**Lemma 2.1.** *Let  $Q$  be a Schur function, with  $Q(\infty) = D$  and assume that  $DD^* < I$ . Set*

$$\begin{aligned} Q_D &= (I - DD^*)^{1/2} (I - QD^*)^{-1} (Q - D) (I - D^*D)^{-1/2} \\ U_D &= (U - VD) (I - D^*D)^{-1/2} \\ V_D &= (V - UD^*) (I - DD^*)^{-1/2} \end{aligned}$$

Then  $Q_D$  is a Schur-function and  $Q$  is a solution of the interpolation problem (2.4) defined by  $\mathcal{A}, U, V$  and  $D$  if and only if  $Q_D$  is a solution of (2.4) defined by  $\mathcal{A}, U_D, V_D$  and 0.

**PROOF.** We show that, if  $\alpha$  is a contractive matrix, then

$$(\alpha - D)(I - D^*D)^{-1} (\alpha^* - D^*) \leq (I - \alpha D^*) (I - DD^*)^{-1} (I - D\alpha^*),$$

Notice first that

$$D(I - D^*D)^{-1} = (I - DD^*)^{-1}D$$

Then,

$$\begin{aligned} &(\alpha - D)(I - D^*D)^{-1} (\alpha^* - D^*) \\ &= -\alpha(I - D^*D)^{-1}D^* - D(I - D^*D)^{-1}\alpha^* + D(I - D^*D)^{-1}D^* + \alpha(I - D^*D)^{-1}\alpha^* \\ &= -\alpha D^*(I - DD^*)^{-1} - (I - DD^*)^{-1}D\alpha^* + (I - DD^*)^{-1}DD^* + \alpha(I - D^*D)^{-1}\alpha^* \\ &= -\alpha D^*(I - DD^*)^{-1} - (I - DD^*)^{-1}D\alpha^* - I + (I - DD^*)^{-1} + \alpha(I - D^*D)^{-1}\alpha^* \\ &\leq -\alpha D^*(I - DD^*)^{-1} - (I - DD^*)^{-1}D\alpha^* - \alpha\alpha^* + (I - DD^*)^{-1} + \alpha(I - D^*D)^{-1}\alpha^* \\ &= -\alpha D^*(I - DD^*)^{-1} - (I - DD^*)^{-1}D\alpha^* + (I - DD^*)^{-1} + \alpha D^*(I - DD^*)^{-1}D\alpha^* \\ &= (I - \alpha D^*)(I - DD^*)^{-1} (I - D\alpha^*) \end{aligned}$$

as wanted. On the other hand  $I - QD^*$  has no zeros on the right half plane, proving that  $Q_D$  is a Schur-function.

Furthermore

$$\begin{aligned} Q_D U_D^* - V_D^* &= (I - DD^*)^{1/2} (I - QD^*)^{-1} \\ &\quad \times [(Q - D) (I - D^*D)^{-1} (U^* - D^*V^*) \\ &\quad \quad - (I - QD^*) (I - DD^*)^{-1} (V^* - DU^*)] \\ &= (I - DD^*)^{1/2} (I - QD^*) [QU^* - V^*] \end{aligned}$$

using that

$$\begin{aligned} (I - D^*D)^{-1} (U^* - D^*V^*) - D^* (I - DD^*)^{-1} (V^* - DU^*) &= U^* \\ D(I - D^*D)^{-1} (U^* - D^*V^*) - (I - DD^*)^{-1} (V^* - DU^*) &= V^* . \end{aligned}$$

Thus  $(Q_D U_D^* - V_D^*) (sI + \mathcal{A}^*)^{-1}$  is analytic on the right half plane if and only if  $(QU^* - V^*) (sI + \mathcal{A}^*)^{-1}$  is analytic on  $\mathbb{C}^+$ .

Straightforward computation gives that if

$$Q = \left( \begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \end{array} \right)$$

then

$$Q_D = \left( \begin{array}{c|c} A_Q + B_Q D^* (I - DD^*)^{-1} C_Q & B_Q (I - D^*D)^{-1/2} \\ \hline (I - DD^*)^{-1/2} C_Q & 0 \end{array} \right) .$$

### 3 State-space realizations of the solutions

It is well known that all solutions of the interpolation Problem 2.1 without constraints on the McMillan-degree can be generated by a linear fractional transformation defined by a  $J$ -inner function  $\Theta$  (See Ball, Gohberg and Rodman [2]). This function has the following realization

$$\Theta = \left( \begin{array}{c|cc} \mathcal{A} & U & V \\ \hline -U^* \mathcal{P}^{-1} & I & 0 \\ V^* \mathcal{P}^{-1} & 0 & I \end{array} \right) \quad (3.1)$$

where  $\mathcal{P} = \mathcal{P}^* > 0$  satisfies the Lyapunov-equation

$$\mathcal{A} \mathcal{P} + \mathcal{P} \mathcal{A}^* + U U^* - V V^* = 0 . \quad (3.2)$$

Then the Schur-function  $Q$  is a solution of the interpolation Problem 2.1 – without any constraint on the MacMillan-degree – if and only if

$$Q := (S\Theta_{12} + \Theta_{22})^{-1} (S\Theta_{11} + \Theta_{21})$$

where

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \quad (3.3)$$

and  $S$  is a Schur-function with  $S(\infty) = D$ . We shall write shortly  $Q = T_\Theta(S)$ .

Especially, if  $u_i$  and  $v_i$  are the rows of  $U$  and  $V$  and

$$\mathcal{A} = \text{diag}\{-\bar{s}_1, \dots, -\bar{s}_n\} ,$$

then the Schur function  $Q$  is a solution to then Nevanlinna-Pick problem

$$Q(s_i) u_i^* = v_i^* .$$

Similarly, if we take  $\mathcal{A}$  lower triangular and  $\mathcal{P}$  diagonal, we obtain a Potapov factorization of  $\Theta$  and therefore a Schur problem.

**Lemma 3.1.** *Let  $\Theta$  be a  $J$ -inner function as in (3.1), and let*

$$S = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right). \quad (3.4)$$

*Then  $Q = T_{\Theta}(S)$  has the following (possibly non-minimal) realization:*

$$Q = \left( \begin{array}{cc|c} \mathcal{A} + V(DU^* - V^*)\mathcal{P}^{-1} & -VC & -U + VD \\ -BU^*\mathcal{P}^{-1} & A & -B \\ \hline (DU^* - V^*)\mathcal{P}^{-1} & -C & D \end{array} \right) \quad (3.5)$$

**PROOF.** We can write

$$\begin{aligned} & S\Theta_{11} + \Theta_{21} \\ &= \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \left( \begin{array}{c|c} \mathcal{A} & U \\ \hline -U^*\mathcal{P}^{-1} & I \end{array} \right) + \left( \begin{array}{c|c} \mathcal{A} & U \\ \hline V^*\mathcal{P}^{-1} & 0 \end{array} \right) \\ &= \left( \begin{array}{cc|c} \mathcal{A} & 0 & U \\ -BU^*\mathcal{P}^{-1} & A & B \\ \hline -DU^*\mathcal{P}^{-1} & C & D \end{array} \right) + \left( \begin{array}{cc|c} \mathcal{A} & 0 & U \\ -BU^*\mathcal{P}^{-1} & A & B \\ \hline V^*\mathcal{P}^{-1} & 0 & 0 \end{array} \right) \\ &= \left( \begin{array}{cc|c} \mathcal{A} & 0 & U \\ -BU^*\mathcal{P}^{-1} & A & B \\ \hline -(DU^* - V^*)\mathcal{P}^{-1} & C & D \end{array} \right) \end{aligned}$$

Similarly,

$$\begin{aligned} & S\Theta_{12} + \Theta_{22} \\ &= \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \left( \begin{array}{c|c} \mathcal{A} & V \\ \hline -U^*\mathcal{P}^{-1} & 0 \end{array} \right) + \left( \begin{array}{c|c} \mathcal{A} & V \\ \hline V^*\mathcal{P}^{-1} & I \end{array} \right) \\ &= \left( \begin{array}{cc|c} \mathcal{A} & 0 & V \\ -BU^*\mathcal{P}^{-1} & A & 0 \\ \hline -DU^*\mathcal{P}^{-1} & C & 0 \end{array} \right) + \left( \begin{array}{cc|c} \mathcal{A} & 0 & V \\ -BU^*\mathcal{P}^{-1} & A & 0 \\ \hline V^*\mathcal{P}^{-1} & 0 & I \end{array} \right) \\ &= \left( \begin{array}{cc|c} \mathcal{A} & 0 & V \\ -BU^*\mathcal{P}^{-1} & A & 0 \\ \hline -(DU^* - V^*)\mathcal{P}^{-1} & C & I \end{array} \right) \end{aligned}$$

So the inverse of the above matrix is easily seen to be:

$$(S\Theta_{12} + \Theta_{22})^{-1} = \left( \begin{array}{cc|c} \mathcal{A} + V(DU^* - V^*)\mathcal{P}^{-1} & -VC & V \\ -BU^*\mathcal{P}^{-1} & A & 0 \\ \hline (DU^* - V^*)\mathcal{P}^{-1} & -C & I \end{array} \right)$$

Therefore,

$$\begin{aligned}
Q &= (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21}) \\
&= \left( \begin{array}{cc|c} \mathcal{A} + V(DU^* - V^*)\mathcal{P}^{-1} & -VC & V \\ -BU^*\mathcal{P}^{-1} & A & 0 \\ \hline (DU^* - V^*)\mathcal{P}^{-1} & -C & I \end{array} \right) \\
&= \left( \begin{array}{cc|c} \mathcal{A} & 0 & U \\ -BU^*\mathcal{P}^{-1} & A & B \\ \hline -(DU^* - V^*)\mathcal{P}^{-1} & C & D \end{array} \right) \\
&= \left( \begin{array}{cccc|c} \mathcal{A} & 0 & 0 & 0 & U \\ -BU^*\mathcal{P}^{-1} & A & 0 & 0 & B \\ -V(DU^* - V^*)\mathcal{P}^{-1} & VC & \mathcal{A} + V(DU^* - V^*)\mathcal{P}^{-1} & -VC & VD \\ 0 & 0 & -BU^*\mathcal{P}^{-1} & A & 0 \\ \hline -(DU^* - V^*)\mathcal{P}^{-1} & C & (DU^* - V^*)\mathcal{P}^{-1} & -C & D \end{array} \right) \\
&=: \left( \begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right)
\end{aligned}$$

Using the change of basis  $T = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$  we get

$$\mathbf{A}' = T\mathbf{A}T^{-1} \quad \mathbf{B}' = T\mathbf{B} \quad \mathbf{C}' = \mathbf{C}T^{-1}$$

which applied to our system yields:

$$\begin{aligned}
\left( \begin{array}{c|c} \mathbf{A}' & \mathbf{B}' \\ \hline \mathbf{C}' & \mathbf{D} \end{array} \right) &= \left( \begin{array}{cccc|c} \mathcal{A} & 0 & 0 & 0 & U \\ -BU^*\mathcal{P}^{-1} & A & 0 & 0 & B \\ 0 & 0 & \mathcal{A} + V(DU^* - V^*)\mathcal{P}^{-1} & -VC & -U + VD \\ 0 & 0 & -BU^*\mathcal{P}^{-1} & A & -B \\ \hline 0 & 0 & (DU^* - V^*)\mathcal{P}^{-1} & -C & D \end{array} \right) \\
&= \left( \begin{array}{cc|c} \mathcal{A} + V(DU^* - V^*)\mathcal{P}^{-1} & -VC & -U + VD \\ -BU^*\mathcal{P}^{-1} & A & -B \\ \hline (DU^* - V^*)\mathcal{P}^{-1} & -C & D \end{array} \right)
\end{aligned}$$

Observe that the feedthrough term  $D$  of  $S$  goes through the linear fractional transformation without any change.

In order to find some connection between the McMillan-degree of  $S$  and  $Q$  we need a state-space description of the inverse linear fractional transformation as well.

**Lemma 3.2.** *Let  $\Theta$  be a  $J$ -inner function as in (3.1) and let*

$$Q = \left( \begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \end{array} \right). \quad (3.6)$$

Then  $S = T_{\Theta}^{-1}(Q)$  has the following (possibly non-minimal) realization:

$$S = \left( \begin{array}{cc|c} A_Q & -B_Q U^* \mathcal{P}^{-1} & B_Q \\ VC_Q & \mathcal{A} + (U - VD)U^* \mathcal{P}^{-1} & VD - U \\ \hline C_Q & (V^* - DU^*)\mathcal{P}^{-1} & D \end{array} \right). \quad (3.7)$$

**PROOF.** Inverting the relation  $Q = T_{\Theta}(S)$  we get:

$$\begin{aligned} Q &= T_{\Theta}(S) = (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21}) \\ (S\Theta_{12} + \Theta_{22})Q &= (S\Theta_{11} + \Theta_{21}) \\ S(\Theta_{12}Q - \Theta_{11}) &= -(\Theta_{22}Q - \Theta_{21}) \\ S &= (\Theta_{22}Q - \Theta_{21})(-\Theta_{12}Q + \Theta_{11})^{-1} \end{aligned}$$

and so we can write

$$\begin{aligned} &\Theta_{22}Q - \Theta_{21} \\ &= \left( \begin{array}{c|c} \mathcal{A} & V \\ \hline V^*\mathcal{P}^{-1} & I \end{array} \right) \left( \begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \end{array} \right) - \left( \begin{array}{c|c} \mathcal{A} & U \\ \hline V^*\mathcal{P}^{-1} & 0 \end{array} \right) \\ &= \left( \begin{array}{cc|c} A_Q & 0 & B_Q \\ \hline VC_Q & \mathcal{A} & VD \\ C_Q & V^*\mathcal{P}^{-1} & D \end{array} \right) - \left( \begin{array}{cc|c} A_Q & 0 & 0 \\ \hline VC_Q & \mathcal{A} & U \\ C_Q & V^*\mathcal{P}^{-1} & 0 \end{array} \right) \\ &= \left( \begin{array}{cc|c} A_Q & 0 & B_Q \\ \hline VC_Q & \mathcal{A} & VD - U \\ C_Q & V^*\mathcal{P}^{-1} & D \end{array} \right) \end{aligned}$$

Similarly,

$$\begin{aligned} &-\Theta_{12}Q + \Theta_{11} \\ &= - \left( \begin{array}{c|c} \mathcal{A} & V \\ \hline -U^*\mathcal{P}^{-1} & 0 \end{array} \right) \left( \begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \end{array} \right) + \left( \begin{array}{c|c} \mathcal{A} & U \\ \hline -U^*\mathcal{P}^{-1} & I \end{array} \right) \\ &= - \left( \begin{array}{cc|c} A_Q & 0 & B_Q \\ \hline VC_Q & \mathcal{A} & VD \\ 0 & -U^*\mathcal{P}^{-1} & 0 \end{array} \right) + \left( \begin{array}{cc|c} A_Q & 0 & 0 \\ \hline VC_Q & \mathcal{A} & U \\ 0 & -U^*\mathcal{P}^{-1} & I \end{array} \right) \\ &= \left( \begin{array}{cc|c} A_Q & 0 & -B_Q \\ \hline VC_Q & \mathcal{A} & -VD + U \\ 0 & -U^*\mathcal{P}^{-1} & I \end{array} \right) \end{aligned}$$

So the inverse of the above matrix is easily seen to be:

$$(-\Theta_{12}Q + \Theta_{11})^{-1} = \left( \begin{array}{cc|c} A_Q & -B_Q U^* \mathcal{P}^{-1} & -B_Q \\ \hline VC_Q & \mathcal{A} + (-VD + U)U^* \mathcal{P}^{-1} & -VD + U \\ 0 & U^* \mathcal{P}^{-1} & I \end{array} \right)$$

Therefore,

$$\begin{aligned}
S &= (\Theta_{22}Q - \Theta_{21})(-\Theta_{12}Q + \Theta_{11})^{-1} \\
&= \left( \begin{array}{cc|c} A_Q & 0 & B_Q \\ VC_Q & \mathcal{A} & VD - U \\ \hline C_Q & V^*\mathcal{P}^{-1} & D \end{array} \right) \left( \begin{array}{cc|c} A_Q & -B_Q U^* \mathcal{P}^{-1} & -B_Q \\ VC_Q & \mathcal{A} + (-VD + U)U^* \mathcal{P}^{-1} & -VD + U \\ \hline 0 & U^* \mathcal{P}^{-1} & I \end{array} \right) \\
&= \left( \begin{array}{cccc|c} A_Q & -B_Q U^* \mathcal{P}^{-1} & 0 & 0 & -B_Q \\ VC_Q & \mathcal{A} + (-VD + U)U^* \mathcal{P}^{-1} & 0 & 0 & -VD + U \\ 0 & B_Q U^* \mathcal{P}^{-1} & A_Q & 0 & B_Q \\ 0 & (VD - U)U^* \mathcal{P}^{-1} & VC_Q & \mathcal{A} & VD - U \\ \hline 0 & DU^* \mathcal{P}^{-1} & C_Q & V^* \mathcal{P}^{-1} & D \end{array} \right) \\
&=: \left( \begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right)
\end{aligned}$$

Using the change of basis  $T = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$  we get

$$\mathbf{A}' = T\mathbf{A}T^{-1} \quad \mathbf{B}' = T\mathbf{B} \quad \mathbf{C}' = \mathbf{C}T^{-1}$$

which applied to our system yields:

$$\begin{aligned}
\left( \begin{array}{c|c} \mathbf{A}' & \mathbf{B}' \\ \hline \mathbf{C}' & \mathbf{D} \end{array} \right) &= \left( \begin{array}{cccc|c} A_Q & -B_Q U^* \mathcal{P}^{-1} & 0 & 0 & -B_Q \\ VC_Q & \mathcal{A} + (-VD + U)U^* \mathcal{P}^{-1} & 0 & 0 & -VD + U \\ 0 & 0 & A_Q & 0 & 0 \\ 0 & 0 & VC_Q & \mathcal{A} & 0 \\ \hline -C_Q & (DU^* - V^*)\mathcal{P}^{-1} & C_Q & V^* \mathcal{P}^{-1} & D \end{array} \right) \\
&= \left( \begin{array}{cc|c} A_Q & -B_Q U^* \mathcal{P}^{-1} & B_Q \\ VC_Q & \mathcal{A} + (U - VD)U^* \mathcal{P}^{-1} & VD - U \\ \hline C_Q & (V^* - DU^*)\mathcal{P}^{-1} & D \end{array} \right)
\end{aligned}$$

concluding the proof of the lemma.

## 4 Analysis of the McMillan-degree

In order to put a bound on the McMillan-degree of the interpolation function  $Q$  obtained from a linear fractional transformation the pole and zero positions of the corresponding functions should be analysed. This can be done for example applying Rouché's theorem or using the state-space realizations derived above. In order to apply Rouché's theorem we need a lemma.

**Lemma 4.1.** *The matrix*

- (i)  $\mathcal{A} + UU^* \mathcal{P}^{-1}$  *is antistable;*

(ii)  $\mathcal{A} - VV^*\mathcal{P}^{-1}$  is stable;

(iii)  $\mathcal{A} + (U - VD)U^*\mathcal{P}^{-1}$  is antistable;

(iv)  $\mathcal{A} - V(V^* - DU^*)\mathcal{P}^{-1}$  is stable.

**PROOF.** First observe that  $\mathcal{P} > 0$  implies that the pair  $(\mathcal{A}, U)$  is controllable. Now from the Lyapunov equation (3.2) we obtain that

$$(\mathcal{A} + UU^*\mathcal{P}^{-1})\mathcal{P} + \mathcal{P}(\mathcal{A}^* + \mathcal{P}^{-1}UU^*) - UU^* - VV^* = 0$$

and

$$(\mathcal{A} - VV^*\mathcal{P}^{-1})\mathcal{P} + \mathcal{P}(\mathcal{A}^* - \mathcal{P}^{-1}VV^*) + UU^* + VV^* = 0$$

proving the (i) and (ii), in view of the above-mentioned controllability.

(iii) Rearranging again the terms in (3.2) we can write that

$$\begin{aligned} (\mathcal{A} + (U - VD)U^*\mathcal{P}^{-1})\mathcal{P} + \mathcal{P}(\mathcal{A}^* + \mathcal{P}^{-1}U(U^* - D^*V^*)) \\ - UU^* - VV^* + VDU^* + UD^*V^* = 0 \end{aligned} \quad (4.1)$$

But

$$\begin{aligned} -UU^* - VV^* + VDU^* + UD^*V^* \\ = -(U - VD)(U^* - D^*V^*) + VDD^*V^* - VV^* \end{aligned}$$

so this latter is negative semidefinite (using the fact that  $DD^* \leq I$ ). Now if

$$\alpha^*(\mathcal{A} + (U - VD)U^*\mathcal{P}^{-1}) = \lambda\alpha^* \quad (4.2)$$

then, multiplying (4.1) by  $\alpha^*$  from the left and  $\alpha$  from the right and using (4.2), we get that

$$2\operatorname{Re} \lambda \alpha^*\mathcal{P}\alpha = \alpha^*[(U - VD)(U^* - D^*V^*) + V(I - DD^*)V^*]\alpha.$$

Thus  $\operatorname{Re} \lambda \geq 0$ . Now if  $\operatorname{Re} \lambda = 0$ , then  $\alpha^*(U - VD) = 0$  too, then (4.2) implies that

$$\alpha^*\mathcal{A} = \lambda\alpha,$$

implying that  $\operatorname{Re} \lambda$  should be negative leading to a contradiction. Thus all eigenvalues of  $\mathcal{A} + (U - VD)U^*\mathcal{P}^{-1}$  are in the open right-half-plane, proving (iii).

(iv) The Lyapunov-equation (3.2) implies that

$$\begin{aligned} (\mathcal{A} - V(V^* - DU^*)\mathcal{P}^{-1})\mathcal{P} + \mathcal{P}(\mathcal{A}^* - \mathcal{P}^{-1}(V - UD^*)V^*) \\ + (U - VD)(U^* - D^*V^*) + V(I - DD^*)V^* = 0 \end{aligned}$$

So an argument similar to the one applied in case (iii) gives that  $\mathcal{A} - V(V^* - DU^*)\mathcal{P}^{-1}$  is a stable matrix.

Denote by  $n_Q$  and  $n_S$  the McMillan-degree of  $Q$  and  $S$ , respectively.

**Proposition 4.1.** *Let  $S$  be a Schur-function and  $Q = T_\Theta(S)$ . Then*

$$n_Q \geq n_S . \quad (4.3)$$

**PROOF.** Using Lemma 4.1 (i) we obtain that the function  $\Theta_{11}$  has stable poles and antistable zeros, because they are determined by the eigenvalues of  $\mathcal{A}$  and  $\mathcal{A} + UU^*\mathcal{P}^{-1}$ , respectively.

The J-inner property of  $\Theta$  implies that

$$\Theta_{11}\Theta_{11}^* = I + \Theta_{12}\Theta_{12}^*$$

along the imaginary axis.

Since  $Q$  is contractive, the inequality

$$\Theta_{11}\Theta_{11}^* > \Theta_{12}QQ^*\Theta_{12}^*$$

holds on the imaginary axis, thus Rouché's theorem can be applied.

Consequently, the function  $\Theta_{11} - \Theta_{12}Q$  should have the same number of unstable zeros as  $\Theta_{11}$ . Since the Lyapunov-equation (3.2) implies that the realization

$$\Theta_{11} = \left( \begin{array}{c|c} \mathcal{A} & U \\ \hline -U^*\mathcal{P}^{-1} & I \end{array} \right)$$

is minimal, we obtain that the number of unstable zeros of  $\Theta_{11} - \Theta_{12}Q$  is  $n$ .

On the other hand if  $n_{1,Q}$  denotes the McMillan-degree of  $\Theta_{11} - \Theta_{12}Q$ , then  $n_{1,Q} \leq n + n_Q$ , and the number of stable zeros of it is  $n_{1,Q} - n$ .

As we have seen from the computation the (stable) poles of  $\Theta_{22}Q - \Theta_{21}$  are cancelled by the the stable zeros of  $(\Theta_{11} - \Theta_{12}Q)^{-1}$ . Thus the poles of the function  $S$  are being originated from the poles of  $(\Theta_{11} - \Theta_{12}Q)^{-1}$ , i.e.

$$n_S \leq n_{1,Q} - n \leq n_Q$$

proving the proposition.

The analysis of the controllability and unobservability subspaces of the realizations (3.5) and (3.7) gives a more detailed picture.

**Proposition 4.2.** *If the pair  $(A, B)$  in the realization (3.4) is controllable then the realization of the function  $Q$  defined in (3.5) is controllable, as well.*

**PROOF.** The identity

$$\mathcal{A} + V(DU^* - V^*)\mathcal{P}^{-1} = -\mathcal{P}\mathcal{A}^*\mathcal{P}^{-1} + (VD - U)U^*\mathcal{P}^{-1}$$

implies that the controllability subspace defined by the pair

$$\left( \left[ \begin{array}{cc} \mathcal{A} + V(DU^* - V^*)\mathcal{P}^{-1} & -VC \\ -BU^*\mathcal{P}^{-1} & A \end{array} \right], \left[ \begin{array}{c} -U + VD \\ -B \end{array} \right] \right)$$

coincide with that determined by

$$\left( \left[ \begin{array}{cc} -\mathcal{P}\mathcal{A}^*\mathcal{P}^{-1} & -VC \\ 0 & A \end{array} \right], \left[ \begin{array}{c} -U + VD \\ -B \end{array} \right] \right)$$

To prove controllability the celebrated P-B-H test will be applied. The orthogonal complement of the controllability subspace is invariant under the adjoint of the state-transition matrix. Thus we can consider an eigenvector belonging to that subspace.

$$[\alpha^*, \beta^*] \begin{bmatrix} U - VD \\ B \end{bmatrix} = 0 \quad (4.4)$$

$$[\alpha^*, \beta^*] \begin{bmatrix} -\mathcal{P}\mathcal{A}^*\mathcal{P}^{-1} & -VC \\ 0 & A \end{bmatrix} = \lambda [\alpha^*, \beta^*] \quad (4.5)$$

The equation (4.5) gives that

$$\lambda \alpha^* = -\alpha^* \mathcal{P} \mathcal{A}^* \mathcal{P}^{-1}, \quad (4.6)$$

$$\lambda \beta^* = -\alpha^* VC + \beta^* A. \quad (4.7)$$

If  $\alpha \neq 0$  then  $\lambda$  should be an eigenvalues of  $-\mathcal{A}^*$ , i.e.  $\lambda$  coincides with one of the interpolation nodes. In particular,  $\text{Re } \lambda > 0$ . Consequently,  $\lambda$  is not an eigenvalue of  $A$ , thus the matrix  $\lambda I - A$  is nonsingular implying that

$$\beta^* = -\alpha^* VC (\lambda I - A)^{-1}. \quad (4.8)$$

Substituting this back to the equation in (4.4)

$$\alpha^* [U - V (D + C(\lambda I - A)^{-1}B)] = 0. \quad (4.9)$$

Shortly,

$$\alpha^* [U - VS(\lambda)] = 0 \quad (4.10)$$

Observe that since  $\lambda$  is in the right half plane the inequality

$$S(\lambda)S(\lambda)^* \leq I \quad (4.11)$$

holds true.

Multiplying the Lyapunov-equation (3.2) by  $\alpha^*$  and  $\alpha$  from the left and right, respectively, we get that

$$-2\text{Re } \lambda \alpha^* \mathcal{P} \alpha + \alpha^* V [S(\lambda)S(\lambda)^* - I] V^* \alpha = 0$$

If  $\alpha$  is nonzero then the first term is strictly negative while the second term is nonpositive leading to a contradiction.

If  $\alpha = 0$  then equations in (4.4) and (4.5) give that

$$\begin{aligned} \beta^* B &= 0 \\ \beta^* A &= \lambda \beta^*. \end{aligned}$$

Now the controllability of the pair  $A, B$  implies that  $\beta = 0$ , concluding our proof.

**Theorem 4.1.** *If the realization (3.4) of  $S$  is observable, then the unobservability subspace of the realization (3.5) is determined by the range of the matrix  $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ , where the range of  $\alpha$  is an invariant subspace of  $\mathcal{A}$ , i.e. there exists a matrix  $\Gamma$  such that*

$$\mathcal{A}\alpha = \alpha\Gamma \quad (4.12)$$

and  $\alpha$  and  $\beta$  satisfy the equation:

$$(S(s)U^* - V^*)\mathcal{P}^{-1}(sI - \mathcal{A})^{-1}\alpha = C(sI - A)^{-1}\beta. \quad (4.13)$$

*Epecially, in this case the dimension of the unobservability subspace in the realization of  $Q$  given by (3.5) is at most  $n$ .*

**PROOF.** Obviously the unobservability subspace of the realization given in (3.5) is determined by the pair

$$\left( [(DU^* - V^*)\mathcal{P}^{-1}, -C], \begin{bmatrix} \mathcal{A} & 0 \\ -BU^*\mathcal{P}^{-1} & A \end{bmatrix} \right).$$

Assume that the column vectors of a matrix  $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$  determine a basis in the unobservability subspace. Then

$$(DU^* - V^*)\mathcal{P}^{-1}\alpha - C\beta = 0 \quad (4.14)$$

$$\begin{bmatrix} \mathcal{A} & 0 \\ -BU^*\mathcal{P}^{-1} & A \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Gamma \quad (4.15)$$

for some matrix  $\Gamma$ .

If a vector  $\gamma$  is in the kernel space of  $\alpha$ , i.e.  $\alpha\gamma = 0$ , then the equation in (4.15) implies that

$$\alpha\Gamma\gamma = \mathcal{A}\alpha\gamma = 0,$$

thus  $\Gamma\gamma \in \text{Ker}(\alpha)$ , so the kernel subspace of  $\alpha$  is  $\Gamma$  invariant. Especially, it contains an eigenvector of  $\Gamma$ . In other words there exists a vector  $\gamma$ , for which the identities

$$\begin{aligned} \alpha\gamma &= 0 \\ \Gamma\gamma &= \mu\gamma \end{aligned}$$

for some  $\mu$  hold. Multiplying the equations (4.14) and (4.15) from the right by  $\gamma$  we obtain that

$$\begin{aligned} C\beta\gamma &= 0 \\ A\beta\gamma &= \mu\beta\gamma. \end{aligned}$$

Thus the vector  $\beta\gamma$  is in the unobservability subspace determined by the pair  $(C, A)$ . Using the assumed observability of the realization  $S$  we get that  $\beta\gamma = 0$ . Consequently

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \gamma = 0.$$

But the column vectors of the matrix  $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$  are linearly independent, thus  $\gamma = 0$ .

The equation (4.15) gives that

$$\begin{aligned} \mathcal{A}\alpha &= \alpha\Gamma, \\ -BU^*\mathcal{P}^{-1}\alpha + A\beta &= \beta\Gamma, \end{aligned}$$

The first relation proves (4.12). Adding  $-s\beta$  to both sides of the second equation and multiplying on the left by  $-(sI - A)^{-1}$  we get

$$(sI - A)^{-1}BU^*\mathcal{P}^{-1}\alpha + \beta = (sI - A)^{-1}\beta(sI - \Gamma).$$

Multiplying from the left by  $C$  and using equation (4.14) we obtain that

$$(D + C(sI - A)^{-1}B)U^*\mathcal{P}^{-1}\alpha - V^*\mathcal{P}^{-1}\alpha = C(sI - A)^{-1}\beta(sI - \Gamma).$$

In other words

$$(S(s)U^* - V^*)\mathcal{P}^{-1}\alpha(sI - \Gamma)^{-1} = C(sI - A)^{-1}\beta, \quad (4.16)$$

Since from (4.12) we have  $\alpha(sI - \Gamma)^{-1} = (sI - \mathcal{A})^{-1}\alpha$ , formula (4.13) follows immediately. At the same time (4.12) implies the last part of the proposition, as well.

Let us introduce the notation:

$$d_{o,Q} = \text{dimension of the unobservability subspace of the realization (3.5).}$$

and recall that

$$\begin{aligned} n &= \text{McMillan degree of } \Theta, \\ n_S &= \text{McMillan degree of } S, \\ n_Q &= \text{McMillan degree of } Q \end{aligned}$$

**Corollary 4.1.** *Assume that the realization of the function  $S$  is minimal. Then*

$$n_Q = n + n_S - d_{o,Q} \quad (4.17)$$

Moreover (cf. Proposition 4.1)

$$d_{o,Q} \leq n \quad \text{i.e.} \quad n_Q \geq n_S.$$

Observe that the poles of the function on the right hand side of (4.13) are among the eigenvalues of the matrix  $A$ , while the function standing on the left hand side has formally poles at the eigenvalues of  $A$  and  $\Gamma$ . Thus the eigenvalues of this latter one should be cancelled. This is again an interpolation type condition where the interpolation nodes are defined by the eigenvalues of  $\Gamma$  forming a subset of the eigenvalues of  $\mathcal{A}$ . These interpolation nodes are in the left half plane  $\mathbb{C}^-$ . This kind of interpolation problems is analyzed in [7].

Note that in the special case when

$$n_Q \leq n$$

the number of the interpolation constraints formulated above on the function  $S$  should be as large as its McMillan-degree  $n_S$ .

Especially, if  $\mathcal{A} = \text{diag}(-\bar{s}_1, \dots, -\bar{s}_n)$  and  $n_Q = n_S$  then these interpolation conditions can be expressed as

$$S(-\bar{s}_j)U^*\mathcal{P}^{-1}e_j = V^*\mathcal{P}^{-1}e_j, \quad i = 1, \dots, n, \quad (4.18)$$

where  $e_j$  denotes the  $j$ -th unit vector.

Now let us analyse the controllability and unobservability subspaces of the realization (3.7) of  $S$  obtained from that of  $Q$  using the inverse linear fractional transformation.

First observe that in the case when  $Q$  is a solution of the interpolation Problem (2.4) then the state space of the realization (3.7) can be further reduced.

**Proposition 4.3.** *Let  $Q$  be a solution of the interpolation Problem (2.4) with realization*

$$Q = \left( \begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \end{array} \right).$$

Assume that  $(C_Q, A_Q)$  is an observable pair. Then  $S = T_{\Theta}^{-1}(Q)$  has the observable realization

$$S = \left( \begin{array}{c|c} A_Q - Y\mathcal{P}^{-1}VC_Q & B_Q - Y\mathcal{P}^{-1}(VD - U) \\ \hline C_Q & D \end{array} \right), \quad (4.19)$$

where  $Y$  is defined by the equation

$$(Q(s)U^* - V^*)(sI + \mathcal{A}^*)^{-1} = C_Q(sI - A_Q)^{-1}Y. \quad (4.20)$$

**PROOF.** Since the function  $(Q(s)U^* - V^*)(sI + \mathcal{A}^*)^{-1}$  is analytic on  $\mathbb{C}^+$ , it might have poles only at the eigenvalues of  $A_Q$ , and vanishes at  $\infty$ , the observability of  $(C_Q, A_Q)$  implies the existence of the matrix  $Y$  in (4.20).

Evaluating the identity

$$Q(s)U^* - V^* = DU^* - V^* + C_Q(sI - A_Q)^{-1}B_QU^* = C_Q(sI - A_Q)^{-1}Y(sI + \mathcal{A}^*)$$

at  $\infty$  the equation

$$DU^* - V^* = C_QY \quad (4.21)$$

is obtained. Subtraction from the previous equation yields:

$$\begin{aligned} C_Q(sI - A_Q)^{-1}B_QU^* &= C_Q(sI - A_Q)^{-1}(Y(sI + \mathcal{A}^*) - (sI - A_Q)Y) \\ &= C_Q(sI - A_Q)^{-1}(Y\mathcal{A}^* + A_QY). \end{aligned}$$

Observability of  $(C_Q, A_Q)$  implies that

$$B_QU^* = Y\mathcal{A}^* + A_QY. \quad (4.22)$$

Let us apply the state transformation defined by the matrix

$$T = \begin{bmatrix} I & Y \\ 0 & \mathcal{P} \end{bmatrix}$$

for the realization (3.7). Using that

$$[C_Q, (V^* - DU^*)\mathcal{P}^{-1}] \begin{bmatrix} I & Y \\ 0 & \mathcal{P} \end{bmatrix} = [C_Q, 0]$$

$$\begin{bmatrix} I & Y \\ 0 & \mathcal{P} \end{bmatrix} \begin{bmatrix} B_Q \\ VD - U \end{bmatrix} = \begin{bmatrix} B_Q - Y\mathcal{P}^{-1}(VD - U) \\ \mathcal{P}^{-1}(VD - U) \end{bmatrix}$$

$$\begin{bmatrix} A_Q & -B_Q U^* \mathcal{P}^{-1} \\ VC_Q & \mathcal{A} + (U - VD)U^* \mathcal{P}^{-1} \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & \mathcal{P} \end{bmatrix} = \begin{bmatrix} I & Y \\ 0 & \mathcal{P} \end{bmatrix} \begin{bmatrix} A_Q - Y\mathcal{P}^{-1}VC_Q & 0 \\ \mathcal{P}^{-1}VC_Q & -\mathcal{A}^* \end{bmatrix}$$

the following realization is obtained:

$$S = \left( \begin{array}{c|c} \frac{A_Q - Y\mathcal{P}^{-1}VC_Q}{C_Q} & \frac{B_Q - Y\mathcal{P}^{-1}(VD - U)}{D} \end{array} \right). \quad (4.23)$$

The realization (4.19) is obviously observable if  $(C_Q, A_Q)$  is an observable pair. This concludes the proof of the proposition.

Now let us analyze the controllability subspace of the realization defined in (4.19).

**Theorem 4.2.** *Assume that the realization of  $Q$  given in (3.6) is minimal. If the functions  $VQ(s) - U$  and  $sI - \mathcal{A}$  have no common left zero-functions then the realization of  $S$  in (4.19) is minimal, especially  $n_S = n_Q$ .*

**PROOF.** Observability follows Proposition 4.3. To check controllability, let us apply the P-H-B test. Assume that there exist a nonzero column vector  $\alpha_Q$  and  $\mu \in \mathbb{C}$  such that

$$\alpha_Q^* (B_Q - Y\mathcal{P}^{-1}(VD - U)) = 0 \quad (4.24)$$

$$\alpha_Q^* (A_Q - Y\mathcal{P}^{-1}VC_Q) = \mu\alpha_Q^*. \quad (4.25)$$

It is immediately obtained that  $\alpha_Q^* Y \neq 0$ . Otherwise the equations  $\alpha_Q^* B_Q = 0$ ,  $\alpha_Q^* A_Q = \mu\alpha_Q^*$  would hold, contradicting to the controllability of  $(A_Q, B_Q)$ .

Computing (4.24) $U^*$  - (4.25) $Y$ , we obtain that

$$\alpha_Q^* (B_Q U^* - Y\mathcal{P}^{-1}VDU^* + Y\mathcal{P}^{-1}UU^* - A_Q Y + Y\mathcal{P}^{-1}VC_Q Y) = -\mu\alpha_Q^* Y.$$

Using (4.21) and (4.22) we get that

$$\alpha_Q^* (Y\mathcal{A}^* - Y\mathcal{P}^{-1}VV^* + Y\mathcal{P}^{-1}UU^*) = -\mu\alpha_Q^* Y.$$

I.e.

$$(\alpha_Q^* Y\mathcal{P}^{-1})(\mu I - \mathcal{A}) = 0 \quad (4.26)$$

On the other hand equations in (4.24) and (4.25) can be arranged into the following form

$$[\alpha_Q^*, -\alpha_Q^* Y \mathcal{P}^{-1}] \begin{bmatrix} A_Q - \mu I & B_Q \\ VC_Q & VD - U \end{bmatrix} = [0, 0] . \quad (4.27)$$

Equations (4.26) and (4.27) indicate that the functions  $sI - \mathcal{A}$  and  $VQ(s) - U$  have a common zero direction at  $s = \mu$ , contradicting our assumption. Thus the realization of  $S$  given in (4.19) is controllable, concluding the proof of the proposition.

Introduce the notation

$$d_{c,S} = \text{dimension of the controllability subspace of the realization (4.19).}$$

According to Proposition 4.3 the realization (4.19) is observable. Consequently

$$n_S = d_{c,S} .$$

**Corollary 4.2.** *Assume that the realization (3.6) of the function  $Q$  is minimal, moreover the matrices  $\mathcal{A}$  and  $A_Q$  have no common eigenvalues and*

$$\text{Ker} \left( \begin{bmatrix} VQ(s) - U \\ sI - \mathcal{A} \end{bmatrix} \right) = \{0\}$$

for all  $s \in \mathbb{C}$ . Then

$$n_S = n_Q .$$

The argument applied in the proof of Proposition 4.2 can be modified to give a more precise description between the McMillan-degree of  $S$  and  $Q$ . This leads to the following proposition which essentially states that if  $n_S < n_Q$  then the function  $Q$  satisfies additional interpolation conditions with nodes at some of the eigenvalues of  $\mathcal{A}$ .

**Theorem 4.3.** *Assume that the realization of  $Q$  given in (3.6) is minimal and using the matrices in the realization of  $S$  given in (4.19) consider the following set of equations:*

$$\beta^* \mathcal{A} = \Gamma \beta^* \quad (4.28)$$

$$[\gamma^*, \beta^*] \begin{bmatrix} A_Q & B_Q \\ VC_Q & VD - U \end{bmatrix} = [\Gamma \gamma^* \quad 0] . \quad (4.29)$$

Then,

- (i) *if the linearly independent column vectors of the  $\alpha_Q$  span the orthogonal complement of the controllability subspace of the realization (4.19), then there exists a matrix  $\Gamma$ , for which*

$$\beta^* = \alpha_Q^* Y \mathcal{P}^{-1}, \quad \gamma^* = \alpha_Q^* \text{ and } \Gamma$$

*solve equations (4.28) and (4.29).*

- (ii) *Conversely, if the matrices  $\beta, \gamma, \Gamma$  solve these equations, where the column vectors of the matrix  $\beta$  are linearly independent, then  $\beta^* = \gamma^* Y \mathcal{P}^{-1}$ , and the column vectors of  $\gamma$  are orthogonal to the controllability subspace.*

**PROOF.** Consider first the controllability subspace of the realization (4.19) and assume that the linearly independent columns of a matrix  $\alpha_Q$  span the orthogonal complement of this controllability subspace. Then due to the fact that this orthogonal complement is  $(A_Q - Y\mathcal{P}^{-1}VC_Q)^*$  invariant there exists a matrix  $\Gamma$  such that the following equations hold:

$$\alpha_Q^* (B_Q - Y\mathcal{P}^{-1}(VD - U)) = 0 \quad (4.30)$$

$$\alpha_Q^* (A_Q - Y\mathcal{P}^{-1}VC_Q) = \Gamma\alpha_Q^* . \quad (4.31)$$

Computing (4.30) $U^*$  - (4.31) $Y$ , we get that

$$\alpha_Q^* (B_Q U^* - Y\mathcal{P}^{-1}VDU^* + Y\mathcal{P}^{-1}UU^* - A_Q Y + Y\mathcal{P}^{-1}VC_Q Y) = -\Gamma\alpha_Q^* Y .$$

Now using (4.21) and (4.22) we obtain that

$$\alpha_Q^* (Y\mathcal{A}^* - Y\mathcal{P}^{-1}VV^* + Y\mathcal{P}^{-1}UU^*) = -\Gamma\alpha_Q^* Y .$$

I.e.

$$(\alpha_Q^* Y\mathcal{P}^{-1})\mathcal{A} = \Gamma(\alpha_Q^* Y\mathcal{P}^{-1}) . \quad (4.32)$$

An immediate consequence of this equation and the controllability of the pair  $(A_Q, B_Q)$  that the row vectors of  $\alpha_Q^* Y\mathcal{P}^{-1}$  are linearly independent. In fact, from equation (4.32) it follows that  $\text{Ker}[\mathcal{P}^{-1}Y^*\alpha_Q]$  is  $\Gamma^*$  invariant. So, let  $\kappa$  be an eigenvector of  $\Gamma^*$ ,  $\Gamma^*\kappa = \mu\kappa$ , for which

$$\mathcal{P}^{-1}Y^*\alpha_Q\kappa = 0 .$$

Then multiplying (4.30) and (4.31) from the right by  $\kappa^*$  we obtain that equations

$$\begin{aligned} \kappa^* \alpha_Q^* B_Q &= 0 \\ \kappa^* \alpha_Q^* A_Q &= \bar{\mu} \kappa^* \alpha_Q^* \end{aligned}$$

hold, implying – due to the controllability of the pair  $(A_Q, B_Q)$  – that  $\alpha_Q\kappa = 0$ . Since the columns are  $\alpha_Q$  are linearly independent,  $\kappa$  should be the zero vector.

On the other hand equations in (4.30) and (4.31) can now be arranged into the following form

$$[\alpha_Q^*, -\alpha_Q^* Y\mathcal{P}^{-1}] \begin{bmatrix} A_Q & B_Q \\ VC_Q & VD - U \end{bmatrix} = [\Gamma\alpha_Q^*, 0] , \quad (4.33)$$

proving (i).

Conversely, if for some matrices  $\beta, \gamma, \Gamma$ , assuming that the column vectors of  $\beta$  are linearly independent, the equations

$$\beta^* \mathcal{A} = \Gamma\beta^* \quad (4.34)$$

$$[\gamma^*, \beta^*] \begin{bmatrix} A_Q & B_Q \\ VC_Q & VD - U \end{bmatrix} = [\Gamma\gamma^* \quad 0] , \quad (4.35)$$

holds, then – using that the row vectors of  $\beta^*$  are linearly independent – equation (4.34) implies that the eigenvalues of  $\Gamma$  form a subset of that of  $\mathcal{A}$ , and furthermore multiplying

(4.35) from the right by  $\begin{bmatrix} -Y \\ U^* \end{bmatrix}$  we obtain that

$$-\Gamma\gamma^* Y = \gamma^* Y\mathcal{A}^* + \beta^* \mathcal{A}\mathcal{P} + \beta^* \mathcal{P}\mathcal{A}^* \quad (4.36)$$

holds. Invoking equation (4.34) we get that

$$(\gamma^*Y + \beta^*\mathcal{P})\mathcal{A} = -\Gamma(\gamma^*Y + \beta^*\mathcal{P}) .$$

Since the matrices  $\Gamma$  and  $-\mathcal{A}$  have no common eigenvalues we get that

$$\gamma^*Y + \beta^*\mathcal{P} = 0 ,$$

i.e.

$$\beta^* = -\gamma^*Y\mathcal{P}^{-1} ,$$

Substituting this into (4.35) the obtained equations

$$\begin{aligned} \gamma^*(B_Q - Y\mathcal{P}^{-1}(VD - U)) &= 0 \\ \gamma^*(A_Q - Y\mathcal{P}^{-1}VC_Q) &= \Gamma\gamma^* \end{aligned}$$

imply that the column vectors of  $\gamma$  are orthogonal to the controllability subspace of the realization given in (4.19), concluding the proof of (ii).

Note that Theorem 4.3 implies again that the inequalities  $n_S \leq n_Q \leq n_S + n$  hold.

On the other hand equation (4.33) can be formulated as the pair  $(\alpha_Q^*Y\mathcal{P}^{-1}, \Gamma)$  forms a left zero pair of the function  $VQ(z) - U$ . Furthermore equation (4.32) implies – using again that the row vectors of  $\alpha_Q^*Y\mathcal{P}^{-1}$  are linearly independent – that the eigenvalues of  $\Gamma$  form a subset of that of  $\mathcal{A}$ . In particular, if  $\mathcal{A} = \text{diag}(-\bar{s}_1, \dots, -\bar{s}_n)$  and assuming – for the sake of simplicity – that  $A_Q$  and  $\mathcal{A}$  has no common eigenvalues, then for some subset  $N_c \subset \{1, \dots, n\}$  of size  $|N_c| = n_Q - n_S$  the function  $Q$  should satisfy the following set of extra interpolation conditions:

$$e_j^*VQ(-\bar{s}_j) = e_j^*U, \quad j \in N_c , \quad (4.37)$$

where  $e_j$  denotes the  $j$ -th unit vector.

## 5 Solutions of the interpolation problem with "low complexity"

Consider now the interpolation problem formulated in Problem 2.2. For any given matrix  $B$  of size  $n \times p$  denote by

$$\begin{aligned} F_B^* &= \left( \begin{array}{c|c} -\mathcal{A}^* & B \\ \hline -U^* & I \end{array} \right) , \\ G_B^* &= \left( \begin{array}{c|c} -\mathcal{A}^* & B \\ \hline -V^* & D \end{array} \right) . \end{aligned}$$

**Theorem 5.1.** *The Schur-function  $Q$  is a solution of the interpolation Problem 2.2 if and only if it has the realization*

$$Q = \left( \begin{array}{c|c} -\mathcal{A}^* + BU^* & B \\ \hline DU^* - V^* & D \end{array} \right). \quad (5.1)$$

*Especially, in this case*

$$Q = G_B^* (F_B^*)^{-1},$$

**PROOF.** Assume that  $Q$  has the realization given in (5.1). Set

$$F^* = F_B^* = \left( \begin{array}{c|c} -\mathcal{A}^* & B \\ \hline -U^* & I \end{array} \right)$$

Then a straightforward computation gives that

$$\begin{aligned} QF^* &= \left( \begin{array}{c|c} -\mathcal{A}^* + BU^* & B \\ \hline DU^* - V^* & D \end{array} \right) \left( \begin{array}{c|c} -\mathcal{A}^* & B \\ \hline -U^* & I \end{array} \right) \\ &= \left( \begin{array}{cc|c} -\mathcal{A}^* & 0 & B \\ -BU^* & -\mathcal{A}^* + BU^* & B \\ \hline -DU^* & DU^* - V^* & D \end{array} \right) \\ &= \left( \begin{array}{c|c} -\mathcal{A}^* & B \\ \hline -V^* & D \end{array} \right) \end{aligned}$$

which is an unstable function, proving the first part. Immediate calculation gives that  $\Xi F^*$  is a stable function.

Conversely, assume that  $\Xi F^*$  is a stable function, for which  $F(\infty) = I$  then  $F^*$  has the form

$$F^* = \left( \begin{array}{c|c} -\mathcal{A}^* & B \\ \hline -U^* & I \end{array} \right)$$

for some matrix  $B$ . Now if  $Q = G^* (F^*)^{-1}$  is a solution then

$$\begin{aligned} G^* - (D - V^* (sI + \mathcal{A}^*)^{-1} B) &= QF^* - (D - V^* (sI + \mathcal{A}^*)^{-1} B) \\ &= Q - D - (QU^* - V^*) (sI + \mathcal{A}^*)^{-1} B. \end{aligned}$$

The function on the left hand side is obviously an unstable function while that on the right hand side – using that  $Q$  is a solution of the interpolation Problem 2.1 – is a stable, strictly proper function. Thus it must be identically zero, proving that

$$G^* = G_B^* = \left( \begin{array}{c|c} -\mathcal{A}^* & B \\ \hline -V^* & D \end{array} \right).$$

Finally, the calculation

$$\begin{aligned}
Q &= G^*(F^*)^{-1} = \left( \begin{array}{c|c} -\mathcal{A}^* & B \\ \hline -V^* & D \end{array} \right) \left( \begin{array}{c|c} -\mathcal{A}^* + BU^* & B \\ \hline U^* & I \end{array} \right) \\
&= \left( \begin{array}{cc|c} -\mathcal{A}^* + BU^* & 0 & B \\ BU^* & -\mathcal{A}^* & B \\ \hline DU^* & -V^* & D \end{array} \right) \\
&= \left( \begin{array}{c|c} -\mathcal{A}^* + BU^* & B \\ \hline DU^* - V^* & D \end{array} \right)
\end{aligned}$$

concludes the proof.

Note that for any  $B$ , the transfer function  $G_B^*(F_B^*)^{-1}$  has the realization given in (5.1) but it is not necessarily a Schur-function, therefore it does not necessarily produce a solution of the interpolation Problem 2.2.

Notice that, if  $Q$  has realization (5.1), the matrix  $Y$  in (4.22) is equal to the identity, and therefore, the representation (4.19) of  $S$  in Proposition 4.3 takes on the form:

$$\begin{aligned}
S &= \left( \begin{array}{c|c} \frac{-\mathcal{A}^* + BU^* - \mathcal{P}^{-1}V(DU^* - V^*)}{DU^* - V^*} & \frac{B - \mathcal{P}^{-1}(VD - U)}{D} \\ \hline \frac{\mathcal{P}^{-1}(\mathcal{A}\mathcal{P} + \mathcal{P}BU^* - VDU^* + UU^*)}{DU^* - V^*} & \frac{B + \mathcal{P}^{-1}(U - VD)}{D} \end{array} \right) \\
&= \left( \begin{array}{c|c} \frac{\mathcal{A} + \mathcal{P}BU^*\mathcal{P}^{-1} + (U - VD)U^*\mathcal{P}^{-1}}{(DU^* - V^*)\mathcal{P}^{-1}} & \frac{\mathcal{P}B + (U - VD)}{D} \\ \hline & \end{array} \right)
\end{aligned}$$

Setting eventually

$$B_S := \mathcal{P}B + U - VD .$$

we get

$$S = \left( \begin{array}{c|c} \frac{\mathcal{A} + B_S U^* \mathcal{P}^{-1}}{(DU^* - V^*) \mathcal{P}^{-1}} & \frac{B_S}{D} \\ \hline & \end{array} \right) , \quad (5.2)$$

In Proposition 5.1 and Theorem 5.2 we provide conditions assuring that the function  $Q$  with realization (5.1) becomes a stable or Schur function, respectively.

**Proposition 5.1.** *Let  $Q$  be as in (5.1), and suppose it has McMillan-degree  $n$ . Then  $Q$  is stable if and only if*

$$B = \mathcal{P}_W^{-1}(U - W)$$

where  $W \in \mathbb{C}^{n \times p}$  satisfies the generalized Pick condition, i.e. the Lyapunov equation

$$\mathcal{A}\mathcal{P}_W + \mathcal{P}_W\mathcal{A}^* + UU^* - WW^* = 0 \quad (5.3)$$

has a unique positive definite solution.

**PROOF.** Assume that  $Q$  with the realization given in (5.1) is a stable function. The assumption concerning its McMillan-degree implies that this realization is minimal, i.e. there exists a matrix  $\mathcal{P}_Q > 0$  for which the equation

$$(-\mathcal{A}^* + BU^*)\mathcal{P}_Q + \mathcal{P}_Q(-\mathcal{A} + UB^*) + BB^* = 0$$

holds. This can be arranged to

$$\mathcal{A}\mathcal{P}_Q^{-1} + \mathcal{P}_Q^{-1}\mathcal{A}^* + UU^* - (U + \mathcal{P}_Q^{-1}B)(U + \mathcal{P}_Q^{-1}B)^* = 0.$$

Introducing the notations

$$\begin{aligned}\mathcal{P}_W &= \mathcal{P}_Q^{-1} \\ W &= U + \mathcal{P}_Q^{-1}B\end{aligned}$$

the equation (5.3) is obtained.

Conversely assume that (5.3) holds. Straightforward calculation gives that

$$\begin{aligned}\mathcal{P}_W(-\mathcal{A}^* - \mathcal{P}_W^{-1}(U - W)U^*)\mathcal{P}_W^{-1} &= \mathcal{A} + W(U^* - W^*)\mathcal{P}_W^{-1} \\ (\mathcal{A} + W(U^* - W^*)\mathcal{P}_W^{-1})\mathcal{P}_W + \mathcal{P}_W(\mathcal{A} + W(U^* - W^*)\mathcal{P}_W^{-1})^* &+ (U - W)(U^* - W^*) = 0.\end{aligned}$$

Now the controllability of the pair

$$(\mathcal{A}^* + \mathcal{P}_W^{-1}(U - W)W^*, \mathcal{P}_W^{-1}(U - W))$$

implies the stability of  $(\mathcal{A} + W(U^* - W^*)\mathcal{P}_W^{-1})$ .

**Theorem 5.2.** *Let  $Q$  be as in (5.1), and suppose it has McMillan-degree  $n$ . Then  $Q$  is a Schur-function if and only if*

$$B = -R^{-1}\left(U - VD - \tilde{V}(I - D^*D)^{\frac{1}{2}}\right)$$

where  $\tilde{V} \in \mathbb{C}^{n \times p}$  and  $R \in \mathbb{C}^{n \times n}$ ,  $R$  is positive definite and the equation

$$AR + RA^* + UU^* - VV^* - \tilde{V}\tilde{V}^* = 0 \tag{5.4}$$

holds.

**PROOF.** According to the bounded real lemma the function  $Q$  is a Schur-function if and only if the equation

$$\begin{aligned}R(A_Q + B_Q D^*(I - DD^*)^{-1}C_Q) + (A_Q + B_Q D^*(I - DD^*)^{-1}C_Q)^* R \\ + C_Q^*(I - DD^*)^{-1}C_Q + RB_Q(I - D^*D)^{-1}B_Q^* R = 0\end{aligned} \tag{5.5}$$

has a positive definite solution  $R$ . Expressing the matrices  $A_Q$  and  $C_Q$  using (5.1) straightforward calculation gives that

$$RA^* + AR + (UU^* - VV^*) - (RB + U - VD)(I - D^*D)^{-1}(B^*R + U^* - D^*V^*) = 0$$

holds. Introducing the notation

$$\tilde{V} = (RB + U - VD)(I - D^*D)^{-\frac{1}{2}}$$

equation (5.4) is obtained.

Based on this a parameterization of stable solutions to the interpolation problem can be obtained using the results in [1], [9].

Finally note that the right null-pair corresponding to the zeros of the function  $I - QQ^*$  inside  $\mathbb{C}^-$  is given by the matrices

$$\left( -\mathcal{A}^* + B(I - D^*D)^{-\frac{1}{2}} \tilde{V}^*, \quad V^* - (I - DD^*)^{-1} D(I - D^*D)^{\frac{1}{2}} \tilde{V}^* \right) .$$

Using the transformation allowed by Lemma 2.1 this has a particularly simple form:

$$\left( -\mathcal{A}^* + B\tilde{V}^*, \quad V^* \right)$$

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