ON THE PARAMETRIZATION OF SCHUR AND POSITIVE REAL FUNCTIONS OF DEGREE n WITH FIXED INTERPOLATING CONDITIONS

A. GOMBANI and G. MICHALETZKY

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INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

On the parametrization of Schur and Positive Real functions of degree n with fixed interpolating conditions

A. Gombani
LADSEB-CNR
Corso Stati Uniti 4
35127 Padova, Italy
e-mail: gombani@ladseb.pd.cnr.it
and
György Michaletzky
Eötvös Loránd University
H-1117 Pázmány Péter stny. 1/C,
Computer and Automation Institute of HAS
H-1111 Kende u. 13-17,
Budapest, Hungary
e-mail: michgy@ludens.elte.hu

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Abstract

We investigate here a parametrization of matrix Schur and positive real functions of degree n with given interpolating conditions. The results are presented for matrix valued functions.

Keywords: Schur parameters, Interpolation, Schur functions, Positive Real Functions.

MSC2000 Number: 30E05, 93B29, 93B30

1 Introduction

We investigate here the problem of parametrizing the set of Schur or positive real function of degree n which satisfies some given interpolating conditions. The problem has a long history: the first results without degree constraints date back to the beginning of last century and are due to Schur [20], Nevanlinna [18] and Pick [19] and, for Positive Real functions, by Carathéodory (see e.g. [10]); more recently, an extensive theory which makes use of linear fractional transformations and reproducing kernel Hilbert spaces has been developed by a large community of researchers: we refer to the work of Ball, Gohberg, Rodman [3] and Dym [8] and the references therein. The problem with degree constraints for positive real functions was first stated by Kalman [16], and a first parametrization of all interpolating functions of given degree, but without control on the positive realness, is due to Kimura and Georgiou (see [17], [11]); more complete but non constructive results about existence of a parametrization of positive real functions with degree constraints were developed in [7]; more recently Byrnes, Georgiou, Gusev and Lindquist ([12], [4], [5], [6]) derived an optimization method to derive a parametrization of the scalar functions which also satisfy the positive real condition. We derive here a different parametrization for square matrix valued functions, with prescribed tangential interpolating condition and fixed degree n, based on a parametrization of inner functions of given degree studied in [1]. The idea is to complete a Schur function Q of dimension $p \times m$ into a tall inner function Q_c (where c is for column) of dimension $(p+m) \times m$. This completion is obtained from a spectral factor of $I_m - Q^*Q$ and it is uniquely determined if we choose, say, the minimum-phase spectral factor. But since in [1] it is given a parametrization of inner functions in terms of tangential Schur interpolating conditions, the interpolation problem is readily solved. The advantage of this approach with respect to the one in [4] is that formulas are explicit and easily computable and easily extend to the multivariable case. A drawback is that the domain of the charts is not as simple (a similar situation can be found in [4] where the Nevanlinna-Pick and the Schur interpolating conditions are considered). A parametrization using the same idea of inner extensions, but with a different technique has been independently derived by Horiguchi [15].

2 Preliminaries and notation

Let F be a rational $p \times m$ matrix of McMillan degree N, whose entries lie in the Hardy space of the right half-plane. We shall denote by $\mathbb{C}^+$ the right half-plane, and by $\mathcal{H}_+^2$ the corresponding Hardy space of vector or matrix valued functions (the proper dimension will be understood from the context). The space $\mathcal{H}_+^2$ is naturally endowed with the scalar product,

$$< F, G > = \frac{1}{2\pi} \mathrm{Tr} \int_{-\infty}^{\infty} F(iy) G(iy)^* \, dy, \tag{1}$$

and we shall denote by $\| \cdot \|_2$ the associated norm. Note that if M is a complex matrix, Tr stands for its trace, M^T for its transpose and M^* for its transpose conjugate. Similarly, we define $\mathcal{H}_+^\infty$ to be Hardy space of essentially bounded functions analytic on the right half plane (with the usual radial limit condition). Let $m \leq p$; a $p \times m$ matrix valued function $Q \in \mathcal{H}_+^\infty$ is called a *Schur* function if $Q(i\omega)^* Q(i\omega) \leq I_m$ for (almost all) $\omega \in \mathbb{R}$; it is called *tall inner* or *rigid* if $Q(i\omega)^* Q(i\omega) = I_m$ for (almost all) $\omega \in \mathbb{R}$.

We assume that we are given a set of interpolation points $s_1, ..., s_n$ in the right half-plane $\mathbb{C}^+$ and interpolating conditions

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{2}$$

with u_i, v_i row vectors in $\mathbb{C}^m$, $\mathbb{C}^p$, respectively, with $\|u_i\| = 1$ and $\|v_i\| \leq 1$ for $i = 1, ..., n$; moreover we assume that we are given a $p \times m$ constant strictly contractive matrix D. Our goal will be to find the Schur functions Q of *fixed degree* n which are solutions to the Nevanlinna-Pick interpolation problem

$$u_i Q(s_i)^* = v_i \qquad i = 1, ..., n \qquad Q(\infty) = D \tag{3}$$

A more general form of the interpolation problem (3) which allows for multiplicities of the interpolation nodes can be defined in the following way (see [3]). Assume that the eigenvalues of the matrix $\mathcal{A}$ are in the open left half plane $\mathbb{C}^-$ and let Γ be any closed curve around the spectrum of $\mathcal{A}$. We want to parametrize the Schur functions Q of *fixed degree* n which satisfy

$$\int_\Gamma Q(s) U^* \left(sI + \mathcal{A}^*\right)^{-1} ds = V^* \qquad Q(\infty) = D \tag{4}$$

Problem (3) is then recovered by choosing $\mathcal{A} := \mathrm{diag}\{-\overline{s}_1, ..., -\overline{s}_n\}$. Notice that another formulation of the above conditions is

$$(Q(s)U^* - V^*)\,(sI + \mathcal{A}^*)^{-1} \qquad \text{is analytic on } \mathbb{C}^+ \tag{5}$$

with the same constraint at infinity. To avoid pathological cases, we assume that the functions $Q(s)u_i^*$ are non constant. Notice that, if D denotes the constant term of Q, this assumption is always satisfied if for all $i = 1, ..., n$ we have $u_i D^* \neq v_i$.

3 Linear fractional transformations: state space representations

Let $J = \begin{bmatrix} I_m & 0 \\ 0 & -I_p \end{bmatrix}$. We say that a rational $(m{+}p) \times (m{+}p)$ matrix valued function Θ is J-inner if $\Theta(i\omega)^* J \Theta(i\omega) = J$ for $\omega \in \mathbb{R}$ and $\Theta(s)^* J \Theta(s) \leq J$ for Re $s > 0$. Let Θ be a $J-$inner function,

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \tag{6}$$

If the $p \times m$ matrix valued function S is inner, then also

$$R := (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21}) \tag{7}$$

is inner. Moreover, if S is a Schur function, then R is a Schur function, as well.

Lemma 3.1 *A rational function Θ is $J-$inner if and only if it admits a realization*

$$\Theta = \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline -\,J\mathcal{B}^*\mathcal{Q}^{-1} & I \end{array} \right)$$

where $(\mathcal{A}, \mathcal{B})$ is a controllable pair and the equation

$$\mathcal{A}\mathcal{Q} + \mathcal{Q}\mathcal{A}^* + \mathcal{B}J\mathcal{B}^* = 0$$

has a unique positive definite solution $\mathcal{Q}$.

Proof. by verification.

Notice that if $J = I$, we get a state space representation for square inner functions.

Corollary 3.1 *Let* $R = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *a* $p \times m$ *tall inner function, and let* D_e *be a unitary extension of* D*, i.e.* $\mathbf{D} = [D, D_e]$ *is a unitary* $p \times p$ *matrix. Then, there exists a unique inner extension* $\mathbf{R}$ *of* R *of the same degree and such that* $\mathbf{R}(\infty) = \mathbf{D}$*, and it is given by*

$$\mathbf{R} = \left(\begin{array}{c|cc} A & B & -P^{-1}C^*D_e \\ \hline C & D & D_e \end{array}\right)$$

where P *is the solution to*

$$A^*P + PA + C^*C = 0.$$

Suppose now $\mathcal{B} = [U_R, V_R]$ where the dimensions are in agreement with the partition of J. Then we can write our realizations as

$$\Theta = \left(\begin{array}{c|cc} \mathcal{A} & U_R & V_R \\ \hline -U_R^*\mathcal{Q}^{-1} & I & 0 \\ V_R^*\mathcal{Q}^{-1} & 0 & I \end{array}\right) \tag{8}$$

where $\mathcal{Q}$ satisfies

$$\mathcal{A}\mathcal{Q} + \mathcal{Q}\mathcal{A}^* + U_RU_R^* - V_RV_R^* = 0 \tag{9}$$

Then it is also well known (see [8], [3]) that R in (7) is a solution of the interpolation problem (4).

The following is a state space representation of R in terms of $\mathcal{A}, U_R, V_R, D$

Lemma 3.2 *Let* Θ *be a* $J-$*inner function as in (8), with* $m = p$*, and let* $S = D$ *be a constant unitary matrix. Then the function* R *given by (7) has realizations:*

$$R = \left(\begin{array}{c|c} \mathcal{A} + V_R(DU_R^* - V_R^*)\mathcal{Q}^{-1} & -U_R + V_RD \\ \hline (DU_R^* - V_R^*)\mathcal{Q}^{-1} & D \end{array}\right) \tag{10}$$

and

$$R = \left(\begin{array}{c|c} -\mathcal{A}^* - \mathcal{Q}^{-1}(U_R - V_RD)U_R^* & \mathcal{Q}^{-1}(U_R - V_RD) \\ \hline -(DU_R^* - V_R^*) & D \end{array}\right) \tag{11}$$

This function is the unique inner function of degree n which solves the interpolation problem

$$\int_\Gamma R(s)U_R^*\left(sI+\mathcal{A}^*\right)^{-1}ds = V_R^* \qquad Q(\infty)=D$$

and has constant term D.

Proof. to see that, if R has realization (10), it solves the interpolation problem, we show it comes from a realization of the linear fractional transformation (7) with $S=D$. In fact, we can write

$$\begin{aligned}
&S\Theta_{11}+\Theta_{21}\\
&= D\left(\begin{array}{c|c} \mathcal{A} & U_R \\ \hline -\,U_R^*\mathcal{Q}^{-1} & I \end{array}\right) + \left(\begin{array}{c|c} \mathcal{A} & U_R \\ \hline V_R^*\mathcal{Q}^{-1} & 0 \end{array}\right)\\
&= \left(\begin{array}{c|c} \mathcal{A} & U_R \\ \hline -\,(DU_R^*-V_R^*)\mathcal{Q}^{-1} & D \end{array}\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&S\Theta_{12}+\Theta_{22}\\
&= D\left(\begin{array}{c|c} \mathcal{A} & V_R \\ \hline -\,U_R^*\mathcal{Q}^{-1} & 0 \end{array}\right) + \left(\begin{array}{c|c} \mathcal{A} & V_R \\ \hline V^*\mathcal{Q}^{-1} & I \end{array}\right)\\
&= \left(\begin{array}{c|c} \mathcal{A} & V_R \\ \hline -\,(DU_R^*-V_R^*)\mathcal{Q}^{-1} & I \end{array}\right)
\end{aligned}$$

So the inverse of the above matrix is easily seen to be:

$$(S\Theta_{12}+\Theta_{22})^{-1} = \left(\begin{array}{c|c} \mathcal{A}+V_R(DU_R^*-V_R^*)\mathcal{Q}^{-1} & V_R \\ \hline (DU_R^*-V_R^*)\mathcal{Q}^{-1} & I \end{array}\right)$$

Therefore,

$$\begin{aligned}
R &= (S\Theta_{12}+\Theta_{22})^{-1}(S\Theta_{11}+\Theta_{21})\\
&= \left(\begin{array}{c|c} \mathcal{A}+V_R(DU_R^*-V_R^*)\mathcal{Q}^{-1} & V_R \\ \hline (DU_R^*-V_R^*)\mathcal{Q}^{-1} & I \end{array}\right)\left(\begin{array}{c|c} \mathcal{A} & U_R \\ \hline -\,(DU_R^*-V_R^*)\mathcal{Q}^{-1} & D \end{array}\right)\\
&= \left(\begin{array}{cc|c} \mathcal{A} & 0 & U_R \\ -V_R(DU_R^*-V_R^*)\mathcal{Q}^{-1} & \mathcal{A}+V_R(DU_R^*-V_R^*)\mathcal{Q}^{-1} & V_RD \\ \hline -\,(DU_R^*-V_R^*)\mathcal{Q}^{-1} & (DU_R^*-V_R^*)\mathcal{Q}^{-1} & D \end{array}\right)\\
&=: \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & D \end{array}\right)
\end{aligned}$$

Using the change of basis $T = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$ we get

$$\mathbf{A}' = T\mathbf{A}T^{-1} \qquad \mathbf{B}' = T\mathbf{B} \qquad \mathbf{C}' = \mathbf{C}T^{-1}$$

which applied to our system yields:

$$\begin{aligned} \left(\begin{array}{c|c} \mathbf{A}' & \mathbf{B}' \\ \hline \mathbf{C}' & D \end{array}\right) &= \left(\begin{array}{cc|c} \mathcal{A} & 0 & U_R \\ 0 & \mathcal{A} + V_R(DU_R^* - V_R^*)\mathcal{Q}^{-1} & -U_R + V_R D \\ \hline 0 & (DU_R^* - V_R^*)\mathcal{Q}^{-1} & D \end{array}\right) \\ &= \left(\begin{array}{c|c} \mathcal{A} + V_R(DU_R^* - V_R^*)\mathcal{Q}^{-1} & -U_R + V_R D \\ \hline (DU_R^* - V_R^*)\mathcal{Q}^{-1} & D \end{array}\right) \end{aligned}$$

As for (11), observe that, in view of (9),

$$\mathcal{Q}^{-1}[\mathcal{A} + V_R(DU_R^* - V_R^*)\mathcal{Q}^{-1}]\mathcal{Q} = -\mathcal{A}^* - \mathcal{Q}^{-1}(U_R - V_R D)U_R^*$$

Then (11) is obtained from (10) by the change of basis induced by $-\mathcal{P}$. The fact that R solves the interpolation problem (4) is well known (see e.g. [3],[8]). For the claim on uniqueness we refer to [1]. ■

4 Main results

We now come back to our original problem (4). Let D be a $p \times m$ isometry (i.e. such that $D^*D = I_m$). We denote with $\mathcal{S}_n^{p,m}(D)$ the set of $p \times m$ Schur functions of degree n whose value at infinity is D. Similarly, we denote by $\mathcal{I}_n^{p,m}(D)$ the set of functions in $\mathcal{S}_n^{p,m}(D)$ which are inner (clearly in this case it must be $D^*D = I_m$).

Let the interpolating conditions (2) $\mathcal{A}, U, V, D$ be given, let $\tilde{D}$ be such that $\tilde{D}^*\tilde{D} = I - D^*D$ and $D_e := \begin{bmatrix} D \\ \tilde{D} \end{bmatrix}$. We denote by $\mathcal{I}_n^c(\mathcal{A}, U, V, D)$ the set of $(m+p) \times m$-dimensional inner functions $Q_c \in \mathcal{I}_n^{p+m,m}(D_e)$ such that

- $Q_c = \begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix}$ and Q satisfies (4)
- Setting
$$[V_Q, \tilde{V}_{\tilde{Q}}] := \left[\int_\Gamma \begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix} U^*(sI + \mathcal{A}^*)^{-1}\right]^*$$

the solution $\mathcal{X}$ to

$$\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^* + UU^* - V_Q V_Q^* - \tilde{V}_{\tilde{Q}}\tilde{V}_{\tilde{Q}}^* = 0 \tag{12}$$

is positive definite.

We denote by $\mathcal{S}_n^{p,m}(\mathcal{A}, U, V, D)$ the set of functions in $\mathcal{S}_n^{p,m}(D)$ which satisfy (4) and such that, for a minimal realization of $Q = \left(\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \end{array}\right)$, the equation:

$$B_Q U^* = \mathcal{Y}\mathcal{A}^* + A_Q \mathcal{Y} \tag{13}$$

has a nonsingular solution.

The reason why we careful about elements on the boundary is that there are functions in $\mathcal{S}_n^{p,m}(D)$ which satisfy (4), but are not in $\mathcal{S}_n^{p,m}(\mathcal{A}, U, V, D)$ (an example can be found in [13]). The set of such functions is clearly algebraic and we shall see that it is constituted of elements for which there exists, in fact, an interpolating solution of degree strictly less than n. It is therefore quite natural that they cannot fit into our parametrization.

So our interpolation problem can be formulated as follows.

Problem 1 *Given $\mathcal{A}, U, V, D$, parametrize the set* $\mathcal{S}_n^{p,m}(\mathcal{A}, U, V, D)$

Notice that D can be assumed to be a positive semidefinite matrix. In fact, given U_0, V, D_0, we can decompose D_0 as

$$D_0 = DT$$

where D is positive semidefinite and T is unitary. Set then $U := U_0 T^*$ and consider then the new problem defined by U, V, D, and let Q be an arbitrary solution. Then the function $Q_0 := QT$ is easily seen to be a solution to the original problem U_0, V, D_0.

It's well known that a stable matrix function Q is Schur if and only if it has a tall inner completion $\tilde{Q}$, i.e.

$$Q_c := \begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix} = \left(\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \\ \tilde{C}_Q & \tilde{D} \end{array}\right)$$

and $Q_c^* Q_c = I$. We would now like to characterize now the set $\mathcal{I}_n^c(\mathcal{A}, U, V, D)$ of all the inner extensions such that the first block is a Schur interpolating function. We need an intermediate lemma, which is a variation of a result of Horiguchi [15].

Lemma 4.1 *The function Q is in $\mathcal{S}_n^{p,m}(\mathcal{A}, U, V, D)$ if and only if, for any $\tilde{Q}$ such that $Q_c = \begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix}$ is tall inner, the function Q_c is in $\mathcal{I}_n^c(\mathcal{A}, U, V, D)$.*

Proof. Suppose $Q = \left(\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \end{array}\right)$ is in $\mathcal{S}_n^{p.m}(\mathcal{A}, U, V, D)$ and let $Q_c = \begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix}$ be a tall inner extension. The interpolation constraints are obviously satisfied, so we only need to show that (12) holds. Given $\mathcal{A}, U, V, D$, it is easy to see that (5) is satisfied if and only if there exists a matrix $\mathcal{Y}$ such that

$$B_Q U^* = \mathcal{Y}\mathcal{A}^* + A_Q \mathcal{Y} \tag{14}$$

$$DU^* - V^* = C_Q \mathcal{Y} \tag{15}$$

In fact, suppose (5) holds and let $\mathcal{Y}$ be the solution to (14). We show that then also (15) holds:

$$\begin{aligned}
&(Q(s)U^* - V^*)\,(sI + \mathcal{A}^*)^{-1} \\
&= (DU^* + C_Q(sI - A_Q)^{-1}B_Q U^* - V^*)(sI + \mathcal{A}^*)^{-1} \\
&= (DU^* - V^*)(sI + \mathcal{A}^*)^{-1} + C_Q(sI - A_Q)^{-1}B_Q U^*(sI + \mathcal{A}^*)^{-1} \\
&= (DU^* - V^*)(sI + \mathcal{A}^*)^{-1} \\
&\qquad + C_Q(sI - A_Q)^{-1}(s\mathcal{Y} + \mathcal{Y}\mathcal{A}^* - s\mathcal{Y} + A_Q\mathcal{Y})(sI + \mathcal{A}^*)^{-1} \\
&= (DU^* - V^*)(sI + \mathcal{A}^*)^{-1} \\
&\qquad + C_Q(sI - A_Q)^{-1}\mathcal{Y} - C_Q\mathcal{Y}(sI + \mathcal{A}^*)^{-1}
\end{aligned}$$

which, since $(sI + \mathcal{A}^*)^{-1}$ has its poles in the right-half plane, entails that its coefficients must vanish and thus (15). The converse is obtained backtracking the above chain of equalities.

Let now $\tilde{Q} = \left(\begin{array}{c|c} A_Q & B_Q \\ \hline C_{\tilde{Q}} & \tilde{D} \end{array}\right)$ be such that $Q_e := \begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix}$ is tall inner. The Bounded Real Lemma asserts that there exists a matrix P such that:

$$A_Q^* P + P A_Q + C_Q^* C_Q + C_{\tilde{Q}}^* C_{\tilde{Q}} = 0 \tag{16}$$

$$P B_Q + C_Q^* D + C_{\tilde{Q}}^* \tilde{D} = 0 \tag{17}$$

$$D^* D + \tilde{D}^* \tilde{D} = 0$$

We can also define $\tilde{V} := U\tilde{D}^* - \mathcal{Y}^* C^*_{\tilde{Q}}$. Then, using (17) and (15), we obtain:

$$\begin{aligned} \mathcal{Y}^* P B_Q &= -\mathcal{Y}^* C^*_Q D + \mathcal{Y}^* C_{\tilde{Q}} \tilde{D} = -(UD^* - V)D - (U\tilde{D}^* - \tilde{V})\tilde{D}^* \\ &= -U + VD + \tilde{V}\tilde{D} \end{aligned} \tag{18}$$

Multiplying (16) by $\mathcal{Y}^*$ and $\mathcal{Y}$, substituting (14) and (15) and then using (18), we obtain:

$$\begin{aligned} 0 &= \mathcal{Y}^*(A^*_Q P + P A_Q + C^*_Q C_Q + C^*_{\tilde{Q}} C_{\tilde{Q}})\mathcal{Y} \\ &= (UB^*_Q P - \mathcal{A}\mathcal{Y}^* P)\mathcal{Y} + \mathcal{Y}^*(PB_Q U^* - P\mathcal{Y}\mathcal{A}^*) + (V - UD^*)(V^* - DU^*) \\ &\qquad + (\tilde{V} - U\tilde{D}^*)(\tilde{V}^* - \tilde{D}U^*) \\ &= -\mathcal{A}\mathcal{Y}^* P\mathcal{Y} - \mathcal{Y}^* P\mathcal{Y}\mathcal{A}^* + VV^* + \tilde{V}\tilde{V}^* + UU^* \\ &\qquad + U(B^*_Q P\mathcal{Y} - D^*V^* - \tilde{D}^*\tilde{V}^*) + (\mathcal{Y}^* PB_Q - VD - \tilde{V}\tilde{D})U^* \\ &= -\mathcal{A}\mathcal{Y}^* P\mathcal{Y} - \mathcal{Y}^* P\mathcal{Y}\mathcal{A}^* + VV^* + \tilde{V}\tilde{V}^* - UU^* \end{aligned}$$

But the above is precisely (12) with $\mathcal{X} = \mathcal{Y}^* P\mathcal{Y}$ positive definite.

Suppose now (14) is not satisfied. Then also $\mathcal{X}$ will be singular and thus $Q_c \notin \mathcal{I}^c_n(\mathcal{A}, U, V, D)$. ■

In conclusion, if a function Q does not satisfy (13), any inner extension will be such that $\mathcal{X}$ in (12) is singular. But this, in turn, implies that there is a solution to the interpolation problem

$$\left(Q_c U^* - \begin{bmatrix} V^* \\ \tilde{V}^* \end{bmatrix}\right)(sI + \mathcal{A}^*)^{-1}$$

which has degree strictly less than n (see [8], [9]). The first block of this solution will then provide a lower degree solution to the original problem. The structure of this set of solution, although very interesting, is beyond the aim of this paper and will be further investigated in [13].

We then have the following result:

Lemma 4.2 *Let Q be a rational function. Then $Q \in \mathcal{S}^{p,m}_n(\mathcal{A}, U, V, D)$ (i.e. is an interpolating Schur function) if and only if Q is the first block of a tall inner function Q_c with realization*

$$Q_c = \begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix} = \left(\begin{array}{c|c} -\mathcal{A}^* - \mathcal{P}^{-1}[U - VD - \tilde{V}\tilde{D}]U^* & \mathcal{P}^{-1}[U - VD - \tilde{V}\tilde{D}] \\ \hline -(DU^* - V^*) & D \\ -(\tilde{D}U^* - \tilde{V}^*) & \tilde{D} \end{array}\right) \tag{19}$$

where $\tilde{V} \in \mathbb{C}^{n\times m}$ *is such that the solution* $\mathcal{P}$ *to the Lyapunov equation*

$$\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^* + UU^* - VV^* - \tilde{V}\tilde{V}^* = 0$$

is positive definite. In other words, $Q \in \mathcal{S}_n^{m,p}(\mathcal{A}, U, V, D)$ *if and only if there exists* $\tilde{V}, \tilde{D}$ *such that* $\begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix} \in \mathcal{I}_n^c(\mathcal{A}, U, V, D)$*.*

Proof. Suppose Q is the upper block of the tall inner function (19). Then Q is clearly Schur; we need to show that it satisfies (4). Observe first that, if we set $D_1 := (I - DD^*)^{1/2}$ and $\tilde{D}_1 := -\tilde{D}^{-*}D^*D_1$, we get that $\mathbf{D} = \begin{bmatrix} D & D_1 \\ \tilde{D} & \tilde{D}_1 \end{bmatrix}$ is a unitary matrix. Then we can extend the matrix Q_c to a square matrix $\mathbf{Q}$:

$$\begin{aligned}
\mathbf{Q} &= \begin{bmatrix} Q & Q_1 \\ \tilde{Q} & \tilde{Q}_1 \end{bmatrix} \\
&= \left(\begin{array}{c|cc} -\mathcal{A}^* - \mathcal{P}^{-1}[U - VD - \tilde{V}\tilde{D}]U^* & \mathcal{P}^{-1}[U - VD - \tilde{V}\tilde{D}] & -\mathcal{P}^{-1}[VD_1 + \tilde{V}\tilde{D}_1] \\ \hline -\,(DU^* - V^*) & D & D_1 \\ -(\tilde{D}U^* - \tilde{V}^*) & \tilde{D} & \tilde{D}_1 \end{array} \right) \\
&= \left(\begin{array}{c|cc} -\mathcal{A}^* - \mathcal{P}^{-1}[U - VD - \tilde{V}\tilde{D}]U^* & \mathcal{P}^{-1}[UD^* - V] & \mathcal{P}^{-1}[U\tilde{D}^* - \tilde{V}] \\ \hline -\,(DU^* - V^*) & I & 0 \\ -(\tilde{D}U^* - \tilde{V}^*) & 0 & I \end{array} \right) \mathbf{D} \qquad (20)
\end{aligned}$$

Setting $\hat{\mathbf{Q}} := \mathbf{Q}\mathbf{D}^*$, we immediately see that $\hat{\mathbf{Q}}$ is inner, in view of Lemma 3.2 and the fact that

$$\begin{aligned}
&\left[-\mathcal{A}^* - \mathcal{P}^{-1}[U - VD - \tilde{V}\tilde{D}]U^*\right]\mathcal{P}^{-1} + \mathcal{P}^{-1}\left[-\mathcal{A}^* - \mathcal{P}^{-1}[U - VD - \tilde{V}\tilde{D}]U^*\right]^* \\
&\qquad + [\mathcal{P}^{-1}(UD^* - V), \mathcal{P}^{-1}(U\tilde{D}^* - \tilde{V})] \begin{bmatrix} (DU^* - V^*)\mathcal{P}^{-1} \\ (\tilde{D}U^* - \tilde{V}^*)\mathcal{P}^{-1} \end{bmatrix} \\
&= \mathcal{P}^{-1}\left[-\mathcal{P}\mathcal{A}^* - UU^* + VDU^* + \tilde{V}\tilde{D}U^* - \mathcal{A}\mathcal{P} - UU^* + UD^*V^* + U\tilde{D}^*\tilde{V}^*\right]\mathcal{P}^{-1} \\
&\qquad + \mathcal{P}^{-1}\Big[UDD^*U^* - VDU^* - UD^*V^* + VV^* \\
&\qquad\qquad + U\tilde{D}\tilde{D}^*U^* - \tilde{V}\tilde{D}U^* - U\tilde{D}^*\tilde{V}^* + \tilde{V}\tilde{V}^*\Big]\mathcal{P}^{-1} \\
&= \mathcal{P}^{-1}\left[-\mathcal{P}\mathcal{A}^* - \mathcal{A}\mathcal{P} - UU^* + VV^* + \tilde{V}\tilde{V}^*\right]\mathcal{P}^{-1} = 0
\end{aligned}$$

from (9). Moreover, in view of (11) $\hat{\mathbf{Q}}$ interpolates the conditions:

$$\int_\Gamma \hat{\mathbf{Q}}(s) \begin{bmatrix} DU^* \\ \tilde{D}U^* \end{bmatrix} = \begin{bmatrix} V^* \\ \tilde{V}^* \end{bmatrix} \qquad \hat{\mathbf{Q}}(\infty) = I_{p+m} \tag{21}$$

But since

$$\hat{\mathbf{Q}}(s) \begin{bmatrix} DU^* \\ \tilde{D}U^* \end{bmatrix} = \hat{\mathbf{Q}}(s) \begin{bmatrix} D & D_1 \\ \tilde{D} & \tilde{D}_1 \end{bmatrix} \begin{bmatrix} U^* \\ 0 \end{bmatrix} = \mathbf{Q}(s) \begin{bmatrix} U^* \\ 0 \end{bmatrix}$$

the first block Q of $\mathbf{Q}$ interpolates the conditions (4). Lemma 4.1 ensures that (13) is satisfied, as wanted.

Conversely, suppose $Q \in \mathcal{S}(\mathcal{A}, U, V, D)$. Then $I - Q^*Q$ has a stable spectral factor $\tilde{Q}$; so $Q_c = \begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix}$ is tall inner and, in view of Corollary 3.1 we can extend it to a square inner matrix $\mathbf{Q}$. Define

$$\tilde{V} := \left(\int_\Gamma \tilde{Q}(s) U^* (sI + \mathcal{A}^*)^{-1} ds \right)^*$$

Backtracking the above reasoning, we see that $\hat{\mathbf{Q}} := \mathbf{Q}\mathbf{D}^*$ interpolates (21); but, again in view of Lemma 3.2 and 4.1, $\hat{\mathbf{Q}}$ (and hence Q) has a realization as in (20), which achieves the proof. ■

Let now $\mathcal{A}, U, V, \tilde{V}$ be given, and let $\mathcal{P}$ be the solution (if it exists and it is unique) to:

$$\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^* + UU^* - VV^* - \tilde{V}\tilde{V}^* = 0$$

In this case, we set

$$\mathcal{L}(\mathcal{A}, U, V, \tilde{V}) := \mathcal{P} \tag{22}$$

Then the above Lemma easily yields a parametrization of the set of all tall inner extensions $\mathcal{I}_n^c(\mathcal{A}, U, V, D)$ of interpolating functions.

Theorem 4.1 (Inner extension parametrization) *Let $\mathcal{A}$ be a stable matrix and $U \in \mathbb{C}^{n\times m}$ and $V \in \mathbb{C}^{n\times p}$ and $D \in \mathbb{C}^{p\times m}$ be given, with each row u_i of norm 1 (i.e. $u_i^* u_i = 1$) and D strictly contractive (i.e. $D^*D < I$); let $\tilde{D} \in \mathbb{C}^{m\times m}$ be such that $D_c := \begin{bmatrix} D \\ \tilde{D} \end{bmatrix}$ is an isometry, and define the set*

$\mathcal{V}_{(\mathcal{A},U,V,D_c)}$ *and the function* $\varphi_{(\mathcal{A},U,V,D_c)}$ *as follows:*

$$\begin{aligned}
\varphi_{(\mathcal{A},U,V,D_c)}(Q_c) : &= \left[\int_\Gamma \tilde{Q}(s)U^*(sI+\mathcal{A})^{-1}ds\right]^* \qquad \text{for } Q_c \in \mathcal{S}_n^{m+p,m}(D_c) \\
\mathcal{V}_{(\mathcal{A},U,V,D_c)} : &= \Big\{ Q_c = \begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix} \in \mathcal{I}_n^c(\mathcal{A},U,V,D_c); \qquad (23) \\
&\quad \mathcal{L}(\mathcal{A},U,V,\varphi_{(\mathcal{A},U,V,D_c)}(Q_c)) > 0, Q_c(\infty) = D_c \Big\} \qquad (24)
\end{aligned}$$

Then the family $(\mathcal{V}_{(\mathcal{A},U,V,D_c)}, \varphi_{(\mathcal{A},U,V,D_c)})$ *forms an atlas for the set* $\mathcal{I}_n^c(\mathcal{A},U,V,D_c)$ *whose topology coincides with the one induced by the topology of* $\mathcal{I}_n^{p+m,m}(D_c)$*.*

Proof. We can assume, without loss of generality, that the matrix $\mathcal{A}$ is of the form:

$$\begin{bmatrix} -\overline{s}_1 & 1 & 0 & \dots & 0 \\ 0 & -\overline{s}_2 & 1 & 0 & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & & -\overline{s}_{n-1} & 1 \\ 0 & \dots & & & -\overline{s}_n \end{bmatrix}$$

where the s_i are not necessarily distinct. In this case it can be shown (see [8], [3]), that this corresponds to the classical Schur problem and that (see [1], [14]), if $\mathbf{D}$ is a $r \times r$ unitary matrix, the set of inner functions $\mathcal{I}_n^{r,r}(\mathbf{D})$ of degree n is a smooth manifold and an atlas is given as follows: for $\mathbf{U}, \mathbf{W} \in \mathbb{C}^{n\times r}$, define the set $\mathcal{V}_{(\mathcal{A},\mathbf{U},\mathbf{D})}$ and the function $\varphi_{(\mathcal{A},\mathbf{U},\mathbf{D})}$ as

$$\begin{aligned}
\mathcal{V}_{(\mathcal{A},\mathbf{U},\mathbf{D})} : &= \{\mathbf{Q} \in \mathcal{I}_n^{r,r}(\mathbf{D}); |\mathbf{u}_i \mathbf{Q}_c(s_i)^*| < 1, i = 1, \dots, n, \} \\
\varphi_{(\mathcal{A},\mathbf{U},\mathbf{D})}(\mathbf{Q}) : &= \left[\int_\Gamma \mathbf{Q}(s)\mathbf{U}^*(sI+\mathcal{A}^*)^{-1}ds\right]^*
\end{aligned}$$

The map φ^{-1} is given by:

$$\varphi^{-1}_{(\mathcal{A},\mathbf{U},\mathbf{D})}(\mathbf{W}) = \left(\begin{array}{c|c} \mathcal{A} + \mathbf{W}(\mathbf{D}\mathbf{U}^* - \mathbf{W}^*)\mathbf{P}^{-1} & -\mathbf{U} + \mathbf{W}\mathbf{D} \\ \hline (\mathbf{D}\mathbf{U}^* - \mathbf{W}^*)\mathbf{P}^{-1} & \mathbf{D} \end{array}\right)$$

where $\mathbf{P}$ is the positive definite matrix satisfying:

$$\mathcal{A}\mathbf{P} + \mathbf{P}\mathcal{A}^* + \mathbf{U}\mathbf{U}^* - \mathbf{W}\mathbf{W}^* = 0$$

From the above, we can now easily derive an atlas for tall inner functions $\mathcal{I}_n^{r,m}(D_c)$, where now D_c is an isometry. In fact, given $\mathbf{D} = [D_c, D_e]$, from

Corollary 3.1, there exists a unique inner completion of a tall inner function Q_c. Moreover, if $m < r$ the subset of $\mathcal{I}_n^{r,r}(\mathbf{D})$ of inner function such that the first m columns have degree n is obviously open. Therefore, the map

$$\begin{aligned} \psi \;\; : \;\; & \mathcal{I}_n^{r,m}(D_c) \mapsto \mathcal{I}_n^{r,r}(\mathbf{D}) \\ & \psi(Q_c) = \mathbf{Q} \end{aligned}$$

is continuous and continuously invertible on its image. We can define the set $\mathcal{V}_{(\mathcal{A},U,D_c)}$ and the function $\varphi_{(\mathcal{A},U,D_c)}$ as

$$\begin{aligned} \varphi_{(\mathcal{A},U,D_c)}(Q_c) : \;\; &= \;\; \varphi_{(\mathcal{A},\mathbf{U},\mathbf{D})}(\psi(Q_c)) \qquad \text{for } Q_c \in \mathcal{S}_n^{m+p,m}(D_c) \qquad (25) \\ \mathcal{V}_{(\mathcal{A},U,D_c)} : \;\; &= \;\; \{Q_c \in \mathcal{I}_n^{r,m}(D_c); \mathcal{L}(\mathcal{A},U,V,\varphi_{(\mathcal{A},U,D_c)}(Q_c)) > 0\} \quad (26) \end{aligned}$$

Consequently,

$$\varphi^{-1}_{(\mathcal{A},U,D_c)}(W_c) \;\; = \;\; \left(\begin{array}{c|c} \mathcal{A} + W_c(D_cU^* - W_c^*)\mathcal{P}^{-1} & -U + W_cD_c \\ \hline (D_cU^* - W_c^*)\mathcal{P}^{-1} & D_c \end{array} \right)$$

where $\mathcal{P}$ satisfies:

$$\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^* + UU^* - W_cW_c^* = 0$$

We claim that the above is an atlas for $\mathcal{I}_n^{r,m}(D_c)$ In fact, from Corollary (3.1) there is a one to one correspondence between the elements of $\mathcal{I}_n^{r,m}(D_c)$ and the elements of for $\mathcal{I}_n^{r,r}(\mathbf{D})$ whose first m columns have rank n and it is given by setting $\mathbf{U} = [U, 0]$, $\mathbf{W} = W_c$ and

$$\mathbf{Q} = \left(\begin{array}{c|cc} \mathcal{A} + W_c(D_cU^* - W_c^*)P^{-1} & -U + W_cD_c & W_cD_e \\ \hline (D_cU^* - W_c^*)\mathcal{P}^{-1} & D_c & D_e \end{array} \right)$$

we clearly see that

$$\varphi^{-1}(\mathcal{A}, U, D_c)(W_c) = \psi^{-1}(\varphi^{-1}_{(\mathcal{A},\mathbf{U},\mathbf{D})}(\mathbf{W}))$$

i.e. (26) is an atlas. To get eventually the claim of the theorem, partition W_c and D_c as:

$$W_c = \left[\begin{array}{c} W \\ \tilde{W} \end{array} \right] \qquad D_c = \left[\begin{array}{c} D \\ \tilde{D} \end{array} \right]$$

and set $\mathcal{W} := \left\{ W_c \in \mathbb{C}^{r \times m}; W = V, \mathcal{P}_{W_c} > 0 \right\}$ where $\mathcal{P}_{W_c}$ is the solution to

$$\mathcal{A}\mathcal{P}_{W_c} + \mathcal{P}_{W_c}\mathcal{A}^* + UU^* - VV^* - \tilde{W}\tilde{W}^* = 0$$

Clearly $\mathcal{W}$ is diffeomorphic to an open dense subset of $\mathbb{C}^{m\times m}$ and obviously $\mathcal{I}_n^c(\mathcal{A},U,V,D)=\varphi^{-1}(\mathcal{W})$, which shows that $\mathcal{I}_n^c(\mathcal{A},U,V,D)$ is a submanifold of $\mathcal{I}_n^{r\times m}(\mathcal{A},U,D)$ ■

The above is a parametrization of all inner completions of Schur interpolants; since each interpolants has many completions, corresponding to the solutions to Bounded Real Lemma equations, this is not a parametrization of $\mathcal{S}_n^c(\mathcal{A},U,V,D)$

Corollary 4.1 *The interior points of $\mathcal{I}_n^c(\mathcal{A},U,V,D)$ have degree exactly n. That is, all zero poles cancellations occur on the boundary of the disk.*

Proof. suppose Q has McMillan degree strictly less than n; then there exists a completion $\tilde{Q}$ such that $Q_c=\begin{bmatrix} Q\\ \tilde{Q}\end{bmatrix}$ is tall inner; moreover, we can assume that, possibly after multiplying by a conjugate inner function, $\tilde{Q}$ is outer. But this implies that the degree of $\tilde{Q}^*\tilde{Q}$ is twice the degree of $\tilde{Q}$. On the other hand, $\deg Q^*Q=\deg\tilde{Q}^*\tilde{Q}$, and this means that $\deg\tilde{Q}\le\deg Q<n$; since moreover Q and $\tilde{Q}$ have the same state, we conclude that $\deg Q_c<n$; but Q_c is tall inner and therefore it is on the boundary of $\mathcal{I}_n^{r,m}(\infty)$ and thus, *a fortiori*, on the boundary of $\mathcal{I}_n^c(\mathcal{A},U,V,D)$ ■

For $Q\in\mathcal{S}_n^{p,m}(D)$ and $\tilde{D}$ given such that $D_c=\begin{bmatrix} D\\ \tilde{D}\end{bmatrix}$ is an isometry, it's well known that there exists a unique *minimum phase* spectral factor $\tilde{Q}$ of $I-Q^*Q$ such that $\tilde{Q}(\infty)=\tilde{D}$.

Theorem 4.2 *Let $\chi_{\tilde{D}}$ be the map*

$$\begin{aligned}&\chi_{\tilde{D}}:\mathcal{S}_n^{p,m}(D)\mapsto\mathcal{S}_n^{m,m}(\tilde{D})\\ &\chi_{\tilde{D}}(Q)=\tilde{Q}\end{aligned}$$

Then

$$\begin{aligned}\varphi_{(\mathcal{A},U,V,D,\tilde{D})}(Q):&=\left[\int_\Gamma\chi_{\tilde{D}}(Q)(s)U^*(sI+\mathcal{A}^*)^{-1}ds\right]^*\qquad\text{for } Q\in\mathcal{S}_n^{p,m}(D)\\ \mathcal{V}_{(\mathcal{A},U,V,D,\tilde{D})}:&=\{Q\in\mathcal{S}_n^{p,m}(\mathcal{A},U,V,D);\\ &\qquad\mathcal{L}(\mathcal{A},U,V,\varphi_{(\mathcal{A},U,V,D,\tilde{D})}(Q))>0\}\end{aligned}$$

is an atlas for $\mathcal{S}_n^{p,m}(\mathcal{A},U,V,D)$. The inverse map $\varphi^{-1}_{(\mathcal{A},U,V,D,\tilde{D})}$ has domain

$$\tilde{\mathcal{W}}=\{\tilde{V}\in\mathbb{C}^{n\times m};\mathcal{P}>0,(-\mathcal{A}^*-\mathcal{P}^{-1}[U\tilde{D}^{-1}-VD\tilde{D}^{-1}-\tilde{V}]\tilde{V}^*)\ \textit{is stable}\}$$

and it is given by:

$$\varphi^{-1}_{(\mathcal{A},U,V,D,\tilde{D})}(\tilde{V}) = \left(\begin{array}{c|c} -\mathcal{A}^* - \mathcal{P}^{-1}[U - VD - \tilde{V}\tilde{D}]U^* & \mathcal{P}^{-1}[U - VD - \tilde{V}\tilde{D}] \\ \hline -\left(DU^* - V^*\right) & D \end{array}\right)$$

Proof. since $\begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix} \in \mathcal{I}^c_n(\mathcal{A}, U, V, D_c)$ we can define the map

$$\begin{aligned} \tau \ &: \ \mathcal{S}^{p,m}_n(\mathcal{A}, U, V, D) \mapsto \mathcal{I}^c_n(\mathcal{A}, U, V, D_c) \\ & \tau(Q) = \begin{bmatrix} Q \\ \chi_{\tilde{D}}(Q) \end{bmatrix} \end{aligned}$$

which is clearly continuous (it's a property of spectral factorization); the inverse, restricted to the subset of $\mathcal{I}^c_n(\mathcal{A}, U, V, D_c)$ of $Q_c = \begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix}$ such that $\tilde{Q}$ is outer, is well defined in view of Corollary 4.1 and is also clearly continuous. This induces a differential structure on $\mathcal{S}^{p,m}_n(\mathcal{A}, U, V, D)$ by means of that of $\mathcal{I}^c_n(\mathcal{A}, U, V, D)$; then the atlas is exactly that of (4.2) restricted to the set of Q_c such that $\tilde{Q}$ is outer. As for the inverse $\varphi^{-1}(\mathcal{A}, U, V, D, \tilde{D})$, the condition on the domain simply reflects the stability of $\tilde{Q}^{-1}$.

From (20), we see that the zero matrix of Q_1 is

$$\begin{aligned} A_Z^{Q_1} &= A_Q + \mathcal{P}^{-1}[VD_1 + \tilde{V}\tilde{D}_1]D_1^{-1}C_Q \\ &= -\mathcal{A}^* - \mathcal{P}^{-1}[U - VD - \tilde{V}\tilde{D}]U^* - \mathcal{P}^{-1}[VD_1 + \tilde{V}\tilde{D}_1]D_1^{-1}(DU^* - V^*) \\ &= -\mathcal{A}^* - \mathcal{P}^{-1}UU^* - \mathcal{P}^{-1}[-\tilde{V}\tilde{D}U^* + \tilde{V}\tilde{D}_1D_1^{-1}DU^* - \tilde{V}\tilde{D}_1D_1^{-1}V^*] + \mathcal{P}^{-1}VV^* \\ &= \mathcal{P}^{-1}[\mathcal{A} + \tilde{V}[\tilde{D}^{-*}U^* - \tilde{D}^{-*}D^*V^* - \tilde{V}^*]\mathcal{P}^{-1}]\mathcal{P} \end{aligned}$$

where we have used the fact that $\tilde{D} - \tilde{D}_1D_1^{-1}D = \tilde{D}^{-*}$ and $\tilde{D}_1D_1^{-1} = -\tilde{D}^{-*}D^*$. Thus the zeros of Q_1 are determined by pair:

$$(A_Z^{Q_1}, B_Z^{Q_1}) = (\mathcal{A} + \tilde{V}[\tilde{D}^{-*}U^* - \tilde{D}^{-*}DV^* - \tilde{V}^*]\mathcal{P}^{-1}, -VD_1 - \tilde{V}\tilde{D}_1)$$

The zeros of $\tilde{Q}$ are the conjugates of those of Q_1. In fact

$$\begin{aligned} A_Z^{\tilde{Q}} &= A_Q - B_Q\tilde{D}^{-1}\tilde{C}_Q = -\mathcal{A}^* - B_Q(U^* + \tilde{D}^{-1}\tilde{C}) = -\mathcal{A}^* - B_Q\tilde{D}^{-1}\tilde{V} = \\ &= -\mathcal{A}^* - \mathcal{P}^{-1}[U\tilde{D}^{-1} - VD\tilde{D}^{-1} - \tilde{V}]\tilde{V}^* = -(A_Z^{Q_1})^* \end{aligned}$$

Thus the zeros of $\tilde{Q}$ are determined by the observable pair:

$$(C_Z^{\tilde{Q}}, A_Z^{\tilde{Q}}) = (-[U^* - \tilde{D}^{-1}\tilde{V}^*], -\mathcal{A}^* - \mathcal{P}^{-1}[U\tilde{D}^{-1} - VD\tilde{D}^{-1} - \tilde{V}]\tilde{V}^*)$$

■

5 Positive real functions

An $m \times m$ matrix valued rational function Y is said to be *positive real* (see [2]) if

- $Y(i\omega) + Y^*(i\omega) \geq 0 \qquad \text{for } \omega \in \mathbb{R}$
- $Y(s)$ is real for $s \in \mathbb{R}^+$
- $Y \in H_+^\infty$ (i.e. Y is stable).

Let D be a positive definite $m \times m$ complex matrix. The set of $m \times m$ Positive Real Functions of degree n such that $Y(\infty) = D$ will be denoted by $Y_n^{m,m}(D)$ Since positive real functions are so important in system and control theory, it's not surprising that the problem of parametrizing positive real functions of fixed degree, which satisfy some given interpolating conditions, has received a lot of attention in recent years (see [4], [7],[5],[6]). We know from the Positive Real Lemma that $Y = \left(\begin{array}{c|c} A_Y & F_Y \\ \hline C_Y & L_Y \end{array}\right)$ is a Positive Real Function if there exist $P > 0, B_Y, D_Y$ such that

$$A_Y P + P A_Y^* + B_Y B_Y^* = 0 \tag{27}$$

$$A_Y P + B_Y D_Y^* = F_Y \tag{28}$$

$$D_Y D_Y^* = L_Y + L_Y^* \tag{29}$$

It's well known that a function Y is positive real if and only if the function Q:

$$Q = (Y + I)^{-1}(Y - I) \tag{30}$$

is Schur. Thus, combining together (30) and (7), we obtain that the $m \times m$ matrix valued function Y is positive real if and only it satisfies:

$$Q = (Y + I)^{-1}(Y - I) = (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21}) \tag{31}$$

for some Θ J-inner of dimension $2m \times 2m$ and S Schur of dimension $m \times m$. Equivalently, solving for Y:

$$\begin{aligned} Y &= (I-Q)^{-1}(I+Q) \\ &= [I-(S\Theta_{12}+\Theta_{22})^{-1}(S\Theta_{11}+\Theta_{21})]^{-1} \\ &\quad \cdot [I+(S\Theta_{12}+\Theta_{22})^{-1}(S\Theta_{11}+\Theta_{21})] \\ &= [S\Theta_{12}+\Theta_{22}-S\Theta_{11}-\Theta_{21}]^{-1} \\ &\quad \cdot [S\Theta_{12}+\Theta_{22}+S\Theta_{11}+\Theta_{21}] \\ &= [S(\Theta_{12}-\Theta_{11})+\Theta_{22}-\Theta_{21}]^{-1} \\ &\quad \cdot [S(\Theta_{12}+\Theta_{11})+\Theta_{22}+\Theta_{21}] \end{aligned}$$

In particular, if Q satisfies (4), from (30) we get that Y satisfies the interpolating conditions:

$$\int_\Gamma (Y(s)+I)^{-1}\left[(Y(s)-I)U^* - (Y+I)V^*\right](sI+\mathcal{A}^*)^{-1}ds = 0$$

(we have used the fact that $\int_\Gamma (sI+\mathcal{A}^*)^{-1}ds = I$). Since Y is positive real, $(Y(s)+I)^{-1}$ is analytic in the right half-plane together with its inverse. Therefore it cannot contribute to change the analyticity properties of the other factors in the integral; thus the above is equivalent to

$$\int_\Gamma (Y(s)(U^*-V^*)(sI+\mathcal{A}^*)^{-1}ds = U^*+V^*$$

Setting

$$\xi := \frac{1}{\sqrt{2}}(U-V) \qquad \eta := \frac{1}{\sqrt{2}}(U+V) \tag{32}$$

we get the following result (see [3]):

Lemma 5.1 *Let $\xi, \eta \in \mathbb{C}^{n\times m}$; then there exists an $m \times m$ positive real function Y such that*

$$\int_\Gamma Y(s)\xi^*(sI+\mathcal{A}^*)^{-1}ds = \eta^* \tag{33}$$

if and only if the equation

$$\mathcal{A}\mathcal{R}+\mathcal{R}\mathcal{A}^*+\xi\eta^*+\eta\xi^* = 0 \tag{34}$$

has a unique positive definite solution $\mathcal{R}$.

Proof. since it is

$$\xi\eta^* + \eta\xi^* = \frac{1}{2}(U-V)(U^*+V^*) + \frac{1}{2}(U-V)(U^*+V^*) = UU^* - VV^*$$

(34) is equivalent to (9). ■

Given $\mathcal{A}, \xi, \eta, D_Y$ with $D_Y + D_Y^* > 0$, we can define $\mathcal{Y}_n^m(\mathcal{A}, \xi, \eta, D_Y)$ as the set of positive real functions satisfying the following interpolation conditions:

$$\int_\Gamma Y(s)\xi^*(sI+\mathcal{A}^*)^{-1}ds = \eta^* \qquad Y(\infty) = D_Y \tag{35}$$

A full-rank $m \times m$ positive real functions Y uniquely determines a stable *minimum-phase* $m \times m$ spectral factor W (which has the properties that $Y + Y^* = W^*W$ and W^{-1} is stable). Observe that, since $D_Y + D_Y^* > 0$, there exists a matrix D_W such that $D_W^* D_W = D_Y + D_Y^*$. Then, similarly to what we did in the Schur function case, we can define the set $\mathcal{W}_n^m(\mathcal{A}, \xi, \tilde{V}, D_W)$ as the set of stable minimum-phase rational functions satisfying:

$$\int_\Gamma W(s)\xi^*(sI+\mathcal{A}^*)^{-1}ds = \tilde{V}^* \qquad W(\infty) = D_W \tag{36}$$

In fact, we have the following state space relation between the Schur and Positive real realizations:

Lemma 5.2 *Let* $Q_c = \begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix}$ *be as in (19) and let* $\xi, \eta, \tilde{V}$ *be determined by the relations (32). Set*

$$D_Y := (I+D)(I-D)^{-1} \qquad D_W := \sqrt{2}\tilde{D}(I-D)^{-1} \tag{37}$$

Then a realization of $\begin{bmatrix} Y \\ W \end{bmatrix}$ *satisfying (35) and (36) is given by:*

$$\begin{bmatrix} Y \\ W \end{bmatrix} = \left(\begin{array}{c|c} -\mathcal{A}^* - \mathcal{R}^{-1}[\xi D_Y + \eta - \tilde{V}D_W]\xi^* & \mathcal{R}^{-1}[\xi D_Y + \eta - \tilde{V}D_W] \\ \hline -\,(D_Y\xi^* - \eta^*) & D_Y \\ -(D_W\xi^* - \tilde{V}^*) & D_W \end{array}\right) \tag{38}$$

Proof. we recall first that, if $\begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix}$ is tall inner and Q is square, then the pair

$$\begin{bmatrix} Y \\ W \end{bmatrix} := \begin{bmatrix} I+Q \\ \sqrt{2}\tilde{Q} \end{bmatrix} (I-Q)^{-1}$$

satisfies $W^*W = Y + Y^*$. In fact,

$$\begin{aligned} & Y^* + Y = (I-Q^*)^{-1}(I+Q^*) + (I+Q)(I-Q)^{-1} \\ = \; & (I-Q^*)^{-1}[(I+Q^*)(I-Q) + (I-Q^*)(I+Q)](I-Q)^{-1} \\ = \; & (I-Q^*)^{-1}[I - Q + Q^* - Q^*Q + I - Q^* + Q - Q^*Q](I-Q)^{-1} \\ = \; & 2(I-Q^*)^{-1}[I - Q^*Q](I-Q)^{-1} \\ = \; & 2(I-Q^*)^{-1}\tilde{Q}^*\tilde{Q}(I-Q)^{-1} = W^*W \end{aligned}$$

Now, if $\begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix} = \left(\begin{array}{c|c} F & G \\ \hline H & D \\ \tilde{H} & \tilde{D} \end{array}\right)$, we can write

$$\begin{aligned} & \begin{bmatrix} Y \\ W \end{bmatrix} := \begin{bmatrix} I+Q \\ \sqrt{2}\tilde{Q} \end{bmatrix} (I-Q)^{-1} \\ = \; & \left(\begin{array}{c|c} F & G \\ \hline H & I+D \\ \sqrt{2}\tilde{H} & \sqrt{2}\tilde{D} \end{array}\right) \left(\begin{array}{c|c} F & -G \\ \hline H & I-D \end{array}\right)^{-1} \\ = \; & \left(\begin{array}{c|c} F & G \\ \hline H & I+D \\ \sqrt{2}\tilde{H} & \sqrt{2}\tilde{D} \end{array}\right) \left(\begin{array}{c|c} F + G(I-D)^{-1}H & G(I-D)^{-1} \\ \hline (I-D)^{-1}H & (I-D)^{-1} \end{array}\right) \\ = \; & \left(\begin{array}{cc|c} F + G(I-D)^{-1}H & 0 & G(I-D)^{-1} \\ G(I-D)^{-1}H & F & G(I-D)^{-1} \\ \hline (I+D)(I-D)^{-1}H & H & (I+D)(I-D)^{-1} \\ \sqrt{2}\tilde{D}(I-D)^{-1}H & \sqrt{2}\tilde{H} & \sqrt{2}\tilde{D}(I-D)^{-1} \end{array}\right) \\ = \; & \left(\begin{array}{cc|c} F + G(I-D)^{-1}H & 0 & G(I-D)^{-1} \\ 0 & F & 0 \\ \hline 2(I-D)^{-1}H & H & (I+D)(I-D)^{-1} \\ \sqrt{2}[\tilde{D}(I-D)^{-1}H + \tilde{H}] & \sqrt{2}\tilde{H} & \sqrt{2}\tilde{D}(I-D)^{-1} \end{array}\right) \\ = \; & \left(\begin{array}{c|c} F + G(I-D)^{-1}H & G(I-D)^{-1} \\ \hline 2(I-D)^{-1}H & D_Y \\ D_W H + \sqrt{2}\tilde{H} & D_W \end{array}\right) \qquad (39) \end{aligned}$$

Now, in our case, noticing that, inverting (32) we get

$$U = \frac{1}{\sqrt{2}}(\xi + \eta) \qquad V = \frac{1}{\sqrt{2}}(-\xi + \eta)$$

we can write:

$$\begin{aligned}
G(I-D)^{-1} &= \mathcal{R}^{-1}(U - VD - \tilde{V}\tilde{D})(I-D)^{-1} \\
&= \mathcal{R}^{-1}\left(\frac{1}{\sqrt{2}}(\xi+\eta) - \frac{1}{\sqrt{2}}(-\xi+\eta)D - \tilde{V}\tilde{D}\right)(I-D)^{-1} \\
&= \mathcal{R}^{-1}\frac{1}{\sqrt{2}}\left[(\xi(I+D) + \eta(I-D) - \sqrt{2}\tilde{V}\tilde{D}\right](I-D)^{-1} \\
&= \mathcal{R}^{-1}\frac{1}{\sqrt{2}}(\xi D_Y + \eta - \tilde{V}D_W) \qquad (40)
\end{aligned}$$

Similarly,

$$\begin{aligned}
(I-D)^{-1}H &= -(I-D)^{-1}(DU^* - V^*) \\
&= -(I-D)^{-1}\frac{1}{\sqrt{2}}[D(\xi^* + \eta^*) - (-\xi^* + \eta^*)] \\
&= -(I-D)^{-1}\frac{1}{\sqrt{2}}[(I+D)\xi^* - (I-D)\eta^*] \\
&= -\frac{1}{\sqrt{2}}[D_Y\xi^* - \eta^*] \qquad (41)
\end{aligned}$$

and

$$\begin{aligned}
D_W H + \sqrt{2}\tilde{H} &= -\sqrt{2}[\tilde{D}(I-D)^{-1}(DU^* - V^*) + \tilde{D}U^* - \tilde{V}^*] \\
&= -\sqrt{2}[\tilde{D}(I-D)^{-1}(DU^* - V^* + U^* - DU^*) - \tilde{V}^*] \\
&= -\sqrt{2}[D_W\xi^* - \tilde{V}^*] \qquad (42)
\end{aligned}$$

Eventually, setting as before $B := \mathcal{P}^{-1}[U - VD - \tilde{V}\tilde{D}]$, we have

$$\begin{aligned}
F + G(I-D)^{-1}H &= -\mathcal{A}^* - BU^* - B(I-D)^{-1}(DU^* - V^*) \\
&= -\mathcal{A}^* - B(I-D)^{-1}(U^* - DU^* + DU^* - V^*) \\
&= -\mathcal{A}^* - \mathcal{R}^{-1}\frac{1}{\sqrt{2}}(\xi D_Y + \eta - \tilde{V}D_W)(U^* - V^*) \\
&= -\mathcal{A}^* - \mathcal{R}^{-1}(\xi D_Y + \eta - \tilde{V}D_W)\xi^* \qquad (43)
\end{aligned}$$

Plugging (40),(41), (42) and (43) into (39) we get

$$\begin{array}{rcl} \begin{bmatrix} Y \\ W \end{bmatrix} & := & \begin{bmatrix} I+Q \\ \sqrt{2}\tilde{Q} \end{bmatrix} (I-Q)^{-1} \\ & = & \left(\begin{array}{c|c} F+G(I-D)^{-1}H & G(I-D)^{-1} \\ \hline 2(I-D)^{-1}H & D_Y \\ D_W H+\sqrt{2}\tilde{H} & D_W \end{array}\right) \\ & = & \left(\begin{array}{c|c} F+G(I-D)^{-1}H & \sqrt{2}G(I-D)^{-1} \\ \hline \sqrt{2}(I-D)^{-1}H & D_Y \\ \frac{1}{\sqrt{2}}(D_W H+\sqrt{2})\tilde{H} & D_W \end{array}\right) \\ & = & \left(\begin{array}{c|c} -\mathcal{A}^* - \mathcal{R}^{-1}[\xi D_Y + \eta - \tilde{V} D_W]\xi^* & \mathcal{R}^{-1}[\xi D_Y + \eta - \tilde{V} D_W] \\ \hline -\,(D_Y\xi^* - \eta^*) & D_Y \\ -(D_W\xi^* - \tilde{V}^*) & D_W \end{array}\right) \end{array}$$

Remark eventually that, since $\tilde{Q}$ and W have the same zeros, $\tilde{Q}$ is stable if and only if W is. ■

Let now $\mathcal{A}, \xi, \eta, \tilde{V}$ be given, and let $\mathcal{R}$ be the solution (if it exists and it is unique) to:

$$\mathcal{A}\mathcal{R} + \mathcal{R}\mathcal{A}^* + \xi\eta^* + \eta\xi^* - \tilde{V}\tilde{V}^* = 0$$

In this case, we set

$$\mathcal{L}(\mathcal{A}, \xi, \eta, \tilde{V}) := \mathcal{R}$$

We then have the following result:

Theorem 5.1 *Let χ_{D_W} be the map*

$$\begin{array}{l} \chi_{D_W} : \mathcal{Y}_n^m(\mathcal{A}, \xi, \eta, D_Y) \mapsto \mathcal{W}_n^m(\mathcal{A}, \xi, \tilde{V}, D_W) \\ \chi_{D_W}(Y) = W \end{array}$$

Then

$$\begin{array}{rcll} \theta_{(\mathcal{A},\xi,\eta,D_Y,D_W)}(Y) : & = & \left[\int_\Gamma \chi_{D_W}(Y)(s)\xi^*(sI+\mathcal{A}^*)ds\right]^* & \text{for } Y \in \mathcal{Y}_n^{m,m}(D_Y) \\ \mathcal{V}_{(\mathcal{A},\xi,\eta,D_Y,D_W)} : & = & \{Y \in \mathcal{Y}_n^m(\mathcal{A}, \xi, \eta, D_Y); & \\ & & \mathcal{L}(\mathcal{A}, \xi, \eta, \theta_{(\mathcal{A},\xi,\eta,D_Y,D_W)}(Y)) > 0\} & (44) \end{array}$$

is an atlas for $\mathcal{Y}_n^m(\mathcal{A}, \xi, \eta, D_Y)$. The inverse map $\theta^{-1}_{(\mathcal{A},\xi,\eta,D_Y,D_W)}$ has domain

$$\tilde{\mathcal{W}} = \{\tilde{V} \in \mathbb{C}^{n\times m}; \mathcal{R} > 0, (-\mathcal{A}^* - \mathcal{R}^{-1}[\xi D_Y + \eta - \tilde{V} D_W]\xi^*) \textit{ is stable}\}$$

and it is given by:

$$\theta^{-1}_{(\mathcal{A},\xi,\eta,D_Y,D_W)}(\tilde{V}) \quad = \quad \left(\begin{array}{c|c} -\mathcal{A}^* - \mathcal{R}^{-1}[\xi D_Y + \eta - \tilde{V} D_W]\xi^* & \mathcal{R}^{-1}[\xi D_Y + \eta - \tilde{V} D_W] \\ \hline -\left(D_Y \xi^* - \eta^*\right) & D_Y \end{array}\right)$$

Proof. suppose $Y \in \mathcal{Y}^m_n(\mathcal{A},\xi,\eta,D_Y)$ and let $Q = (Y+I)^{-1}(Y-I)$; in view of Theorem 4.2, there exists a unique $\tilde{Q}$ such that $\left[\begin{array}{c} Q \\ \tilde{Q} \end{array}\right] \in \mathcal{I}^c_n(\mathcal{A},U,\tilde{V},D_c)$; since $Y = (I-Q)^{-1}(I+Q)$, we obtain

$$\begin{aligned} Y + Y^* \quad &= \quad (I-Q)^{-1}(I+Q) + (I+Q)(I-Q)^{-1} \\ &\quad\ = 2(I-Q^*)(I-Q^*Q)(I-Q)^{-1} = 2(I-Q^*)\tilde{Q}^*\tilde{Q}(I-Q)^{-1} \end{aligned}$$

where $\tilde{Q}$ is the stable outer factor of $I - Q^*Q$. Then, setting $W := \sqrt{2}\tilde{Q}(I-Q)^{-1}$, we have $\frac{1}{\sqrt{2}}(I-Q^*)W^* = \tilde{Q}^*$; thus

$$\int_\Gamma \tilde{Q}(s)U^*(sI+\mathcal{A}^*)^{-1}ds = \tilde{V}^*$$

is equivalent to

$$\frac{1}{\sqrt{2}}\int_\Gamma W(s)(I-Q(s))U^*(sI+\mathcal{A}^*)^{-1}ds = \tilde{V}^*$$

Using the alternative formulation (5) we get

$$\left(\frac{1}{\sqrt{2}}W(s)(I-Q(s))U^* - \tilde{V}^*\right)(sI+\mathcal{A}^*)^{-1} \qquad \text{is analytic in } \mathbb{C}^+$$

which is equivalent to

$$\left(\frac{1}{\sqrt{2}}W(s)\left[(U^*-V^*) - (Q(s)U^*-V^*)\right] - \tilde{V}^*\right)(sI+\mathcal{A}^*)^{-1} \qquad \text{is analytic in } \mathbb{C}^+$$

or, using (32)

$$\Big(W(s)[\xi^* - (Q(s)U^*-V^*)] - \tilde{V}^*\Big)(sI+\mathcal{A}^*)^{-1} \qquad \text{is analytic in } \mathbb{C}^+$$

But now, from (4), $(Q(s)U^*-V^*)(sI+\mathcal{A}^*)^{-1}$ is analytic in $\mathbb{C}^+$ and thus we get

$$\Big(W(s)\xi^* - \tilde{V}^*\Big)(sI+\mathcal{A}^*)^{-1} \qquad \text{is analytic in } \mathbb{C}^+$$

which, in view of the equivalence between (4) and (5) is (36). Therefore $W \in \mathcal{W}_n^m(\mathcal{A}, \xi, \tilde{V}, \tilde{D}_Y)$. A reasoning completely similar to that of Theorem 4.1 concludes the proof.

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