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HANKEL NORM APPROXIMATION PROBLEM**

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Grassmannian approach to the Hankel norm approximation problem

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1 Introduction

A unifying framework to strictly contractive extension problems was initiated by Dym and Gohberg [5]. This has further been developed by Ball, Gohberg and Kaashoek [2] and is called the Grassmannian approach for extension problems. The solution to the strictly contractive extension problem in this abstract framework, when applied to a certain concrete case, yielded a complete characterization of solutions to the sub-optimal Nehari problem. The sub-optimal Hankel norm approximation problem is a more general problem, and the Nehari problem can be considered to be a special case of this problem (an excellent exposition of this can be found, for instance in Young [11]). However, this more general sub-optimal Hankel norm approximation problem does not fit a priori in the abstract framework of [2]. So one wonders if the abstract framework in [2] can be suitably enlarged so as to include the sub-optimal Hankel norm approximation problem as a special case as well.

In this article, we provide such a setting in §2. In §3, we give the statement of the strictly contractive extension problem. Here in our more general setting (as compared to the one in [2]), we get a family of problems (indexed by the nonnegative integers l) and in the special case when $l = 0$, we get the strictly contractive extension problem of [2]. Thus our new setting can be thought of as a more refined version of [2]. In the next section (§4) we will give a characterization of all solutions to the strictly contractive extension problem (of order l for k), under certain assumptions. Finally in the last section, we apply these results to two concrete extension problems:

1. the sub-optimal Hankel norm approximation problem for time-invariant infinite-dimensional linear systems, and
2. the sub-optimal Hankel norm approximation problem for time-varying, periodic, finite-dimensional linear systems.

Although the solutions to the above special problems were known before (see Sasane [10] and Kaashoek and Kos [6]), our framework enables one to deal with both of the above (and possibly other extension problems hitherto unknown) as special cases of our solution given in §4. Thus this article, in some sense, captures the essence of the proofs given in [10] and [6].

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2 Basic Objects

We will use the following set up. Just as in Ball, Gohberg and Kaashoek [2], the basic objects are \mathcal{N} , \mathcal{N}_1 and \mathcal{N}_+ (which is a part of \mathcal{N}_1)¹. However, starting from 2, owing to the introduction of a “stratification” of \mathcal{N}_1 , we obtain a much more intricate structure. Notwithstanding this fact, we have conformity with [2] in the sense that if we ignore the underlying stratification, we obtain a framework similar to the one in [2]. In the following, assumed properties appear in boldface, while consequences of the properties or other auxiliary definitions are displayed in ordinary font.

1. Let \mathcal{N} be a unital C^* -algebra with unit e .

An element $g \in \mathcal{N}$ is said to be *positive definite*, denoted by $g >_{\mathcal{N}} 0$ if $g = a^*a$ for some invertible $a \in \mathcal{N}$.

An element $g \in \mathcal{N}$ is said to be *strictly contractive* if $e - g^*g >_{\mathcal{N}} 0$, which is in turn equivalent with $\|g\| < 1$.

Note that

- (a) $g \in \mathcal{N}, \|g\| < 1 \Rightarrow (e - g)^{-1} \in \mathcal{N}$
- (b) $a + a^* >_{\mathcal{N}} 0 \Rightarrow a$ is invertible.

2. There exists a family of subsets $(\mathcal{N}_{1,l})_{l \in \{0,1,2,\dots\}}$ of \mathcal{N} such that

- (a) $\mathcal{N}_+ := \mathcal{N}_{1,0} \subset \mathcal{N}_{1,1} \subset \mathcal{N}_{1,2} \subset \dots$.
- (b) $\mathcal{N}_{1,l}\mathcal{N}_{1,p} \subset \mathcal{N}_{1,l+p}$.

Let $\mathcal{N}_1 = \cup_{l \geq 0} \mathcal{N}_{1,l}$. **We also assume that**

$$\mathcal{N}_1 + \mathcal{N}_1 \subset \mathcal{N}_1.$$

Note that 2b implies that

$$\mathcal{N}_1\mathcal{N}_1 \subset \mathcal{N}_1.$$

In particular, we have

$$\mathcal{N}_1\mathcal{N}_+ \subset \mathcal{N}_1;$$

thus \mathcal{N}_1 is a right-module over \mathcal{N}_+ .

Also we denote the set $\mathcal{N}_{1,l} \setminus \mathcal{N}_{1,l-1}$ by $\mathcal{N}_{1,[l]}$. Clearly we have $\mathcal{N}_{1,[p]} = \mathcal{N}_{1,[l]}$ iff $p = l$

3. Let $m \in \mathcal{N}_+$, and let m be an invertible element of \mathcal{N} , that is $m \in G\mathcal{N}$, the set of invertible elements of \mathcal{N} . Let us define the *index function* $\nu : \mathcal{N}_+ \cap G\mathcal{N} \rightarrow \{0, 1, 2, \dots\} \cup \{+\infty\}$ as follows:

$$\nu(m) = \inf_l \{l \mid m^{-1} \in \mathcal{N}_{1,l}\}.$$

$\nu(m)$ is called the *Nyquist index* of m . In the case of transfer functions, this is the number of zeros in the open right half plane. Note that if for all $l \geq 0$, we have that $m^{-1} \notin \mathcal{N}_{1,l}$, then $\nu(m) = +\infty$. Also it is clear that $\nu(e) = 0$ and that ν satisfies the following inequality:

$$\nu(mn) \leq \nu(m) + \nu(n),$$

for elements m and n in \mathcal{N}_+ that are invertible in \mathcal{N} . **We now make the assumption that in fact our set-up is such that ν satisfies**

$$\nu(mn) = \nu(m) + \nu(n).$$

For the usual transfer function algebras, the index function is simply the number of zeros in the open right half plane for stable functions.

¹See also the remark at the end of the third paragraph on page 25 of [2].

4. Relations between \mathcal{N}_+ and \mathcal{N}_1 .

If $m, n \in \mathcal{N}_+$, then the pair (m, n) is said to be *right coprime (over \mathcal{N}_+)* if there exist $x, y \in \mathcal{N}_+$ such that

$$xm - yn = e.$$

Let $g \in \mathcal{N}$ and (m, n) be right coprime such that m is invertible as an element of \mathcal{N} . If $g = nm^{-1}$, then we say that (m, n) forms a *right coprime factorization of g (over \mathcal{N}_+)*.

(a) **Existence of a right coprime factorization.**

If $k \in \mathcal{N}_{1,[l]}$, then there exists a right coprime factorization formed by a right coprime pair (m, n) such that m has Nyquist index l , that is, $\nu(m) = l$.

(b) **If**

- i. m, n are coprime $\in \mathcal{N}_+$,
 - ii. m is invertible as an element of \mathcal{N} , and
 - iii. $\nu(m) = l$,
- then $nm^{-1} \in \mathcal{N}_{1,[l]}$.

5. **Homotopic invariance of the index.**

If

- (a) $n, m \in \mathcal{N}_+$, with $\nu(m) = l$,
- (b) $tn + m$ is invertible as an element of \mathcal{N} for $0 \leq t \leq 1$,

then $\nu(n + m) = l$.

This is the usual Nyquist criterion in the case of transfer functions. Note that this implies AXIOM (A) on page 25 of Ball et al. [2]. Indeed since

- (a) $g, e \in \mathcal{N}_+$, with $\nu(e) = 0$,
- (b) $-tg + e$ is invertible as an element of \mathcal{N} (indeed, $\|tg\| = |t|\|g\| < 1$),

it follows that $\nu(e - g) = 0$. Thus by 4b above, $e(e - g)^{-1} = (e - g)^{-1} \in \mathcal{N}_{1,0} = \mathcal{N}_+$.

3 Problem statement and the assumption of the key J -spectral factorization problem

Fix $k \in \mathcal{N}$. Then $f \in \mathcal{N}$ is said to be a *strictly contractive extension of order l for k* if the following hold:

1. $f - k \in \mathcal{N}_{1,l}$,
2. $\|f\| < 1$.

Given $k \in \mathcal{N}$, we wish to derive a linear fractional representation of all strictly contractive extensions of order l for k . In order to do this, we will assume that there exists a 2×2 matrix

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

with entries in \mathcal{N} that satisfies the following additional conditions:

- S1. (*Bistability.*) $\Theta \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix} = \begin{bmatrix} e & k \\ 0 & e \end{bmatrix} \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix}$
- S2. Θ_{22} is invertible as an element in \mathcal{N} and $\nu(\Theta_{22}) = l$
- S3. (*Spectral factorization.*) $\Theta^* \begin{bmatrix} -e & 0 \\ 0 & e \end{bmatrix} \Theta = \begin{bmatrix} -p & 0 \\ 0 & q \end{bmatrix}$, where p and q are positive definite elements of \mathcal{N} .

4 Characterization of all solutions

Theorem 4.1 *Let $k \in \mathcal{N}$. If Θ satisfies S1, S2, S3, then given any $h \in \mathcal{N}_+$ such that $q - h^*ph$ is positive definite, f defined by*

$$f = (\Theta_{11}h + \Theta_{12})(\Theta_{21}h + \Theta_{22})^{-1} \quad (=: \mathcal{F}_\Theta(h))$$

is a strictly contractive extension of order l for k .

Proof First we will show that $\Theta_{21}h + \Theta_{22}$ is an element belonging to \mathcal{N}_+ , it is invertible as an element of \mathcal{N} and $\nu(\Theta_{21}h + \Theta_{22}) = l$.

1. From S1, it follows that there exist $g_1, g_2 \in \mathcal{N}_+$ such that

$$\Theta \begin{bmatrix} e \\ 0 \end{bmatrix} = \begin{bmatrix} e & k \\ 0 & e \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

and so $\Theta_{21} (= g_2) \in \mathcal{N}_+$. Similarly, there exist $h_1, h_2 \in \mathcal{N}_+$ such that

$$\Theta \begin{bmatrix} 0 \\ e \end{bmatrix} = \begin{bmatrix} e & k \\ 0 & e \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

and so $\Theta_{22} (= h_2) \in \mathcal{N}_+$. Thus $\Theta_{21}h + \Theta_{22} \in \mathcal{N}_+$.

2. Let $h \in \mathcal{N}_+$ be such that $q - h^*ph$ is positive definite. Since $p >_{\mathcal{N}} 0$ and $q >_{\mathcal{N}} 0$, we may consider their square roots $p^{\frac{1}{2}}$ and $q^{\frac{1}{2}}$, respectively. Put

$$\widehat{\Theta} = \begin{bmatrix} \widehat{\Theta}_{11} & \widehat{\Theta}_{12} \\ \widehat{\Theta}_{21} & \widehat{\Theta}_{22} \end{bmatrix} = \Theta \begin{bmatrix} p^{-\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{bmatrix}, \quad J = \begin{bmatrix} e & 0 \\ 0 & -e \end{bmatrix}.$$

Then S3 gives $\widehat{\Theta}^* J \widehat{\Theta} = J$, and from S2 it follows that $\widehat{\Theta}_{22} = \Theta_{22}q^{-\frac{1}{2}}$ is invertible. Applying Theorem 2.1 in Ball et al. [1], we obtain that $\widehat{\Theta}_{22}^{-1}\widehat{\Theta}_{21}$ is strictly contractive. In other words,

$$\|q^{\frac{1}{2}}\Theta_{22}^{-1}\Theta_{21}p^{-\frac{1}{2}}\| < 1. \quad (1)$$

Since $q - h^*ph$ is positive definite, we also have

$$\|p^{\frac{1}{2}}hq^{-\frac{1}{2}}\| < 1. \quad (2)$$

From (1) and (2), it follows that $q^{\frac{1}{2}}\Theta_{22}^{-1}\Theta_{21}hq^{-\frac{1}{2}}$ is strictly contractive. Hence $\Theta_{21}h + \Theta_{22} = \Theta_{22}q^{-\frac{1}{2}} \left[q^{\frac{1}{2}}\Theta_{22}^{-1}\Theta_{21}hq^{-\frac{1}{2}} + e \right] q^{\frac{1}{2}}$ is invertible, and $f = \mathcal{F}_\Theta(h)$ is a well-defined element of \mathcal{N} .

3. We have

- (a) $h, e \in \mathcal{N}_+$, with $\nu(e) = 0$,
- (b) $\Theta_{21}, \Theta_{22} \in \mathcal{N}_+$, with $\nu(\Theta_{22}) = l$, and
- (c) $\Theta_{21}h + \Theta_{22}e = \Theta_{21}h + \Theta_{22}$ is invertible as an element of \mathcal{N} ,

and so $\nu(\Theta_{21}h + \Theta_{22}) = l$.

4. From S1, we have the existence of $\alpha_1, \alpha_2 \in \mathcal{N}_+$ such that

$$\begin{bmatrix} e & -k \\ 0 & e \end{bmatrix} \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} h \\ e \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.$$

Consequently

$$\Theta_{11}h + \Theta_{12} - k(\Theta_{21}h + \Theta_{22}) = \alpha_1.$$

Thus

$$(\Theta_{11}h + \Theta_{12})(\Theta_{21}h + \Theta_{22})^{-1} - k = \alpha_1(\Theta_{21}h + \Theta_{22})^{-1},$$

that is, $f - k = \alpha_1(\Theta_{21}h + \Theta_{22})^{-1}$. Since

- (a) $\alpha_1 \in \mathcal{N}_+$, $\Theta_{21}h + \Theta_{22} \in \mathcal{N}_+$,
- (b) $\Theta_{21}h + \Theta_{22}$ is invertible as an element of \mathcal{N} , and
- (c) $\nu(\Theta_{21}h + \Theta_{22}) = l$,

it follows that $\alpha_1(\Theta_{21}h + \Theta_{22})^{-1} \in \mathcal{N}_{1,l}$, that is $f - k \in \mathcal{N}_{1,l}$.

5. Finally, we now show that f is strictly contractive. First of all, note that

$$\begin{bmatrix} f \\ e \end{bmatrix} = \begin{bmatrix} (\Theta_{11}h + \Theta_{12})(\Theta_{21}h + \Theta_{22})^{-1} \\ (\Theta_{21}h + \Theta_{22})(\Theta_{21}h + \Theta_{22})^{-1} \end{bmatrix} = \Theta \begin{bmatrix} h \\ e \end{bmatrix} x^{-1},$$

where $x = \Theta_{21}h + \Theta_{22}$. Hence

$$\begin{aligned} e - f^*f &= \begin{bmatrix} f^* & e \end{bmatrix} \begin{bmatrix} -e & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} f \\ e \end{bmatrix} \\ &= (x^{-1})^* \begin{bmatrix} h^* & e \end{bmatrix} \Theta^* \begin{bmatrix} -e & 0 \\ 0 & e \end{bmatrix} \Theta \begin{bmatrix} h \\ e \end{bmatrix} x^{-1} \\ &= (x^{-1})^* \begin{bmatrix} h^* & e \end{bmatrix} \begin{bmatrix} -p & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} h \\ e \end{bmatrix} x^{-1} \\ &= (x^{-1})^* [q - h^*ph] x^{-1} >_{\mathcal{N}} 0. \end{aligned}$$

This completes the proof. ■

Theorem 4.2 *Let $k \in \mathcal{N}$. Suppose that Θ satisfies S1, S2, S3 and that Θ is invertible. Then given any strictly contractive extension f of order l for k , there exists a $h \in \mathcal{N}_+$ such that $q - h^*ph$ is positive definite and*

$$f = \mathcal{F}_{\Theta}(h) = (\Theta_{11}h + \Theta_{12})(\Theta_{21}h + \Theta_{22})^{-1}.$$

Proof Suppose that f is a strictly contractive extension of order l of k .

1. Let $c := \Theta_{12}\Theta_{22}^{-1}$ and $w := f - c$. From Theorem 4.1, we know that c is a strictly contractive extension of order l for k (indeed, $c = \mathcal{F}_{\Theta}(0)$). Thus $w = f - c = (f - k) + (k - c) \in \mathcal{N}_1 + \mathcal{N}_1 \subset \mathcal{N}_1$.
2. S1 and S3 imply that

$$\begin{bmatrix} -p & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix} = \begin{bmatrix} -\Theta_{11}^* & \Theta_{21}^* \\ -\Theta_{12}^* & \Theta_{22}^* \end{bmatrix} \begin{bmatrix} e & k \\ 0 & e \end{bmatrix} \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix},$$

and so

$$\Theta_{11}^* \mathcal{N}_+ \subset p\mathcal{N}_+, \tag{3}$$

$$\Theta_{12}^* \mathcal{N}_+ \subset q\mathcal{N}_+. \tag{4}$$

Let (n, m) form a right coprime factorization of $w \in \mathcal{N}_1$. Thus $\Theta_{12}^* w = \Theta_{12}^* nm^{-1}$. Since $\Theta_{12}^* n \in \Theta_{11}^* \mathcal{N}_+ \subset p\mathcal{N}_+$, it follows that $\Theta_{12}^* n = q\tilde{n}$ for some $\tilde{n} \in \mathcal{N}_+$. Hence $\Theta_{12}^* w = q\tilde{n}m^{-1} \in q\mathcal{N}_1$, and so $q - \Theta_{12}^* w \Theta_{22} \in q\mathcal{N}_1$.

3. Now we show that $q - \Theta_{12}^* w \Theta_{22}$ is invertible. Since $f = w + c$, we have $e - f^*f = e - c^*c - w^*w - c^*w - w^*c$. Using $c = \Theta_{12}\Theta_{22}^{-1}$ and S3 we obtain $e - c^*c = (\Theta_{22}^{-1})^* q \Theta_{22}^{-1}$. Therefore

$$\begin{aligned} \Theta_{22}^*(e - f^*f)\Theta_{22} + \Theta_{22}^*w^*w\Theta_{22} + q &= 2q - \Theta_{12}^*w\Theta_{22} - \Theta_{22}^*w^*\Theta_{12} \\ &= (q - \Theta_{12}^*w\Theta_{22}) + (q - \Theta_{12}^*w\Theta_{22})^*. \end{aligned}$$

Since $e - f^*f$ and q are positive definite in \mathcal{N} , it follows that

$$(q - \Theta_{12}^*w\Theta_{22}) + (q - \Theta_{12}^*w\Theta_{22})^* >_{\mathcal{N}} 0.$$

Thus $q - \Theta_{12}^*w\Theta_{22}$ is invertible.

4. Define $a = q^{-\frac{1}{2}}(q - \Theta_{12}^* w \Theta_{22}) q^{-\frac{1}{2}}$. Then $a + a^* \succ_{\mathcal{N}} 0$, and $a \in \mathcal{M}_1 := q^{\frac{1}{2}} \mathcal{N}_1 q^{-\frac{1}{2}}$. We note that \mathcal{M}_1 is a subalgebra of \mathcal{N} , and so AXIOM (A) holds for \mathcal{M}_1 .

[Indeed, let $a \in \mathcal{M}_1$ be strictly contractive. Then $r := q^{-\frac{1}{2}} a q^{\frac{1}{2}} \in \mathcal{N}_1$ and $r_{\text{spec}}(r) < 1$. Thus $\|r^n\| < 1$ and so by AXIOM (A) (for \mathcal{N}_1), it follows that $(e - r^n)^{-1} \in \mathcal{N}_+$. Moreover $e + r + \dots + r^{n-1} \in \mathcal{N}_1$, and therefore $(e - r)^{-1} = (e - r^n)(e + r + \dots + r^{n-1}) \in \mathcal{N}_1$. Thus $(e - q^{-\frac{1}{2}} a q^{\frac{1}{2}})^{-1} \in \mathcal{N}_1$ and hence $(e - a)^{-1} = q^{\frac{1}{2}} (e - q^{-\frac{1}{2}} a q^{\frac{1}{2}})^{-1} q^{-\frac{1}{2}} \in \mathcal{M}_1$.]

Now we use the following result from Ball et al. [2] (Lemma 2.1, page 26):

Lemma 4.3 *Let \mathcal{M}_1 be a subalgebra of \mathcal{N} for which AXIOM (A) holds. If $a \in \mathcal{M}_1$ is such that $a + a^* \succ_{\mathcal{N}} 0$, then a is invertible and $a^{-1} \in \mathcal{M}_1$.*

Thus $a^{-1} \in \mathcal{M}_1$ and so $(q - \Theta_{12}^* w \Theta_{22})^{-1} = q^{-\frac{1}{2}} a^{-1} q^{-\frac{1}{2}} \in \mathcal{N}_1 q^{-1}$. Define

$$h = p^{-1} \Theta_{11}^* w \Theta_{22} (q - \Theta_{12}^* w \Theta_{22})^{-1} q. \quad (5)$$

Since $\Theta_{11}^* \mathcal{N}_+ \subset p \mathcal{N}_+$, it follows that $p^{-1} \Theta_{11}^* n \in \mathcal{N}_+$, and so $p^{-1} \Theta_{11}^* n m^{-1} \in \mathcal{N}_1$. Thus $p^{-1} \Theta_{11}^* w \Theta_{22} \in \mathcal{N}_1 \mathcal{N}_+ \subset \mathcal{N}_1$. Finally since $(q - \Theta_{12}^* w \Theta_{22})^{-1} q \in \mathcal{N}_1$, it follows that $h \in \mathcal{N}_1 \mathcal{N}_1 \subset \mathcal{N}_1$.

5. From S3 we obtain

$$\begin{bmatrix} -\Theta_{11}^* & \Theta_{21}^* \\ -\Theta_{12}^* & \Theta_{22}^* \end{bmatrix} \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} -p & 0 \\ 0 & q \end{bmatrix}.$$

Thus

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} -p^{-1} & 0 \\ 0 & q^{-1} \end{bmatrix} \begin{bmatrix} -\Theta_{11}^* & \Theta_{21}^* \\ -\Theta_{12}^* & \Theta_{22}^* \end{bmatrix} \begin{bmatrix} e & k \\ 0 & e \end{bmatrix} \begin{bmatrix} n_1 \\ n_+ \end{bmatrix} = \begin{bmatrix} e & k \\ 0 & e \end{bmatrix} \begin{bmatrix} n_1 \\ n_+ \end{bmatrix},$$

$n_1 \in \mathcal{N}_1$, $n_+ \in \mathcal{N}_+$. In particular, we have

$$(\Theta_{21} p^{-1} \Theta_{11}^* - \Theta_{22} q^{-1} \Theta_{12}^*) n_1 = 0, \quad n_1 \in \mathcal{N}_1; \quad (6)$$

$$(\Theta_{11} p^{-1} \Theta_{11}^* - \Theta_{12} q^{-1} \Theta_{12}^*) n_1 = n_1, \quad n_1 \in \mathcal{N}_1. \quad (7)$$

We know that $w \Theta_{22} (q - \Theta_{12}^* w \Theta_{22})^{-1} q \in \mathcal{N}_1$. Thus (5) and (6) yield

$$\begin{aligned} & \Theta_{21} h + \Theta_{22} \\ &= \Theta_{21} p^{-1} \Theta_{11}^* w \Theta_{22} (q - \Theta_{12}^* w \Theta_{22})^{-1} q + \Theta_{22} \\ &= \Theta_{22} q^{-1} \Theta_{12}^* w \Theta_{22} (q - \Theta_{12}^* w \Theta_{22})^{-1} q + \Theta_{22} \\ &= \Theta_{22} (q - \Theta_{12}^* w \Theta_{22})^{-1} q. \end{aligned}$$

It follows that $\Theta_{21} h + \Theta_{22}$ is invertible, and thus using (5) and (7) we obtain

$$\begin{aligned} & (\Theta_{11} h + \Theta_{12})(\Theta_{21} h + \Theta_{22})^{-1} \\ &= (\Theta_{11} p^{-1} \Theta_{11}^* w \Theta_{22} (q - \Theta_{12}^* w \Theta_{22})^{-1} q + \Theta_{12}) q^{-1} (q - \Theta_{12}^* w \Theta_{22}) \Theta_{22}^{-1} \\ &= (\Theta_{12} q^{-1} \Theta_{12}^* w \Theta_{22} (q - \Theta_{12}^* w \Theta_{22})^{-1} q + \Theta_{12}) q^{-1} (q - \Theta_{12}^* w \Theta_{22}) \Theta_{22}^{-1} \\ &= \Theta_{12} q^{-1} \Theta_{12}^* w + w + \Theta_{12} q^{-1} (q - \Theta_{12}^* w \Theta_{22}) \Theta_{22}^{-1} \\ &= w + \Theta_{12} \Theta_{22}^{-1} \\ &= w + c \\ &= f. \end{aligned}$$

Hence $f = \mathcal{F}_{\Theta}(h)$.

6. Also, we have

$$\begin{aligned}
0 <_{\mathcal{N}} e - f^* f &= \begin{bmatrix} f^* & e \end{bmatrix} \begin{bmatrix} -e & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} f \\ e \end{bmatrix} \\
&= ((\Theta_{21}h + \Theta_{22})^{-1})^* \begin{bmatrix} h^* & e \end{bmatrix} \Theta^* \begin{bmatrix} -e & 0 \\ 0 & e \end{bmatrix} \Theta \begin{bmatrix} h \\ e \end{bmatrix} (\Theta_{21}h + \Theta_{22})^{-1} \\
&= ((\Theta_{21}h + \Theta_{22})^{-1})^* \begin{bmatrix} h^* & e \end{bmatrix} \begin{bmatrix} -p & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} h \\ e \end{bmatrix} (\Theta_{21}h + \Theta_{22})^{-1} \\
&= ((\Theta_{21}h + \Theta_{22})^{-1})^* (q - h^* p h) (\Theta_{21}h + \Theta_{22})^{-1}.
\end{aligned}$$

Thus $q - h^* p h$ is positive definite.

7. Finally, since $h \in \mathcal{N}_1$, we have $h \in \mathcal{N}_{1,[d]}$ for some $d \in \{0, 1, 2, \dots\}$ and h has a right coprime factorization (m_1, n_1) such that $\nu(m_1) = d$. Let $x, y \in \mathcal{N}_+$ be such that $xm_1 - yn_1 = e$.

Define

$$V = \begin{bmatrix} e & -k \\ 0 & e \end{bmatrix} \Theta.$$

From S1 it follows that

$$V \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix} = \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix},$$

and so the entries of V , namely $V_{11}, V_{12}, V_{21}, V_{22}$, all belong to \mathcal{N}_+ .

We have

$$\begin{aligned}
f - k &= (\Theta_{11}h + \Theta_{12})(\Theta_{21}h + \Theta_{22})^{-1} - k \\
&= ((V_{11} + kV_{21})h + V_{12} + kV_{22})(V_{21}h + V_{22})^{-1} - k \\
&= (V_{11}h + V_{12} + k(V_{21}h + V_{22}))(V_{21}h + V_{22})^{-1} - k \\
&= (V_{11}h + V_{12})(V_{21}h + V_{22})^{-1} \\
&= (V_{11}n_1m_1^{-1} + V_{12})(V_{21}n_1m_1^{-1} + V_{22})^{-1} \\
&= (V_{11}n_1 + V_{12}m_1)(V_{21}n_1 + V_{22}m_1)^{-1}.
\end{aligned}$$

Since Θ is invertible, let

$$\Lambda := \Theta^{-1} \begin{bmatrix} e & k \\ 0 & e \end{bmatrix}.$$

Then

$$\Lambda \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix} = \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix},$$

and so the entries of Λ , namely $\Lambda_{11}, \Lambda_{12}, \Lambda_{21}, \Lambda_{22}$, all belong to \mathcal{N}_+ . Thus $-y\Lambda_{12} + x\Lambda_{22} \in \mathcal{N}_+$ and $y\Lambda_{11} - x\Lambda_{21} \in \mathcal{N}_+$. It is easy to check that

$$(-y\Lambda_{12} + x\Lambda_{22})(V_{21}n_1 + V_{22}m_1) - (y\Lambda_{11} - x\Lambda_{21})(V_{11}n_1 + V_{12}m_1) = xm_1 - yn_1 = e$$

and so it follows that $(V_{21}n_1 + V_{22}m_1, V_{11}n_1 + V_{12}m_1)$ forms a right coprime factorization of $f - k$. Furthermore, $V_{21}n_1 + V_{22}m_1$ is invertible as an element of \mathcal{N} and

$$\nu(V_{21}n_1 + V_{22}m_1) = \nu(V_{22}) + \nu(m_1) = \nu(\Theta_{22}) + \nu(m_1) = l + d.$$

Consequently, $f - k \in \mathcal{N}_{1,[l+d]}$. But we know that $f - k \in \mathcal{N}_{1,l}$ and so $d = 0$. Thus $h \in \mathcal{N}_{1,[0]} = \mathcal{N}_+$. This completes the proof. ■

5 Examples

5.1 Example I: The sub-optimal Hankel norm approximation problem for time-invariant, infinite-dimensional, linear systems

1. $\underline{\mathcal{N}}$: We take \mathcal{N} to be the algebra of continuous functions on the one point compactified imaginary axis $C_0 \cup \{\infty\}$ (that is, the set of continuous functions k defined on $i\mathbb{R}$ for which the limits $\lim_{\omega \pm \infty} k(i\omega)$ exist and are equal).
2. \mathcal{N}_+ denotes the space of functions $f : \overline{\mathbb{C}_+} \rightarrow \mathbb{C}$ defined in the closed right half-plane such that f is analytic in \mathbb{C}_+ and bounded in the closed right half-plane (that is in $\overline{\mathbb{C}_+} = \mathbb{C}_+ \cup i\mathbb{R} \cup \{\infty\}$). $\mathcal{N}_{1,l}$ is the subset of functions $f : \overline{\mathbb{C}_+} \rightarrow \mathbb{C}$ defined in the closed right half-plane such that f can be decomposed into the sum of two functions g and h , where h is in \mathcal{N}_+ and g is a strictly proper rational function with at most l poles and they are all contained in \mathbb{C}_+ .

It is easily verified that $\mathcal{N}_+ = \mathcal{N}_{1,0} \subset \mathcal{N}_{1,1} \subset \mathcal{N}_{1,2} \subset \dots$ and that $\mathcal{N}_{1,l}\mathcal{N}_{1,p} \subset \mathcal{N}_{1,l+p}$. Finally, if $\mathcal{N}_1 = \cup_{l \geq 0} \mathcal{N}_{1,l}$, then we also have $\mathcal{N}_1 + \mathcal{N}_1 \subset \mathcal{N}_1$.

3. From the proof of Lemmas 2.5.2 and 2.5.3 in Sasane [10], it follows that the index of an element m in $\mathcal{N}_+ \cap G\mathcal{N}$ is simply the number of zeros of m (all of which lie in the open right half plane).

For any $m, n \in \mathcal{N}_+$, (e, mn) is coprime (in the sense of transfer functions; see page 53 of [10]). From the proof of Lemma 2.5.4 in [10], it follows that $\nu(mn) = \nu(m) + \nu(n)$.

4. 4a of §2 follows from the proof of Lemma 2.5.2 in [10], and 4b follows from the proof of Lemma 2.5.4 in [10].
5. This is an easy consequence of Lemma A.1.18 of Curtain and Zwart [4] and can be seen as follows. Given $n, m \in \mathcal{N}_+$ with $\nu(m) = l$ and such that $tn + m$ is invertible as an element of \mathcal{N} for $0 \leq t \leq 1$, we first show that $\nu(m) = \nu(n+m)$ under the assumption that $\|nm^{-1}\| < 1$. From the proof of Lemma 2.5.10 in [10], it follows that $tn + m$ has only finitely many zeros, and they are all contained in \mathbb{C}_+ . Let $\delta > 0$ be small enough so that $\mathbb{C}_{\delta,+}$ contains all the zeros of m and $n + m$. Let $\zeta > 0$ and consider the function $\varphi(s, t) = tn(\zeta + s) + m(\zeta + s)$. Then

- (a) $\varphi : i\mathbb{R} \times [0, 1] \rightarrow \mathbb{C}$ is a continuous function.
- (b) $\varphi(\bullet, 0), \varphi(\bullet, 1)$ are meromorphic (in fact analytic!) functions in an open set containing $\overline{\mathbb{C}_+}$.
- (c) For all $t \in [0, 1]$, $\varphi(i\omega, t) \neq 0$ for all $\omega \in \mathbb{R}$. Indeed, this follows from the fact that $\|nm^{-1}\| < 1$.

Since $tn + m$ invertible, we also have that for all $t \in [0, 1]$, $\varphi(\infty, t) \neq 0$.

Thus the Nyquist indices of $\varphi(\bullet, 0)$ and $\varphi(\bullet, 1)$ are the same. Being analytic, this means that their number of zeros in \mathbb{C}_+ are the same. In other words m and $n + m$ have the same number of zeros in $\mathbb{C}_{\zeta,+}$. But the choice of $0 < \zeta < \delta$ was arbitrary. Hence it follows that $\nu(m) = \nu(n + m)$ in the case when $\|nm^{-1}\| < 1$.

In the general case, we proceed as follows. For $t \in (0, 1)$, we have

$$\begin{aligned} \nu(n + m) &= \nu\left(\frac{1}{t}m(tnm^{-1} + e)\right) \\ &= \nu\left(\frac{1}{t}m\right) + \nu(tnm^{-1} + e) \\ &= \nu(m) + \nu(tnm^{-1} + e). \end{aligned}$$

But for t small enough, $\|tnm^{-1}\| < 1$. Hence from the above we obtain

$$\nu(tnm^{-1} + e) = \nu(e) = 0.$$

This completes the proof of the homotopic invariance of the index.

Statement of the problem: Let k be the transfer function of an infinite dimensional system with generating operators (A, B, C) such that

1. A is the exponentially stable, infinitesimal generator of a strongly continuous semigroup $(e^{tA})_{t \geq 0}$ on the Hilbert space X ,
2. $B \in \mathcal{L}(\mathbb{C}, X)$, and
3. $C \in \mathcal{L}(X, \mathbb{C})$.

Then $k \in \mathcal{N}_+$. Let l be a nonnegative integer and suppose that the following condition holds:

$$\sigma_{l+1}(H_k) < 1 < \sigma_l(H_k),$$

where $\sigma_r(\bullet)$ denotes the r th singular value of a bounded linear operator, and $H_k : L_2(0, \infty) \rightarrow L_2(0, \infty)$ denotes the Hankel operator corresponding to the kernel function $\kappa(t) = Ce^{tA}B$ (of the integral operator k), and is given by:

$$(H_k \varphi)(t) = \int_0^\infty \kappa(t+s)\varphi(s)ds, \quad t \geq 0.$$

Then find $f \in \mathcal{N}$, which is a *strictly contractive extension of order l for k* , that is, such that the following conditions hold:

1. $f - k \in \mathcal{N}_{1,l}$,
2. $\|f\| < 1$.

Define

$$\Lambda(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -CL_B \\ B^* \end{bmatrix} (I - L_C L_B)^{-1} (sI + A^*)^{-1} \begin{bmatrix} C^* & L_C B \end{bmatrix}, \quad (8)$$

where L_B and L_C denote the controllability and observability Gramians, respectively, of the infinite-dimensional system given by the triple (A, B, C) (see for instance Theorem 4.1.23 on page 160 of Curtain and Zwart [4]).

Then from Chapter 4 of [10], it follows that Θ defined by

$$\Theta = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \Lambda^{-1} \quad (9)$$

satisfies S1, S2 and S3 (with $p = q = e = 1$). Moreover Θ is invertible. Hence applying the results from §4, we obtain the following theorem:

Theorem 5.1 *Let*

1. A be the exponentially stable, infinitesimal generator of a strongly continuous semigroup $(e^{tA})_{t \geq 0}$ on the Hilbert space X ,
2. $B \in \mathcal{L}(\mathbb{C}, X)$, and
3. $C \in \mathcal{L}(X, \mathbb{C})$.

Let $k = C(\bullet I - A)^{-1}B \in \mathcal{N}_+$ and suppose that there exists a l such that $\sigma_{l+1}(H_k) < 1 < \sigma_l(H_k)$, where $\sigma_r(H_k)$'s denote the Hankel singular values. Let Θ be given by (9).

Then $f \in \mathcal{N}$ such that $f - k \in \mathcal{N}_{1,l}$ and $\|f\| < 1$ iff

$$f = (\Theta_{11}h + \Theta_{12})(\Theta_{21}h + \Theta_{22})^{-1},$$

for some $h \in \mathcal{N}_+$ such that $\|h\| < 1$.

5.2 Example II: The sub-optimal Hankel norm approximation problem for periodic, finite-dimensional, linear systems

1. \mathcal{N} :

In this section we apply the results of the previous section to integral operators $T = T_\Sigma$ from $L_2(\mathbb{R})$ into $L_2(\mathbb{R})$ which are the input-output operators of (time-varying) periodic systems of the form:

$$\Sigma \begin{cases} \left(\frac{d}{dt}x\right)(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t) + D(t)u(t), \end{cases} \quad t \in \mathbb{R}, \quad (10)$$

where $A \in C_{T_0}(\mathbb{R})^{n \times n}$, $B \in C_{T_0}(\mathbb{R})^{n \times 1}$, $C \in C_{T_0}(\mathbb{R})^{1 \times n}$, $D \in C_{T_0}(\mathbb{R})^{1 \times 1}$, where $C_{T_0}(\mathbb{R})$ denotes the space of continuous functions on \mathbb{R} with period T_0 . The differential equation

$$\left(\frac{d}{dt}x\right)(t) = A(t)x(t), \quad t \in \mathbb{R}, \quad (11)$$

is assumed to have a dichotomy P_Σ . This means that there exists a unique projection P_Σ in \mathbb{C}^n and constants $M \geq 0$ and $0 < a < 1$ such that

$$\|\gamma_A(t, s)\| \leq Ma^{|t-s|}, \quad t, s \in \mathbb{R},$$

where

$$\gamma_A(t, s) = \begin{cases} U(t)P_\Sigma U(s)^{-1} & \text{if } t \geq s, \\ -U(t)(I - P_\Sigma)U(s)^{-1} & \text{if } t < s, \end{cases}$$

and $U(t)$ is the evolution matrix (normalized to I at $t = 0$) associated with the differential equation (11). The input-output operator $T_\Sigma : L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})$ of the system Σ is given by

$$(T_\Sigma \varphi)(t) = D(t)\varphi(t) + \int_{-\infty}^{\infty} C(t)\gamma_A(t, s)B(s)\varphi(s)ds, \quad t \in \mathbb{R}.$$

The class of operators $T = T_\Sigma$, with Σ as above will be our \mathcal{N} .

It is easily verified that 1 of §2 is satisfied.

2. $\mathcal{N}_{1,l} = \{T \in \mathcal{N} \mid T = T_\Sigma \text{ with } \Sigma \text{ as in (10) and } \text{rank}(P_\Sigma) \leq l\}$. It is easy to check that $\mathcal{N}_{1,l}\mathcal{N}_{1,p} \subset \mathcal{N}_{1,l+p}$ (see equation (1.13) on page 441 of Kaashoek and Kos [6]) and with $\mathcal{N}_1 := \cup_{l \geq 0} \mathcal{N}_{1,l}$ we also have

$$\mathcal{N}_1 + \mathcal{N}_1 \subset \mathcal{N}_1.$$

This verifies 2 of §2.

3. First we check that ν matches with the Fredholm index function for invertible T_Σ 's. (For preliminaries concerning the Fredholm index, we refer the reader to §1.3 of [6]. For an invertible $m = T_\Sigma \in \mathcal{N}_+$, we have $\text{rank}(P_\Sigma) = 0$ and by the definition of ν , we also know that $\text{rank}(P_{\Sigma^{-1}}) = \nu(m)$. Now let $\tilde{T}_\Sigma : L_2(0, \infty) \rightarrow L_2(0, \infty)$ denote the compression of the input-output operator T_Σ to the half line, that is,

$$(\tilde{T}_\Sigma \varphi)(t) = (T_\Sigma \tilde{\varphi})(t), \quad t \geq 0, \quad \varphi \in L_2(0, \infty),$$

with $\tilde{\varphi}(t) = \varphi(t)$, $t \geq 0$ and $\tilde{\varphi}(t) = 0$, $t < 0$. Consequently,

$$\text{index}(\tilde{T}_\Sigma) = \text{rank}(P_{\Sigma^{-1}}) - \text{rank}(P_\Sigma) = \nu(m) - 0 = \nu(m).$$

Using the fact that

$$\text{index}(\tilde{T}_{\Sigma_1 \Sigma_2}) = \text{index}(\tilde{T}_{\Sigma_1}) + \text{index}(\tilde{T}_{\Sigma_2}),$$

we obtain

$$\nu(mn) = \nu(m) + \nu(n).$$

This verifies 3 of §2.

4. From Kalman's structure theorem (Theorem 5-6 in Kalman [7]), there exists a continuously differentiable and invertible coordinate transformation $S(\cdot)$ such that if $\bar{x}(t) = S(t)x(t)$, then the given input-output map T_Σ has a realization (which is continuous and periodic with period T_0) of the following type:

$$\begin{aligned}\bar{A}(t) &= \begin{bmatrix} \bar{A}_{AA}(t) & \bar{A}_{AB}(t) & \bar{A}_{AC}(t) & \bar{A}_{AD}(t) \\ 0 & \bar{A}_{BB}(t) & 0 & \bar{A}_{BD}(t) \\ 0 & 0 & \bar{A}_{CC}(t) & \bar{A}_{CD}(t) \\ 0 & 0 & 0 & \bar{A}_{DD}(t) \end{bmatrix} \\ \bar{B}(t) &= \begin{bmatrix} \bar{B}_A(t) \\ \bar{B}_B(t) \\ 0 \\ 0 \end{bmatrix} \\ \bar{C}(t) &= [0 \quad \bar{C}_B(t) \quad 0 \quad \bar{C}_D(t)].\end{aligned}$$

In other words the input-output maps of (10) and the system with the realization $(\bar{A}, \bar{B}, \bar{C})$ are the same. Furthermore, the quadruple $(\bar{A}_{BB}, \bar{B}_B, \bar{C}_B, D)$ is a *minimal realization* of T_Σ , that is, it is a completely controllable and observable realization (see [7]). Let us denote the system with the minimal and truncated realization $\bar{\Sigma}_B$. In Brunovsky [3], it is shown that a completely controllable system with n states and of period T_0 is controllable over the interval $[t, t + nT_0]$ for all t . This means that the controllability Gramian $W(t, t + nT_0) = U_{\bar{A}_{BB}}(t) \int_t^{t+nT_0} U_{\bar{A}_{BB}}(s)^{-1} \bar{B}_B(s) \bar{B}_B(s)^* U_{\bar{A}_{BB}}(s)^{-*} ds U_{\bar{A}_{BB}}(t)^*$ is invertible and periodic with period T_0 , and hence

$$0 < \epsilon_1 I \leq W(t, t + nT_0) \leq \epsilon_2 I < +\infty \quad (12)$$

for all t and some positive numbers ϵ_1, ϵ_2 (independent of t). Now (12) is a sufficient condition for (periodic) stabilizability (see for instance the proof of Theorem 14.7 in Rugh [9]). A dual analysis shows that the pair $(\bar{A}_{BB}, \bar{C}_B)$ is detectable as well. Hence the input-output operator T_Σ has a stabilizable and detectable realization. Finally from Theorem 4.1 of Ravi et al. [8], it follows that 4 of §2 holds.

Remark. It is at this point that we have made the choice of considering *periodic* systems as opposed to the more general class of time-varying systems considered in Kaashoek and Kos [6]. Indeed, a theory of coprime factorizations exists for time-varying systems starting from the assumption that the system possesses a bounded, stabilizable and detectable realization, see [8]. But there seems to be a lacuna in the knowledge about the realization theory of general input-output operators about the existence of bounded, stabilizable and detectable realizations. By restricting oneself to periodic systems, one obtains the desired coprime factorizations. This should not necessarily be considered as a severe restriction, since very often the time-varying systems that arise out of practical engineering applications are indeed periodic or may be approximated by periodic systems.

5. For $t \in (0, 1)$, we have

$$\begin{aligned}\nu(n+m) &= \nu\left(\frac{1}{t}m(tnm^{-1} + e)\right) \\ &= \nu\left(\frac{1}{t}m\right) + \nu(tnm^{-1} + e) \\ &= \nu(m) + \nu(tnm^{-1} + e).\end{aligned}$$

But for t small enough, we can ensure that the input-output operator corresponding to tnm^{-1} has norm < 1 . Hence from the stability of the index of Fredholm operators under "small" perturbations, it follows that

$$\nu(tnm^{-1} + e) = \nu(e) = 0.$$

Thus 5 of §2 holds.

Statement of the problem: Let k be a lower triangular integral operator in \mathcal{N} , that is, the system is stable: $k \in \mathcal{N}_+$. Let l be a nonnegative integer. Suppose that the following condition holds:

$$\sup_{\tau \in \mathbb{R}} \sigma_{l+1}(H_k(\tau)) < 1 < \inf_{\tau \in \mathbb{R}} \sigma_l(H_k(\tau)),$$

where $\sigma_r(\bullet)$ denotes the r th singular value of a bounded linear operator, and for $\tau \in \mathbb{R}$, $H_k(\tau) : L_2(0, \infty) \rightarrow L_2(0, \infty)$ denotes the generalized Hankel operator corresponding to (the kernel function κ of) the integral operator k , and is given by:

$$(H_k(\tau)\varphi)(t) = \int_0^\infty \kappa(t + \tau, \tau - s)\varphi(s)ds, \quad t \geq 0,$$

which is T_0 -periodic in τ . Then find $f \in \mathcal{N}$, which is a *strictly contractive extension of order l for k* , that is, such that the following conditions hold:

1. $f - k \in \mathcal{N}_{1,l}$,
2. $\|f\| < 1$.

Let k have the uniformly observable forward stable realization $\Sigma(k) = (A_k, B_k, C_k, 0)$. Define

$$G_k(t) = U_k(t)^{-*} \int_t^\infty U_k(s)^* C_k(s)^* C_k(s) U_k(s) ds U_k(t)^{-1}, \quad t \in \mathbb{R}, \quad (13)$$

$$Z_k(t) = G_k(t)^{-1} - U_k(t) \int_{-\infty}^t U_k(s)^{-1} B_k(s) B_k(s)^* U_k(s)^{-*} ds U_k(t)^*, \quad t \in \mathbb{R}. \quad (14)$$

Let

$$A_\Theta = \begin{bmatrix} A_k(t) & 0 \\ 0 & -A_k(t)^* \end{bmatrix}, \quad (15)$$

$$B_\Theta = \begin{bmatrix} I - G_k(t)^{-1} Z_k(t)^{-1} & G_k(t)^{-1} Z_k(t)^{-1} \\ Z_k(t)^{-1} & -Z_k(t)^{-1} \end{bmatrix} \begin{bmatrix} -G_k(t)^{-1} C_k(t)^* & 0 \\ 0 & B_k(t) \end{bmatrix} \quad (16)$$

$$C_\Theta = \begin{bmatrix} C_k(t) & 0 \\ 0 & B_k(t)^* \end{bmatrix} \quad (17)$$

$$D_\Theta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (18)$$

for $t \in \mathbb{R}$, and let Θ be the input-output operator of the system $\Sigma(\Theta)$, where

$$\Sigma(\Theta) \begin{cases} \left(\frac{d}{dt}x\right)(t) & = A_\Theta(t)x(t) + B_\Theta(t)u(t), \\ y(t) & = C_\Theta(t)x(t) + D_\Theta(t)u(t), \end{cases} \quad t \in \mathbb{R}. \quad (19)$$

S1. Part 1. Lemma 5.2 (page 454 of Kaashoek and Kos [6]) it follows that the entries of $\begin{bmatrix} e & -k \end{bmatrix} \Theta$ are upper triangular integral operators and so $\begin{bmatrix} e & -k \end{bmatrix} \Theta \subset \begin{bmatrix} \mathcal{N}_+ & \mathcal{N}_+ \end{bmatrix}$. Thus

$$\begin{bmatrix} e & -k \end{bmatrix} \Theta \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix} \subset \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix}. \quad (20)$$

Also, we have

$$\Theta_{21} = -B_k^* \left(\frac{d}{dt} + A_k^* \right)^{-1} Z_k^{-1} G_k^{-1} C_k^*,$$

$$\Theta_{22} = I - B_k^* \left(\frac{d}{dt} + A_k^* \right)^{-1} Z_k^{-1} B_k,$$

and since A_k has dichotomy I , it follows that $\Theta_{21}, \Theta_{22} \in \mathcal{N}_+$. hence

$$\begin{bmatrix} 0 & e \end{bmatrix} \Theta \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix} \subset \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix}. \quad (21)$$

Combining (20) and (21) we obtain

$$\begin{bmatrix} e & -k \\ 0 & e \end{bmatrix} \Theta \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix} \subset \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix}. \quad (22)$$

Part 2. To prove the reverse inclusion, we note that we have to prove

$$\Theta^{-1} \begin{bmatrix} e & k \\ 0 & e \end{bmatrix} \Theta \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix} \subset \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix}. \quad (23)$$

From Lemma 5.2 (page 454 of of Kaashoek and Kos [6]) it follows that the entries of $\begin{bmatrix} k^* & e \end{bmatrix} \Theta$ are lower triangular integral operators and so

$$\Theta^* \begin{bmatrix} k \\ -e \end{bmatrix} \in [\mathcal{N}_+ \quad \mathcal{N}_+]. \quad (24)$$

Also, we know that

$$\begin{aligned} \Theta_{11}^* &= I + C_k^* G_k^{-*} (I - Z_k^{-*} G_k^{-*}) \left(\frac{d}{dt} + A_k^* \right)^{-1} C_k^*, \\ \Theta_{12}^* &= -B_k^* Z_k^{-*} G_k^{-*} \left(\frac{d}{dt} + A_k^* \right)^{-1} C_k^*, \end{aligned}$$

and since A_k has dichotomy I , it follows that

$$\Theta^* \begin{bmatrix} e \\ 0 \end{bmatrix} \in [\mathcal{N}_+ \quad \mathcal{N}_+]. \quad (25)$$

Combining (24) and (25) we obtain

$$\Theta^*(-J) \begin{bmatrix} e & k \\ 0 & e \end{bmatrix} \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix} \subset \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix}. \quad (26)$$

Finally, using this and the fact that $\Theta^* J \Theta = J$, we have

$$\Theta^{-1} \begin{bmatrix} e & k \\ 0 & e \end{bmatrix} \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix} = (-J) \Theta^*(-J) \begin{bmatrix} e & k \\ 0 & e \end{bmatrix} \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix} \subset (-J) \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix} = \begin{bmatrix} \mathcal{N}_+ \\ \mathcal{N}_+ \end{bmatrix}.$$

This completes the proof that S1 holds.

S2. This is precisely the content of Lemma 5.3 on page 455 of Kaashoek and Kos [6].

S3. Here $p = q = e$. This is proved in Lemma 5.2 on page 454 of Kaashoek and Kos [6].

We remark that Θ is also invertible. Indeed, we have $\Theta^* J \Theta = J$ and $\Theta J \Theta^* = J$ (this follows from Theorem 2.1 of Ball et al. [1]), so that Θ has both a right inverse and a left inverse, and so it is invertible.

Hence the results of §4 apply and we have the following:

Theorem 5.2 *Consider the linear, periodic, finite-dimensional system given by (10), where $A \in C_{T_0}(\mathbb{R})^{n \times n}$, $B \in C_{T_0}(\mathbb{R})^{n \times 1}$, $C \in C_{T_0}(\mathbb{R})^{1 \times n}$, $D \in C_{T_0}(\mathbb{R})^{1 \times 1}$. Let $k \in \mathcal{N}_+$ denote the input-output operator of this system, and suppose that $\sup_{\tau \in \mathbb{R}} \sigma_{l+1}(H_k(\tau)) < 1 < \inf_{\tau \in \mathbb{R}} \sigma_l(H_k(\tau))$, where $H_k(\tau)$ denotes the generalized Hankel operator corresponding to the integral operator k . Let Θ be the input-output operator of the system given by (19).*

Then $f \in \mathcal{N}$ such that $f - k \in \mathcal{N}_{1,l}$ and $\|f\| < 1$ iff

$$f = (\Theta_{11}h + \Theta_{12})(\Theta_{21}h + \Theta_{22})^{-1},$$

for some $h \in \mathcal{N}_+$ such that $\|h\| < 1$.

Finally we remark that the problem of Hankel norm approximation for infinite-dimensional, periodic systems is open and we conjecture that it should be possible to solve it by using the result from §4.

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