REPRESENTATION OF EQUILIBRIUM SOLUTIONS TO THE TABLE PROBLEM FOR GROWING SANDPILES

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Representation of Equilibrium Solutions to the Table Problem for Growing Sandpiles

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Abstract

In the dynamical theory of granular matter the so-called table problem consists in studying the evolution of a heap of matter poured continuously onto a bounded domain $\Omega \subset \mathbb{R}^2$. The mathematical description of the table problem, at an equilibrium configuration, can be reduced to a boundary value problem for a system of partial differential equations. The analysis of such a system, also connected with other mathematical models such as the Monge-Kantorovich problem, is the object of this paper. Our main result is an integral representation formula for the solution, in terms of the boundary curvature and of the normal distance to the cut locus of Ω.

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1 Introduction

In recent years, the attention of many authors has been focussed on the system of partial differential equations

$$\begin{cases} -\mathrm{div}\,(vDu) = f & \text{in } \Omega \\ |Du| - 1 = 0 & \text{in } \{v > 0\} \end{cases} \tag{1}$$

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in a given domain $\Omega \subset \mathbb{R}^n$.

For instance, system (1) arises in the Monge-Kantorovich theory as necessary conditions to be satisfied by an optimal mass transfer plan, see [13], [2] and [16]. In a related framework, system (1) characterizes the limit, as $p \to \infty$, of the p-Laplace equation $-\mathrm{div}\,(|Du|^{p-2}Du) = f$, see [10], [20], [6] and [15]. Furthermore, the above system has been applied to an idealized model for compression molding in [5], and to shape optimization in [8].

Another interesting example of application of (1) stems from granular matter theory, see [4], [23], [6] and [15]. Recently, Hadeler and Kuttler [18] proposed a new model to study the evolution of a sandpile created by pouring dry matter onto a 'table'. In such a model, built on previous work by Boutreux and de Gennes [9], the table is represented by a bounded domain $\Omega \subset \mathbb{R}^2$, and the matter source by a function $f(t,x) \geq 0$. The physical description of the growing heap is based on the introduction of the so-called *standing* and *rolling layers*. The former collects the amount of matter that remains at rest, the latter represents matter moving down along the surface of the standing layer—eventually falling down when the base of the heap touches the boundary of the table.

As pointed out in [18], system (1) is related to equilibrium configurations that may occur in physical models with constant source. To explain this connection, let us denote by $u(x)$ and $v(x)$, respectively, the heights of the standing and rolling layers at a point $x \in \Omega$, for an equilibrium configuration. For physical reasons, the slope of the standing layer cannot exceed a given constant—typical of the matter under consideration—that we normalize to 1. Consequently, the standing layer must vanish on the boundary of the table. So, $|Du| \leq 1$ in Ω and $u = 0$ on $\partial\Omega$. Also, in the region where v is positive, the standing layer has to be 'maximal', for otherwise more matter would roll down there to rest. On the other hand, the rolling layer results from transporting matter, poured by the source, along the surface of the standing layer at a speed that is assumed proportional to the slope Du, with constant equal to 1. The above considerations lead to the boundary value problem

$$\begin{cases} -\mathrm{div}\,(vDu) = f & \text{in } \Omega \\ |Du| - 1 = 0 & \text{in } \{v > 0\} \\ |Du| \leq 1 \quad u, v \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

An interesting representation formula for the solution of (2), in 1 space dimension, was presented in [18]. The main purpose of the present work is to obtain a similar formula in two space dimensions.

To describe our results more precisely, let us denote by $d : \overline{\Omega} \to \mathbb{R}$ the distance function from the boundary of Ω and by Σ the singular set of d, that is, the set of points $x \in \Omega$ at which d is not differentiable. Such a set is also called the cut locus of Ω. Introducing the projection $\Pi(x)$ of x onto $\partial\Omega$ in the usual way, Σ is also the set of points x at which $\Pi(x)$ is not a singleton. Since d is Lipschitz continuous, Σ has Lebesgue measure zero. In our analysis, a major role will be played by the normal distance to $\overline{\Sigma}$, that is the function

$$\tau(x) = \min \left\{ t \geq 0 \ : \ x + tDd(x) \in \overline{\Sigma} \right\} \qquad \forall x \in \overline{\Omega} \backslash \overline{\Sigma} \,.$$

It is well-known that the eikonal equation $|Du| = 1$ does not possess global smooth solutions in general, neither does the conservation law $-\mathrm{div}\,(vDu) = f$. Thus, before stating our main result, we must explain what we mean by a solution of (2). We say that a pair (u, v) of *continuous* functions in Ω is a solution of problem (2) if

- $u = 0$ on $\partial\Omega$, $\|Du\|_{\infty,\Omega} \leq 1$, and u is a viscosity solution of

$$|Du| = 1 \quad \text{in} \quad \{x \in \Omega \ : \ v(x) > 0\}$$

- $v \geq 0$ in Ω and, for every test function $\phi \in \mathcal{C}_c^\infty(\Omega)$,

$$\int_\Omega v(x)\langle Du(x), D\phi(x)\rangle dx = \int_\Omega f(x)\phi(x) dx \,.$$

Notice that the maximality of the standing layer, justified above by physical considerations, is now ensured by typical properties of viscosity solutions.

Our main result is the following.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary of class $\mathcal{C}^2$ and $f \geq 0$ be a continuous function in Ω. Then, a solution of system* (2) *is given by the pair (u, v), where $u = d$ in Ω, $v = 0$ on $\overline{\Sigma}$ and*

$$v(x) = \int_0^{\tau(x)} f(x + tDd(x)) \, \frac{1 - (d(x) + t)\kappa(x)}{1 - d(x)\kappa(x)} \, dt \qquad \forall x \in \Omega \backslash \overline{\Sigma} \,. \tag{3}$$

Here, $\kappa(x)$ denotes the curvature of $\partial\Omega$ at the point $\Pi(x)$.

Moreover, the above solution is unique in the following sense: if (u', v') is another solution of (2)*, then $v' = v$ in Ω and $u' = d$ in $\{x \in \Omega : v > 0\}$.*

In the proof of the above theorem, we will first show that the pair (d, v) is indeed a solution of the boundary value problem. This will also provide an existence result for (2). Incidentally, we note that an existence result for problem (2) is obtained in [8] in any space dimension (see also [23], [13] and [2] for existence results concerning system (1)).

Then, we will show that the solution of (2) is unique. We recall that uniqueness results for system (1), with Neuman boundary conditions for u, are known in the literature (see [2] and [16]). However, the boundary value problem (2) has never been addressed explicitly.

As it has already been pointed out, the main novelty of the present paper is, in our opinion, the representation formula (3) for the solution of problem (2). In fact, we are not aware of any similar representation formula for solutions of the Monge-Kantorovich system (1). Another noteworthy aspect of our result is that we do provide a continuous solution, instead of just a measure or a function in $L^1(\Omega)$ as is generally expected for v. So, Theorem 1.1 could also be viewed as a regularity result.

The main technical tools we use in this paper, are the results of [3] and [1] that describe the propagation of singularities of semi-concave functions. We also need a Lipschitz regularity result for the normal distance to $\overline{\Sigma}$ proved in [19] (see also [21]). To help the reader, we have provided a simple proof of such a result for the 2-dimensional case in the Appendix.

Except for the Appendix that we have just summarized, the rest of the paper is devoted to the proof of Theorem 1.1: preliminary properties of the distance function are collected in section 2, the proof that the pair (d, v) is a solution of (2) is given in section 3, uniqueness is shown in section 4.

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2 Preliminaries

In this section we collect our notations and some properties of the distance function.

For any $s \in \mathbb{R}$, we set $[s]_+ = \max\{s, 0\}$ and $[s]_- = \max\{-s, 0\}$. We denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the Euclidean scalar product and norm in $\mathbb{R}^2$, respectively. For any $x \in \mathbb{R}^2$ and $r > 0$, $B_r(x)$ stands for the open ball with center x and radius r. For any pair $x, y \in \mathbb{R}^2$ we denote by $]x, y[$ and $[x, y]$, respectively, the open and closed line segment of extreme points x and y.

For any given set $K \subset \mathbb{R}^2$ we define $\operatorname{diam} K = \sup\{|y - x| : x, y \in K\}$.

If $K \subset \mathbb{R}^2$ is closed, we set, for any point $x \in \mathbb{R}^2$,

$$d_K(x) = \min_{y \in K} |y - x| \quad \text{and} \quad \Pi_K(x) = \{y \in K \ : \ d_K(x) = |y - x|\} \ .$$

For any measurable set $A \subset \mathbb{R}^2$, we denote by $|A|$ the Lebesgue measure of A. If $u : A \to \mathbb{R}$ is a bounded measurable function, then $\|u\|_{\infty,A}$ stands for the essential supremum of u in A. If A is open and u is Lipschitz continuous, then, by Rademacher's Theorem, u is differentiable a. e. in A. In this case, we denote by $\|Du\|_{\infty,A}$ the number $\sup\{|Du(x)| \ : \ x \in A\,,\ \exists Du(x)\}$, and by $D^*u(x)$ the set of limiting gradients of u at x defined as

$$D^*u(x) = \{\lim_n Du(x_n) : \ A \ni x_n \to x,\ \exists Du(x_n)\} \ .$$

As usual, the superdifferential of u at a point $x \in A$ is the set

$$D^+u(x) = \left\{ p \in \mathbb{R}^n \mid \limsup_{h \to 0} \frac{u(x+h) - u(x) - \langle p, h\rangle}{|h|} \leq 0 \right\} ,$$

while the subdifferential D^-u is given by the formula $D^-u(x) = -D^+(-u)(x)$.

Definition 2.1 *We say that u is a viscosity solution of the eikonal equation $|Du| = 1$ in an open set $\Omega \subset \mathbb{R}^2$ if, for any $x \in \Omega \subset \mathbb{R}^2$, we have*

$$\begin{array}{lcl} p \in D^-u(x) & \Rightarrow & |p| \geq 1 \\ p \in D^+u(x) & \Rightarrow & |p| \leq 1 \end{array}$$

The results we give below are standard for the eikonal equation. For their proof, we refer the reader to [7], and precisely to Theorem 5.9 and Remark 5.10 in Chapter II, as well as to Proposition 3.12 in Chapter IV .

Proposition 2.2 *Let u be a Lipschitz continuous viscosity solution of the eikonal equation $|Du| = 1$ in Ω. Then, the following statements hold true.*

(a) *For every $x \in \Omega$*

$$u(x) = \min_{y \in \partial\Omega} \Big\{ \ u(y) + |y - x| \ \Big\} = \min_{y \in \partial\Omega,\ ]x,y[\subset \Omega} \Big\{ \ u(y) + |y - x| \ \Big\} .$$

(b) *Let $x \in \Omega$ and let $y \in \partial\Omega$ be such that $u(x) = u(y) + |x - y|$. Then, for any $z \in]x, y[$, u is differentiable at z and $Du(z) = (x - y)/|x - y|$.*

(c) *Let u be differentiable at a point $x \in \Omega$ and set*

$$\bar{t} = \inf\{t > 0 \ : \ x - tDu(x) \notin \Omega\} \, .$$

Then, $y := x - \bar{t}Du(x) \in \partial\Omega$ and $u(x) = u(y) + |x - y|$.

Remark 2.3 The representation formula in (a) implies, as is well-known, that u is locally semi-concave in Ω, i.e., for any convex set $\Omega' \subset\subset \Omega$ there is a constant $C \in \mathbb{R}$ such that $x \mapsto u(x) - C|x|^2$ is concave in Ω'. Consequently,

$$D^+u(x) = \operatorname{co} D^*u(x) \qquad \forall x \in \Omega \tag{4}$$

and the set-valued map $x \mapsto D^+u(x)$ is upper semicontinuous in Ω, that is, for every $x \in \Omega$,

$$\Omega \ni x_n \to x \, , \quad D^+u(x_n) \ni p_n \to p \quad (n \to \infty) \quad \Rightarrow \quad p \in D^+u(x) \, .$$

Moreover, Du is a vector-valued function of locally bounded variation in Ω, see, e. g., [14, p. 240]. Thus, Du is also approximately differentiable a.e. in Ω (see [14, p. 233]), that is, for a. e. $x \in \Omega$ there exists a linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that, for each $\varepsilon > 0$,

$$\lim_{r \downarrow 0} \frac{1}{r^2} \left| B_r(x) \cap \left\{ y \in \Omega : \frac{|Du(y) - Du(x) - L(y - x)|}{|y - x|} > \varepsilon \right\} \right| = 0 \, .$$

Throughout the paper we assume that

$$\Omega \text{ is a connected bounded open subset of } \mathbb{R}^2 \text{ with } \mathcal{C}^2 \text{ boundary.} \tag{5}$$

For simplicity, we will abbreviate d for $d_{\partial\Omega}$ and Π for $\Pi_{\partial\Omega}$. Whenever x has a unique projection onto $\partial\Omega$, with a minor abuse of notation, we will identify the set $\Pi(x)$ with its unique element.

Remark 2.4 We recall that the distance function d is the unique viscosity solution of the eikonal equation $|Du| = 1$ in Ω, with boundary condition $u = 0$ in $\partial\Omega$. Equivalently, d is the largest function such that $\|Du\|_{\infty,\Omega} \leq 1$ and $u = 0$ on $\partial\Omega$.

We will denote by Σ the singular set of the distance from $\partial\Omega$ or—briefly, the singular set of Ω—that is, the set of all points $x \in \Omega$ at which d fails to be differentiable. Equivalently, Σ is also the set of points x at which $\Pi(x)$ is not a singleton. Since d is Lipschitz continuous, Σ has Lebesgue measure 0.

A standard—yet important—consequence of assumption (5) is that d is $\mathcal{C}^2$ in $\overline{\Omega} \backslash \overline{\Sigma}$. The result we recall below can be proved using the classical method of characteristics.

Proposition 2.5 *Let $x \in \overline{\Omega}\backslash\overline{\Sigma}$ and let $t > 0$ be such that $x + sDd(x) \notin \overline{\Sigma}$ for every $s \in [0, t)$. Then, for every $s \in [0, t)$,*

(a) $\qquad d(x + sDd(x)) = d(x) + s$

(b) $\qquad Dd(x + sDd(x)) = Dd(x)$

(c) $\qquad \Pi(x + sDd(x)) = \Pi(x)$

In view of points (a) or (c) in the above proposition, $x + sDd(x) \in \overline{\Sigma}$ for some $s > 0$. So, the map $\tau : \overline{\Omega} \to [0, \operatorname{diam}\Omega/2]$

$$\tau(x) = \begin{cases} \min\left\{t \geq 0 \ : \ x + tDd(x) \in \overline{\Sigma}\right\} & \forall x \in \overline{\Omega}\backslash\overline{\Sigma} \\ 0 & \forall x \in \overline{\Sigma} \end{cases} \tag{6}$$

is well defined. Such a map measures the distance of a point x to the set $\overline{\Sigma}$ along the direction of $Dd(x)$ (which differs from the distance of x to $\overline{\Sigma}$, in general). In this paper, it will be called the *normal distance to* $\overline{\Sigma}$. In the literature, τ is often referred to as the distance to the cut locus of Ω.

Hereafter, for any $x \in \partial\Omega$, we denote by $\kappa(x)$ the curvature of $\partial\Omega$ at x, with the sign convention that $\kappa \geq 0$ if Ω is convex. Also, we will label in the same way the extension of κ to $\overline{\Omega} \setminus \Sigma$ given by

$$\kappa(x) = \kappa(\Pi(x)) \qquad \forall x \in \overline{\Omega} \setminus \Sigma\,. \tag{7}$$

In the result below, $p \otimes q$ stands for the tensor product of two vectors $p, q \in \mathbb{R}^2$, defined as $(p \otimes q)(x) = p\,\langle q, x\rangle\,,\ \forall x \in \mathbb{R}^2$.

Proposition 2.6 *For any $x \in \overline{\Omega}$ and any $y \in \Pi(x)$ we have $\kappa(y)d(x) \leq 1$. If, in addition, $x \in \overline{\Omega}\backslash\overline{\Sigma}$, then*

$$\kappa(x)d(x) < 1 \qquad \textit{and} \qquad D^2d(x) = -\frac{\kappa(x)}{1 - \kappa(x)d(x)}q \otimes q$$

where q is any unit vector such that $\langle q, Dd(x)\rangle = 0$.

Remark 2.7 Owing to assumption (5), we have

$$\sup_{y,z\in\partial\Omega\,,\ y\neq z} \frac{|Dd(z) - Dd(y)|}{|z - y|} < \infty\,. \tag{8}$$

So, in view of Proposition 2.6, we will denote the above supremum by $\operatorname{Lip}(\kappa)$.

Proof—Let $x \in \Omega$ and $y \in \Pi(x)$. Then the disk of center x and radius $d(x)$ is tangent to $\partial\Omega$ at y. Therefore, we have either $\kappa(y) \leq 0$ or $1/\kappa(y) \geq d(x)$. So, $\kappa(y)d(x) \leq 1$.

If we assume, next, that $x \notin \overline{\Sigma}$, then y belongs to the projection of $x + sDd(x)$ for $s > 0$ sufficiently small. Thus $\kappa(y)d(x + sDd(x)) \leq 1$. Since $d(x + sDd(x)) = d(x) + s$ and $\kappa(x) = \kappa(y)$ by definition, we have $\kappa(x)d(x) < 1$. For the last assertion see [17, Lemma 14.17]. □

We will need a more detailed description of the singular set Σ. Let us recall that $\Sigma = \Sigma^1 \cup \Sigma^2$, where Σ^i $(i = 1, 2)$ is the set of points $x \in \Sigma$ with *magnitude* i, that is, such that the dimension of (the convex set) $D^+d(x)$ is equal to i. Let us also define the set of (regular) *conjugate points* Γ as

$$\Gamma = \{x \in \Omega \backslash \Sigma \ : \ d(x)\kappa(x) = 1\} \ .$$

Notice that a point $x \in \Omega\backslash\Sigma$ belongs to Γ if and only if

$$\Pi(x) = \left\{x - \frac{1}{\kappa(x)} Dd(x)\right\} .$$

Proposition 2.8 *Under assumption* (5)*, we have* $\overline{\Sigma} \subset \Omega$ *and* $\overline{\Sigma} = \Sigma \cup \Gamma$.

Proof— Let $x \in \Sigma$ and y, z be two distinct elements of $\Pi(x)$. Then

$$x = y + d(x)Dd(y) = z + d(x)Dd(z) \ . \tag{9}$$

Therefore, recalling Remark 2.7,

$$|y - z| = d(x)|Dd(y) - Dd(z)| \leq d(x)K|y - z|$$

for some constant $K > 0$ independent of x. We have thus proved that $d(x) \geq 1/K$ for every $x \in \Sigma$. So, $\overline{\Sigma} \subset \Omega$. Furthermore, the inclusion $\Gamma \subset \overline{\Sigma}$ is a straightforward consequence of the strict inequality in Proposition 2.6.

In order to prove the inclusion $\overline{\Sigma} \subset \Sigma \cup \Gamma$, let $\{x_n\}$ be a sequence of singular points converging to a point $x \in \Omega\backslash\Sigma$. We claim that $d(x)\kappa(x) = 1$. To see this, let y_n and z_n be two distinct points in $\Pi(x_n)$. Then, both $\{y_n\}$ and $\{z_n\}$ must converge to $\Pi(x)$ as $n \to \infty$. Also, passing to a subsequence,

$$\lim_{n\to\infty} \frac{y_n - z_n}{|y_n - z_n|} = \theta$$

for some unit vector $\theta \in \mathbb{R}^2$. From identity (9) applied to x_n, y_n and z_n, we have

$$0 = \frac{y_n - z_n}{|y_n - z_n|} + d(x_n)\frac{Dd(y_n) - Dd(z_n)}{|y_n - z_n|} \ .$$

Hence, taking the limit as $n \to \infty$ we conclude that $0 = \theta + d(x)D^2 d(\Pi(x))\theta$. Therefore, $-1/d(x)$ is a nonzero eigenvalue of $D^2 d(\Pi(x))$, a matrix of the form $-\kappa(x) q \otimes q$ by Proposition 2.6. So, $-1/d(x) = -\kappa(x)$, as claimed. $\square$

The following result ensures that segments of minimal length joining a point to $\partial\Omega$, contain no singular or conjugate points in their interior.

Proposition 2.9 *Let $x \in \Omega$ and $y \in \Pi(x)$. Then $\overline{\Sigma} \cap]y, x[= \emptyset$.*

Proof—We already know that $\Sigma \cap]y, x[= \emptyset$ by Proposition 2.2(b), and that $\kappa(y)d(x) \leq 1$ by Proposition 2.6. Since $\kappa(y)d(z) < 1$ for every $z \in]y, x[$, we conclude that $z \notin \Gamma$. $\square$

The following proposition, that will be crucial to our analysis, is an adaptation of some of the results of [1] describing propagation of singularities for semiconcave functions.

Proposition 2.10 *Let $x_0 \in \Sigma$, and let p_0, q_0 be two distinct limiting gradients at x_0 such that the segment $[p_0, q_0]$ is a face of $D^+ d(x_0)$. Let n_0 be a nonzero vector satisfying*

$$\langle p, n_0 \rangle \leq \langle p_0, n_0 \rangle = \langle q_0, n_0 \rangle \qquad \forall p \in D^+ d(x_0) \, .$$

Then, there exist a number $\eta > 0$ and a Lipschitz arc $\boldsymbol{x} : [0, \eta] \to \Omega$ such that

$$\boldsymbol{x}(0) = x_0 \, , \qquad \dot{\boldsymbol{x}}(0) = -n_0 \, , \qquad \boldsymbol{x}(s) \in \Sigma \quad \forall s \in [0, \eta] \, . \tag{10}$$

Moreover, $\boldsymbol{x}(s_n) \in \Sigma^1$ for some sequence $s_n \downarrow 0$, and

$$D^+ d(\boldsymbol{x}(s_n)) = [p_n, q_n] \qquad \forall n \geq 0 \tag{11}$$

where $p_n \to p_0$ and $q_n \to q_0$ as $n \to \infty$.

Proof—The existence of a singular arc $\boldsymbol{x}$ satisfying (10) follows from Lemma 4.5 and Theorem 4.2 of [1], where a bound of the form $\operatorname{diam} D^+ d(\boldsymbol{x}(s)) \geq \delta$ is also deduced for some $\delta > 0$ and every $s \in [0, \eta]$.

To prove the last part of the conclusion, we note that, for any $\epsilon > 0$, $\mathcal{H}^1(\boldsymbol{x}([0, \epsilon])) > 0$ because $\dot{\boldsymbol{x}}(0) \neq 0$ and $\boldsymbol{x}$ is Lipschitz continuous. Since Σ^2 is at most countable (see, e.g., [12]), we conclude that $\mathcal{H}^1(\boldsymbol{x}([0, \epsilon]) \cap \Sigma^1) > 0$ for any $\epsilon > 0$. Consequently, there exists a sequence $s_n \downarrow 0$ such that $\boldsymbol{x}(s_n) \in \Sigma^1$ for every $n \in \mathbb{N}$. Let us set $D^+ d(\boldsymbol{x}(s_n)) = [p_n, q_n]$, choosing p_n so that $\langle p_n, p_0 \rangle \geq \langle q_n, p_0 \rangle$. Notice that, in particular, $|p_n - q_n| \geq \delta$. Now, let us consider converging subsequences of $\{p_n\}$ and $\{q_n\}$ (labelled like the original sequences) and denote by p^* and q^*, respectively, their limits. Applying [3, Theorem 2.1], we deduce that

$$p^*, q^* \in \arg\max_{p\in D^+d(x_0)} \langle p, n_0\rangle = [p_0, q_0]\,.$$

Since p^* and q^* belong to $D^*d(x_0)$, we conclude that $p^*, q^* \in \{p_0, q_0\}$. Moreover, $|p^* - q^*| \geq \delta$ and $\langle p^*, p_0\rangle \geq \langle q^*, p_0\rangle$. This forces $(p, q) = (p_0, q_0)$. □

Remark 2.11 Elementary geometric arguments show that $D^+d(x_0)$ possesses a 1-dimensional exposed face—and so, owing to Proposition 2.10, x_0 is the inital point of a nonconstant Lipschitz singular arc—if and only if $D^+d(x_0)$ fails to cover the closed unit ball $\overline{B}_1$. On the other hand, in view of (4), we have that $D^+d(x_0) = \overline{B}_1$ if and only if $\partial D^+d(x_0) = D^*d(x_0)$. By [1, Theorem 6.2], the last identity is necessary and sufficient for Σ to be a singleton or, equivalently, for Ω to coincide with $B_R(x_0)$, where $R = d(x_0)$. In fact, the equivalence between Σ being a singleton and the identity $\Omega = B_R(x_0)$ follows from a classical result of Motzkin's [22].

In conclusion, either $\Omega = B_R(x_0)$ or every singularity propagates along Lipschitz arcs. Moreover, by the last part of Proposition 2.10, Σ^1 is dense in Σ in the latter case.

We conclude this section with a regularity result for the normal distance τ.

Theorem 2.12 *Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary of class $\mathcal{C}^{2,1}$. Then the map τ defined in* (6) *is Lipschitz continuous on $\partial\Omega$.*

The first author became aware of the above property from [21]. A proof of this result for $\mathcal{C}^\infty$ smooth submanifolds of an n-dimensional smooth manifold is given in [19]. In the Appendix, we provide an independent proof of Theorem 2.12, based on Proposition 2.10.

Hereafter, we will denote by $\mathrm{Lip}(\tau)$ the Lipschitz seminorm of τ on $\partial\Omega$. Since $x \mapsto x + \tau(x)Dd(x)$ maps $\partial\Omega$ onto $\overline{\Sigma}$, a straightforward application of Theorem 2.12 is that the 1-dimensional Hausdorff measure of $\overline{\Sigma}$ is finite:

Corollary 2.13 *Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary of class $\mathcal{C}^{2,1}$. Then,*

$$\mathcal{H}^1(\overline{\Sigma}) \leq k_\Omega\, \mathcal{H}^1(\partial\Omega) < \infty$$

where $k_\Omega \geq 0$ is a constant depending on $\mathrm{Lip}(\tau)$ *and on the seminorm* $\mathrm{Lip}(\kappa)$ *defined in Remark 2.7.*

For less regular domains the Lipschitz continuity of τ may fail, but continuity is preserved as we show below.

Lemma 2.14 *Assume* (5)*. Then the map* τ*, extended to* 0 *on* $\overline{\Sigma}$*, is continuous in* $\overline{\Omega}$*.*

Proof—We only need to show that τ is upper semicontinuous in $\overline{\Omega}$, lower semicontinuity being a direct consequence of $\overline{\Sigma}$ is closed. For this purpose, consider a sequence $\{x_n\}$ in $\overline{\Omega} \setminus \overline{\Sigma}$, converging to some point $x \in \overline{\Omega}$, and suppose by contradiction $t^* := \lim_n \tau(x_n) > \tau(x)$. In particular, this implies that t^* is positive. We can also assume, without loss of generality, that $\{Dd(x_n)\}$ converges, say to p. Let $\bar{t} \in (\tau(x), t^*)$. Then, d is differentiable at $x_n + \bar{t}Dd(x_n)$ by definition. Thus, for n large enough,

$$\Pi(x_n + \bar{t}Dd(x_n)) = \Pi(x_n) = \{x_n - d(x_n)Dd(x_n)\}\ .$$

Taking the limit as $n \to \infty$, we obtain $x - d(x)p \in \Pi(x + \bar{t}p)$. So, $x + \tau(x)p$ belongs to the interior of the segment $[x - d(x)p, x + \bar{t}p]$. Since $x + \tau(x)p \in \overline{\Sigma}$, this contradicts Proposition 2.9. □

We conclude this section with an approximation result. Roughly speaking, we need to make sure that both the singular set and the normal distance are stable for convergence in the $\mathcal{C}^2$ topology. We begin by defining the signed distance from $\partial\Omega$ as

$$\boldsymbol{d}_\Omega(x) = \begin{cases} d_{\partial\Omega}(x) & \text{if} \quad x \in \overline{\Omega} \\ -d_{\partial\Omega}(x) & \text{if} \quad x \in \mathbb{R}^2 \setminus \overline{\Omega}\,. \end{cases}$$

We say that a sequence of sets $\{\Omega_n\}$, satisfying (5), converges to Ω in the $\mathcal{C}^2$ topology if the boundary of the Ω_n converge to the boundary of Ω for the Hausdorff distance and if $\boldsymbol{d}_{\Omega_n}$, $D\boldsymbol{d}_{\Omega_n}$ and $D^2\boldsymbol{d}_{\Omega_n}$ converge to $\boldsymbol{d}_\Omega$, $D\boldsymbol{d}_\Omega$ and $D^2\boldsymbol{d}_\Omega$, uniformly in a neighbourhood of $\partial\Omega$.

Proposition 2.15 *Let* $\{\Omega_n\}$ *be a sequence of sets satisfying* (5)*. For any* $n \in \mathbb{N}$*, denote by* Σ_n *and* τ_n*, respectively, the singular set of* Ω_n *and the normal distance to* $\overline{\Sigma}_n$*. If* $\{\Omega_n\}$ *converges to* Ω *in the* $\mathcal{C}^2$ *topology, then* $\{\overline{\Sigma}_n\}$ *converges to* $\overline{\Sigma}$ *in the Hausdorff topology, and* $\{\tau_n\}$ *converges to* τ *uniformly on all compact subsets of* Ω*.*

Proof—Let us prove, first, that the upper limit of $\{\overline{\Sigma}_n\}$ is contained in $\overline{\Sigma}$. For this it suffices to show that, if a sequence $\{x_n\}$ in Σ_n converges to a point $x \in \overline{\Omega}$, then x belongs to $\overline{\Sigma}$. Indeed, let y_n and z_n be two distinct projections of x_n onto $\partial\Omega_n$. Without loss of generality we can assume that both $\{y_n\}$ and $\{z_n\}$ converge to points of $\Pi(x)$, say y and z respectively. If

$y \neq z$, then x belongs to Σ and our claim follows. So, suppose $x \in \overline{\Omega} \setminus \Sigma$ and $y = z$. Since $y_n + \boldsymbol{d}_{\Omega_n}(x_n) D\boldsymbol{d}_{\Omega_n}(y_n) = z_n + \boldsymbol{d}_{\Omega_n}(x_n) D\boldsymbol{d}_{\Omega_n}(z_n)$, we have

$$\frac{y_n - z_n}{|y_n - z_n|} = -\boldsymbol{d}_{\Omega_n}(x_n) \frac{D\boldsymbol{d}_{\Omega_n}(y_n) - D\boldsymbol{d}_{\Omega_n}(z_n)}{|y_n - z_n|} \,. \tag{12}$$

The sequence in the left-hand side above will converge, up to replacement with a subsequence, to some unit vector $\theta \in \mathbb{R}^2$. Then, passing to the limit in (12) we obtain $\theta = -\boldsymbol{d}_\Omega(x) D^2 \boldsymbol{d}_\Omega(y)\theta$. Hence, recalling the structure of the hessian matrix $D^2 d(y)$, we conclude that $d(x)\kappa(x) = d(x)\kappa(y) = 1$. Therefore, x belongs to $\overline{\Sigma}$.

Now, let us prove that the lower limit of the sequence $\{\overline{\Sigma}_n\}$ contains $\overline{\Sigma}$. For this, it suffices to show that $\Sigma \subset \liminf \Sigma_n$. Let $x \in \Omega \setminus \liminf \Sigma_n$. Then, there exists a subsequence $\{\Sigma_{n_k}\}$ of $\{\Sigma_n\}$ such that, for some $\varepsilon > 0$, $B_\varepsilon(x) \subset \Omega \setminus \Sigma_{n_k}$. We claim that $B_{\varepsilon/2}(x) \cap \Sigma = \emptyset$. For let $z \in B_{\varepsilon/2}(x)$ and set $y_k = \Pi_{\partial\Omega_{n_k}}(z)$. Since

$$z + \varepsilon \frac{z - y_k}{2\boldsymbol{d}_{\Omega_{n_k}}(z)} \in B_{\varepsilon/2}(z) \subset B_\varepsilon(x) \subset \Omega \setminus \Sigma_{n_k} \,,$$

y_k is also the unique projection of $z + \varepsilon(z - y_k)/2\boldsymbol{d}_{\Omega_{n_k}}(z)$ onto $\partial\Omega_{n_k}$. Now, a subsequence of $\{y_k\}$ will converge to some point $y \in \partial\Omega$ belonging to both $\Pi(z)$ and $\Pi(z + \varepsilon(z - y)/2\boldsymbol{d}_\Omega(z))$. Therefore, $z \notin \overline{\Sigma}$ owing to Proposition 2.9, and our claim is proved as well as the convergence of $\overline{\Sigma}_n$ to $\overline{\Sigma}$.

We omit the proof that $\{\tau_n\}$ converges to τ, because the reasoning has much in common with the proof Lemma 2.14. □

3 Existence

In this section we prove that the pair (d, v), where

$$v(x) = \int_0^{\tau(x)} f(x + tDd(x)) \frac{1 - (d(x) + t)\kappa(x)}{1 - d(x)\kappa(x)} dt \qquad \forall x \in \Omega \setminus \overline{\Sigma} \tag{13}$$

and $v \equiv 0$ on $\overline{\Sigma}$, is a solution of system (2). We begin with two preliminary results, the former describing continuity and differentiability properties of v, the latter providing an approximatimation result for the characteristic function of a compact set, in the spirit of capacity theory.

Proposition 3.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary of class $\mathcal{C}^2$ and $f \geq 0$ be a continuous function in Ω. Then, v is a locally bounded* *continuous function in* Ω. *Moreover, in any set* $\Omega_\varepsilon := \{x \in \Omega \ : \ d(x) > \varepsilon\}$, $\varepsilon > 0$, v *satisfies the bound*

$$0 \leq v(x) \leq \|f\|_{\infty,\Omega_\varepsilon}\Big[1 + \|[\kappa]_-\|_{\mathcal{C}(\partial\Omega)}\mathrm{diam}\,\Omega\Big]\tau(x) \qquad \forall x \in \Omega_\varepsilon, \tag{14}$$

where $\|[\kappa]_-\|_{\mathcal{C}(\partial\Omega)} := \max_{x\in\partial\Omega}[\kappa(x)]_-$.

If, in addition, $\partial\Omega$ *is of class* $\mathcal{C}^{2,1}$ *and* f *is Lipschitz continuous in* Ω, *then* v *is locally Lipschitz continuous in* $\Omega\backslash\overline{\Sigma}$ *and satisfies*

$$-\mathrm{div}\,(v(x)Dd(x)) = f(x) \tag{15}$$

at each point $x \in \Omega\backslash\overline{\Sigma}$ *at which* v *is differentiable.*

Remark 3.2 Since d is $\mathcal{C}^2$ in $\Omega\backslash\overline{\Sigma}$, equality (15) reads

$$\langle Dv(x), Dd(x)\rangle + v(x)\Delta d(x) + f(x) = 0\,. \tag{16}$$

Moreover, a straightforward consequence of Proposition 3.1 is that the equality $-\mathrm{div}\,(vDd) = f$ holds in the sense of distributions in $\Omega\backslash\overline{\Sigma}$ as soon as f is Lipschitz and $\partial\Omega$ of class $\mathcal{C}^{2,1}$.

Proof—We note, first, that the maps Dd, τ and κ are continuous in $\Omega\backslash\overline{\Sigma}$ since Ω has a $\mathcal{C}^2$ boundary. Hence, when f is continuous, so is v in $\Omega\backslash\overline{\Sigma}$.

Let us now prove that v is continuous on $\overline{\Sigma}$. Observe that, for any $x \notin \overline{\Sigma}$, the term

$$\frac{1-(d(x)+t)\kappa(x)}{1-d(x)\kappa(x)} = \frac{1-d(x+tDd(x))\kappa(x)}{1-d(x)\kappa(x)} \qquad 0 < t < \tau(x)$$

is nonnegative by Proposition 2.6. A simple computation shows that it is also bounded by $1 + \|[\kappa]_-\|_{\mathcal{C}(\partial\Omega)}\tau(x)$. This proves (14) recalling that $x + tDd(x) \in \Omega_\varepsilon$ whenever $x \in \Omega_\varepsilon$ and $0 \leq t \leq \tau(x)$. The continuity of v on $\overline{\Sigma}$ an immediate consequence of (14).

Next, let $\partial\Omega$ be of class $\mathcal{C}^{2,1}$ and f be Lipschitz. Then, Theorem 2.12 ensures that τ is Lipschitz on $\partial\Omega$. Therefore, $\tau = \tau\circ\Pi - d$ is locally Lipschitz in $\overline{\Omega}\backslash\overline{\Sigma}$, as well as v.

Finally, let us check the validity of (15) at every differentiability point x for v in the open set $\Omega\backslash\overline{\Sigma}$. We note that, at any such point x,

$$\langle Dv(x), Dd(x)\rangle = \frac{d}{d\lambda}v(x+\lambda Dd(x))_{|\lambda=0}\,.$$

But $\tau(x+\lambda Dd(x)) = \tau(x) - \lambda$ and $d(x+\lambda Dd(x)) = d(x)+\lambda$ for $\lambda > 0$ sufficiently small. So,

$$\begin{aligned} v(x+\lambda Dd(x)) &= \int_0^{\tau(x)-\lambda} f\Big(x+(t+\lambda)Dd(x)\Big)\frac{1-(d(x)+\lambda+t)\kappa(x)}{1-(d(x)+\lambda)\kappa(x)}\,dt \\ &= \int_\lambda^{\tau(x)} f(x+tDd(x))\frac{1-(d(x)+t)\kappa(x)}{1-(d(x)+\lambda)\kappa(x)}\,dt\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle Dv(x), Dd(x)\rangle &= -f(x) + \int_0^{\tau(x)} f(x+tDd(x))\frac{1-(d(x)+t)\kappa(x)}{(1-d(x)\kappa(x))^2}\,\kappa(x)\,dt \\ &= -f(x) - v(x)\Delta d(x) \end{aligned}$$

where we have taken into account the identity

$$\Delta d(x) = -\frac{\kappa(x)}{1-d(x)\kappa(x)} \qquad \forall x \in \Omega\backslash\overline{\Sigma}\,,$$

that follows from Proposition 2.6. We have thus obtained (16)—an equivalent version of (15)—and completed the proof. □

Proposition 3.3 *Let K be a compact subset of $\mathbb{R}^2$ such that $\mathcal{H}^1(K) < \infty$. Then, there exists a sequence $\{\xi_k\}$ of functions in $W^{1,1}(\mathbb{R}^2)$ with compact support, such that*

(a) $0 \le \xi_k \le 1$ *for every* $k \in \mathbb{N}$*;*

(b) $\mathrm{dist}(\mathrm{spt}(\xi_k), K) \to 0$ *as* $k \to \infty$*;*

(c) $K \subset \mathrm{int}\{x \in \mathbb{R}^2 : \xi_k(x) \ge 1\}$ *for every* $k \in \mathbb{N}$*;*

(d) $\xi_k \to 0$ *in* $L^1(\mathbb{R}^2)$ *as* $k \to \infty$*;*

(e) $\int_{\mathbb{R}^2} |D\xi_k|dx \le C$ *for every* $k \in \mathbb{N}$ *and some constant* $C > 0$*.*

The standard notations dist, spt and int stand for distance (between two sets), support (of a function) and interior (of a set), respectively. We give a proof of the proposition for the reader's convenience.

Proof—Since $\mathcal{H}^1(K) < \infty$, for any fixed $k \in \mathbb{N}$ there exists a sequence of points $\{x_i^{(k)}\}_{i\in\mathbb{N}}$ in K and a sequence of radii $\{r_i^{(k)}\}_{i\in\mathbb{N}}$ such that

- $0 < r_i^{(k)} \le \frac{1}{k}$ and $\sum_i r_i^{(k)} \le C\left(\mathcal{H}^1(K) + \frac{1}{k}\right)$;

- $K \subset \mathrm{int}\Big(\bigcup_i B_{r_i^{(k)}}(x_i^{(k)})\Big)$.

for some constant $C > 0$. Now, define, for any $x \in \mathbb{R}^2$,

$$\xi_i^{(k)}(x) = \Big[1 - \frac{1}{r_i^{(k)}}\left(|x - x_i^{(k)}| - r_i^{(k)}\right)_+\Big]_+$$
$$\xi_k(x) = \sup_{i\in\mathbb{N}} \xi_i^{(k)}(x)$$

and observe that

$$\mathrm{spt}(\xi_i^{(k)}) = \overline{B}_{2r_i^{(k)}}(x_i^{(k)})$$
$$\mathrm{spt}(D\xi_i^{(k)}) = \overline{B}_{2r_i^{(k)}}(x_i^{(k)}) \backslash B_{r_i^{(k)}}(x_i^{(k)})$$

Then, $\xi_k \in L^1(\mathbb{R}^2)$ since $0 \le \xi_k \le 1$ and ξ_k has compact support. Moreover, an easy computation shows that $\int_{\mathbb{R}^2} |D\xi_i^{(k)}|dx = 3\pi r_i^{(k)}$. So, applying [13, Lemma 2, p.148], we have

$$\int_{\mathbb{R}^2} |D\xi_k|dx \le \sup_i \int_{\mathbb{R}^2} |D\xi_i^{(k)}|dx \le \sum_i \int_{\mathbb{R}^2} |D\xi_i^{(k)}|dx \le C\Big(\mathcal{H}^1(K) + \frac{1}{k}\Big)$$

for some constant $C > 0$. Therefore, $\xi_k \in W^{1,1}(\mathbb{R}^2)$ and (e) holds true. Properties (b) and (c) are true by construction. Finally, (d) follows by Lebesgue's Theorem because $0 \le \xi_k \le 1$ and $\xi_k(x) = 0$ for any point $x \notin K$ and k large enough. □

Proof of Theorem 1.1[Part 1: Existence]—We will prove that the pair (d, v), with v defined by (13), is a solution of system (2). Let us point out, to begin with, that d is a viscosity solution of the eikonal equation in Ω, and so, a fortiori, in the open set $\{x \in \Omega : v(x) > 0\}$. Therefore, what actually remains to be shown is that

$$\int_\Omega f\phi\, dx = \int_\Omega v\langle Dd, D\phi\rangle dx \qquad \forall \phi \in \mathcal{C}_c^\infty(\Omega)\,. \tag{17}$$

Let us assume, first, that f is Lipschitz and Ω has a $\mathcal{C}^{2,1}$ boundary. Since $\mathcal{H}^1(\overline{\Sigma}) < \infty$ by Proposition 2.13, we can apply Proposition 3.3 with $K = \overline{\Sigma}$ to construct a sequence $\{\xi_k\}$ enjoying properties (a), (b), (c) and (d). Let $\phi \in \mathcal{C}_c^\infty(\Omega)$ be a test function, and set $\phi_k = \phi(1 - \xi_k)$. Notice that, for k large enough, $\mathrm{spt}(\phi_k) \subset\subset \Omega\backslash\overline{\Sigma}$. This follows from (a), (b) and from the fact that $\overline{\Sigma} \subset \Omega$ (see Proposition 2.8). Then, Proposition 3.1 and Rademacher's

Theorem imply that $-\mathrm{div}\,(vDd) = f$ a. e. in $\Omega\backslash\overline{\Sigma}$. So, multiplying this equation by ϕ_k and integrating by parts, we obtain

$$\int_\Omega f\phi_k dx = \int_\Omega v(1-\xi_k)\langle Dd, D\phi\rangle dx - \int_\Omega v\phi\langle Dd, D\xi_k\rangle dx\,. \qquad (18)$$

We claim that the rightmost term above goes to 0 as $k\to\infty$. Indeed,

$$\begin{aligned}\Big|\int_\Omega v\phi\langle Dd, D\xi_k\rangle dx\Big| &\le \|\phi\|_{\infty,\Omega}\,\|v\|_{\infty,\mathrm{spt}(\xi_k)}\int_\Omega |D\xi_k|dx\\ &\le C\,\|\phi\|_{\infty,\Omega}\,\|v\|_{\infty,\mathrm{spt}(\xi_k)}\end{aligned}$$

where C is the constant provided by Proposition 3.3 (d). Now, using property (a) of the proposition and the fact that v is a continuous function vanishing on $\overline{\Sigma}$, we conclude that $\|v\|_{\infty,\mathrm{spt}(\xi_k)}\to 0$ as $k\to\infty$. This proves our claim. The conclusion (17) immediately follows since, in view of (a) and (c), the integrals $\int_\Omega f\phi_k dx$ and $\int_\Omega v(1-\xi_k)\langle Dd, D\phi\rangle dx$ converge to $\int_\Omega f\phi dx$ and $\int_\Omega v\langle Dd, D\phi\rangle dx$—respectively—as $k\to\infty$.

Finally, the extra assumptions that $\partial\Omega$ be of class $\mathcal{C}^{2,1}$ and f be Lipschitz in Ω, can be easily removed by an approximation argument based on the lemma below. □

Let $\{\Omega_n\}$ be a sequence of open domains, with $\mathcal{C}^{2,1}$ boundary, converging to Ω in the $\mathcal{C}^2$ topology, and let $\{f_n\}$ be a sequence of Lipschitz functions in Ω_n converging to f, uniformly on all compact subsets of Ω. Denote by Σ_n and τ_n, respectively, the singular set of Ω_n and the normal distance to $\overline{\Sigma}_n$. Define $v_n(x) = 0$ for every $x\in\overline{\Sigma}_n$ and

$$v_n(x) = \int_0^{\tau_n(x)} f_n(x + tD\boldsymbol{d}_{\Omega_n}(x))\frac{1-(\boldsymbol{d}_{\Omega_n}(x)+t)\kappa_n(x)}{1-\boldsymbol{d}_{\Omega_n}(x)\kappa_n(x)}dt \quad \forall x\in\Omega_n\setminus\overline{\Sigma}_n\,,$$

where $\kappa_n(x)$ stands for the curvature of $\partial\Omega_n$ at the projection of x.

Lemma 3.4 *$\{v_n\}$ converges to v in $L^1_{loc}(\Omega)$.*

Proof—Since, owing to (14), the sequence $\{v_n\}$ is locally uniformly bounded in Ω, it suffices to prove that it converges uniformly to v on every compact subset of Ω. For this, recall that, on account of Proposition 2.15, $\{\overline{\Sigma}_n\}$ converges to $\overline{\Sigma}$ in the Hausdorff topology and $\{\tau_n\}$ converges to τ uniformly on all compact subsets of Ω. Then, our assumptions imply that $\{\kappa_n\}$ converges to κ uniformly on every compact subset of $\Omega\backslash\overline{\Sigma}$, and so does $\{v_n\}$ to v. To complete the proof it suffices to combine the above local uniform convergence in $\Omega\backslash\overline{\Sigma}$ with the estimate

$$0\le v_n(x)\le \|f_n\|_{\infty,\Omega_\varepsilon}\left(1+\|[\kappa_n]_-\|_{\mathcal{C}(\partial\Omega_n)}\mathrm{diam}\,\Omega_n\right)\tau_n(x) \qquad \forall x\in\Omega_\varepsilon,$$

that allows to estimate v_n on any neighborhood of $\overline{\Sigma}$. $\square$

4 Uniqueness

In this section we will prove that, if (u', v') is a solution of system (2), then v' is given by (3) and $u' \equiv d$ in $\Omega_{v'} := \{x \in \Omega : v'(x) > 0\}$. We begin by showing the last statement.

Proposition 4.1 *If (u', v') is a solution of system (2), then $u' \equiv d$ in $\Omega_{v'}$.*

Proof—Since $\|Du'\|_{\infty,\Omega} \leq 1$ and $u' = 0$ on $\partial\Omega$, we have that $u' \leq d$ in Ω because, in view of Remark 2.4 , d is the largest function with such properties. Moreover, since u' solves the eikonal equation in $\Omega_{v'}$, Proposition 2.2 ensures that

$$u'(x) = \min_{y \in \partial\Omega_{v'},\,]x,y[\subset \Omega_{v'}} \Big\{ u'(y) + |y - x| \Big\} \qquad \forall x \in \Omega_{v'} \,.$$

We will argue by contradiction, supposing $u'(x_0) < d(x_0)$ for some point $x_0 \in \Omega_{v'}$. Without loss of generality, x_0 may be assumed to be a point of differentiability of u', and of approximate differentiability of Du' (see Remark 2.3). Let then $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that, for any $\varepsilon > 0$,

$$\lim_{r \downarrow 0} \frac{1}{r^2} \Big| B_r(x_0) \cap \Big\{ x \in \Omega : \frac{|Du'(x) - Du'(x_0) - L(x - x_0)|}{|x - x_0|} > \varepsilon \Big\} \Big| = 0 \,. \quad (19)$$

Moreover, let $y_0 \in \partial\Omega_{v'}$ be such that

$$]x_0, y_0[\subset \Omega_{v'} \quad \text{and} \quad u'(x_0) = u'(y_0) + |y_0 - x_0| \,.$$

Notice that $y_0 \notin \partial\Omega$ because, otherwise, one would have $u'(y_0) = 0$, and so $u'(x_0) = |y_0 - x_0|$, in contrast with $u'(x_0) < d(x_0)$.

Next, let us fix

$$0 < \varepsilon < \min \Big\{ 1, \frac{v'(x_0)}{16[1 + (1 + \|L\|)\mathrm{diam}\,\Omega]} \Big\} \,. \quad (20)$$

We claim that there exists $\rho > 0$ such that the balls $B_\rho(x_0)$ and $B_\rho(y_0)$ are both contained in Ω, and

$$|p - Du'(x_0)| \leq 1/2 \qquad \forall p \in D^+u'(x),\ \forall x \in B_\rho(x_0) \quad (21)$$

$$v'(x) \geq v'(x_0)/2 \qquad \forall x \in B_\rho(x_0) \quad (22)$$

$$v'(y) \leq \varepsilon \qquad \forall y \in B_\rho(y_0) \quad (23)$$

Indeed, (21) follows from the upper semicontinuity of D^+u' (see Remark 2.3), while (22) and (23) can be obtained by a simple continuity argument since $v'(x_0) > 0$ and $v'(y_0) = 0$.

Let us set, for the sake of brevity, $e_2 = Du'(x_0)$ and let $e_1 \in \mathbb{R}^2$ be such that $\{e_1, e_2\}$ is a direct orthonormal basis of $\mathbb{R}^2$.

From (19) it follows that, for every sufficiently small $r > 0$, there exists a point $x_r \in B_r(x_0)$ of differentiability for u such that

$$\begin{array}{ll} (i) & |Du'(x_r) - Du'(x_0) - L(x_r - x_0)| \leq \varepsilon r \\ (ii) & \langle e_2, x_r - x_0 \rangle \ < \ 0 \\ (iii) & \langle e_1, x_r - x_0 \rangle \ > \ r/2. \end{array} \tag{24}$$

Now, fix $y_1 \in]x_0, y_0[\cap B_{\rho/2}(y_0)$, and let $r > 0$ and be so small that

$$y_r := x_r - |x_0 - y_1| Du'(x_r) \in B_\rho(y_0) \quad \text{and} \quad \mathrm{co}\,\{x_0, x_r, y_1, y_r\} \subset \Omega_{v'}\,.$$

Such a number r exists because $[x_0, y_1] \subset \Omega_{v'}$ and

$$\lim_{r \downarrow 0} y_r \ = \ x_0 - |x_0 - y_1| Du'(x_0) = y_1\,,$$

since $x_r \to x_0$ and $Du'(x_r) \to Du'(x_0)$ as $r \downarrow 0$.

Finally, let us set $x_1 = \Pi_{[x_0, y_1]}(x_r)$ and $\mathcal{C} = \mathrm{co}\,\{x_1, y_1, x_r, y_r\}$. We point out that, because of (24)(ii), x_1 belongs to the open segment $]x_0, y_1[$. The convex set $\mathcal{C}$ is a quadrilateral with sides $[x_1, x_r]$, $[x_r, y_r]$, $[y_r, y_1]$ and $[y_1, x_1]$. Moreover, u' is differentiable at any point $x \in [y_1, x_1]$ and $Du'(x) = Du'(x_0)$, as guaranteed by Proposition 2.2(b). Similarly, combining properties (c) and (b) of the same proposition, shows that u is differentiable at any point $x \in [x_r, y_r]$ and $Du'(x) = Du'(x_r)$.

Our next step would be to integrate the equation $-\mathrm{div}\,(v'Du) = f$ over $\mathcal{C}$ and apply the Divergence Theorem. This reasoning needs the following approximation argument to be made rigorous. For any $\sigma > 0$, consider the test function

$$\psi_\sigma(x) := \left[1 - \frac{1}{\sigma} d_{\mathcal{C}}(x)\right]_+ \qquad x \in \mathbb{R}^2\,,$$

an element of $W^{1,\infty}(\mathbb{R}^2)$ with support $\mathcal{C}_\sigma := \{x \in \mathbb{R}^2 : d_{\mathcal{C}}(x) \leq \sigma\}$. Observe that, for σ sufficiently small, $\psi_\sigma \in W^{1,\infty}_c(\Omega)$. Also, $\mathrm{spt}(D\psi_\sigma) = \overline{\mathcal{C}_\sigma \backslash \mathcal{C}}$. Thus,

$$\int_\Omega f \psi_\sigma dx = \int_\Omega v' \langle Du', D\psi_\sigma \rangle dx = \int_{\mathcal{C}_\sigma \backslash \mathcal{C}} v' \langle Du', D\psi_\sigma \rangle dx\,. \tag{25}$$

In the right-hand side of the above equality, we split the integration domain as $\mathcal{C}_\sigma \backslash \mathcal{C} = E_1(\sigma) \cup E_2(\sigma) \cup E_3(\sigma) \cup E_4(\sigma)$, where

$$\begin{aligned} E_1(\sigma) &= \{x \in E : \Pi_{\mathcal{C}}(x) \in]x_1, y_1[\} \\ E_2(\sigma) &= \{x \in E : \Pi_{\mathcal{C}}(x) \in]x_r, y_r[\} \\ E_3(\sigma) &= \{x \in E : \Pi_{\mathcal{C}}(x) \in [y_1, y_r]\} \\ E_4(\sigma) &= \{x \in E : \Pi_{\mathcal{C}}(x) \in [x_1, x_r]\} \quad , \end{aligned}$$

and proceed to estimate the integrals

$$\mathcal{E}_i(\sigma) := \int_{E_i(\sigma)} v' \langle Du', D\psi_\sigma \rangle dx \qquad i = 1, \ldots, 4 .$$

To find an upper bound for $\mathcal{E}_1(\sigma)$, observe that $|E_1(\sigma)| \leq \sigma |y_1 - x_1|$ and $D\psi_\sigma = -e_1/\sigma$ on $E_1(\sigma)$. Therefore, recalling that $Du'(x_0) = e_2$,

$$\begin{aligned} |\mathcal{E}_1(\sigma)| = \frac{1}{\sigma} \Big| \int_{E_1(\sigma)} v' \langle Du', e_1 \rangle dx \Big| &= \frac{1}{\sigma} \Big| \int_{E_1(\sigma)} v' \langle Du' - Du'(x_0), e_1 \rangle dx \Big| \\ &\leq \frac{1}{\sigma} |E_1(\sigma)| \, \|v'\|_{\infty, \mathcal{C}_\sigma} \|Du' - Du'(x_0)\|_{\infty, E_1(\sigma)} \\ &\leq |y_1 - x_1| \, \|v'\|_{\infty, \mathcal{C}_\sigma} \|Du' - Du'(x_0)\|_{\infty, E_1(\sigma)} . \end{aligned} \tag{26}$$

Moreover, since u' is continuously differentiable at every point $x \in]y_0, x_0[$ and satisfies $Du'(x) = Du'(x_0)$, we have

$$\omega_1(\sigma) := \|Du' - Du'(x_0)\|_{\infty, E_1(\sigma)} \to 0 \quad \text{as} \quad \sigma \downarrow 0 .$$

Similarly,

$$|\mathcal{E}_2(\sigma)| \leq |y_r - x_r| \, \|v'\|_{\infty, \mathcal{C}_\sigma} \, \omega_2(\sigma) , \tag{27}$$

where $\omega_2(\sigma) := \|Du' - Du'(x_0)\|_{\infty, E_2(\sigma)} \to 0$ as $\sigma \downarrow 0$.

Next, to bound $\mathcal{E}_3(\sigma)$ we note that $E_3(\sigma) \subset B_\rho(y_0)$ for $\sigma > 0$ small enough. So, in view of (23), $|\mathcal{E}_3(\sigma)| \leq \varepsilon |E_3(\sigma)|/\sigma$ because $|D\psi_\sigma| \leq 1/\sigma$ and $|Du'| \leq 1$. Since $|E_3(\sigma)| \leq 2\sigma(|y_1 - y_r| + 2\sigma)$, we finally get the estimate

$$|\mathcal{E}_3(\sigma)| \leq 2\varepsilon(|y_1 - y_r| + 2\sigma) . \tag{28}$$

The reasoning we need to estimate $\mathcal{E}_4(\sigma)$ is just slightly longer than the previous ones. Let us split $E_4(\sigma)$ in two parts, $E_4'(\sigma)$ and $E_4''(\sigma)$, where

$$\begin{aligned} E_4'(\sigma) &= \{x \in E_4(\sigma) : \Pi_{\mathcal{C}}(x) \in]x_1, x_r[\} \\ E_4''(\sigma) &= \{x \in E_4(\sigma) : \Pi_{\mathcal{C}}(x) \in \{x_1, x_2\}\} . \end{aligned}$$

By choosing $\sigma > 0$ so small that $E_4(\sigma) \subset B_\rho(x_0)$, we have $|Du' - e_2| \le 1/2$ a.e. in $E_4(\sigma)$ owing to (21). Therefore,

$$\langle Du', D\psi_\sigma\rangle \le \langle e_2, D\psi_\sigma\rangle + \frac{1}{2\sigma} \le -\frac{1}{2\sigma} \quad \text{a. e. in} \quad E_4'(\sigma)$$

because, on such a set, $D\psi_\sigma = -e_2/\sigma$. Now, by (22),

$$\begin{aligned}\mathcal{E}_4(\sigma) &\le \int_{E_4'(\sigma)} v'\langle Du', D\psi_\sigma\rangle\, dx + \frac{|E_4''(\sigma)|}{\sigma}\, \|v'\|_{\infty,\mathcal{C}_\sigma} \\ &\le -\frac{1}{2\sigma}\,\frac{v'(x_0)}{2}|E_4''(\sigma)| + 2\pi\sigma\|v'\|_{\infty,\mathcal{C}_\sigma} \\ &\le -\frac{v'(x_0)}{4}\,|x_1 - x_r| + 2\pi\sigma\|v'\|_{\infty,\mathcal{C}_\sigma} \qquad (29)\end{aligned}$$

Now, plugging estimates (26), (27), (28) and (29) into (25), we obtain

$$\begin{aligned}0 \le \int_\Omega f\psi_\sigma\, dx \le 2\varepsilon\Big(|y_1 - y_r| + 2\sigma\Big) - \frac{v'(x_0)}{4}\,|x_1 - x_r| \\ +\|v'\|_{\infty,\mathcal{C}_\sigma}\Big[\,|y_1 - x_1|\omega_1(\sigma) + |y_r - x_r|\omega_2(\sigma) + 2\pi\sigma\,\Big]\end{aligned}$$

Hence, letting $\sigma \downarrow 0$,

$$0 \le 2\varepsilon|y_1 - y_r| - \frac{v'(x_0)}{4}|x_1 - x_r|\,. \qquad (30)$$

Since, owing to (24)(i), $|Du'(x_r) - Du'(x_0)| \le \varepsilon r + \|L\||x_r - x_0|$, we have

$$\begin{aligned}|y_1 - y_r| \;&=\; \Big|\,x_r - |x_0 - y_1|Du'(x_r) - \Big(x_0 - |x_0 - y_1|Du'(x_0)\Big)\,\Big| \\ &\le\; |x_0 - y_1|\,(\varepsilon r + \|L\||x_r - x_0|) + |x_r - x_0|\end{aligned}$$

But $|x_r - x_0| \le r$ and, by (24(iii)), $|x_1 - x_r| \ge r/2$. So,

$$\varepsilon r + \|L\||x_r - x_0| \le 2(\varepsilon + \|L\|)|x_1 - x_r|$$

and

$$|y_1 - y_r| \le 2\,\left[\,1 + |x_0 - y_1|\,(\varepsilon + \|L\|)\,\right]\,|x_1 - x_r|\,. \qquad (31)$$

Combining (30) and (31), we obtain

$$0 \le \Big\{4\varepsilon\Big[\,1 + |x_0 - y_1|\,(\varepsilon + \|L\|)\,\Big] - \frac{v'(x_0)}{4}\,\Big\}|x_1 - x_r|\,,$$

which is in contrast with (20). We have reached a contradiction assuming $u'(x_0) < d(x_0)$. So, $u' \equiv d$ and the proof is complete. □

Our next task is to show that v' is given by the representation formula (3). We will do this in the next two propositions: the first one computes v' away from the singular set, the second one on $\overline{\Sigma}$.

Proposition 4.2 *Let (d, v') be a solution of system (2). Then, for any $z_0 \in \Omega \setminus \overline{\Sigma}$ and $\theta \in (0, \tau(z_0))$, we have*

$$v'(z_0) - \frac{1-(d(z_0)+\theta)\kappa(z_0)}{1-d(z_0)\kappa(z_0)} v'(z_0 + \theta Dd(z_0)) = \int_0^\theta f(z_0 + tDd(z_0)) \frac{1-(d(z_0)+t)\kappa(z_0)}{1-d(z_0)\kappa(z_0)} dt\,.$$

Proof—Let $z_0 \in \Omega \setminus \overline{\Sigma}$, $\theta \in (0, \tau(z_0))$ and set $x_0 = z_0 + \theta Dd(z_0)$. Notice that $[z_0, x_0] \subset \Omega \setminus \overline{\Sigma}$ and $Dd(z) = Dd(z_0)$ for $z \in [z_0, x_0]$ by Proposition 2.2(b).

Let us use—once again—a coordinate system that simplifies the notation: we set $e_2 = Dd(z_0)$ and choose e_1 such that $\{e_1, e_2\}$ is a direct orthonormal basis of $\mathbb{R}^2$. Also, fix $r > 0$ so small that $x_r := x_0 + re_1 \notin \overline{\Sigma}$ and $\langle Dd(x_r), e_2 \rangle > 0$. Let then $\bar{t} > 0$ be such that the point $z_r := x_r - \bar{t} Dd(x_r)$ satisfies $\langle z_r - z_0, e_2 \rangle = 0$. We note that $\bar{t}$ is given by

$$\bar{t} = \frac{\langle x_r - z_0, e_2 \rangle}{\langle Dd(x_r), e_2 \rangle} = \frac{|x_0 - z_0|}{\langle Dd(x_r), e_2 \rangle}\,. \tag{32}$$

Finally, let us possibly reduce $r > 0$ in order to ensure that the domain $D_r := \operatorname{co}\{x_0, x_r, z_r, z_0\}$ be contained in $\Omega \setminus \overline{\Sigma}$ and d be of class $\mathcal{C}^2$ in a neighbourhood of D_r.

Integrating by parts the equation $-\operatorname{div}(v'Dd) = f$ on D_r, we obtain

$$\int_{D_r} f\,dx = -\int_{\partial D_r} v' \langle Dd, \nu \rangle d\mathcal{H}^1 \tag{33}$$

where ν is the outward unit normal to ∂D_r. The above right-hand side amounts to

$$\int_{\partial D_r} v' \langle Dd, \nu \rangle d\mathcal{H}^1 = \int_{[x_0, x_r]} v' \langle Dd, e_2 \rangle d\mathcal{H}^1 + \int_{[z_0, z_r]} v' \langle Dd, -e_2 \rangle d\mathcal{H}^1 \tag{34}$$

because

$$\int_{[z_0, x_0]} v' \langle Dd, \nu \rangle d\mathcal{H}^1 = \int_{[z_0, x_0]} v' \langle e_2, -e_1 \rangle d\mathcal{H}^1 = 0$$

and, similarly, $\langle Dd, \nu\rangle = 0$ on $[z_r, x_r]$. Moreover, we have

$$\int_{D_r} f dx = \int_0^{|z_0 - x_0|} dt \int_0^{l_t} f(z_0 + te_2 + se_1) ds \tag{35}$$

where

$$l_t = \Big(1 - \frac{t}{|z_0 - x_0|}\Big)|z_0 - z_r| + \frac{t}{|z_0 - x_0|}|x_0 - x_r| \,.$$

Our next step will be to compute $\lim_{r\downarrow 0} \frac{1}{r}\int_{D_r} f dx$. Aiming at this, let us recall that, in view of Proposition 2.6,

$$D^2 d(x_0) = \gamma_0(e_1 \otimes e_1) \quad \text{where} \quad \gamma_0 = -\frac{\kappa(x_0)}{1 - \kappa(x_0)d(x_0)} \,.$$

Hence,

$$\frac{1}{r}\frac{\langle Dd(x_r), e_1\rangle}{\langle Dd(x_r), e_2\rangle} = \frac{1}{r}\frac{\langle Dd(x_0) + rD^2d(x_0)e_1 + \mathrm{o}(r), e_1\rangle}{\langle Dd(x_0) + rD^2d(x_0)e_1 + \mathrm{o}(r), e_2\rangle} = \frac{\gamma_0 + \epsilon(r)}{1 + \epsilon(r)}$$

where $\epsilon(r) \to 0$ as $r \downarrow 0$. Since

$$\frac{|z_0 - z_r|}{r} = 1 - |x_0 - z_0|\,\frac{1}{r}\frac{\langle Dd(x_r), e_1\rangle}{\langle Dd(x_r), e_2\rangle} = 1 - |x_0 - z_0|\,\frac{\gamma_0 + \epsilon(r)}{1 + \epsilon(r)} \,, \tag{36}$$

we obtain

$$\begin{aligned} \lim_{r\to 0^+}\frac{l_t}{r} &= \Big(1 - \frac{t}{|z_0 - x_0|}\Big)(1 - \gamma_0|x_0 - z_0|) + \frac{t}{|z_0 - x_0|} \\ &= 1 - \gamma_0|x_0 - z_0| + t\gamma_0 \end{aligned}$$

Therefore, in view of (35), we conclude that

$$\lim_{r\downarrow 0}\frac{1}{r}\int_{D_r} f dx = \int_0^{|z_0 - x_0|} f(z_0 + te_2)\Big(1 - \gamma_0|x_0 - z_0| + t\gamma_0\Big) dt \,. \tag{37}$$

We now turn to the evaluation of $\lim_{r\downarrow 0}\frac{1}{r}\int_{\partial D_r} v'\langle Dd, \nu\rangle$. Since Dd is continuous at x_0 and $Dd(x_0) = e_2$, we have

$$\lim_{r\downarrow 0}\frac{1}{r}\int_{[x_0, x_r]} v'\langle Dd, e_2\rangle d\mathcal{H}^1 = v'(x_0) \,.$$

A similar continuity argument and (36) show that

$$\lim_{r\downarrow 0}\frac{1}{r}\int_{[z_0, z_r]} v'\langle Dd, -e_2\rangle d\mathcal{H}^1 = -v'(z_0)(1 - \gamma_0|x_0 - z_0|) \,.$$

Then, recalling (33), (34) and (37), we conclude that

$$\lim_{r\downarrow 0} -\frac{1}{r}\int_{\partial D_r} v'\langle Dd,\nu\rangle d\mathcal{H}^1 = v'(z_0)(1-\gamma_0|x_0-z_0|) - v'(x_0)$$
$$= \int_0^{|z_0-x_0|} f(z_0+te_2)\Big(1-\gamma_0|x_0-z_0|+t\gamma_0\Big)dt\,,$$

whence, since $|z_0-x_0|=\theta$,

$$v'(z_0) - \frac{v'(x_0)}{1-\gamma_0\theta} = \int_0^\theta f(z_0+te_2)\Big(1+\frac{t\gamma_0}{1-\gamma_0\theta}\Big)dt\,. \tag{38}$$

Finally, recalling the definition of γ_0 and using the equality $d(x_0)=d(z_0)+\theta$, we have

$$1-\gamma_0\theta = 1+\frac{\kappa(x_0)\theta}{1-d(x_0)\kappa(x_0)} = \frac{1-d(z_0)\kappa(x_0)}{1-d(x_0)\kappa(x_0)}$$

and

$$\frac{\gamma_0}{1-\gamma_0\theta} = -\frac{\kappa(x_0)}{1-d(z_0)\kappa(x_0)}\,.$$

In view of the above identities and of the fact that $\kappa(x_0)=\kappa(z_0)$, (38) can be recasted as

$$v'(z_0) - \frac{1-d(x_0)\kappa(z_0)}{1-d(z_0)\kappa(z_0)}v'(x_0) = \int_0^\theta f(z_0+te_2)\frac{1-(d(z_0)+t)\kappa(z_0)}{1-d(z_0)\kappa(z_0)}dt\,.$$

The last formula yields the conclusion. □

The following result is reminiscent of [13, Proposition 7.1].

Proposition 4.3 *If (d,v') is a solution of system (2), then $v'=0$ on $\overline{\Sigma}$.*

Proof—Let us assume, first, that Σ is a singleton, say $\{x_0\}$. Then, by a classical result of Motzkin's [22] (see also Remark 2.11), Ω is the disk $B_R(x_0)$ with $R=d(x_0)$. Integrating the equation $-\mathrm{div}\,(v'Dd)=f$ on $B_r(x_0)$, for $0<r<R$, gives

$$\int_{B_r(x_0)} f dx = -\int_{\partial B_r(x_0)} v'\langle Dd,\nu\rangle d\mathcal{H}^1\,,$$

where ν is the outward unit normal to $\partial B_r(x_0)$. Since $\langle Dd,\nu\rangle=-1$, we have

$$0 = \lim_{r\downarrow 0}\frac{1}{r}\int_{B_r(x_0)} f dx = \lim_{r\downarrow 0}\frac{1}{r}\int_{\partial B_r(x_0)} v' d\mathcal{H}^1 = 2\pi v'(x_0)\,.$$

Thus, $v'(x_0) = 0$.

Suppose, next, that Σ is not a singleton. Then, again by Remark 2.11, the set Σ^1 of singular points with magnitude 1 is dense in Σ. Since v' is continuous, it suffices to prove that v' vanishes on Σ^1. So, let $x_0 \in \Sigma^1$ and $D^+d(x_0) = [p_0, q_0]$ with $p_0 \neq q_0$. Then, by Proposition 2.10, there exists a Lipschitz arc $\boldsymbol{x} : [0, \eta] \to \Sigma$ such that $\boldsymbol{x}(0) = x_0$, $\dot{\boldsymbol{x}}(0) \neq 0$, and

$$\langle \dot{\boldsymbol{x}}(0), p_0 - q_0 \rangle = 0 \,. \tag{39}$$

Moreover, $\boldsymbol{x}(s_n) \in \Sigma^1$ for some sequence $s_n \downarrow 0$, and

$$D^+d(\boldsymbol{x}(s_n)) = [p_n, q_n] \quad \text{with} \quad p_n \to p_0 \,,\ q_n \to q_0 \,. \tag{40}$$

Since Σ has Lebesgue measure zero, we have, by Fubini's Theorem,

$$\mathcal{H}^1\Big([x_0 - \alpha s_n p_0, \boldsymbol{x}(s_n) - \alpha s_n p_n] \cap \Sigma\Big) = 0 \quad \text{for a. e. } \alpha \in [1, 2] \,,$$

provided n is sufficiently large. Let $\alpha_n \in [1, 2]$ be such that

$$\mathcal{H}^1\Big([x_0 - \alpha_n s_n p_0, \boldsymbol{x}(s_n) - \alpha_n s_n p_n] \cap \Sigma\Big) = 0 \,.$$

In the same way, let $\beta_n \in [1, 2]$ be such that

$$\mathcal{H}^1\Big([x_0 - \beta_n s_n q_0, \boldsymbol{x}(s_n) - \beta_n s_n q_n] \cap \Sigma\Big) = 0 \,.$$

Let us set, for every $n \in \mathbb{N}$,

$$I_p^n :=]x_0 - \alpha_n s_n p_0, \boldsymbol{x}(s_n) - \alpha_n s_n p_n[\,, \quad I_q^n :=]x_0 - \beta_n s_n q_0, \boldsymbol{x}(s_n) - \beta_n s_n q_n[\,,$$

and let us denote by $\overline{I}_p^n$ (resp. $\overline{I}_q^n$) the closure of I_p^n (resp. I_q^n).

Now, for $n \in \mathbb{N}$ large enough define the domain

$$D_n := \mathrm{co}\,([x_0, \boldsymbol{x}(s_n)] \cup \overline{I}_p^n) \cup \mathrm{co}\,([x_0, \boldsymbol{x}(s_n)] \cup \overline{I}_q^n)$$

and consider, for $\sigma > 0$, the function

$$\psi_\sigma^n(x) = \Big[1 - \frac{1}{\sigma} d_{D_n}(x)\Big]_+ \qquad x \in \Omega \,.$$

Notice that, for n large enough, $\psi_\sigma^n \in W_c^{1,\infty}(\Omega)$. Therefore, using ψ_σ^n as test function for the equation $-\mathrm{div}\,(v' Dd) = f$, we have

$$\int_\Omega f \psi_\sigma^n dx = \int_\Omega v' \langle Dd, D\psi_\sigma^n \rangle dx \,.$$

In order to estimate the right-hand side, observe that the support of $D\psi_\sigma^n$ is given by the closure of the set $A^n(\sigma) := \{x \in \Omega \backslash D_n : d_{D_n}(x) < \sigma\}$. This set can be represented as the disjoint union $A_p^n(\sigma) \cup A_q^n(\sigma) \cup \widetilde{A}^n(\sigma)$, where

$$A_p^n(\sigma) = \{x \in A(\sigma) : \Pi_{D_n}(x) \in I_p^n\}\,, \quad A_q^n(\sigma) = \{x \in A(\sigma) : \Pi_{D_n}(x) \in I_q^n\}\,.$$

Then, the gradient of d_{D_n} is constant on $A_p^n(\sigma)$, say $Dd_{D_n} \equiv \nu_p^n$. Similarily, $Dd_{D_n} \equiv \nu_q^n$ on $A_q^n(\sigma)$. Now, observe that

$$\begin{aligned} & -\int_\Omega f\psi_\sigma dx \\ & \quad = \int_{A_p^n(\sigma)} \frac{v'}{\sigma}\langle Dd, \nu_p^n\rangle dx + \int_{A_q^n(\sigma)} \frac{v'}{\sigma}\langle Dd, \nu_q^n\rangle dx + \int_{\widetilde{A}^n(\sigma)} v'\langle Dd, D\psi_\sigma^n\rangle dx \end{aligned} \tag{41}$$

We will pass to the limit as $\sigma \downarrow 0$ in the above identity. We have

$$\lim_{\sigma\downarrow 0}\int_\Omega f\psi_\sigma^n dx = \int_{D_n} f dx\,.$$

Moreover, arguing as in the proof of Proposition 4.1, we find

$$\lim_{\sigma\downarrow 0}\int_{\widetilde{A}^n(\sigma)} v'\langle Dd, D\psi_\sigma^n\rangle dx = 0\,.$$

In order to estimate the term

$$\int_{A_p^n(\sigma)} \frac{v'}{\sigma}\langle Dd, \nu_p^n\rangle dx = \frac{1}{\sigma}\int_{I_p^n} d\mathcal{H}^1(y)\int_0^\sigma v'(y+t\nu_p^n)\langle Dd(y+t\nu_p^n), \nu_p^n\rangle dt\,,$$

recall that $\mathcal{H}^1(I_p^n \cap \Sigma) = 0$, and so Dd is continuous at $\mathcal{H}^1$-almost every point of I_p^n. Therefore,

$$\lim_{\sigma\downarrow 0}\int_{A_p^n(\sigma)} \frac{v'}{\sigma}\langle Dd, \nu_p^n\rangle dx = \int_{I_p^n} v'(y)\langle Dd(y), \nu_p^n\rangle d\mathcal{H}^1(y)\,.$$

Similarly,

$$\lim_{\sigma\downarrow 0}\int_{A_q^n(\sigma)} \frac{v'}{\sigma}\langle Dd, \nu_q^n\rangle dx = \int_{I_q^n} v'(y)\langle Dd(y), \nu_q^n\rangle d\mathcal{H}^1(y)\,.$$

Thus, passing to the limit as $\sigma \downarrow 0$ in (41), we conclude that

$$-\int_{D_n} f = \int_{I_p^n} v'(y)\langle Dd(y), \nu_p^n\rangle d\mathcal{H}^1(y) + \int_{I_q^n} v'(y)\langle Dd(y), \nu_q^n\rangle d\mathcal{H}^1(y)\,. \tag{42}$$

Our final step will be to divide both sides of (42) by s_n and to take the limit as $n \to \infty$. For this we need two preliminary remarks. The first one is that, for every sequence $\{y_n\}_n$ such that $y_n \in I_p^n$ and d is differentiable at y_n, $Dd(y_n)$ converges to p_0 as $n \to \infty$. For let $\lambda_n \in [0,1]$ be such that

$$\begin{aligned} y_n &= \lambda_n(x_0 - \alpha_n s_n p_0) + (1-\lambda_n)(\boldsymbol{x}(s_n) - \alpha_n s_n p_n) \\ &= \lambda_n(x_0 - \alpha_n s_n p_0) + (1-\lambda_n)(x_0 + s_n \dot{\boldsymbol{x}}(0) + \mathrm{o}(s_n) - \alpha_n s_n p_n) \end{aligned}$$

and suppose $\lambda_n \to \lambda^* \in [0,1]$ and $\alpha_n \to \alpha^* \in [1,2]$ as $n \to \infty$ (which always holds, up to subsequences). Then,

$$\lim_{n\to\infty} \frac{y_n - x_0}{s_n} = -\alpha^* \lambda^* p_0 + (1-\lambda^*)\dot{\boldsymbol{x}}(0) =: \theta^* \,.$$

But $\min\{\langle p, \theta^* \rangle : p \in D^+ d(x_0)\}$ is attained at p_0, since, in view of (39), $\langle \dot{\boldsymbol{x}}(0), p \rangle = \langle \dot{\boldsymbol{x}}(0), p_0 \rangle$ for every $p \in [p_0, q_0]$. Thus, by [3, Theorem 2.1], $Dd(y_n) \to p_0$ as claimed.

The second remark we need, to proceed with our computation, is that

$$\lim_{n\to\infty} \nu_p^n = -\frac{p_0 - q_0}{|p_0 - q_0|} \,. \tag{43}$$

Indeed, by definition,

$$\langle \nu_p^n, \boldsymbol{x}(s_n) - \alpha_n s_n p_n - (x_0 - \alpha_n s_n p_0) \rangle = 0$$

where

$$\boldsymbol{x}(s_n) - x_0 + \alpha_n s_n (p_0 - p_n) = s_n \dot{\boldsymbol{x}}(0) + \mathrm{o}(s_n) \tag{44}$$

in view of (40). Thus, ν_p^n is nearly orthogonal to $\dot{\boldsymbol{x}}(0)$, and so

$$\nu_p^n = \rho_0(p_0 - q_0) + \varepsilon_n \quad \text{with} \quad \lim_{n\to\infty} \varepsilon_n = 0 \quad \text{and} \quad |\rho_0| = \frac{1}{|p_0 - q_0|} \,.$$

Moreover, $\langle \nu_p^n, p_0 \rangle \leq 0$ for n large enough, because ν_p^n is an outward normal to the set $\mathrm{co}\,([x_0, \boldsymbol{x}(s_n)] \cup \overline{I}_p^n)$ at the point $x_0 - \alpha_n s_n p_0$ and x_0 belongs to such a set. Therefore, $\rho_0 < 0$ and (43) follows.

We are now ready for our final step. Dividing both sides of (42) by s_n and taking the limit as $n \to \infty$, we obtain

$$0 = \lim_{n\to\infty} \frac{1}{s_n} \Big\{ \int_{I_p^n} v'(y) \langle Dd(y), \nu_p^n \rangle d\mathcal{H}^1(y) + \int_{I_q^n} v'(y) \langle Dd(y), \nu_q^n \rangle d\mathcal{H}^1(y) \Big\} \,.$$

Since $\mathcal{H}^1(I_p^n) = |\boldsymbol{x}(s_n) - \alpha_n s_n p_n - (x_0 - \alpha_n s_n p_0)| = s_n|\dot{\boldsymbol{x}}(0)| + \mathrm{o}(s_n)$ on account of (44), we have

$$\lim_{n\to\infty} \frac{1}{s_n} \int_{I_p^n} v'(y)\langle Dd(y), \nu_p^n\rangle d\mathcal{H}^1(y) = -v'(x_0)|\dot{\boldsymbol{x}}(0)|\Big\langle p_0, \frac{p_0 - q_0}{|p_0 - q_0|}\Big\rangle .$$

By a similar argument,

$$\lim_{n} \frac{1}{s_n} \int_{I_q^n} v'(y)\langle Dd(y), \nu_q^n\rangle d\mathcal{H}^1(y) = v'(x_0)|\dot{\boldsymbol{x}}(0)|\Big\langle q_0, \frac{p_0 - q_0}{|p_0 - q_0|}\Big\rangle .$$

Thus,

$$\begin{aligned} 0 &= v'(x_0)|\dot{\boldsymbol{x}}(0)|\Big\{ -\Big\langle p_0, \frac{p_0 - q_0}{|p_0 - q_0|}\Big\rangle + \Big\langle q_0, \frac{p_0 - q_0}{|p_0 - q_0|}\Big\rangle\Big\} \\ &= -v'(x_0)|\dot{\boldsymbol{x}}(0)|\,|p_0 - q_0| \end{aligned}$$

Since $\dot{\boldsymbol{x}}(0) \neq 0$ and $p_0 \neq q_0$, we have finally obtained that $v'(x_0) = 0$. □

We are now ready to complete the proof of our main result.

Proof of Theorem 1.1[Part 2: Uniqueness]—Let (u', v') is a solution of system (2). Then, $u' \equiv d$ in $\Omega_{v'} := \{x \in \Omega : v'(x) > 0\}$ by Proposition 4.1. In particular, (d, v') is also a solution of (2). So, owing to Proposition 4.3, $v' = 0$ on $\overline{\Sigma}$. Now, let $x \in \Omega \setminus \overline{\Sigma}$. In view of Proposition 4.2, we have

$$\begin{aligned} v'(x) - &\frac{1 - (d(x) + \theta)\kappa(x)}{1 - d(x)\kappa(x)}\, v'(x + \theta Dd(x)) \\ &= \int_0^\theta f(x + tDd(x))\, \frac{1 - (d(x) + t)\kappa(x)}{1 - d(x)\kappa(x)}\, dt \end{aligned}$$

for each $\theta \in (0, \tau(x))$. Since v' is continuous and vanishes on $\overline{\Sigma}$, the left-hand side above converges to $v'(x)$ as $\theta \uparrow \tau(x)$. So, $v'(x)$ coincides with $v(x)$, given by (3), and the proof is complete. □

A Appendix: Proof of Theorem 2.12

We already know that the normal distance τ, defined in (6), is continuous in $\overline{\Omega}$, see Proposition 2.14. In this section we will prove that, if Ω has a $\mathcal{C}^{2,1}$ boundary, then τ is also Lipschitz continuous on $\partial\Omega$ (Theorem 2.12). The main step of the proof is the following preliminary result.

Lemma A.1 *Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary of class $\mathcal{C}^{2,1}$. Then, every $x \in \partial\Omega$ has a neighborhood, U, such that*

$$\tau(y) \leq \tau(x) + K(\operatorname{diam}\Omega)^2|y-x| \qquad \forall y \in \partial\Omega \cap U\,, \tag{45}$$

where

$$K = \sup_{\substack{x, y \in \partial\Omega \\ x \neq y}} \max\left\{\frac{|\kappa(y)-\kappa(x)|}{|y-x|}, \frac{|Dd(y)-Dd(x)-D^2d(x)(y-x)|}{|y-x|^2}\right\}.$$

Proof—Let $x \in \partial\Omega$ be fixed. We will analyse, first, the simpler case $\tau(x)\kappa(x) = 1$. Recalling that $\tau(x) \leq \operatorname{diam}\Omega/2$, we have $\kappa(x) \geq 2/\operatorname{diam}\Omega$. Let U be an open neighborhood of x such that $\kappa(y) > 1/\operatorname{diam}\Omega$ for every $y \in U$. Then, for every $y \in \partial\Omega \cap U$,

$$\tau(y) \leq \frac{1}{\kappa(y)} \leq \frac{1}{\kappa(x)} + \frac{\kappa(y)-\kappa(x)}{\kappa(y)\kappa(x)} \leq \tau(x) + \frac{K}{2}(\operatorname{diam}\Omega)^2|y-x|$$

and (45) is proved.

Now, suppose $\tau(x)\kappa(x) < 1$ and define $\bar{x} = x + \tau(x)Dd(x)$. We claim that $Dd(x)$ must be isolated in $D^*d(\bar{x})$. For suppose $Dd(x) = \lim_k p_k$ for some sequence $\{p_k\}$ in $D^*d(\bar{x})$ satisfying $p_k \neq Dd(x)$ for every k. Then, $p_k = Dd(x_k)$, where $x_k = \bar{x} - d(\bar{x})p_k \neq \bar{x} - d(\bar{x})Dd(x) = x$ is a sequence of boundary points $\{x_k\}$ converging to x. We can also assume, without loss of generality, that $(x_k - x)/|x_k - x|$ converges to some unit vector θ. Hence,

$$\theta = \lim_{k\to\infty}\frac{x_k - x}{|x_k - x|} = -d(\bar{x})\lim_{k\to\infty}\frac{Dd(x_k)-Dd(x)}{|x_k - x|} = -d(\bar{x})D^2d(x)\theta\,.$$

Therefore, recalling that the nonzero eigenvalue of $D^2d(x)$ is given by $-\kappa(x)$, we obtain $-\kappa(x) = -1/d(\bar{x}) = -1/\tau(x)$ in contrast with $\tau(x)\kappa(x) < 1$. So, our claim is proved.

Hereafter, we denote by $\mathcal{R}$ the rotation matrix

$$\mathcal{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and by $\{e_1, e_2\}$ the orthonormal basis of $\mathbb{R}^2$ given by

$$e_1 = \mathcal{R}^{-1}Dd(x) \qquad e_2 = Dd(x)\,.$$

We split the reasoning into several steps.

Step 1: *constructing a singular arc.*
We want to construct a Lipschitz arc $\boldsymbol{x} : [0,\eta] \to \Omega$ such that

$$\boldsymbol{x}(0) = \bar{x}\,, \quad |\dot{\boldsymbol{x}}(0)| = 1\,, \quad \langle \dot{\boldsymbol{x}}(0), e_1\rangle > 0\,, \quad \boldsymbol{x}(s) \in \Sigma \quad \forall s \in [0,\eta]\,. \tag{46}$$

Suppose, first, $\bar{x} \in \Sigma^2$. Since e_2 is isolated in $D^*d(\bar{x})$, there are two distinct vectors $p_1, p_2 \in D^*d(\bar{x})$ such that the segments $[p_1, e_2]$ and $[p_2, e_2]$ are contained in $\partial D^+d(\bar{x})$. Let n_1 and n_2 be outward unit normals to $D^+d(\bar{x})$ exposing the faces $[p_1, e_2]$ and $[p_2, e_2]$ respectively, i.e.

$$\max_{p \in D^+d(\bar{x})} \langle p, n_i\rangle = \langle p_i, n_i\rangle = \langle e_2, n_i\rangle \qquad i = 1, 2\,.$$

We claim that

$$e_2 = \lambda_1 n_1 + \lambda_2 n_2 \tag{47}$$

for suitable numbers $\lambda_1, \lambda_2 > 0$. Indeed, the normal cone to $D^+d(\bar{x})$ at e_2 is generated by $\{n_1, n_2\}$. Since e_2 belongs to such a cone, $e_2 = \lambda_1 n_1 + \lambda_2 n_2$ with $\lambda_1, \lambda_2 \geq 0$. If $\lambda_1 = 0$, then $\lambda_2 = 1$ and $e_2 = n_2$. Therefore, $\langle p_2, n_2\rangle = \langle e_2, n_2\rangle = 1$, which implies $p_2 = n_2 = e_2$ in contrast with the definition of p_2. So, $\lambda_1 > 0$. Similarly, $\lambda_2 > 0$; our claim is thus proved.

Now, observe that, on account of (47), $0 = \lambda_1\langle n_1, e_1\rangle + \lambda_2\langle n_2, e_1\rangle$. So, either $\langle n_1, e_1\rangle < 0$ or $\langle n_2, e_1\rangle < 0$. Suppose $\langle n_1, e_1\rangle < 0$, and apply Proposition 2.10 to the face $[p_1, e_2]$ of $D^+d(\bar{x})$, with normal n_1, to construct a Lipschitz arc $\boldsymbol{x} : [0,\eta] \to \Omega$ such that

$$\boldsymbol{x}(0) = \bar{x}\,, \qquad \dot{\boldsymbol{x}}(0) = -n_1\,, \qquad \boldsymbol{x}(s) \in \Sigma \quad \forall s \in [0,\eta]\,.$$

Since $\langle n_1, e_1\rangle < 0$, we have $\langle \dot{\boldsymbol{x}}(0), e_1\rangle > 0$, which proves (46).

To complete the proof of this step it suffices to note that the case $\bar{x} \in \Sigma^1$ can be treated by a similar—yet simpler—argument.

Step 2: *intersecting the singular arc.*
We want to construct a neighborhood of x, U, such that, for any boundary point $y \in \partial\Omega \cap U$ satisfying $\langle y - x, e_1\rangle > 0$, there exist $s_y, \rho_y > 0$ with

$$\boldsymbol{x}(s_y) = y + \rho_y Dd(y) \tag{48}$$

$$\lim_{y \to x} s_y = 0 \tag{49}$$

(where the limit is taken for $y \in \partial\Omega \cap U$ such that $\langle y - x, e_1\rangle > 0$).

Let V be an open neighborhood of x such that $\partial\Omega \cap V$ is the trace of a regular curve $h \mapsto \boldsymbol{y}(h)$, $-r < h < r$, with $\boldsymbol{y}(0) = x$ and $\dot{\boldsymbol{y}}(0) = e_1$.

Then, $\boldsymbol{y}(h) = x + he_1 + \mathrm{o}(h)$, where the standard notation $\mathrm{o}(h)$ denotes—hereafter—a (scalar- or vector-valued) map such that $\mathrm{o}(h)/h \to 0$ as $h \to 0$. Moreover, for every $y \in \partial\Omega \cap V$ satisfying $\langle y - x, e_1 \rangle > 0$,

$$\exists! \quad h_y \in (0, r) \quad \text{such that} \quad y = \boldsymbol{y}(h_y)\,. \tag{50}$$

Now, for $0 < h < r$, consider the map $\phi_h : [0, \eta] \to \mathbb{R}$

$$\phi_h(s) = \langle \boldsymbol{x}(s) - \boldsymbol{y}(h), \mathcal{R}Dd(\boldsymbol{y}(h)) \rangle \qquad \forall s \in [0, \eta]\,,$$

where $\boldsymbol{x}$ is the singular arc of Step 1. Since $D^2 d(x) = -\kappa(x) e_1 \otimes e_1$, we have

$$\begin{aligned}
\phi_h(0) &= \langle \bar{x} - \boldsymbol{y}(h), \mathcal{R}Dd(\boldsymbol{y}(h)) \rangle \\
&= \langle x + \tau(x) e_2 - (x + he_1 + h\varepsilon(h)), -e_1 + h\mathcal{R}D^2 d(x) e_1 \rangle + \mathrm{o}(h) \\
&= h(1 - \tau(x)\kappa(x)) + \mathrm{o}(h)\,.
\end{aligned}$$

But $1 - \tau(x)\kappa(x) > 0$. So, $\phi_h(0) > 0$ for h small enough, say $0 < h < r_0$. Moreover,

$$\begin{aligned}
\phi_h(s) &= \\
&= \langle \tau(x) e_2 + s\dot{\boldsymbol{x}}(0) - he_1, -e_1 + h\mathcal{R}D^2 d(x) e_1 \rangle + \mathrm{o}(h) + \mathrm{o}(s) \\
&= -s \langle \dot{\boldsymbol{x}}(0), e_1 \rangle + h(1 - \tau(x)\kappa(x)) + \mathrm{o}(h) + \mathrm{o}(s)
\end{aligned} \tag{51}$$

Since $\langle \dot{\boldsymbol{x}}(0), e_1 \rangle > 0$, there exists $\bar{s} \in (0, \eta]$ such that

$$\phi_h(s) \leq -\frac{s}{2} \langle \dot{\boldsymbol{x}}(0), e_1 \rangle + h(1 - \tau(x)\kappa(x)) + \mathrm{o}(h) \qquad \forall s \in [0, \bar{s}] \tag{52}$$

So, there exists $\bar{r} \in (0, r_0]$ such that $\phi_h(\bar{s}) < 0$ for every $h \in [0, \bar{r}]$. This proves that, for any $h \in [0, \bar{r}]$, there exists $s(h) \in (0, \bar{s})$ such that

$$\phi_h(s(h)) = \langle \boldsymbol{x}(s(h)) - \boldsymbol{y}(h), \mathcal{R}Dd(\boldsymbol{y}(h)) \rangle = 0\,. \tag{53}$$

Furthermore, recalling (52),

$$0 < s(h) \leq \frac{2}{\langle \dot{\boldsymbol{x}}(0), e_1 \rangle} \Big[h(1 - \tau(x)\kappa(x)) + \mathrm{o}(h) \Big] \qquad \forall h \in [0, \bar{r}]\,, \tag{54}$$

so that $s(h) \to 0$ as $h \downarrow 0$.

Next, observe that, in view of (50), equality (53) can be expressed in intrinsic terms saying that for any point $y \in \partial\Omega$ of a suitable neighborhood of x, say $U \subset V$, satisfying $\langle y - x, e_1 \rangle > 0$, there exists $s_y := s(h_y) > 0$ such that $\langle \boldsymbol{x}(s_y) - y, \mathcal{R}Dd(y)\rangle = 0$. Consequently, $\boldsymbol{x}(s_y) = y + \rho_y Dd(y)$ for some $\rho_y \in \mathbb{R}$, and (48) will be proved if we show $\rho_y > 0$. To this end, observe that

$$h_y = |y - x| + \mathrm{o}(|y - x|) \tag{55}$$

as $\partial\Omega \cap U \ni y \to x$ satisfying $\langle y - x, e_1\rangle > 0$. Also, in view of the above formula and (54),

$$0 < s_y \leq C|y - x| \tag{56}$$

for some constant $C > 0$. So, (49) is proved. Furthermore,

$$\lim_{y \to x} \boldsymbol{x}(s_y) = \bar{x} = x + \tau(x)Dd(x)\,,$$

so that $\rho_y \to \tau(x)$ as $y \to x$. Hence, $\rho_y > 0$ for y sufficiently close to x, which completes the proof of this step.

Step 3: *an estimate for s_y.*
We claim that

$$s_y = \frac{1 - \tau(x)\kappa(x)}{\langle \dot{\boldsymbol{x}}(0), e_1\rangle}|y - x| + \mathrm{o}(|y - x|) \tag{57}$$

as $\partial\Omega \cap U \ni y \to x$ satisfying $\langle y - x, e_1\rangle > 0$. Indeed, (51) yields

$$0 = -s_y\langle \dot{\boldsymbol{x}}(0), e_1\rangle + h_y(1 - \tau(x)\kappa(x)) + \mathrm{o}(h_y) + \mathrm{o}(s_y)\,.$$

The above identity yields the desired result thanks to (55) and (56).

Step 4: *an upper bound for ρ_y.*
We claim that

$$\rho_y \leq \tau(x) + \frac{1 - \tau(x)\kappa(x)}{\langle \dot{\boldsymbol{x}}(0), e_1\rangle}|y - x| + \mathrm{o}(|y - x|) \tag{58}$$

as $\partial\Omega \cap U \ni y \to x$ satisfying $\langle y - x, e_1\rangle > 0$. Indeed, returning to the parametric representation of $\partial\Omega$ introduced in Step 2, we have, for every $h \in [0, \bar{r}]$,

$$\begin{aligned}
\rho_{\boldsymbol{y}(h)} &= |\boldsymbol{x}(s(h)) - \boldsymbol{y}(h)| = \langle \boldsymbol{x}(s(h)) - \boldsymbol{y}(h), Dd(\boldsymbol{y}(h))\rangle \\
&= \langle \tau(x)e_2 + s(h)\dot{\boldsymbol{x}}(0) - he_1 + \mathrm{o}(h), e_2 + hD^2d(x)(e_1) + \mathrm{o}(h)\rangle \\
&= \tau(x) + s(h)\langle \dot{\boldsymbol{x}}(0), e_2\rangle + \mathrm{o}(h)
\end{aligned}$$

since $0 < s(h) \leq Ch$. In intrinsic notation, $\rho_y = \tau(x) + s_y\langle \dot{\boldsymbol{x}}(0), e_2\rangle + \mathrm{o}(h_y)$ for every $y \in \partial\Omega \cap U$ satisfying $\langle y - x, e_1\rangle > 0$. Since $|\langle \dot{\boldsymbol{x}}(0), e_2\rangle| \leq 1$, our claim follows in view of (55) and (57).

Step 5: *a global bound.*
We will now derive the estimate

$$\frac{1-\tau(x)\kappa(x)}{\langle \dot{\boldsymbol{x}}(0), e_1\rangle} \le \frac{K}{2}(\operatorname{diam}\Omega)^2 \tag{59}$$

that is a delicate one, since $\dot{\boldsymbol{x}}(0) = -n_1$ and $e_1 = \mathcal{R}^{-1}Dd(x)$ also depend on x. Let p_1 and n_1 be as in Step1. Then, the point $z := \bar{x} - \tau(x)p_1$ belongs to $\Pi(\bar{x})$. Moreover, $Dd(z) = p_1$. So,

$$z - x = -\tau(x)(Dd(z) - Dd(x)) = \tau(x)(e_2 - p_1)\ .$$

Also,

$$|Dd(z) - (Dd(x) + D^2d(x)(z-x))| \le K|z-x|^2\ .$$

Therefore, recalling that $D^2d(x) = -\kappa(x)e_1 \otimes e_1$,

$$\begin{aligned} |(I - \tau(x)\kappa(x)e_1 \otimes e_1)(p_1 - e_2)| &\le K\tau^2(x)|p_1 - e_2| \\ &\le \frac{K}{4}(\operatorname{diam}\Omega)^2|p_1 - e_2|^2\ . \end{aligned}$$

Since the matrix $I - \tau(x)\kappa(x)e_1 \otimes e_1$ is positive definite with eigenvalues 1 and $1 - \tau(x)\kappa(x) > 0$, this proves that

$$(1 - \tau(x)\kappa(x))|p_1 - e_2| \le \frac{K}{4}(\operatorname{diam}\Omega)^2|p_1 - e_2|^2\ .$$

Now, recall that $p_1 \neq e_2$ to conclude

$$1 - \tau(x)\kappa(x) \le \frac{K}{4}(\operatorname{diam}\Omega)^2|p_1 - e_2|\ . \tag{60}$$

Next, the identity $\langle \dot{\boldsymbol{x}}(0), p_1 - e_2\rangle = 0$ implies that $\dot{\boldsymbol{x}}(0) = \lambda\mathcal{R}(p_1 - e_2)$ for some $\lambda \in \mathbb{R}$ satisfying $|\lambda| = 1/|p_1 - e_2|$. Therefore,

$$\langle \dot{\boldsymbol{x}}(0), e_1\rangle = |\lambda\langle \mathcal{R}(p_1 - e_2), e_1\rangle| = \frac{|\langle p_1 - e_2, e_2\rangle|}{|p_1 - e_2|} = \frac{1 - \langle p_1, e_2\rangle}{|p_1 - e_2|}\ .$$

Since $|p_1 - e_2|^2 = 2(1 - \langle p_1, e_2\rangle)$, we have $\langle \dot{\boldsymbol{x}}(0), e_1\rangle = |p_1 - e_2|/2$. Combining the last equality and (60) proves our claim (59).

Step 6: *conclusion.*
Possibly reducing the neighborhood U of x that we found in the previous steps, the above construction shows that, for every $y \in U \cap \partial\Omega$ satisfying $\langle y - x, e_1\rangle > 0$,

$$\tau(y) \le \rho_y \le \tau(x) + K(\operatorname{diam}\Omega)^2|y - x|\ .$$

By a similar reasoning, there exists another neighborhood U' of x such that, for every $y \in U' \cap \partial\Omega$ satisfying $\langle y - x, e_1 \rangle < 0$,

$$\tau(y) \le \tau(x) + K(\operatorname{diam}\Omega)^2 |y - x| \ .$$

Putting these estimates together completes the proof of the lemma. □

We are now ready to prove the Lipschitz continuity of τ.

Proof of Theorem 2.12—The conclusion will follow by known results in nonsmooth analysis, once we will have extended estimate (45) to the ε-neighborhood $\Omega^\varepsilon := \{x \in \Omega \ : \ 0 < d(x) < \varepsilon\}$ of $\partial\Omega$. In fact, let $\varepsilon > 0$ be such that $d \in \mathcal{C}^{2,1}(\Omega^\varepsilon)$. We claim that a constant $C > 0$ exists so that every $x \in \Omega^\varepsilon$ has a ball $B_\rho(x) \subset \Omega^\varepsilon$ such that

$$\tau(y) \le \tau(x) + C|y - x| \qquad \forall y \in B_\rho(x) \, . \tag{61}$$

To show this, observe that for every $y \in \Omega^\varepsilon$ such that $\Pi(y)$ is in the neighborhood U of $\Pi(x)$ provided by Lemma A.1, we have, in view of (45),

$$\begin{aligned}
\tau(y) &= \tau(\Pi(y)) - d(y) \\
&\le \ \tau(\Pi(x)) + K(\operatorname{diam}\Omega)^2 |\Pi(y) - \Pi(x)| - d(y) \\
&\le \ \tau(x) + K(\operatorname{diam}\Omega)^2 \|D\Pi\|_{\infty,\Omega^\varepsilon} |y - x| + d(x) - d(y) \, .
\end{aligned}$$

Our claim (61) follows with $C = K(\operatorname{diam}\Omega)^2 \|D\Pi\|_{\infty,\Omega^\varepsilon} + 1$.

Next, we will derive the bound

$$|p| \le C \qquad \forall p \in \partial_P \tau(x) \quad \forall x \in \Omega^\varepsilon \, , \tag{62}$$

where $\partial_P \tau(x)$ denotes the proximal subgradient of τ at x and C is the constant that appears in (61). Then, by [11, Theorem 7.3, p. 52], such an estimate will imply that τ is Lipschitz in Ω^ε, and so on $\partial\Omega$ as well. To check (62), recall that a vector $p \in \mathbb{R}^2$ belongs to $\partial_P \tau(x)$ if and only if there exist numbers $\sigma, \eta > 0$ such that

$$\tau(y) \ge \tau(x) + \langle p, y - x \rangle - \sigma |y - x|^2 \qquad \forall y \in B_\eta(x) \, ,$$

see [11, Theorem 2.5, p. 33]. Now, combine the above inequality with (61) to obtain

$$\langle p, y - x \rangle \le C|y - x| + \sigma |y - x|^2$$

whenever $|y - x| < \min\{\rho, \eta\}$. This implies (62) and completes the proof. □

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