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Genericity of Algebraically Observable Polynomial Systems*

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Abstract

E. Sontag has introduced the concept of algebraic observability for n-dimensional polynomial systems. It is a stronger notion than the usual concept of observability and implies the existence of a polynomial expression of the state variables in terms of a finite number of derivatives of the output function. We prove that algebraic observability is a generic property for polynomial systems of bounded degrees. Explicit geometric characterizations of algebraic observability via polynomial embeddings are derived and it is shown that the state variables of an algebraically observable system can be expressed as a polynomial in the first $2n+1$ derivatives of the output.

Keywords: Polynomial systems, observability, observers, embeddings, genericity.

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1 Introduction

The complete states of dynamical systems arising in biology, economics, population dynamics and in fact in mechanical systems are seldom available for measurement. Measurements are made of observed quantities and from these measurements it is desired to reconstruct the complete set of states of the system. In many, but not all, mechanical systems controls may be utilized to make the states reconstructions feasible. However, for systems arising in biology or medicine it is often impossible, infeasible or immoral to effect the states with direct control. Thus one is faced with the problem of when a set of measurements suffices to reconstruct the state. At the level of mathematics there are many ways of specifying how to reconstruct the state. In this paper we look at a very natural problem of when we can reconstruct the state of a system that is described by polynomials, using only polynomials in the measurement data. We will prove that that this a generic property for polynomial systems, provided the number of measurements is sufficiently large. While at first glance it would seem more natural to ask for a reconstruction of the state using algebraic functions, more thought reveals that algebraic functions require the solution of polynomial equations which may be difficult to construct.

More precisely, we consider single-output polynomial systems of the form

$$\dot{x} = f(x), \; y = h(x) \tag{1.1}$$

on $\mathbb{K}^n$, where $f \in \mathbb{K}^n[x_1, \ldots, x_n], h \in \mathbb{K}[x_1, \ldots, x_n]$ are polynomials, and $\mathbb{K}$ denotes the field of real and complex numbers, respectively. For such systems various concepts of observability have been proposed in the literature, that can all be expressed in terms of conditions on the Lie–derivatives of the system. The best well-known notions are those of observability and algebraic observability, respectively. While the set-theoretic definition of observability is usually a too weak concept to work with, it is Sontag's definition of algebraically observable systems that seems particularly suited to polynomial systems. Recall, that the ith Lie–derivative is recursively defined as

$$L_f^i h = L_f(L_f^{i-1} h), \; L_f^0 h := h.$$

Here $L_f h(x) = dh(x) f(x), x \in \mathbb{K}^n$, denotes the directional derivative of h with respect to the vector field f. Thus $L_f^i h$ is a polynomial for all $i \in \mathbb{N}_0$.

The standard definition of *observability* of (1.1) then requires that the state vector $x \in \mathbb{K}^n$ is uniquely determined from knowledge of the Lie–derivatives $L_f^i h, i \in \mathbb{N}_0$. It is immediately shown by the Hilbert Basis Theorem, that (1.1) is observable if and only if there exists a finite integer $N \in \mathbb{N}_0$ such that for all $x, y \in \mathbb{K}^n$:

$$L_f^i h(x) = L_f^i h(y) \quad for \; i = 0, \cdots, N \; \Rightarrow \; x = y.$$

Moreover, (1.1) is called *algebraically observable* (in the sense of Sontag; see [16]) if the *observation algebra* $Q_{f,h} \subset \mathbb{K}[x]$

$$Q_{f,h} := \{p(h, L_f h, \cdots, L_f^N h) \mid p \in \mathbb{K}[y_0, \cdots, y_N], \; N \in \mathbb{N}_0\}$$

satisfies

$$Q_{f,h} \ = \mathbb{K}[x].$$

Equivalently, (1.1) is algebraically observable if and only if there exists a finite integer $N \in \mathbb{N}_0$ and a polynomial $p \in \mathbb{K}[y_0, \cdots, y_N]$ such that

$$x \ = \ p(h(x), \cdots, L_f^N h(x)) \quad \forall x \in \mathbb{K}^n \tag{1.2}$$

Since the ith time–derivative $y^{(i)}$ of the output function $y = h(x)$ along a trajectory x of (1.1) is given as $y^{(i)} = L_f^i h$, the above condition implies that *the state vector x can be obtained from the first $N + 1$ derivatives of the output function via a suitable polynomial.*

Algebraic observability is stronger than observability. The system

$$\dot{x} = 0 \quad y = x^3 \tag{1.3}$$

is an example of an observable system that is not algebraically observable.

In this paper we show that algebraically observable systems are generic. Of course, this implies that observable polynomial systems are generic, too. More precisely, we show the following results.

Theorem 1.1. *Let $d_0, \cdots, d_n \in \mathbb{N}$ be given. Let*

$$\mathcal{P}(n, d) = \{(f, h) \in \mathbb{K}^n[x] \times \mathbb{K}[x] \mid \deg h \le d_0, \deg f_i \le d_i, i = 1, \cdots, n\}. \tag{1.4}$$

There exists a nonempty Zariski–open subset Ω of polynomial systems of bounded degrees such that every system $(f, h) \in \Omega$ is algebraically observable. Thus the set of algebraically observable polynomial systems of bounded degrees is an open and dense subset of the finite dimensional vector space $\mathcal{P}(n, d)$.

Theorem 1.2. *Let $d_0, \cdots, d_n \in \mathbb{N}$ be given and $N \ge 2n$. There exists a non-empty Zariski–open subset $\Omega \subset \mathcal{P}(n, d)$ of polynomial systems of bounded degrees such that for every $(f, h) \in \Omega$ the following holds true: For every pair $(f, h) \in \Omega$ there exists a polynomial $\pi \in \mathbb{K}[y_0, \cdots, y_N]$ with*

$$x \ = \ \pi(h(x), \cdots, , L_f^N h(x)) \quad \forall x \in \mathbb{K}^n \tag{1.5}$$

The next result shows that algebraic observability happens generically in families of polynomial systems.

Theorem 1.3. *Let $d_0, \cdots, d_n, D \in \mathbb{N}$ be given, $X \subset \mathbb{K}^p$ an irreducible algebraic subvariety, and let $\mathcal{F}$ denote the finite dimensional $\mathbb{K}$–vector space of polynomial maps $\Gamma : X \to \mathcal{P}(n, d)$ of degree $\le D$. Then there exists an open and dense subset $\Omega \subset \mathcal{F}$ such that for each polynomial map $\Gamma \in \Omega$ the following holds true: There exists an open and dense subset $\mathcal{U} \subset X$ such that for each parameter value $u \in \mathcal{U}$ the system $\Gamma(u)$ is algebraically observable.*

A similar genericity statement holds as well for smoothly parametrized polynomial maps of bounded degrees, with respect to the strong Whitney topology on $C^\infty(\mathbb{K}^p, \mathcal{P}(n, d))$. In particular, it is a generic property for polynomial systems of bounded degrees, that the state can be polynomially expressed in terms of the first $N+1$ derivatives $y, \cdots, y^{(N)}$ of the output, provided $N \ge 2n$. Similar results have been shown by F. Takens and D. Aeyels for smooth systems, and by Gauthier and Kupka for real analytic systems; see [1, 2, 10, 18]. There is also a very nice recent survey paper by E. Sontag [17] with potential applications to biology in mind. Our results however do not follow from these earlier contributions.

2 Preliminaries

In this section we recall some basic definitions and facts from complex algebraic geometry. See e.g. the books by Mumford [14] and Hartshorne [11] for further details.

Let $\mathbb{C}$ denote the field of complex numbers. A complex algebraic subvariety $V \subset \mathbb{C}^n$ is the zero set

$$V = \{z \in \mathbb{C}^n \mid p_1(z) = \cdots = p_r(z) = 0\}$$

of a finite number of polynomials $p_1, \cdots, p_r \in \mathbb{C}[z_1, \cdots, z_n]$. V is also called a Zariski–closed subset of $\mathbb{C}^n$ and its complement $U = \mathbb{C}^n - V$ a Zariski–open subset. In particular, Zariski–open subsets of $\mathbb{C}^n$ are open and dense with respect to the standard Euclidean topology of $\mathbb{C}^n$. A *constructible subset* $X \subset \mathbb{C}^n$ is a Boolean combination of a finite number of Zariski–closed subsets. More explicitly, a subset of $\mathbb{C}^n$ is locally closed if it is the intersection of a Zariski–open and a Zariski–closed subset. Any union of finitely many locally closed subsets is a constructible set. The class of constructible subsets has nice properties. The union, the intersection and the Zariski–closure of finitely many constructible subsets is constructible. Moreover, the complement $V - W$ of two constructible subsets is again constructible. If $f : \mathbb{C}^n \to \mathbb{C}^m$ is a *polynomial map*, i.e. if $f = (f_1, \cdots, f_m)$ with $f_1, \cdots, f_m \in \mathbb{C}[z_1, \cdots, z_n]$ polynomials, then f maps constructible subsets $X \subset \mathbb{C}^n$ onto constructible subsets $Y = f(X) \subset \mathbb{C}^m$. Moreover, for the Zariski–closure $\overline{f(X)}$ in $\mathbb{C}^m$

$$dim(\overline{f(X)}) \leq dim(X)$$

holds for all constructible subsets $X \subset \mathbb{C}^n$. In particular,

$$dim(\overline{X}) = dim(X)$$

for each constructible subset $X \subset \mathbb{C}^n$ and corresponding Zariski–closure $\overline{X}$. Finally, a constructible subset $X \subset \mathbb{C}^n$ is Zariski–closed if and only if X is closed in the Euclidean topology of $\mathbb{C}^n$.

Complex polynomial maps are special holomorphic functions and one can therefore apply differential geometric techniques to study such maps. The following definition is well–known from differential geometry.

Definition 2.1. *A polynomial map $f : \mathbb{C}^n \to \mathbb{C}^N$ is called a proper embedding if and only if the following conditions hold:*

(i) f is injective

(ii) f is an immersion, i.e. the derivative $df(z) : \mathbb{C}^n \to \mathbb{C}^N$ is an injective linear map, for all $z \in \mathbb{C}^n$.

(iii) $f : \mathbb{C}^n \to \mathbb{C}^N$ is proper, i.e. the preimage $f^{-1}(K)$ is compact for each compact subset $K \subset \mathbb{C}^m$.

The above conditions on f imply that the image $f(\mathbb{C}^n)$ is a closed complex submanifold of $\mathbb{C}^N$. Even more is true as the following characterization shows; see [11].

Proposition 2.2. *A polynomial map $f : \mathbb{C}^n \to \mathbb{C}^N$ is a proper embedding if and only if the following conditions hold:*

(1) f is an injective immersion

(2) $f(\mathbb{C}^n)$ is a Zariski–closed subset of $\mathbb{C}^N$ and f maps $\mathbb{C}^n$ algebraically isomorphic to $f(\mathbb{C}^n)$. Equivalently, $f(\mathbb{C}^n)$ is Zariski–closed and there exists a polynomial map $\pi : \mathbb{C}^N \to \mathbb{C}^n$ with

$$\pi(f(z)) \; = \; z, \quad \forall z \in \mathbb{C}^n. \tag{2.1}$$

The above result allows for a purely algebraic reformulation, which leads to the standard algebraic definition of a (proper) embedding.

Proposition 2.3. *A polynomial map $f : \mathbb{C}^n \to \mathbb{C}^N$ is a proper embedding if and only if:*

$$f^*(\mathbb{C}[y_1, \cdots, y_N]) \; = \mathbb{C}[x_1, \cdots, x_n], \tag{2.2}$$

where

$$f^*(\mathbb{C}[y]) \; = \{p(f(x)) \in \mathbb{C}[x] \mid p \in \mathbb{C}[y]\}. \tag{2.3}$$

If N is sufficiently large, then there is a beautiful and more explicit recent characterization of complex polynomial proper embeddings due to Z. Jelonek that shows that they are unknotted maps. See [12] for a proof of the next result.

Theorem 2.4. *Let $N \geq 2n$. A complex polynomial map $f : \mathbb{C}^n \to \mathbb{C}^N$ is a proper embedding if and only if there exists a polynomial automorphism $F : \mathbb{C}^N \to \mathbb{C}^N$ with $F|_{\mathbb{C}^n \times \{0\}} = f$. Thus in these dimensions, proper embeddings always extend to polynomial automorphisms on $\mathbb{C}^N$.*

The following two examples serve to illustrate some important phenomena.

Example 2.5. 1. The polynomial map $f : \mathbb{C} \to \mathbb{C}^2, f(z) = (z^2, z^3)$ is injective and proper. The image is the Zariski–closed subset

$$Y = f(\mathbb{C}) = \{(x, y) \in \mathbb{C}^2 \mid x^3 = y^2\}. \tag{2.4}$$

However, f is not a proper embedding as Y is not algebraically isomorphic to $\mathbb{C}$. In fact, the inverse $f^{-1} : Y \to \mathbb{C}, f^{-1}(x, y) = y/x$ is rational and not polynomial.

2. This example shows that the above characterization does not hold for *real* polynomials. Let $f : \mathbb{R} \to \mathbb{R}^2, f(x) = (1 + x^2, x(1 + x^2))$. Then f is an injective, proper immersion and therefore defines an embedding of $\mathbb{R}$. However,

$$f(\mathbb{R}) = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 = v^3, v \geq 1\} \tag{2.5}$$

is not a real algebraic subvariety of $\mathbb{R}^2$. The induced map on complex points $f : \mathbb{C} \to \mathbb{C}^2$ has the image set

$$f(\mathbb{C}) = \{(u, v) \in \mathbb{C}^2 \mid u^2 + v^2 = v^3\}, \tag{2.6}$$

which is Zariski–closed, but not algebraically isomorphic to $\mathbb{C}$. In fact, f is not injective, although it defines a proper immersion on $\mathbb{C}$. Therefore $f : \mathbb{C} \to \mathbb{C}^2$ is not a proper embedding and the inverse map $f^{-1} : f(\mathbb{C}) \to \mathbb{C}, f^{-1}(u, v) = v/u$ is rational, but not polynomial.

The above example shows that the above characterizations of proper embeddings make essential use of the fact that we work over an algebraically closed field. For our purposes the following sufficient condition for a real algebraic map to be an embedding will suffice.

Proposition 2.6. *Let $f : \mathbb{R}^n \to \mathbb{R}^N$ be a real algebraic polynomial map such that the extension $f : \mathbb{C}^n \to \mathbb{C}^N$ is a proper, injective immersion on $\mathbb{C}^n$. Then*

$$f^*(\mathbb{R}[y_1, \cdots, y_N]) = \mathbb{R}[x_1, \cdots, x_n]. \tag{2.7}$$

In particular, there exists a real polynomial $\pi \in \mathbb{R}[y_1, \cdots, y_N]$ with

$$\pi(f(x)) = x, \quad \forall x \in \mathbb{R}^n. \tag{2.8}$$

Proof. From the above,

$$f^*(\mathbb{C}[y_1, \cdots, y_N]) = \mathbb{C}[x_1, \cdots, x_n]. \tag{2.9}$$

For any $\pi \in \mathbb{R}[y_1, \cdots, y_N] \subset \mathbb{C}[y_1, \cdots, y_N]$ there exists $F \in \mathbb{C}[y_1, \cdots, y_N]$ with $F \circ f = \pi$. But then the real polynomial

$$G := (F + \overline{F})/2 \in \mathbb{R}[y_1, \cdots, y_N] \tag{2.10}$$

satisfies $G \circ f = \pi$. The result follows. □

The crucial condition for a complex polynomial map to qualify as an algebraic embedding is the properness. It is usually hard to verify. For the sake of completeness we state a well–known algebraic characterization.

Proposition 2.7. *A polynomial map $f : \mathbb{C}^n \to \mathbb{C}^N$ is proper if and only if it is a finite morphism, i.e. if and only if the polynomial ring $\mathbb{C}[x_1, \cdots, x_n]$ is an integral ring extension over $\mathbb{C}[y_1, \cdots, y_N]$.*

Recall that a ring extension $\mathbb{C}[x]$ of $f^*(\mathbb{C}[y]) \subset \mathbb{C}[x]$ is called integral, if every element $p \in \mathbb{C}[x]$ satisfies a monic equation

$$p^m + a_{m-1}p^{m-1} + \cdots + a_0 = 0 \tag{2.11}$$

with coefficients $a_i \in f^*(\mathbb{C}[y])$. Equivalently, as the integral elements of $\mathbb{C}[x]$ with respect to $f^*(\mathbb{C}[y])$ forming a subring of $\mathbb{C}[x]$, we conclude:

Corollary 2.8. *A polynomial map $f : \mathbb{C}^n \to \mathbb{C}^N$ is proper if and only if every component x_i satisfies a monic polynomial equation*

$$x_i^m + a^{(i)}_{m_i-1}(f(x))x_i^{m_i-1} + \cdots + a_0(f(x)) = 0 \tag{2.12}$$

for all $i = 1, \cdots, n$, $x \in \mathbb{C}^n$, $a_j^{(i)} \in \mathbb{C}[y]$.

Thus the properness of f is equivalent to the state variables $x_1, \cdots, x_n$ being monic algebraic functions of $y_1, \cdots, y_N = f_1(x), \cdots, f_N(x)$.

3 Genericity of Observability

In this section we prove our main results concerning genericity of algebraic observability. Let

$$\mathcal{P}_{m,d}(n) = \{g \in \mathbb{K}^m[x] \mid \deg(g) \leq d\} \tag{3.1}$$

Thus $\mathcal{P}_{m,d}(n)$ is a finite–dimensional $\mathbb{K}$–vector space. Let $\Phi^N(f,h) : \mathbb{K}^n \to \mathbb{K}^N$ be defined by

$$\Phi^N(f,h)(x) = (h(x), \cdots, L_f^{N-1}h(x)). \tag{3.2}$$

Before stating and proving our main result we first show some lemmas.

Lemma 3.1. *For any integers $N, n \geq 1$ consider*

$$X = \{(g_1, \cdots, g_N, x, y) \in \mathcal{P}_{n,d}^N(2n) \times \mathbb{K}^{2n} \mid g_i(x,y)^t(x-y) = 0,\ i = 1, \cdots, N, x \neq y\}. \tag{3.3}$$

Then the Zariski–closure $\overline{X}$ is a closed affine algebraic subvariety of $\mathcal{P}_{n,d}^N(2n) \times \mathbb{K}^{2n}$ and X consists entirely of nonsingular points of $\overline{X}$.

Proof. Let $U := \{(x,y) \in \mathbb{K}^{2n} \mid x \neq y\}$. The map

$$\begin{array}{rcl} \phi : \mathcal{P}_{n,d}^N(2n) \times U & \to & \mathbb{K}^{nN} \times (\mathbb{K}^n - \{0\}) \\ (g_1, \cdots, g_N, x, y) & \mapsto & (g_1(x,y), \cdots, g_N(x,y), x-y) \end{array} \tag{3.4}$$

is a submersion. Moreover, 0 is a regular value for the map

$$\begin{array}{rcl} \psi : \mathbb{K}^{nN} \times (\mathbb{K}^n - \{0\}) & \to & \mathbb{K}^N \\ (u_1, \cdots, u_N, z) & \mapsto & (u_1^t z, \cdots, u_N^t z). \end{array} \tag{3.5}$$

Therefore 0 is a regular value for the composed map $\psi \circ \phi$ and the result follows, as $X = (\psi \circ \phi)^{-1}(0)$. □

Lemma 3.2. *For any integers $N > 2n,\ n \geq 1$ the set*

$$S = \{(g_1, \cdots, g_N) \in \mathcal{P}_{n,d}^N(2n) \mid \exists x \neq y \text{ with } g_i(x,y)^t(x-y) = 0, \forall\ i = 1, \cdots, N\}. \tag{3.6}$$

is a constructible algebraic set of dimension

$$dim(S) \leq dim(X) < N dim(\mathcal{P}_{n,d}(2n)) \tag{3.7}$$

Proof. S is the projection of the algebraic set X and therefore constructible. Moreover,

$$dim(S) \leq dim(X) = N dim(\mathcal{P}_{n,d}(2n)) + 2n - N < N dim(\mathcal{P}_{n,d}(2n)). \tag{3.8}$$

Q.E.D. □

We now prove

Theorem 3.3. *The set of polynomial systems*

$$V_{n,d} = \{(f,h) \in \mathcal{P}_{n,d}(n) \times \mathcal{P}_{1,d}(n) \mid \Phi^N(f,h) \text{ is an injective immersion}\} \tag{3.9}$$

is the complement of a closed proper algebraic subvariety of $\mathcal{P}_{n,d} \times \mathcal{P}_{1,d}$, provided $N > 2n$.

Proof. Consider the polynomial map

$$\Phi : \mathcal{P}_{n,d}(n) \times \mathcal{P}_{1,d}(n) \to \mathcal{P}^N_{1,D}(n)$$

defined by $\Phi(f,h) = (h, \cdots, L_f^{N-1}h)$, $D := N(d-1)+1$.

There exist unique polynomials $g_1, \cdots, g_N$ in $2n$ variables, such that for all x, y

$$L_f^i h(x) - L_f^i h(y) = g_i(x,y)^t (x-y).$$

Then $\Phi(f,h) : \mathbb{K}^n \to \mathbb{K}^N$ is injective if and only if $(g_1, \cdots, g_N) \notin S$. Note that

$$\deg(L_f^i h) \leq (i+1)d - i \ \leq D$$

and therefore $\deg(g_i) \leq (i+1)(d-1) \ \leq D$. We conclude that (f,h) is observable if and only if $\Phi(f,h) \notin S$. Therefore the set of observable systems in $\mathcal{P}_{n,d}(n) \times \mathcal{P}_{1,d}(n)$ is the complement of a constructible set of codimension at least $N-2n$. For $N-2n > 0$ there exists therefore a nonempty Zariski–open (and hence open and dense) set of systems (f,h) which is observable.

For the immersion property consider similarly the algebraic map

$$\Psi : \mathcal{P}_{n,d}(n) \times \mathcal{P}_{1,d}(n) \to \mathcal{P}^N_{n,D}(n)$$

defined by $\Psi(f,h) = (dh, dL_f h, \cdots, dL_f^{N-1}h)^t$. Let

$$\Sigma = \{(\pi_1, \cdots, \pi_N) \in \mathcal{P}^N_{n,D}(n) \mid \exists x \in \mathbb{K}^n \ \textit{such that} \ rk(\pi_1^t(x), \cdots, \pi_N^t(x)) < n\}$$

and

$$\tilde{\Sigma} = \{(\pi_1, \cdots, \pi_N, x) \in \mathcal{P}^N_{n,d}(n) \times \mathbb{K}^n \mid rk(\pi_1^t(x), \cdots, \pi_N^t(x)) < n\}$$

Obviously, Σ is the projection of the algebraic set $\tilde{\Sigma}$ onto first factor. Moreover, it is easily seen (using the linearity in the first component) that

$$(\pi_1, \cdots, \pi_N, x) \mapsto (\pi_1^t(x), \cdots, \pi_N^t(x))$$

is a submersion. Since the set of matrices

$$\{A \in \mathbb{K}^{N \times n} \mid rkA < n\}$$

is algebraic of codimension $N - n + 1$ we conclude that

$$\begin{aligned} dim(\tilde{\Sigma}) &= dim(\mathcal{P}^N_{n,D}(n)) + n - (N-n+1) \\ &= dim(\mathcal{P}^N_{n,D}(n)) + 2n - (N+1) \end{aligned} \tag{3.10}$$

Therefore

$$dim(\Sigma) \leq dim(\tilde{\Sigma}) < dim(\mathcal{P}^N_{n,D}(n)),$$

provided $N+1 > 2n$. This implies that

$$\{(f,h) \in \mathcal{P}_{n,d}(n) \times \mathcal{P}_{1,d}(n) \mid \Phi(f,h) \ \textit{is an immersion}\} = \Psi^{-1}(\mathcal{P}^N_{n,D}(n) - \Sigma)$$

contains a Zariski–open subset for $N+1 > 2n$. Finally note that for $N > n$ the set $V_{n,d}$ is nonempty, as it contains the observable linear systems. Therefore $V_{n,d}$ contains an open and dense subset, provided $N > 2n$. This completes the proof. □

We now turn to the analysis of the properness assumption. We first derive a sufficient condition for properness that has the advantage of being more easily verified than the algebraic characterization. For any polynomial $p \in \mathbb{C}[x]$ let $p^{top} \in \mathbb{C}[x]$ denote the leading term, i.e. if

$$p = p_0 + \cdots + p_d$$

is the canonical decomposition into homogeneous components p_i of degree i, then $p^{top} = p_d$. Thus for e.g. $p(x,y) = 1 + x + y^2 + x^3y - xy^3$ we have $p^{top}(x,y) = x^3y - xy^3$.

Proposition 3.4. *Let $f = (f_1, \cdots, f_N) : \mathbb{C}^n \to \mathbb{C}^N$ be a polynomial map with degrees $d_i := \deg f_i > 0$ for $i = 1, \cdots, N$. If the preimages of f_i^{top} satisfy*

$$(f_1^{top})^{-1}(0) \cap \cdots \cap (f_N^{top})^{-1}(0) \ = \{0\} \quad (or = \emptyset) \tag{3.11}$$

then f is proper.

Proof. Suppose f is not proper, i.e. there exists a sequence of points $z_k \in \mathbb{C}^n$ and a unit vector e and a constant $M > 0$ with

$$r_k := |z_k| \to \infty \qquad e_k := z_k/|z_k| \to e$$

$$|f(z_k)| \leq M \quad \forall k \in \mathbb{N}.$$

Here $|\cdot|$ denotes the Euclidean norm. In particular, for $i = 1, \cdots, N$

$$|f_i(z_k)| \leq M \quad \forall k \in \mathbb{N}.$$

But

$$|f_i(z)| = |f_i^{top}(z) + g_i(z)|$$

where $g_i = f_i - |f_i^{top}$ has degree $\deg g_i < \deg f_i = d_i$. Thus for all $k \in \mathbb{N}$

$$\begin{aligned} |f_i(r_k e_k) &= \quad |r_k^{d_i} f_i^{top}(e_k) + g_i(r_k e_k)| \\ &= r_k^{d_i} |f_i^{top}(e_k) + r_k^{-d_i} g_i(r_k e_k)| \end{aligned} \tag{3.12}$$

Since $f_i^{top}(e_k) \to f_i^{top}(e)$ and $r_k^{-d_i} g_i(r_k e_k) \to 0$ we conclude that $f_i^{top}(e) = 0$ for all i with $d_i > 0$. By assumption therefore $f_1^{top}(e) = \cdots = f_N^{top}(e) = 0$, contradiction. □

We can now complete the proof of our main result.

Theorem 3.5. *Let $d_0, \cdots, d_n \in \mathbb{N}$ be given. There exists a nonempty Zariski–open subset Ω of polynomial systems of bounded degrees*

$$\mathcal{P}_{\mathbb{K}}(n,d) = \{(f,h) \in \mathbb{K}^n[x] \times \mathbb{K}[x] \mid \deg h \leq d_0, \deg f_i \leq d_i, i = 1, \cdots, n\} \tag{3.13}$$

such that every system $(f,h) \in \Omega$ is algebraically observable. Thus the set of algebraically observable polynomial systems of bounded degrees is an open and dense subset of the finite dimensional vector space $\mathcal{P}_{\mathbb{K}}(n,d)$.

Proof. Choose $N > 2n$. We first consider the complex case $\mathbb{K} = \mathbb{C}$. Note that the complex algebraic set

$$\tilde{W} = \{(\pi_1, \cdots, \pi_N, x) \in \mathcal{P}_{1,d}^N(n) \times \mathbb{C}^n - \{0\} \mid \pi_1^{top}(x) = \cdots = \pi_N^{top}(x) \ = 0\} \tag{3.14}$$

has codimension N and therefore

$$W = \{(\pi_1, \cdots, \pi_N) \in \mathcal{P}_{1,d}^N(n) \mid (\pi_1^{top})^{-1}(0) \cap \cdots \cap (\pi_N^{top})^{-1}(0) \ = \{0\} \quad (or = \emptyset)\} \tag{3.15}$$

is the complement of a constructible set of codimension at least $N - n$. Therefore it must contain a Zariski–open subset. This shows that for $N > n$ also the preimage

$$\Phi^{-1}(W) = \{(f, h) \in \mathcal{P}(n, d) \mid ((h^{top})^{-1}(0) \cap \cdots \cap ((L_f^{N-1}h)^{top})^{-1}(0) \ = \{0\} \quad (or = \emptyset)\} \tag{3.16}$$

contains a Zariski–open subset, which is nonempty as the linear observable pairs are contained in it.

Altogether we see that

$$\{(f, h) \in \mathcal{P}_{n,d}(n) \times \mathcal{P}_{1,d}(n) \ |\Phi(f, h) \ is\ an\ embedding\} = \Phi^{-1}(V_{n,d}) \cap \Phi^{-1}(W)$$

contains the intersection of nonempty Zariski–open subsets, provided $N > 2n$. By the embedding characterization of algebraic observability, Proposition 2.3, it follows that the class of complex algebraically observable systems contains an open and dense subset $\Omega_{\mathbb{C}}$. This completes the proof in the complex case. For the real case note that $\Omega_{\mathbb{C}}$ contains the linear observable systems and therefore it contains real points. Since $\mathcal{P}_{\mathbb{R}}(n, d)$ is nonsingular, irreducible this implies that the set of real points of $\Omega_{\mathbb{C}}$, i.e. the subset $\Omega_{\mathbb{R}}$, is dense in $\mathcal{P}_{\mathbb{R}}(n, d)$. This completes the proof.

□

The proof of Theorem 1.3 follows easily from the above one. In fact, the arguments above show for $N > 2n$ the existence of a proper algebraic, Zariski closed subvariety $V \subset \mathcal{P}_{\mathbb{K}}(n, d)$ such that the complement $\Omega_{\mathbb{K}} = \mathcal{P}_{\mathbb{K}}(n, d) - V$ consists of algebraically observable systems. The set of polynomial maps $\Gamma : X \to \mathcal{P}_{\mathbb{K}}(n, d)$ of degree $\leq D$ with $\Gamma(X) \subset V$ is then a Zariski-closed proper algebraic subvariety of the affine space of all maps

$$\Gamma : X \to \mathcal{P}_{\mathbb{K}}(n, d)$$

of degree $\leq D$. Thus its complement is open and dense and has the claimed property. This completes the proof of Theorem 1.3.

4 Application to Tracking Observers

In this section we explain a simple application to observer design. The approach here follows that by [13, 9], see also [3], in the smooth or real analytic case. Given a polynomial system on $\mathbb{K}^n$

$$\dot{x} = f(x), \ y = h(x) \tag{4.1}$$

a *polynomial tracking observer* for (4.1) is a system on $\mathbb{K}^N$

$$\dot{z} = \alpha(z) + \beta(y), \;\; \tilde{x} = \gamma(z) \tag{4.2}$$

together with an embedding $\phi : \mathbb{K}^n \to \mathbb{K}^N$, $z = \phi(x)$, such that for all initial states $x(0) \in \mathbb{K}^n$, $z(0) \in \mathbb{K}^N$

$$z(0) = \phi(x(0)) \;\Rightarrow\; z(t) = \phi(x(t)) \quad \forall\, t \in \mathbb{R}.$$

Let $\mathcal{C} \subset \mathbb{K}^n$ denote a positively invariant compact subset for (4.1). We say that (4.2) is a *locally asymptotic observer* for $\mathcal{C}$, if there exists an $\epsilon > 0$ such that for all initial conditions $x(0) \in \mathbb{K}^n$, $z(0) \in \mathbb{K}^N$

$$|z(0) - \phi(x(0))| < \epsilon \;\Rightarrow |z(t) - \phi(x(t))| \to 0 \quad for\ t \to \infty$$

The embedding dimension N is called the order of the observer. The idea behind this definition is that information about the initial state $x(0)$ of the system may be available only through knowledge of $\phi(x(0))$ and one wants to deduce information about the state trajectory $x(t)$ through that of the observer state $z(t)$.

We prove the generic existence of polynomial tracking observers.

Theorem 4.1. *Let $N > 2n$. Then, for a generic set of polynomial systems (4.1), there exist a* polynomial *tracking observer of order N. Moreover, for every compact positively invariant subset $\mathcal{C}$ of $\mathbb{K}^n$, there exists a locally asymptotic observer (4.2) for (4.1).*

Proof. To prove the theorem we assume that (4.1) satisfies the generic property of algebraic observability. For $N > 2n$ let $\Phi^N(f,h) : \mathbb{K}^n \to \mathbb{K}^N$ denote the embedding

$$\Phi^N(f,h)(x) = (h(x), \cdots, L_f^{N-1}h(x)). \tag{4.3}$$

By algebraic observability, there exists a polynomial map $\psi : \mathbb{K}^N \to \mathbb{K}^n$ such that for all x

$$x = \psi(h(x), \cdots, L_f^{N-1}h(x)). \tag{4.4}$$

Therefore there exists a polynomial $\phi : \mathbb{K}^N \to \mathbb{K}$ such that for all x

$$L_f^N h(x) = \phi(h(x), \cdots, L_f^{N-1}h(x)). \tag{4.5}$$

Consider now the associated polynomial system

$$\dot{z} = (A - JC)z + Jy + b\phi(z), \;\; \tilde{x} = Cz \tag{4.6}$$

where $\phi(z)$ is the above polynomial and

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & \dots & 1 \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix} \in \mathbb{K}^N, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{K}^N,$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{K}^{1\times N}, \quad J = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \in \mathbb{K}^N,$$

is such that the characteristic polynomial of $A - JC$

$$\chi_J(s) = s^N + a_1 s^{N-1} + \cdots + a_N$$

is Hurwitz. It is immediately verified by inspection that Φ^N maps solution of (4.1) to solutions of (4.6). Moreover, (4.6) is a tracking observer for (4.1). Let $\mathcal{C} \subset \mathbb{K}^n$ denote a fixed compact positively invariant subset for (4.1). Then the image $\Phi^N(\mathcal{C})$ is a positively invariant subset for (4.6). By compactness of $\Phi^N(\mathcal{C})$ and standard estimates from stability theory, there exists an output injection matrix J such that $\chi_J(s)$ is Hurwitz and (4.6) has the desired attractivity property around $\Phi^N(\mathcal{C})$. Thus (4.6) is a locally asymptotic observer. The result follows. □

In order to efficiently construct such observers it would be useful to have an explicit bound on degree of the polynomial ϕ in terms of the degrees of f, h. Probably such bounds can be provided using tools from commutative algebra, or using similar methods as in [7], but we are not aware of any concrete estimates.

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