# OUTPUT NULLING SUBSPACES AND SYSTEM INTERCONNECTIONS

P. FUHRMANN

REPORT No. 38, 2002/2003, spring

ISSN 1103-467X
ISRN IML-R- -38-02/03- -SE+spring

INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

# OUTPUT NULLING SUBSPACES AND SYSTEM INTERCONNECTIONS *

P.A. Fuhrmann[†]
Department of Mathematics
Ben-Gurion University of the Negev
Beer Sheva, Israel

October 26, 2003

## Abstract

The present paper points out some connections between geometric control and behaviors. Specifically, given an input/output system, it focuses on the polynomial characterization of output nulling subspaces on the one hand and the analysis of system interconnection from the behavioral point of view. Toeplitz operators and Wiener-Hopf factorizations play an important role.

**Key words: Behaviors, controllability, geometric control, feedback, output nulling subspaces, system interconnections, McMillan degree, skew primeness, doubly unimodular embeddings, factorization theory**

feedback/interconnection.tex

---

[*] Partially supported by ISF under Grant No. 235/01

[†] Earl Katz Family Chair in Algebraic System Theory

# 1 Introduction

Linear system theory has been developed in a wide variety of settings, and as a result there is no universal agreement as tomwhat constitutes an appropriate definition. It has been convincingly argued by J.C. Willems [1989,1991] that there should be a basic, representation free, definition. As a result, the theory of behaviors came into being, see Polderman and Willems [1997] for the basics. There are some clear advantages to taking this approach: it enables the handling of linear systems not described by input/output considerations, it does not require an apriori distinction between inputs and outputs, it is ideally suited to the study of system interconnections and elimination theory can be used to simplify the handling of some problems. However, the system and control area is an applied field and as such needs efficient computational methods. At the end of the day, one wants methods that fit the setup in which the original problem is described. All the different, representation dependent, techniques developed in the past are still there and one would like to understand their relation to behaviors. Among the past approaches to the study of linear systems one finds transfer functions, state space theory, module theory, polynomial matrix descriptions and $H^\infty$, to name the most important ones.

In the present paper we focus on a specific problem and that is the analysis of system interconnections in the behavioral context. The importance of the problem, see Willems [1997] for an introduction, cannot be overly emphasized as it encompasses also the study of feedback in the setting of behaviors. Our object is to gain insight by showing the relation of feedback interconnections to several other problems. First among them is geometric control theory and, more specifically, the analysis of output nulling subspaces. At the same time it is related to the concept of skew primeness and to the analysis and description of all autonomous subbehaviors of a given behavior. In a way, this is not very surprising as in all these contexts the use of polynomial methods is preeminent. In fact, the main tools we shall use are polynomial matrix factorizations, polynomial and rational model theory, introduced in Fuhrmann [1976], and the extension of the basic theorems to behaviors, following Fuhrmann [2002]. We proceed to give some background information on the problems mantioned.

Geometric control theory, initiated by Basile and Marro [1973] and Wonham and Morse, see Wonham [1979], deals mostly with objects that, directly or indirectly, relate to the zeros of linear systems. When described in state space terms this relation is not particularly obvious. However, when adopting a functional approach, as in Emre and Hautus [1980] or Fuhrmann and Willems [1980], the characterizations are given in terms of numerator polynomial matrices in a matrix fraction description or in terms of polynomial system matrices arising in polynomial matrix descriptions. In these cases, the relation to zeros is much more clearly evident. All the objects mentioned are, in general, rectangular polynomial matrices. Contrary to the case of square nonsingular polynomial matrices that describe the pole structure, and hence the dynamics, of the system, the rectangular case is much more delicate. The study of output nulling subspaces originated in geometric control in the analysis of the disturbance decoupling problem. A polynomial characterization of the maximal output nulling subspace $\mathcal{V}^*$ was given in Emre and Hautus [1980] and extended in Fuhrmann and Willems [1980]. The maximal reachability output nulling subspace $\mathcal{R}^*$ was characterized, in polynomial terms, in Fuhrmann [1981]. These characterizations use the numerator polynomial in a left matrix fraction representation of the transfer function. There are in general many $\mathbb{F}[z]$-module structures that can be imposed on $\mathcal{V}^*$ using feedback maps that make it invariant and they induce corresponding module structures on $\mathcal{R}^*$. However, the module structure, and the corresponding invariant factors, on the quotient $\mathcal{V}^*/\mathcal{R}^*$ are fixed. These invariant factors are Morse's transmission polynomials. A similar situation occurs in the study of behaviors. A behavior $\mathcal{B}$ has a unique maximal reachable subbehavior $\mathcal{B}_r$ and this corresponds to a factorization of the polynomial in the kernel representation of $\mathcal{B}$.

Skew primeness has been introduced as an important tool in system theory by Wolovich [1978]. In the definition of skew primeness, at least one of the matrices is allowed to be rectangular. While the definition of skew primeness is algebraic, it has interesting geometric interpretation in terms of module structures. In this connection, see Khargonekar, Georgiou and Özgüler [1983].

Switching to the behavioral setting, a behavior is defined as a space of permissible trajectories. It is one of the basic results in behavioral theory that a behavior can be characterized as the set of solutions of an autoregressive (AR) system of differential/difference equations. These equations are determined by a, not necessarily unique, rectangular polynomial matrix, so it is an underdetermined system which means that there are some free variables. Thus, in general, behaviors are infinite dimensional as linear spaces. The exceptions are the autonomous behaviors, namely the finite dimensional ones. An autonomous behavior is determined by an essentially unique, nonsingular polynomial matrix. Since subbehaviors are characterized via factorizations, factorization theory becomes a key object in this study.

That there are some clear links between all the problems mentioned above, has been pointed out in special cases. However a systematic analysis of these connections seems still missing, and our object in this paper is to partially remedy this situation.

The examples mentioned give a strong indication that geometric control, far from being a dead topic, has the potential, in conjunction with polynomial model techniques, of developing synergetic relation with behavioral theory, to the benefit of both.

The paper is structured as follows. In Section 2 we present the basic results on polynomial and rational models. Also, behaviors are introduced with the time set being $\mathbb{Z}_+$. Section 3 is devoted to the explanation of the relation between the algebraic definition of skew primeness and its geometric interpretaion. This connection has been explored in Khargonekar, Georgiou and Özgüler [1982]. Section 4 is an in depth analysis of the characterization of all autonomous, i.e. finite dimensional, subbehaviors of a given behavior. This is done via the algebra of doubly unimodular embeddings. We study first the case of reachable behaviors and use that for the study of the general case. This problem is related to the problem of parametrizing all shift module structures on certain vectorial polynomial spaces. The analysis of the nonreachable case is closely related to skew primeness. Also, it is essentially equivalent, given a finite dimensional linear system, to the study of output nulling subspaces. This is taken up in Section 5. The exposition here is based on extending results of Fuhrmann [1981] to the proper case. Parts of this analysis are in turn based on Khargonekar and Emre [1982]. Output nulling subspaces are of course related to zeros of linear systems. There are more connections here whose analysis, due to space limitations, is omitted. Finally, in Section 6, we present some results concerning the McMillan degree of the feedabck connection of two linear systems. The characterization of which polynomial matrix extensions can be described by feedback is still open, see Willems [1997] and Trentelman [2002]. We hope the methods developed in this paper may contribute to the understanding of feedback interconnections. It is worth mentioning that Lomadze [2001] contains a geometric approach to the problem of feedback interconnection. Because of the great difference in language, it is quite difficult to compare the respective results.

A word of warning is in order. This paper is, in its present state, incomplete. So it is to be regarded as an advanced draft. The author is grateful for any comments that may improve the final version.

Finally, it is a pleasure to gratefully acknowledges the hospitality and support of the Mittag-Leffler Institute in Stockholm where much work on this paper was done.

# 2 Preliminaries

## 2.1 Polynomial and rational models

We begin by giving a concise introduction to polynomial models. Let $\mathbb{F}$ denote an arbitrary field. We will denote by $\mathbb{F}^m$ the space of all $m$-vectors with coordinates in $\mathbb{F}$. Let $\pi_+$ and $\pi_-$ denote the projections of $\mathbb{F}^m((z^{-1}))$ the space of truncated Laurent series on $\mathbb{F}^m[z]$ and $z^{-1}\mathbb{F}^m[[z^{-1}]]$, the space of formal power series vanishing at infinity, respectively. Since

$$\mathbb{F}^m((z^{-1})) = \mathbb{F}^m[z] \oplus z^{-1}\mathbb{F}^m[[z^{-1}]] \tag{1}$$

$\pi_+$ and $\pi_-$ are complementary projections. Given a nonsingular polynomial matrix $D$ in $\mathbb{F}^{m\times m}[z]$ we define two projections $\pi_D$ in $\mathbb{F}^m[z]$ and $\pi^D$ in $z^{-1}\mathbb{F}^m[[z^{-1}]]$ by

$$\pi_D f = D\pi_- D^{-1} f \qquad for\ f \in \mathbb{F}^m[z] \tag{2}$$

$$\pi^D h = \pi_- D^{-1}\pi_+ D h \qquad for\ h \in z^{-1}\mathbb{F}^m[[z^{-1}]] \tag{3}$$

and define two linear subspaces of $\mathbb{F}^m[z]$ and $z^{-1}\mathbb{F}^m[[z^{-1}]]$ by

$$X_D = \operatorname{Im}\pi_D. \tag{4}$$

and

$$X^D = \operatorname{Im}\pi^D. \tag{5}$$

We refer to $X_D$ as a **polynomial model** whereas to $X^D$ as a **rational model**. It is of great importance to have an easy characterization of elements of polynomial or rational model. The following propsition is immediate from the definitions and we omit the proof.

**Proposition 2.1** *Given a nonsingular polynomial matrix* $D \in \mathbb{F}^{m\times m}[z]$. *Then*

1. *An element $f$ of $\mathbb{F}^m[z]$ belongs to $X_D$ if and only if $\pi_+ D^{-1} f = 0$, i.e. if and only if $D^{-1} f$ is a strictly proper, rational vector function. Equivalently, if the Toeplitz map $\overline{\mathcal{T}}_{D^{-1}} : \mathbb{F}^m[z] \longrightarrow \mathbb{F}^m[z]$ is defined by*
$$\overline{\mathcal{T}}_{D^{-1}} f = \pi_+ D^{-1} h, \tag{6}$$
*then*
$$X_D = \operatorname{Ker} \overline{\mathcal{T}}_{D^{-1}}. \tag{7}$$

2. *$h \in X^D$ if and only if $\pi_- D h = 0$, i.e. if and only if $h$ is in the kernel of the Toeplitz map $\mathcal{T}_D : z^{-1}\mathbb{F}^m[[z^{-1}]] \longrightarrow z^{-1}\mathbb{F}^m[[z^{-1}]]$ defined by $\mathcal{T}_D h = \pi_- D h$. We shall also, for reasons of compatibility with behavioral theory usage, write $\sigma = S_-$ and $D(\sigma) = \mathcal{T}_D$. Thus we have*
$$X^D = \operatorname{Ker} \mathcal{T}_D = \operatorname{Ker} D(\sigma). \tag{8}$$

As a consequence, we have also the following description of the polynomial model $X_D$, namely
$$X_D = \left\{ f \in \mathbb{F}^m[z] | f = Dh, \;\; h \in z^{-1}\mathbb{F}^m[[z^{-1}]]. \right\} \tag{9}$$
The advantage of this characterization is that it makes sense for an arbitrary, rectangular, $p \times m$ polynomial matrix $V$. Thus, following Emre and Hautus [1980], we define
$$X_V = \left\{ f \in \mathbb{F}^p[z] | f = Vh, \;\; h \in z^{-1}\mathbb{F}^m[[z^{-1}]] \right\}. \tag{10}$$
This definition extends the concept of a polynomial model.

For a nonsingular polynomial matrix $D(z)$, we turn $X_D$ into an $\mathbb{F}[z]-$module by defining
$$p \cdot f = \pi_D p f \qquad for \; p \in \mathbb{F}[z], \; f \in X_D. \tag{11}$$
Since $\operatorname{Ker} \pi_D = D\mathbb{F}^m[z]$ it follows that $X_D$ is isomorphic to the quotient module $\mathbb{F}^m[z]/D\mathbb{F}^m[z]$. Similarly, we introduce in $X^D$ a module structure by
$$p \cdot h = \pi_- p h \qquad for \; p \in \mathbb{F}[z], \; h \in X^D. \tag{12}$$

In $X_D$ we will focus on a special map $S_D$, a generalization of the classical companion matrix, which corresponds to the action of the identity polynomial $z$, i.e.,
$$S_D f = \pi_D z f \qquad for \qquad f \in X_D. \tag{13}$$
It is easily checked that
$$S_D f = z f(z) - D(z)\xi_f, \tag{14}$$
where the constant vector $\xi_f$ depends linearly on $f$. In fact we have
$$\xi_f = \pi_+ z D(z)^{-1} f = (D^{-1} f)_{-1}. \tag{15}$$
It follows from (13) that the module structure in $X_D$ is identical to the module structure induced by $S_D$ through $p \cdot f = p(S_D) f$. With this definition the study of $S_D$ is identical to

the study of the module structure of $X_D$. In particular the invariant subspaces of $S_D$ are just the submodules of $X_D$.

Similarly, we introduce in $X^D$ a module structure, given by

$$S_D h = \pi_- z h \text{ for } h \in X^D. \tag{16}$$

The polynomial model $X_D$ and the rational model $X^D$ are isomorphic, with the isomorphism $\rho_D : X^D \longrightarrow X_D$ given by $f \mapsto D^{-1}f$, given by

$$S_D \rho_D = \rho_D S^D. \tag{17}$$

One would like to immitate the procedure of imposing an $\mathbb{F}[z]$-module structure on the space $X_V$, with $V(z)$ rectangular. Here the situation is more complex and lies at the heart of this paper. Such a module structure can indeed be defined, however uniqueness is lost. We will study this question in full detail in Section 4.

The next theorem is of great importance as it connects factorization theory to to the geometry of invariant subspaces.

**Theorem 2.1** *Let $D \in \mathbb{F}^{m\times m}[z]$ be a nonsingular polynomial matrix.*

1. *A subset $M$ of $X_D$ is a submodule, or equivalently an $S_D$ invariant subspace, if and only if $M = D_1 X_{D_2}$ for some factorization $D = D_1 D_2$ with $D_i$ $m \times m$, necessarily nonsingular, polynomial matrices.*
2. *A subset $M$ of $X^D$ is a submodule, or equivalently an $S^D$ invariant subspace, if and only if $M = X^{D_2}$ for some factorization $D = D_1 D_2$ with $D_i$ $m \times m$, necessarily nonsingular, polynomial matrices.*

As a consequence of Theorem 2.1, in a representation of a rational model $X^D$, the polynomial matrix $D$ is uniquely determined up to a left unimodular factor.

The next theorem defines the shift realization, see Fuhrmann [1976,1977].

**Theorem 2.2** *Let $G = VT^{-1}U+W$ be a representation of a proper, $p\times m$ rational function. In the state space $X_T$ a system is defined by*

$$\begin{cases} A = S_T \\ B\xi = \pi_T U\xi, \\ Cf = (VT^{-1}f)_{-1} \\ D = G(\infty). \end{cases} \tag{18}$$

*Then $G = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right)$; this realization is observable if and only if $V$ and $T$ are right coprime and it is reachable if and only if $T$ and $U$ are left coprime.*

We follow Willems [1991] in defining a **dynamical system** $\Sigma$ as a triple

$$\Sigma = (T, W, \mathcal{B}), \tag{19}$$

where $T \subset \mathbf{R}$ is the **time axis**, $W$ is an abstract set called the **signal alphabet** and $\mathcal{B} \subset W^T$ is called the **behavior**. The elements of $\mathcal{B}$ are called the **trajectories** of the system.

This definition is very general and is representation free. In the context of this paper we will identify $T$ with $\mathbf{Z}_+$, the set of positive integers, assume $\mathbb{F}$ is an arbitrary field and take $W = \mathbb{F}^m$. We identify $W^T$ with $z^{-1}\mathbb{F}F^m[[z^{-1}]]$. The space $z^{-1}\mathbb{F}^m[[z^{-1}]]$ has a standard $\mathbb{F}[z]$-module structure induced by the **left** or **backward shift operator** $S_-$ or $\sigma$ defined by

$$S_-h = \sigma h = \pi_- zh, \qquad h \in z^{-1}\mathbb{F}^m[[z^{-1}]]. \tag{20}$$

Recall that $\pi_-$ is the projection of $\mathbb{F}^m((z^{-1}))$ onto $z^{-1}\mathbb{F}^m[[z^{-1}]]$ corresponding to the direct sum representation

$$\mathbb{F}^m((z^{-1})) = \mathbb{F}^m[z] \oplus z^{-1}\mathbb{F}^m[[z^{-1}]] \tag{21}$$

and that the complementary projection is denoted by $\pi_+$.

As an $\mathbb{F}[z]$ module, the space $z^{-1}\mathbb{F}^m[[z^{-1}]]$ has a multitude of submodules, i.e. linear, shift invariant subspaces. In this class we single out a special, small, subclass which is determined by the extra property of completeness.

**Definition 2.1** *In $z^{-1}\mathbb{F}^m[[z^{-1}]]$ we define the projections $P_n, n \in \mathbf{Z}_+$ by*

$$P_n \sum_{i=1}^{\infty} \frac{h_i}{z^i} = \sum_{i=1}^{n} \frac{h_i}{z^i}. \tag{22}$$

*We say that a subset $\mathcal{B} \subset z^{-1}\mathbb{F}^m[[z^{-1}]]$ is* **complete** *if for any $w = \sum_{i=1}^{\infty} w_i z^{-i} \in z^{-1}\mathbb{F}^m[[z^{-1}]]$ and for each positive integer $N$, $P_N w \in P_N(\mathcal{B})$ implies $w \in \mathcal{B}$. A* **behavior** *in our context is defined as a linear, shift invariant and complete subspace of $z^{-1}\mathbb{F}^m[[z^{-1}]]$.*

The principal characterization of behaviors is due to Willems [1986].

**Theorem 2.3** *A subset $\mathcal{B} \subset z^{-1}\mathbb{F}^m[[z^{-1}]]$ is a behavior if and only if it admits a* **kernel representation**, *i.e. there exists a $p \times m$ polynomial matrix $P(z)$ for which*

$$\mathcal{B} = \operatorname{Ker} P(\sigma) = \{h \in z^{-1}\mathbb{F}^m[[z^{-1}]] | \pi_- Ph = P(\sigma)h = 0\}. \tag{23}$$

We can identify $z^{-1}\mathbb{F}^m[[z^{-1}]]$ with the dual space of $\mathbb{F}^m[z]$, see Fuhrmann [1981] for the corresponding duality theory. Of course, elements of $\mathbb{F}^m[z]$ are linear functionals on $z^{-1}\mathbb{F}^m[[z^{-1}]]$. This makes $z^{-1}\mathbb{F}^m[[z^{-1}]]$ into a topological space. The topology, called the $w^*$ topology, is the weakest topology in which all functionals from $\mathbb{F}^m[z]$ are continuous. It can be seen, see Fuhrmann [2002], that completeness of a behavior is equivalent to its closure in this topology.

# 3 Skew primeness

The usual notions of left and right coprimeness of polynomial matrices have, among other, a geometric interpretation in terms of direct sums of polynomial models. The question of when an invariant subspace $\mathcal{V} \subset X_D$ has a complementary subspace relates to the notion of skew primeness. Skew primeness was introduced in Wolovich [1978] and studied in depth in Khargonekar, Georgiou and Özgüler [1983]. We begin by making the following definition.

**Definition 3.1**

1. *Let $E$ be a $p \times p$ nonsingular polynomial matrix and $U$ a $p \times m$ polynomial matrix. The pair $(E, U)$ is called* **left skew-prime** *if there exist a $p \times m$ polynomial matrix $\overline{U}$ and an $m \times m$ polynomial matrix $\overline{E}$ such that*
   - *(a) $EU = \overline{U}\,\overline{E}$,*
   - *(b) $E, \overline{U}$ are left coprime, and*
   - *(c) $U, \overline{E}$ are right coprime.*

   *In this case we will say that the pair $\overline{U}, \overline{E}$ is a* **skew complement** *of $(E, U)$.*

2. *Let $V$ be a $p \times m$ polynomial matrix. A factorization $V = E_1 V_1$ is called a* **left skew-prime factorization** *of $V$ if $(E_1, V_1)$ is a left skew-prime pair. We will say that a left skew-prime factorization $V = E_1 V_1$ is a* **maximal left skew-prime factorization** *if the nontrivial invariant factors of $V$ and $E_1$ are equal.*
   *The corresponding notions of right skew primeness are analogously defined.*

**Theorem 3.1** *Let $U$ be a full row rank polynomial matrix and let*

$$U = E_\rho U_\rho \tag{24}$$

*be an internal/external factorization. Let $U = \overline{U}_\rho \overline{E}_\rho$ be a complementary factorization, i.e. we have $\overline{U}_\rho, E_\rho$ left coprime and $U_\rho, \overline{E}_\rho$ right coprime. Let*

$$\begin{aligned} \mathcal{B} &= \operatorname{Ker} U(\sigma) \\ \mathcal{B}_\rho &= \operatorname{Ker} U_\rho(\sigma). \end{aligned} \tag{25}$$

*Then we have*

1. *$\operatorname{Ker} U_\rho(\sigma)$ and $\operatorname{Ker} \overline{E}_\rho(\sigma)$ are subbehaviors of $\mathcal{B}$.*
2. *We have*
   $$\operatorname{Ker} U_\rho(\sigma) \cap \operatorname{Ker} \overline{E}_\rho(\sigma) = \{0\}. \tag{26}$$
   *and*
   $$\mathcal{B} = \operatorname{Ker} U_\rho(\sigma) \oplus \operatorname{Ker} \overline{E}_\rho(\sigma) \tag{27}$$

**Proof:**

1. We have
   $$\operatorname{Ker} U_\rho(\sigma) \cap \operatorname{Ker} \overline{E}_\rho(\sigma) = \operatorname{Ker} \begin{pmatrix} U_\rho(\sigma) \\ \overline{E}_\rho(\sigma) \end{pmatrix}.$$

   The right coprimeness of $U_\rho, \overline{E}_\rho$ implies (26).

2. By coprimeness, there exist appropriately sized polynomial matrices $X, Y, \overline{X}, \overline{Y}$ (modification included) such that

$$\begin{aligned} \begin{pmatrix} E_\rho & \overline{U}_\rho \\ Y & X \end{pmatrix} \begin{pmatrix} \overline{X} & -U_\rho \\ -\overline{Y} & \overline{E}_\rho \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} \overline{X} & -U_\rho \\ -\overline{Y} & \overline{E}_\rho \end{pmatrix} \begin{pmatrix} E_\rho & \overline{U}_\rho \\ Y & X \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \end{aligned} \tag{28}$$

   In particular, we have

$$\overline{X} E_\rho - U_\rho Y = I. \tag{29}$$

   Conversely, assume there exist $\overline{X}, Y$ such that (29) holds. Complete $\begin{pmatrix} \overline{X} & -U_\rho \end{pmatrix}$ and $\begin{pmatrix} E_\rho \\ Y \end{pmatrix}$ to unimodular matrices satisfying (28).

$$\begin{pmatrix} \overline{X} & -U_\rho \\ -\overline{Y} & \overline{E}_\rho \end{pmatrix} \begin{pmatrix} E_\rho & \overline{U}_\rho \\ Y & X \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{30}$$

   In particular, we have the skew prime factorizations

$$U = E_\rho U_\rho = \overline{U}_\rho \overline{E}_\rho. \tag{31}$$

∎

**Theorem 3.2** *Let $U$ be a full row rank polynomial matrix and let*

$$U = E_\alpha U_\alpha \tag{32}$$

*be a factorization with $E_\alpha$ nonsingular. The following statements are equivalent.*

1. *$E_\alpha$ and $U_\alpha$ are left skew prime.*

2. *The Sylvester equation*

$$X E_\alpha - U_\alpha \overline{Y} = I \tag{33}$$

   *has a polynomial solution.*

3. *The submodule $E_\alpha X_{U_\alpha}$ of $X_U$ is an $\mathbb{F}[z]-$direct summand, i.e. it has a complementary submodule.*

**Proof:**
(1) ⇔ (2)
Assume $U = E_\alpha U_\alpha$ is a left skew prime factorization. Thus there exist $\overline{U}_\alpha, \overline{E}_\alpha$ for which $U = E_\alpha U_\alpha = \overline{U}_\alpha \overline{E}_\alpha$ with $E_\alpha, \overline{U}_\alpha$ left coprime and $U_\alpha, \overline{E}_\alpha$ right coprime. By a standard technique, there exists a doubly coprime factorization

$$\begin{pmatrix} E_\alpha & -\overline{U}_\alpha \\ -\overline{Y} & \overline{X} \end{pmatrix} \begin{pmatrix} X & U_\alpha \\ Y & \overline{E}_\alpha \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} X & U_\alpha \\ Y & \overline{E}_\alpha \end{pmatrix} \begin{pmatrix} E_\alpha & -\overline{U}_\alpha \\ -\overline{Y} & \overline{X} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

From the second equation, we obtain the Sylvester equation (33).

To prove the converse, assume that

$$XE_\alpha - U_\alpha \overline{Y} = I \tag{34}$$

has a polynomial solution. Let $\overline{E}_\alpha^{-1} Y$ be a left coprime factorization of $\overline{Y} E_\alpha^{-1}$. Thus we have

$$\overline{E}_\alpha \overline{Y} = Y E_\alpha. \tag{35}$$

Applying Theorems 4.6 & 4.7 in Fuhrmann [1976], we conclude that the polynomial models $X_{E_\alpha}$ and $X_{\overline{E}_\alpha}$ are isomorphic.
Multiplying the equality (34) by $E_\alpha$ on the left and by $E_\alpha^{-1}$ on the right, we get

$$\begin{aligned} I &= E_\alpha X - E_\alpha U_\alpha \overline{Y} E_\alpha^{-1} \\ &= E_\alpha X - E_\alpha U_\alpha \overline{E}_\alpha^{-1} Y \end{aligned}$$

Since $E_\alpha U_\alpha \overline{E}_\alpha^{-1} Y$ is a polynomial matrix and $\overline{E}_\alpha, Y$ are left coprime, there exists a polynomial matrix $\overline{U}_\alpha$ for which $E_\alpha U_\alpha \overline{E}_\alpha^{-1} = \overline{U}_\alpha$ or, equivalently, that

$$E_\alpha U_\alpha = \overline{U}_\alpha \overline{E}_\alpha. \tag{36}$$

Substituting back, we get $E_\alpha X - \overline{U}_\alpha Y = I$, i.e. $E_\alpha, \overline{U}_\alpha$ are left coprime. Applying again Theorems 4.6 & 4.7 in Fuhrmann [1976], (36) and the left coprimeness of $E_\alpha, \overline{U}_\alpha$ show that $X_{\overline{E}_\alpha}$ is the homorphic image of $X_{E_\alpha}$. But since we know that the two polynomial models are isomorphic, necessarily $\overline{E}_\alpha, U_\alpha$ are right coprime. Thus, (36) is a left skew prime factorization.

(1) ⇔ (3)
Let $U = E_\rho U_\rho$ be a maximal left skew prime factorization. Since $U_\rho$ is polynomially right invertible, there exists a factorization

$$E_\rho = E_\alpha E_\beta \tag{37}$$

for some, necessarily unique, nonsingular polynomial matrix $E_\beta$. We claim that the factorization (37) is a left skew prime factorization. By the equivalence of statements (1) and

(2), there exists a polynomial solution of the Sylvester equation $XE_\alpha + U_\alpha Y = I$. However we have $U_\alpha = E_\beta U_\rho$ and so $XE_\alpha + E_\beta(U_\rho Y) = I$ which shows that $E_\alpha, E_\beta$ are left skew prime. Since both are nonsingular, there exists a factorization $E_\rho = \overline{E}_\beta \overline{E}_\alpha$ for which $X_{E\rho} = E_\alpha X_{E_\beta} \oplus \overline{E}_\beta X_{\overline{E}_\alpha}$. Since we prove, in Theorem 4.4, that $X_{U_\alpha} = X_{E_\beta} \oplus E_\beta X_{U_\rho}$ we have

$$\begin{aligned} X_U &= E_\alpha X_{E_\beta} \oplus \overline{E}_\beta X_{\overline{E}_\alpha} \oplus E_\alpha E_\beta X_{U_\rho} \\ &= E_\alpha [X_{E_\beta} \oplus E_\beta X_{U_\rho}] \oplus \overline{E}_\beta X_{\overline{E}_\alpha} \\ &= E_\alpha X_{U_\alpha} \oplus \overline{E}_\beta X_{\overline{E}_\alpha} \end{aligned}$$

This shows that $E_\alpha X_{U_\alpha}$ is a direct summand.

■

We point out that although we refer to Theorem 4.4 which is proved later in the paper, there is no circular reasoning here.

# 4 Autonomous subbehaviors

Assume that a linear system is given through its behavior $\mathcal{B} \subset z^{-1}\mathbb{F}^m[[z^{-1}]]$ which has the AR representation $\mathcal{B} = \operatorname{Ker} U(\sigma) = X^U$, with $U(z)$ a $p \times m$ polynomial matrix which, without loss of generality, we assume to be of full row rank. Left primeness of $U$ is well known to be a characterization of controllable behaviors, see Willems [1989]. By permuting variables, we can assume without loss of generality that $U(z)$ has the representation $U(z) = \begin{pmatrix} Q(z) & P(z) \end{pmatrix}$ with $Q^{-1}P$ a proper rational function. The behavior $\mathcal{B}$ has a natural $\mathbb{F}[z]$-module structure given, for $p \in \mathbb{F}[z]$ and $h \in \mathcal{B}$, by

$$p \cdot h = \pi_- ph = p(\sigma)h. \tag{38}$$

Our intention is to study the set of all autonomous subbehaviors of $\mathcal{B}$. Since $X_U = \{f \in \mathbb{F}^p[z] | f = Uh, h \in z^{-1}\mathbb{F}^m[[z^{-1}]]\}$, we have $X_U = UX^U$, or $X_{\begin{pmatrix} Q & P \end{pmatrix}} = \begin{pmatrix} Q & P \end{pmatrix} X^{\begin{pmatrix} Q & P \end{pmatrix}}$. Now $X_{\begin{pmatrix} Q & P \end{pmatrix}}$ is finite dimensional as a linear space, whereas, if $p < m$, $\mathcal{B} = X^{\begin{pmatrix} Q & P \end{pmatrix}}$ is infinite dimensional. In particular, $M_{\begin{pmatrix} Q & P \end{pmatrix}} | X^{\begin{pmatrix} Q & P \end{pmatrix}}$ has an infinite dimensional kernel. We will focus now on autonomous subbehaviors of $\mathcal{B}$.

## 4.1 The reachable case

Left primeness of $U(z) = \begin{pmatrix} Q(z) & P(z) \end{pmatrix}$ is well known to be a characterization of reachable behaviors, see Willems [1989] or Fuhrmann [2002]. By permuting variables, we can assume without loss of generality that $U(z)$ has the representation $U(z) = \begin{pmatrix} Q(z) & P(z) \end{pmatrix}$ with $Q^{-1}P$ a proper rational function. Let $\overline{PQ}^{-1}$ be a right coprime factorization of $Q^{-1}P$. Define the multiplication operator $M_{\begin{pmatrix} Q & P \end{pmatrix}}$ on $z^{-1}\mathbb{F}^m[[z^{-1}]]$ by

$$M_{\begin{pmatrix} Q & P \end{pmatrix}} h = \begin{pmatrix} Q & P \end{pmatrix} h, \qquad h \in z^{-1}\mathbb{F}^m[[z^{-1}]]. \tag{39}$$

Clearly, $M_{\begin{pmatrix} Q & P \end{pmatrix}} h \in \mathbb{F}^p[z]$ if and only if $h \in \mathrm{Ker} \begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix}$ and in that case we have

$$X_{\begin{pmatrix} Q & P \end{pmatrix}} = M_{\begin{pmatrix} Q & P \end{pmatrix}} X^{\begin{pmatrix} Q & P \end{pmatrix}}. \tag{40}$$

Now $X_{\begin{pmatrix} Q & P \end{pmatrix}}$ is finite dimensional as a linear space, whereas, if $p < m$, $\mathcal{B} = X^{\begin{pmatrix} Q & P \end{pmatrix}}$ is infinite dimensional. In particular, $M_{\begin{pmatrix} Q & P \end{pmatrix}} | X^{\begin{pmatrix} Q & P \end{pmatrix}}$ has an infinite dimensional kernel.

For later use, we prove the following lemma.

**Lemma 4.1** *Let $U$ be a $p \times m$ polynomial matrix and let $E$ be an $m \times m$ nonsingular polynomial matrix. Then $UX^E \subset X_U$ if and only if there exists factorization $U = U'E$ for some polynomial matrix $U'$.*

**Proof:** If there exists a factorization $U = U'E$, then clearly

$$UX^E = U'EX^E = U'X_E \subset X_{U'E} = X_U.$$

Conversely, assume $UX^E = X_U$. Since $X^E = E^{-1}X_E$, we have $UE^{-1}f \in X_U \subset \mathbb{F}^p[z]$ for all $f \in X_E$. On the other hand, for $f \in E\mathbb{F}^m[z]$, we have $f = Eg$ and hence $UE^{-1}f = UE^{-1}Eg = Ug \in \mathbb{F}^p[z]$. Since $\mathbb{F}^m[z] = X_E \oplus E\mathbb{F}^m[z]$, it follows that $UE^{-1}f$ is a polynomial for all $f \in \mathbb{F}^m[z]$. This implies $U' = UE^{-1}$ is polynomial and hence $U = U'E$ follows. ∎

Before doing that we note the existence of doubly coprime factorizations. This is well known, see Rosenbrock [1970] or Özgüler [1994].

**Proposition 4.1** *Given the coprime factorizations*

$$Q^{-1}P = \overline{P}\,\overline{Q}^{-1}, \tag{41}$$

*then there exists a doubly coprime factorization of the form*

$$\begin{gathered} \begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix} \begin{pmatrix} B_0 & -\overline{P} \\ A_0 & \overline{Q} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\ \begin{pmatrix} B_0 & -\overline{P} \\ A_0 & \overline{Q} \end{pmatrix} \begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{gathered} \tag{42}$$

*This doubly coprime factorization is unique if we add the requirement that*

$$\overline{A}_0 Q^{-1} = \overline{Q}^{-1} A_0 \tag{43}$$

*is strictly proper.*

**Proof:** Existence of the doubly coprime factorization (42) is standard and follows from the solutions to the Bezout equations characterizing coprimeness.

To prove uniqueness, assume that we have two doubly coprime factorizations, both satisfying (43). Thus, in particular, we have $QB_0 + PA_0 = I$ and $QB_1 + PA_1 = I$. By subtraction, we have $Q(B_1 - B_0) + P(A_1 - A_0) = 0$. In turn, this implies

$$B_1 - B_0 = Q^{-1}P(A_0 - A_1) = \overline{PQ}^{-1}(A_0 - A_1).$$

As $\overline{P}, \overline{Q}$ are right coprime, it follows that $A_0 - A_1 = \overline{Q}S$ for some polynomial matrix $S$. Thus, by assumption, $\overline{Q}^{-1}(A_0 - A_1) = S$ is simultaneously both strictly proper as well as polynomial. Necessarily $A_0 = A_1$ and in turn also $B_0 = B_1$ ∎

Doubly coprime factorization are closely related to projections and direct sum representations. This is based on the fact that an $m \times m$ unimodular polynomial matrix $U(z)$ induces an isomorphism on $z^{-1}\mathbb{F}^m[[z^{-1}]]$, given by $U(\sigma)$. The analysis of this is given by the following.

**Proposition 4.2** *Given the doubly coprime factorization (42) with* $Q \in \mathbb{F}^{p\times p}[z]$ *nonsingular and* $P \in \mathbb{F}^{p\times(m-p)}[z]$ *. Define polynomial matrices* $\Pi_1, \Pi_2 \in \mathbb{F}^{m\times m}[z]$ *by*
*maps* $\Pi_1, \Pi_2 : z^{-1}\mathbb{F}^m[[z^{-1}]] \longrightarrow z^{-1}\mathbb{F}^m[[z^{-1}]]$ *by*

$$\begin{aligned} \Pi_1(z) &= \begin{pmatrix} -\overline{P}(z) \\ \overline{Q}(z) \end{pmatrix} \begin{pmatrix} -\overline{A}_0(z) & \overline{B}_0(z) \end{pmatrix} \\ \Pi_2(z) &= \begin{pmatrix} B_0(z) \\ A_0(z) \end{pmatrix} \begin{pmatrix} Q(z) & P(z) \end{pmatrix}. \end{aligned} \tag{44}$$

*The maps* $\Pi_1(\sigma), \Pi_2(\sigma) : z^{-1}\mathbb{F}^m[[z^{-1}]] \longrightarrow z^{-1}\mathbb{F}^m[[z^{-1}]]$ *are projections in* $z^{-1}\mathbb{F}^m[[z^{-1}]]$ *and satisfy*

$$\begin{aligned} \operatorname{Ker}\Pi_1(\sigma) &= \operatorname{Ker}\begin{pmatrix} -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix} = \operatorname{Im}\begin{pmatrix} B_0(\sigma) \\ A_0(\sigma) \end{pmatrix} \\ \operatorname{Im}\Pi_1(\sigma) &= \operatorname{Im}\begin{pmatrix} -\overline{P}(\sigma) \\ \overline{Q}(\sigma) \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix} \end{aligned} \tag{45}$$

$$\begin{aligned} \operatorname{Ker}\Pi_2(\sigma) &= \operatorname{Ker}\begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix} = \operatorname{Im}\begin{pmatrix} -\overline{P}(\sigma) \\ \overline{Q}(\sigma) \end{pmatrix} \\ \operatorname{Im}\Pi_2(\sigma) &= \operatorname{Im}\begin{pmatrix} B_0(\sigma) \\ A_0(\sigma) \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix}. \end{aligned} \tag{46}$$

*Moreover, we have*

$$\begin{aligned} \operatorname{Ker}\Pi_1(\sigma) \cap \operatorname{Ker}\Pi_2(\sigma) &= \{0\} \\ \operatorname{Im}\Pi_1(\sigma) \oplus \operatorname{Im}\Pi_2(\sigma) &= z^{-1}\mathbb{F}^m[[z^{-1}]] \end{aligned} \tag{47}$$

**Proof:** To see that $\Pi_1$ is a projection we use the Bezout equation $\overline{A}_0(z)\overline{P}(z)+\overline{B}_0(z)\overline{Q}(z) = I$ which implies the operator equation $\overline{A}_0(\sigma)\overline{P}(\sigma) + \overline{B}_0(\sigma)\overline{Q}(\sigma) = I$. The same holds for $\Pi_2$.

Now $\begin{pmatrix} -\overline{P}(\sigma) \\ \overline{Q}(\sigma) \end{pmatrix}$ is, by the right coprimeness of $\overline{P}, \overline{Q}$, an injective map, hence $\operatorname{Ker}\Pi_1 = \operatorname{Ker}\begin{pmatrix} -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix}$. Clearly, $\operatorname{Im}\begin{pmatrix} B_0(\sigma) \\ A_0(\sigma) \end{pmatrix} \subset \operatorname{Ker}\begin{pmatrix} -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix}$. To prove the converse inclusion, we use the second equation in (42). Given, appropriately decomposed, $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in z^{-1}\mathbb{F}^m[[z^{-1}]]$, we can write

$$\begin{aligned} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} &= \begin{pmatrix} B_0(\sigma) & -\overline{P}(\sigma) \\ A_0(\sigma) & \overline{Q}(\sigma) \end{pmatrix}\begin{pmatrix} Q(\sigma) & P(\sigma) \\ -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\ &= \begin{pmatrix} B_0(\sigma) & -\overline{P}(\sigma) \\ A_0(\sigma) & \overline{Q}(\sigma) \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} B_0(\sigma) \\ A_0(\sigma) \end{pmatrix} g_1 + \begin{pmatrix} -\overline{P}(\sigma) \\ \overline{Q}(\sigma) \end{pmatrix} g_2. \end{aligned}$$

If $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \operatorname{Ker}\begin{pmatrix} -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix}$, noting that $\begin{pmatrix} -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix}\begin{pmatrix} B_0(\sigma) \\ A_0(\sigma) \end{pmatrix} = 0$ and $\begin{pmatrix} -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix}\begin{pmatrix} -\overline{P}(\sigma) \\ \overline{Q}(\sigma) \end{pmatrix} = I$, it follows that $g_2 = 0$ and therefore $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} B_0(\sigma) \\ A_0(\sigma) \end{pmatrix} g_1 \in \operatorname{Im}\begin{pmatrix} B_0(\sigma) \\ A_0(\sigma) \end{pmatrix}$. The other identities are proved similarly.

Since $\operatorname{Ker}\Pi_1 = \operatorname{Ker}\begin{pmatrix} -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix}$ and $\operatorname{Ker}\Pi_1 = \operatorname{Ker}\begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix}$, we have, using unimodularity, $\operatorname{Ker}\Pi_1 \cap \operatorname{Ker}\Pi_2 = \operatorname{Ker}\begin{pmatrix} Q(\sigma) & P(\sigma) \\ -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix} = \{0\}$. Analogously, we have $\operatorname{Im}\Pi_1 \oplus \operatorname{Im}\Pi_2 = \operatorname{Im}\begin{pmatrix} B_0(\sigma) & -\overline{P}(\sigma) \\ A_0(\sigma) & \overline{Q}(\sigma) \end{pmatrix} = z^{-1}\mathbb{F}^m[[z^{-1}]]$.

∎

Next we characterize the set of autonomous subbehaviors of a given, controllable, behavior. This is a special case, however more specific, of Theorem 6 in Willems [1997].

**Proposition 4.3** *Given a controllable behavior $\mathcal{B}$ which has the representation*

$$\mathcal{B} = \operatorname{Ker}\begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix} = X^{\begin{pmatrix} Q & P \end{pmatrix}}, \tag{48}$$

*with $Q$ nonsingular, $Q, P$ left coprime and $Q^{-1}P$ proper. Then a subset $\mathcal{B}_a \subset \mathcal{B}$ is an autonomous subbehavior if and only if it has a representation*

$$\mathcal{B}_a = \operatorname{Ker}\begin{pmatrix} Q(\sigma) & P(\sigma) \\ -\overline{A}(\sigma) & \overline{B}(\sigma) \end{pmatrix} = X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \tag{49}$$

*for some nonsingular extension $\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}$ of $\begin{pmatrix} Q & P \end{pmatrix}$.*

**Proof:** If $\mathcal{B}_a$ has the representation (49), then it is clearly an autonomous subbehavior of $\mathcal{B}$.

Conversely, if $\mathcal{B}_a$ is an autonomous subbehavior of $\mathcal{B}$ then it can be represented as $\mathcal{B}_a = \operatorname{Ker} \begin{pmatrix} A_{11}(\sigma) & A_{12}(\sigma) \\ A_{21}(\sigma) & A_{22}(\sigma) \end{pmatrix}$ for some factorization of the form

$$\begin{pmatrix} Q & P \end{pmatrix} = \begin{pmatrix} L_1 & L_2 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{50}$$

the matrix on the right being nonsingular. The left primeness of $\begin{pmatrix} Q & P \end{pmatrix}$ implies the left primeness of $\begin{pmatrix} L_1 & L_2 \end{pmatrix}$. Since left primeness implies that all invariant factors are trivial, there exist unimodular matrices $U, V$ such that

$$\begin{aligned} \begin{pmatrix} Q & P \end{pmatrix} &= U \begin{pmatrix} I & 0 \end{pmatrix} V \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} V \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \end{aligned}$$

Redefining the blocks $A_{ij}$, we can assume without loss of generality that

$$\begin{pmatrix} Q & P \end{pmatrix} = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{51}$$

Thus, with $\overline{A} = -A_{21}$ and $\overline{B} = A_{22}$, the result follows.

∎

Actually, we can be even more specific as far as the representation (49) is concerned and state the following.

**Proposition 4.4** *Given a controllable behavior $\mathcal{B} = X^{\begin{pmatrix} Q & P \end{pmatrix}}$, with $Q$ nonsingular, $Q, P$ left coprime and $Q^{-1}P$ proper. Then a subset $\mathcal{B}_a \subset \mathcal{B}$ is an autonomous subbehavior if and only if it has a representation*

$$\mathcal{B}_a = \operatorname{Ker} \begin{pmatrix} Q(\sigma) & P(\sigma) \\ -\overline{A}(\sigma) & \overline{B}(\sigma) \end{pmatrix} = X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \tag{52}$$

*where* $\begin{pmatrix} -\overline{A} & \overline{B} \end{pmatrix} = Q_1 \begin{pmatrix} -\overline{A}_0 & \overline{B}_0 \end{pmatrix}$ *for some unimodular extension* $\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}$ *of* $\begin{pmatrix} Q & P \end{pmatrix}$.

**Proof:** If $\mathcal{B}_a$ has the representation (52), then it is clearly an autonomous subbehavior of $\mathcal{B}$.

To prove the converse, assume that $\mathcal{B}_a \subset \mathcal{B}$ is an autonomous subbehavior. By Proposition 4.3, it has a representation of the form $\mathcal{B}_a = \operatorname{Ker} \begin{pmatrix} Q(\sigma) & P(\sigma) \\ -A'(\sigma) & B'(\sigma) \end{pmatrix} = X^{\begin{pmatrix} Q & P \\ -A' & B' \end{pmatrix}}$. Let $\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}$ of $\begin{pmatrix} Q & P \end{pmatrix}$ be an arbitrary unimodular extension of $\begin{pmatrix} Q & P \end{pmatrix}$. Such an extension exists by the left coprimeness of $Q, P$. Let $\begin{pmatrix} B_0 & -\overline{P} \\ A_0 & \overline{Q} \end{pmatrix}$ be its inverse. We compute

$$\begin{pmatrix} Q & P \\ -A' & B' \end{pmatrix}\begin{pmatrix} B_0 & -\overline{P} \\ A_0 & \overline{Q} \end{pmatrix} = \begin{pmatrix} I & 0 \\ S & R \end{pmatrix},$$

with $R$ necessarily nonsingular. Therefore

$$\begin{pmatrix} Q & P \\ -A' & B' \end{pmatrix} = \begin{pmatrix} I & 0 \\ S & R \end{pmatrix}\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix} = \begin{pmatrix} Q & P \\ SQ - R\overline{A}_0 & SP + R\overline{B}_0 \end{pmatrix}.$$

Now a behavior in an AR representation is unchanged if the polynomial matrix is left multiplied by a unimodular one. Taking $\begin{pmatrix} I & 0 \\ -S & I \end{pmatrix}$ as a left multiplier we have

$$\begin{pmatrix} I & 0 \\ -S & I \end{pmatrix}\begin{pmatrix} Q & P \\ SQ - R\overline{A}_0 & SP + R\overline{B}_0 \end{pmatrix} = \begin{pmatrix} Q & P \\ -R\overline{A}_0 & R\overline{B}_0 \end{pmatrix}.$$

Setting $Q_1 = R$ completes the proof. ∎

There is a natural partial order on the set of all autonomous subbehaviors of $X^{\begin{pmatrix} Q & P \end{pmatrix}}$.

**Proposition 4.5** *Given two autonomous subbehaviors* $X^{\begin{pmatrix} Q & P \\ -\overline{A}_i & \overline{B}_i \end{pmatrix}} \subset X^{\begin{pmatrix} Q & P \end{pmatrix}}$, $i = 1, 2$. *Then we have the inclusion*

$$X^{\begin{pmatrix} Q & P \\ -\overline{A}_1 & \overline{B}_1 \end{pmatrix}} \subset X^{\begin{pmatrix} Q & P \\ -\overline{A}_2 & \overline{B}_2 \end{pmatrix}} \tag{53}$$

*if and only if there exist polynomial matrices* $Y, Z$ *for which*

$$\begin{pmatrix} Q & P \\ -\overline{A}_2 & \overline{B}_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ Z & Y \end{pmatrix}\begin{pmatrix} Q & P \\ -\overline{A}_1 & \overline{B}_1 \end{pmatrix} \tag{54}$$

*If* $\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}$ *is the unique unimodular extension for which* $\overline{A}_0 Q^{-1} = \overline{Q}^{-1} A_0$ *is strictly proper, and without loss of generality we take* $\begin{pmatrix} -\overline{A}_i & \overline{B}_i \end{pmatrix} = Q_i \begin{pmatrix} -\overline{A}_0 & \overline{B}_0 \end{pmatrix}$, *then (53) holds if and only if*

$$Q_2 = Y Q_1, \tag{55}$$

*i.e.* $Q_1$ *is a right factor of* $Q_2$.

**Proof:** The inclusion (53) is equivalent to the existence of a factorization

$$\begin{pmatrix} Q & P \\ -\overline{A}_2 & \overline{B}_2 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} Q & P \\ -\overline{A}_1 & \overline{B}_1 \end{pmatrix}$$

The doubly coprime factorization (42) implies

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & Q_1 \end{pmatrix} = \begin{pmatrix} Q & P \\ -\overline{A}_2 & \overline{B}_2 \end{pmatrix}\begin{pmatrix} B_0 & -\overline{P} \\ A_0 & \overline{Q} \end{pmatrix} = \begin{pmatrix} I & 0 \\ Z_2 & Y_2 \end{pmatrix},$$

with $Y, Z$ appropriately defined, and necessarily polynomial, the factorization (54) holds. If we use the canonical extension, then the equality

$$\begin{pmatrix} I & 0 \\ 0 & Q_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ Z & Y \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Q_1 \end{pmatrix}$$

implies $Z = 0$ and $Q_2 = Y Q_1$.

∎

For an easier understanding of the next theorem, it is convenient to refer to the following diagram.

$$\begin{tikzcd}[ampersand replacement=\&, column sep=huge, row sep=huge]
X_{\begin{pmatrix} Q \& P \end{pmatrix}} \arrow[d, "S_{\begin{pmatrix} Q \& P \\ -\overline{A} \& \overline{B} \end{pmatrix}}"'] \& X_{\begin{pmatrix} Q \& P \\ -\overline{A} \& \overline{B} \end{pmatrix}} \arrow[l, "M_{\begin{pmatrix} Q \& P \end{pmatrix}}"'] \arrow[d, "S_{\begin{pmatrix} Q \& P \\ -\overline{A} \& \overline{B} \end{pmatrix}}"] \& X^{Q_1} \arrow[l, "\begin{pmatrix} -\overline{P}(\sigma) \\ \overline{Q}(\sigma) \end{pmatrix}"'] \arrow[d, "S^{Q_1}"] \\
X_{\begin{pmatrix} Q \& P \end{pmatrix}} \& X_{\begin{pmatrix} Q \& P \\ -\overline{A} \& \overline{B} \end{pmatrix}} \arrow[l, "M_{\begin{pmatrix} Q \& P \end{pmatrix}}"'] \& X^{Q_1} \arrow[l, "\begin{pmatrix} -\overline{P}(\sigma) \\ \overline{Q}(\sigma) \end{pmatrix}"']
\end{tikzcd}$$

**Diagram 4.1**

**Theorem 4.1** *Given the* $p \times m$, *full row rank, left prime polynomial matrix*

$$R(z) = \begin{pmatrix} Q(z) & P(z) \end{pmatrix} \tag{56}$$

*with* $Q^{-1}P$ *a proper rational function, and let (42) be the unique doubly coprime factorization. For an arbitrary* $(m-p) \times (m-p)$ *nonsingular polynomial matrix* $Q_1$, *define*

$$\overline{A} = Q_1 \overline{A}_0, \qquad \overline{B} = Q_1 \overline{B}_0. \tag{57}$$

*Then*

1. *We have*

$$\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix} \begin{pmatrix} B_0 & -\overline{P} \\ A_0 & \overline{Q} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & Q_1 \end{pmatrix} \tag{58}$$

2. *In particular, we have*

$$\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix} \begin{pmatrix} -\overline{P} \\ \overline{Q} \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} Q_1 \tag{59}$$

3. *For the inverse of* $\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}$, *we have*

$$\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}^{-1} = \begin{pmatrix} B_0 & -\overline{P} \\ A_0 & \overline{Q} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Q_1^{-1} \end{pmatrix} \tag{60}$$

4. *The following identity*

$$\begin{pmatrix} 0 & -\overline{A}_0 & \overline{B}_0 \\ 0 & Q & P \\ -I & -\overline{A} & -\overline{B} \end{pmatrix} \begin{pmatrix} Q_1 & 0 & -I \\ -\overline{P} & B_0 & 0 \\ \overline{Q} & A_0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{61}$$

*is a doubly unimodular embedding of* $\begin{pmatrix} 0 & Q & P \\ -I & -\overline{A} & -\overline{B} \end{pmatrix}$, *and* $\begin{pmatrix} Q_1 \\ -\overline{P} \\ \overline{Q} \end{pmatrix}$ *as well as of* $\begin{pmatrix} Q_1 & 0 & -I \end{pmatrix}$ *and* $\begin{pmatrix} -\overline{A}_0 & \overline{B}_0 \\ Q & P \\ -\overline{A} & -\overline{B} \end{pmatrix}$.

*In particular, we have also*

$$\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} Q & P \\ -\overline{A} & -\overline{B} \end{pmatrix} = Q_1 \begin{pmatrix} -\overline{A}_0 & \overline{B}_0 \end{pmatrix} \tag{62}$$

5. (a) *The map* $Z : X^{Q_1} \longrightarrow X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$ *defined by*

$$Zh = \begin{pmatrix} -\overline{P}(\sigma) \\ \overline{Q}(\sigma) \end{pmatrix} h \tag{63}$$

*is a continuous, bijective behavior homomorphism.*
*That means that if* $S^{Q_1} : X^{Q_1} \longrightarrow X^{Q_1}$ *is the restriction of the backward shift operator* $S_- = \sigma$ *to the submodule* $X^{Q_1}$ *and* $S^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$ *is similarly defined, then the right side of diagram 4.1 is commutative.*

(b) *The inverse map* $Z^{-1} : X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \longrightarrow X^{Q_1}$ *is given by*

$$Z^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \tag{64}$$

6. *We have*

(a)

$$X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} = \begin{pmatrix} -\overline{P}(\sigma) \\ \overline{Q}(\sigma) \end{pmatrix} X^{Q_1}. \tag{65}$$

(b) *We have*

$$X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$$
$$= \left\{ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} B_0 f_1 - \overline{P} Q_1^{-1} f_2 \\ A_0 f_1 + \overline{Q} Q_1^{-1} f_2 \end{pmatrix} \in z^{-1}\mathbb{F}^m[[z^{-1}]] | f_1 \in \mathbb{F}^p[z], f_2 \in \mathbb{F}^{m-p}[z] \right\} \tag{66}$$

*Moreover, in the representation*

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} B_0 f_1 - \overline{P} Q_1^{-1} f_2 \\ A_0 f_1 + \overline{Q} Q_1^{-1} f_2 \end{pmatrix}, \tag{67}$$

*the polynomial vectors* $f_1, f_2$ *are uniquely determined.*

7. *Let* $M_{\begin{pmatrix} Q & P \end{pmatrix}} : X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \longrightarrow X_{\begin{pmatrix} Q & P \end{pmatrix}}$ *be the multiplication map by the polynomial matrix* $\begin{pmatrix} Q(z) & P(z) \end{pmatrix}$, *then*

   (a) $M_{\begin{pmatrix} Q & P \end{pmatrix}} | X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$ *is injective if and only if all left Wiener-Hopf factorization indices of* $\overline{Q}Q_1^{-1}$ *are nonnegative.*

   (b) $M_{\begin{pmatrix} Q & P \end{pmatrix}} | X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$ *is surjective if and only if all left Wiener-Hopf factorization indices of* $\overline{Q}Q_1^{-1}$ *are nonpositive.*

   (c) $M_{\begin{pmatrix} Q & P \end{pmatrix}} | X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$ *is bijective if and only if all left Wiener-Hopf factorization indices of* $\overline{Q}Q_1^{-1}$ *are zero.*
   *In this case* $M^{-1}_{\begin{pmatrix} Q & P \end{pmatrix}} : X_{\begin{pmatrix} Q & P \end{pmatrix}} \longrightarrow X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$ *is given, for* $f \in X_{\begin{pmatrix} Q & P \end{pmatrix}}$ *by*

$$M^{-1}_{\begin{pmatrix} Q & P \end{pmatrix}} f = \begin{pmatrix} B_0 f + \overline{P} Q_1^{-1} \overline{\mathcal{T}}^{-1}_{\overline{Q}Q_1^{-1}} A_0 f \\ A_0 f - \overline{Q} Q_1^{-1} \overline{\mathcal{T}}^{-1}_{\overline{Q}Q_1^{-1}} A_0 f \end{pmatrix}, \tag{68}$$

   *where the Toeplitz operator* $\overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}} : \mathbb{F}^{m-p}[z] \longrightarrow \mathbb{F}^{m-p}[z]$ *is defined by*

$$\overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}} f = \pi_+ \overline{Q} Q_1^{-1} f. \tag{69}$$

8. (a) *We have the equality*

$$\{0\} = [\mathrm{Ker} \begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix} \cap \mathrm{Ker} \begin{pmatrix} Q(z) & P(z) \end{pmatrix}] \cap X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \tag{70}$$

   *if and only if all left Wiener-Hopf factorization indices of* $\overline{Q}Q_1^{-1}$ *are nonnegative.*

   (b) *We have the equality*

$$\mathrm{Ker} \begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix} = [\mathrm{Ker} \begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix} \cap \mathrm{Ker} \begin{pmatrix} Q(z) & P(z) \end{pmatrix}] + X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \tag{71}$$

   *if and only if all left Wiener-Hopf factorization indices of* $\overline{Q}Q_1^{-1}$ *are nonpositive.*

(c) *We have the direct sum representation*

$$\mathrm{Ker}\begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix} = [\mathrm{Ker}\begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix} \cap \mathrm{Ker}\begin{pmatrix} Q(z) & P(z) \end{pmatrix}] \oplus X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \tag{72}$$

*if and only if all left Wiener-Hopf factorization indices of* $\overline{Q}Q_1^{-1}$ *are zero.*

9. *Assume* $\overline{A}, \overline{B}$ *are defined by (57) and that all left Wiener-Hopf factorization indices of* $\overline{Q}Q_1^{-1}$ *are zero. Define the map* $S_{\left(\begin{smallmatrix} Q & P \\ -\overline{A} & \overline{B} \end{smallmatrix}\right)} : X_{\left(\begin{smallmatrix} Q & P \end{smallmatrix}\right)} \longrightarrow X_{\left(\begin{smallmatrix} Q & P \end{smallmatrix}\right)}$ *as follows. Let* $f \in X_{\begin{pmatrix} Q & P \end{pmatrix}}$ *be given by* $f = M_{\begin{pmatrix} Q & P \end{pmatrix}} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ *with* $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$ *and let* $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}_{-1} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$. *Define*

$$S_{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} f = z(Qh_1 + Ph_2) - (Q\eta_1 + P\eta_2) \tag{73}$$

*Then the left part of Diagram 4.1 is commutative and we have the isomorphism*

$$S_{\left(\begin{smallmatrix} Q & P \\ -\overline{A} & \overline{B} \end{smallmatrix}\right)} \simeq S_{Q_1}. \tag{74}$$

*We call the* $\mathbb{F}[z]$*-module structure defined in (73) the* **shift module structure** *induced by* $\left(\begin{smallmatrix} Q & P \\ -\overline{A} & \overline{B} \end{smallmatrix}\right)$.

**Proof:**

1. Follows from the doubly coprime factorization (42).
2. Follows from (58).
3. Follows from (58).
4. Equation (42) leads to

$$\begin{pmatrix} 0 & -\overline{A}_0 & \overline{B}_0 \\ 0 & Q & P \\ -I & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -I \\ -\overline{P} & B_0 & 0 \\ \overline{Q} & A_0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

Multiplying by $\left(\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0 \\ Q_1 & 0 & I \end{smallmatrix}\right)$ on the left and by $\left(\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -Q_1 & 0 & I \end{smallmatrix}\right)$ on the right, we get (61).

5. (a) Follows, using (59), from the characterization of continuous behavior homomorphisms given in Theorem 3.4 in Fuhrmann [2003]. The invertibility of the map $Z$ follows from the characterizations, in terms of doubly unimodular embeddings, of injectivity and surjectivity of continuous behavior homomorphisms given in Theorem 4.8 in Fuhrmann [2002]. In fact, it suffices to show that there exists a doubly unimodular embedding for $\left(\begin{smallmatrix} 0 & Q & P \\ -I & -\overline{A} & \overline{B} \end{smallmatrix}\right), \left(\begin{smallmatrix} Q_1 \\ -\overline{P} \\ \overline{Q} \end{smallmatrix}\right)$.

   Incidentally, this shows that $\left(\begin{smallmatrix} 0 & Q & P \\ -I & -\overline{A} & \overline{B} \end{smallmatrix}\right)$ is a MLA of $\left(\begin{smallmatrix} Q_1 \\ -\overline{P} \\ \overline{Q} \end{smallmatrix}\right)$.

(b) Follows, using (62), by applying the characterization of invertible behavior homomorphisms given in Theorem 4.8 in Fuhrmann [2002]. We incorporate also a direct proof. Let $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$.

We claim that $\begin{pmatrix} -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X^{Q_1}$. Indeed,

$$\begin{aligned} Q_1(\sigma) \begin{pmatrix} -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} &= \begin{pmatrix} -Q_1(\sigma)\overline{A}_0(\sigma) & Q_1(\sigma)\overline{B}_0(\sigma) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\ &= \begin{pmatrix} -\overline{A}(\sigma) & \overline{B}(\sigma) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 0. \end{aligned}$$

Here we used the inclusion $X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \subset X^{\begin{pmatrix} -\overline{A} & \overline{B} \end{pmatrix}}$.

Applying Proposition 4.2, and noting that $X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \subset X^{\begin{pmatrix} Q & P \end{pmatrix}}$, implies

$$\begin{pmatrix} -\overline{P}(\sigma) \\ \overline{Q}(\sigma) \end{pmatrix} \begin{pmatrix} -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

For $h \in X^{Q_1}$, we compute

$$\begin{pmatrix} -\overline{A}_0(\sigma) & \overline{B}_0(\sigma) \end{pmatrix} \begin{pmatrix} -\overline{P}(\sigma) \\ \overline{Q}(\sigma) \end{pmatrix} h = (\overline{A}_0(\sigma)\overline{P}(\sigma) + \overline{B}_0(\sigma)\overline{Q}(\sigma))h = h.$$

6. (a) Follows from part 4.

(b) Let $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ be strictly proper. Then $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$ if and only if there exist appropriately sized polynomial vectors $f_1, f_2$ such that

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Using (60) and the fact that $\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}^{-1} = \begin{pmatrix} B_0 & -\overline{P} \\ A_0 & \overline{Q} \end{pmatrix}$, it follows that

$$\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}^{-1} = \begin{pmatrix} B_0 & -\overline{P} \\ A_0 & \overline{Q} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Q_1^{-1} \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} B_0 & -\overline{P} \\ A_0 & \overline{Q} \end{pmatrix} \begin{pmatrix} f_1 \\ Q_1^{-1} f_2 \end{pmatrix} = \begin{pmatrix} B_0 f_1 - \overline{P} Q_1^{-1} f_2 \\ A_0 f_1 + \overline{Q} Q_1^{-1} f_2 \end{pmatrix}.$$

Conversely, if, for vector polynomials $f_1, f_2$, we have $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} B_0 f_1 - \overline{P}Q_1^{-1}f_2 \\ A_0 f_1 + \overline{Q}Q_1^{-1}f_2 \end{pmatrix}$, then $\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, i.e. $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$.

To prove uniqueness of the representation (67), it suffices to show that $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} B_0 f_1 - \overline{P}Q_1^{-1}f_2 \\ A_0 f_1 + \overline{Q}Q_1^{-1}f_2 \end{pmatrix}$ implies $f_1 = 0$ and $f_2 = 0$. Multiplying the above equality by $\begin{pmatrix} Q & P \end{pmatrix}$, we get

$$0 = \begin{pmatrix} Q & P \end{pmatrix}\begin{pmatrix} B_0 f_1 \\ A_0 f_1 \end{pmatrix} = f_1.$$

In turn, we get $\begin{pmatrix} \overline{P}Q_1^{-1}f_2 \\ \overline{Q}Q_1^{-1}f_2 \end{pmatrix} = 0$ and hence $f_2 = 0$ and uniqueness follows.

7. Note first that we have $X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \subset X^{\begin{pmatrix} Q & P \end{pmatrix}} = \operatorname{Ker} \begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix}$. Hence $M_{\begin{pmatrix} Q & P \end{pmatrix}}$ is a well defined map from $X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$ into $X_{\begin{pmatrix} Q & P \end{pmatrix}}$.

   (a) First, we investigate the injectivity of the multiplication map $M_{\begin{pmatrix} Q & P \end{pmatrix}}$. Assume $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \cap \operatorname{Ker} M_{\begin{pmatrix} Q & P \end{pmatrix}}$. In this case we have, using the representation (66),

   $$0 = \begin{pmatrix} Q & P \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} Q & P \end{pmatrix}\begin{pmatrix} B_0 f_1 - \overline{P}Q_1^{-1}f_2 \\ A_0 f_1 + \overline{Q}Q_1^{-1}f_2 \end{pmatrix} = (QB_0 + PA_0)f_1 = f_1.$$

   So, necessarily, $f_1 = 0$ and $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} -\overline{P} \\ \overline{Q} \end{pmatrix} Q_1^{-1}f_2$. In particular, $h_2 = \overline{Q}Q_1^{-1}f_2$ is strictly proper and hence $f_2 \in \operatorname{Ker} \overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}}$ follows. By Theorem 3.3 in Fuhrmann and Helmke [2001], the Toeplitz operator $\overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}}$ is injective if and only if all the left Wiener-Hopf factorization indices of $\overline{Q}Q_1^{-1}$ are nonnegative. Thus it follows that $f_2 = 0$ and we have injecivity.

   To prove the converse, assume that not all the left Wiener-Hopf factorization are nonnegative. Thus there exists a nonzero $f_2 \in \operatorname{Ker} \overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}}$. Consider $\begin{pmatrix} -\overline{P}(\sigma) \\ \overline{Q}(\sigma) \end{pmatrix} Q_1^{-1}f_2$ $X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$. Thus there exists a polynomial vector $f_1$ for which

   $$0 \neq \begin{pmatrix} B_0 \\ A_0 \end{pmatrix} f_1 + \begin{pmatrix} -\overline{P} \\ \overline{Q} \end{pmatrix} Q_1^{-1}f_2 \in X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}.$$

   Applying $\begin{pmatrix} Q & P \end{pmatrix}$ we get $f_1 = 0$, so necessarily $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} -\overline{P}Q_1^{-1}f_2 \\ \overline{Q}Q_1^{-1}f_2 \end{pmatrix}$. Clearly, $h_2$ is strictly proper and nonzero by construction. On the other hand

   $$h_1 = -\overline{P}Q_1^{-1}f_2 = -\overline{PQ}^{-1}\overline{Q}Q_1^{-1}f_2 = -(\overline{PQ}^{-1})h_2,$$

so $h_1$ is strictly proper as $h_2$ is and $\overline{PQ}^{-1} = Q^{-1}P$ is proper. We conclude therefore that $\operatorname{Ker} M_{\begin{pmatrix} Q & P \end{pmatrix}}|X^{\begin{pmatrix} Q & P \\ -A & B \end{pmatrix}} \neq \{0\}$. This proves the necessity part.

(b) Assume that all left Wiener-Hopf factorization indices of $\overline{Q}Q_1^{-1}$ are nonpositive. We show that for any $f_1 \in X_{\begin{pmatrix} Q & P \end{pmatrix}}$ we can find an element $f_2 \in X_{Q_1}$ such that $f_1 = M_{\begin{pmatrix} Q & P \end{pmatrix}} Z Q_1^{-1} f_2$. Now $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X^{\begin{pmatrix} Q & P \\ -A & B \end{pmatrix}}$ if and only if

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} B_0 f_1 - \overline{P}Q_1^{-1} f_2 \\ A_0 f_1 + \overline{Q}Q_1^{-1} f_2 \end{pmatrix}. \tag{75}$$

In particular, we must have $h_2 = A_0 f_1 + \overline{Q}Q_1^{-1} f_2$. Applying the projection $\pi_+$ to this equality, we obtain

$$-A_0 f_1 = \pi_+ \overline{Q}Q_1^{-1} f_2 = \overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}} f_2.$$

Now, see Fuhrmann and Helmke [2001], the Toeplitz operator $\overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}}$ is surjective if and only if all left Wiener-Hopf factorization indices of $\overline{Q}Q_1^{-1}$ are nonpositive. So, by our assumption, equation (75) is solvable. Indeed, if $\overline{Q}Q_1^{-1} = \Gamma\Delta U$ is left Wiener-Hopf factorization, with all factorization indices nonpositive, i.e. with $\Delta^{-1}$ polynomial, then a right inverse of $\overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}}$ is given by $g \mapsto U^{-1}\Delta^{-1}\pi_+\Gamma^{-1}g$, for we can check that

$$\pi_+\Gamma\Delta U U^{-1}\Delta^{-1}\pi_+\Gamma^{-1}g = \pi_+\Gamma\pi_+\Gamma^{-1}g = \pi_+\Gamma\Gamma^{-1}g = g.$$

Thus we conclude that $f_2 = -U^{-1}\Delta^{-1}\pi_+\Gamma^{-1}A_0 f_1$. Using (75), we compute

$$\begin{array}{rcl} h_1 & = & B_0 f_1 - \overline{P}Q_1^{-1} f_2 = B_0 f_1 - \overline{PQ}^{-1}\overline{Q}Q_1^{-1} f_2 \\ & = & B_0 f_1 - \overline{PQ}^{-1}(h_2 - A_0 f_1) \\ & = & -\overline{PQ}^{-1} h_2 + (B_0 f_1 - Q^{-1}PA_0 f_1) \\ & = & -\overline{PQ}^{-1} h_2 + Q^{-1}(QB_0 - PA_0) f_1 \\ & = & -Q^{-1}P h_2 + Q^{-1} f_1 \end{array}$$

This shows that $h_1$ is strictly proper. Since $h_2 = A_0 f_1 + \overline{Q}Q_1^{-1} f_2$, applying $\pi_+$ we get

$$0 = \pi_+(A_0 f_1 + \overline{Q}Q_1^{-1} f_2) = A_0 f_1 + \overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}} f_2$$

which shows that $h_2 = A_0 f_1 + \overline{Q}Q_1^{-1} f_2$ is also strictly proper.

(c) Follows from parts (a) and (b).
Since we assume that all left Wiener-Hopf factorization indices of $\overline{Q}Q_1^{-1}$ are zero, the Toeplitz operator $\overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}}$ is invertible. Let $f \in X_{\begin{pmatrix} Q & P \end{pmatrix}}$ and $f = Qh_1 + Ph_2$, with $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X^{\begin{pmatrix} Q & P \\ -A & B \end{pmatrix}}$. By the representation (66), we have

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} B_0 f_1 - \overline{P}Q_1^{-1} f_2 \\ A_0 f_1 + \overline{Q}Q_1^{-1} f_2 \end{pmatrix}$$

Since $f = Qh_1 + Ph_2$, we necessarily have $f_1 = f$. Applying the projection $\pi_+$, we have $\pi_+ h_2 = 0 = A_0 f + \pi_+ \overline{Q}Q_1^{-1} f = A_0 f + \overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}} f_2$, i.e. we have $f_2 = -\overline{\mathcal{T}}^{-1}_{\overline{Q}Q_1^{-1}} A_0 f$. This proves (68).

8. (a) Since $X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \subset \operatorname{Ker} \begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix}$, we have

$$\begin{aligned} &[\operatorname{Ker} \begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix} \cap \operatorname{Ker} \begin{pmatrix} Q(z) & P(z) \end{pmatrix}] \cap X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \\ &\qquad = \operatorname{Ker} \begin{pmatrix} Q(z) & P(z) \end{pmatrix} \cap X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}. \end{aligned}$$

So (70) is equivalent to $M_{\begin{pmatrix} Q & P \end{pmatrix}} | X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$ being injective. Hence, the result follows from part 7.(a).

(b) Assume all left Wiener-Hopf factorization indices of $\overline{Q}Q_1^{-1}$ are nonpositive. Thus the Toeplitz operator $\overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}}$ is surjective. Given $f_1 \in X_{\begin{pmatrix} Q & P \end{pmatrix}}$, we show there exists a polynomial vector $f_2$ such that

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} B_0 f_1 - \overline{P}Q_1^{-1} f_2 \\ A_0 f_1 + \overline{Q}Q_1^{-1} f_2 \end{pmatrix} \tag{76}$$

is strictly proper. In fact, applying the projection $\pi_+$ to the equality $h_2 = A_0 f_1 + \overline{Q}Q_1^{-1} f_2$, we get $A_0 f_1 + \overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}} f_2 = 0$. Thus by the surjectivity of $\overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}}$, this equation is solvable for $f_2$, though the solution is in general not uniquely determined. This means that there exists a strictly proper $h_2$ such that $h_2 = A_0 f_1 + \overline{Q}Q_1^{-1} f_2$ holds. For this $h_2$, we compute

$$\begin{aligned} h_1 &= B_0 f_1 - \overline{P}Q_1^{-1} f_2 = B_0 f_1 - \overline{PQ}^{-1} \overline{Q} Q_1^{-1} f_2 \\ &= B_0 f_1 - Q^{-1} P (h_2 - A_0 f_1) = Q^{-1}(Q B_0 f_1 + P A_0) f_1 - Q^{-1} P h_2 \\ &= Q^{-1} f_1 - Q^{-1} P h_2, \end{aligned}$$

which is clearly strictly proper. Now $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ defined by (76) is, by (**??**), in $X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$, and clearly $M \begin{pmatrix} Q & P \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = f_1$, i.e. $M_{\begin{pmatrix} Q & P \end{pmatrix}}$ is surjective. Necessarily, the equality (71) follows.

To prove the converse, we will show that the equality (71) implies the surjectivity of the Toeplitz operator $\overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}}$. It suffices to show that for every $f \in \mathbb{F}^p[z]$, we can find an $f_2 \in \mathbb{F}^{m-p}[z]$ such that $f = \overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}} f_2$. Note that by the doubly coprime factorization (42), we have $A_0 Q = \overline{QA}_0$ with $A_0, \overline{Q}$ left coprime and $Q, \overline{A}_0$ right coprime. This, by Theorem 4.7 in Fuhrmann [1976], implies the invertibility of the intertwining map $Z : X_Q \longrightarrow X_{\overline{Q}}$ defined by $Zf = \pi_{\overline{Q}} A_0 f$. Since $\overline{Q}$ is assumed to be column proper, we have the direct sum representation

$\mathbb{F}^{m-p}[z] = X_{\overline{Q}} \oplus \overline{Q}\mathbb{F}^{m-p}[z]$. As a result, every $f \in \mathbb{F}^{m-p}[z]$ has a representation of the form $A_0 f_1 + \overline{Q}\phi$ for some $\phi \in \mathbb{F}^{m-p}[z]$. Now, by equality (71), we clearly have $X_{\begin{pmatrix} Q & P \end{pmatrix}} = M_{\begin{pmatrix} Q & P \end{pmatrix}} X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$, i.e. for every $f_1 \in X_{\begin{pmatrix} Q & P \end{pmatrix}}$, there exists an $f_2$ such that $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} B_0 f_1 - \overline{P}Q_1^{-1} f_2 \\ A_0 f_1 + \overline{Q}Q_1^{-1} f_2 \end{pmatrix}$ is strictly proper. In particular, we have $A_0 f_1 + \pi_+ \overline{Q}Q_1^{-1} f_2 = 0$, i.e. $\overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}} f_2 = -A_0 f_1$ is solvable for every $f_1 \in X_{\begin{pmatrix} Q & P \end{pmatrix}}$. Let now $f \in \mathbb{F}^p[z]$ be arbitrary, and having a representation $f = -A_0 f_1 + \overline{Q}\phi$. Clearly,

$$\overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}}(f_2' + Q_1\phi) = \overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}} f_2' + \overline{Q}\phi = f.$$

Thus the Toeplitz operator $\overline{\mathcal{T}}_{\overline{Q}Q_1^{-1}}$ is surjective, hence by Theorem 3.3 in Fuhrmann and Helmke [2001], necessarily all left Wiener-Hopf factorization indices of $\overline{Q}Q_1^{-1}$ are nonpositive.

(c) Follows from parts (a) and (b).

9. With $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$ and $\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}_{-1}$, we compute

$$\begin{aligned} M_{\begin{pmatrix} Q & P \end{pmatrix}} S^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} &= M_{\begin{pmatrix} Q & P \end{pmatrix}} \sigma \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\ &= \begin{pmatrix} Q & P \end{pmatrix} \left[ z \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right] = z(Qh_1 + Ph_2) - (Q\eta_1 + P\eta_2) \\ &= S_{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}(Qh_1 + Ph_2) = S_{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} M_{\begin{pmatrix} Q & P \end{pmatrix}} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \end{aligned}$$

The isomorphism (73) follows from part (7c).

∎

Given a left coprime pair $Q, P$, with $Q^{-1}P$ proper, we saw that

$$X_{\begin{pmatrix} Q & P \end{pmatrix}} = M_{\begin{pmatrix} Q & P \end{pmatrix}} X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$$

for some nonsingular extension $\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}$ satisfying $\begin{pmatrix} -\overline{A} & \overline{B} \end{pmatrix} = Q_1 \begin{pmatrix} -\overline{A}_0 & \overline{B}_0 \end{pmatrix}$ with $\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}$ unimodular and all left Wiener-Hopf factorization indices of $\overline{Q}Q_1^{-1}$ are nonpositive. If all left Wiener-Hopf factorization indices of $\overline{Q}Q_1^{-1}$ are zero, as is the case when

$\overline{Q}Q_1^{-1}$ is biproper, then we have the isomorphism, as linear spaces, $X_{\begin{pmatrix} Q & P \end{pmatrix}} \simeq X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$. Now $X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$, as a rational model, or equivalently as an autonomous behavior, has a natural module structure induced by the backward shift $\sigma = S_-$, namely

$$S^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} = S_- | X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}. \tag{77}$$

We can use this module structure to induce a module structure on $X_{\begin{pmatrix} Q & P \end{pmatrix}}$. Of course, this module structure depends on the nature of the extension of $\begin{pmatrix} Q & P \end{pmatrix}$. The characterization of this module structure is given by the following.

**Proposition 4.6** *Under the preceeding assumptions on the extension, let* $S_{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} : X_{\begin{pmatrix} Q & P \end{pmatrix}} \longrightarrow X_{\begin{pmatrix} Q & P \end{pmatrix}}$ *be defined by the requirement that the left side of diagram 4.1 is commutative, i.e. by the identity*

$$S_{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} M_{\begin{pmatrix} Q & P \end{pmatrix}} = M_{\begin{pmatrix} Q & P \end{pmatrix}} S^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}. \tag{78}$$

*Then,*

1. *For* $f \in X_{\begin{pmatrix} Q & P \end{pmatrix}}$ *given by*

$$f = M_{\begin{pmatrix} Q & P \end{pmatrix}} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \qquad \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}, \tag{79}$$

   *we have*

$$\begin{aligned} S_{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} f &= S_{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\ &= z \begin{pmatrix} Q & P \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - \begin{pmatrix} Q & P \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \end{aligned} \tag{80}$$

   *with* $\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}_{-1}$.

2. *This can also be rewritten as*

$$S_{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} f = (A - BK) f, \tag{81}$$

   *where* $A = S_Q : X_Q \longrightarrow X_Q$ *is defined by*

$$S_Q f = \pi_Q z f = z f(z) - Q(z) \xi_f \tag{82}$$

*with* $\xi_f = (Q^{-1}f)_{-1}$, $f = (Qh_1 + Ph_2)$, $\begin{pmatrix} Q & P \end{pmatrix} = Q\begin{pmatrix} I & D \end{pmatrix} + \begin{pmatrix} 0 & R \end{pmatrix}$, *and* $Q^{-1}R$ *strictly proper. The map* $B : \mathbb{F}^m \longrightarrow X_Q = X_{\begin{pmatrix} Q & P \end{pmatrix}}$ *is defined by*

$$\begin{aligned} B\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \pi_Q \begin{pmatrix} Q & P \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \\ &= \pi_Q[Q\begin{pmatrix} I & D \end{pmatrix} + \begin{pmatrix} 0 & R \end{pmatrix}]\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & R \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = R\eta_2 \end{aligned} \tag{83}$$

*The map* $K : X_Q = X_{\begin{pmatrix} Q & P \end{pmatrix}} \longrightarrow \mathbb{F}^m$ *is defined by*

$$Kf = KM_{\begin{pmatrix} Q & P \end{pmatrix}}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}_{-1} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \tag{84}$$

**Proof:** ∎

In the reachable case there are no constraints on the characteristic polynomial of $S_{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$, i.e. on $d(z) = \det \begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}$, other than the degree constraints. The situation changes in the nonreachable case, when $Q, P$ are no longer left coprime, i.e. have a nontrivial, nonsingular common left factor. The determinant of such a factor divides the characteristic polynomial $d$. Thus the transmission zeros, arising out of the g.c.l.d. of $Q, P$ provide additional constraints. This we proceed to study next.

## 4.2 The nonreachable case

Based on the analysis of the reachable behaviors, we are in a position to analyse the general case.

**Theorem 4.2** *Given a behavior* $\mathcal{B}$ *with a kernel representation* $\mathcal{B} = \mathrm{Ker}\begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix}$, *with* $Q(z)$ *nonsingular and* $Q^{-1}P$ *proper. Let* $E_\rho = g.c.l.d.(Q, P)$ *and let* $\begin{pmatrix} Q(z) & P(z) \end{pmatrix} = E_\rho(z)\begin{pmatrix} Q_\rho(z) & P_\rho(z) \end{pmatrix}$. *Then* $\mathcal{B}_s \subset \mathcal{B}$ *is an autonomous subbehavior if and only if there exists a factorization* $E_\rho = E_\alpha E_\beta$ *and we have* $\mathcal{B}_s = \mathrm{Ker}\begin{pmatrix} E_\beta(\sigma)Q_\rho(\sigma) & E_\beta(\sigma)P_\rho(\sigma) \\ -\overline{A}(\sigma) & \overline{B}(\sigma) \end{pmatrix}$ *for some nonsingular extension* $\begin{pmatrix} E_\beta Q_\rho & E_\beta P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}$.

**Proof:** Assume we have the factorization $E_\rho = E_\alpha E_\beta$, then $\begin{pmatrix} Q(z) & P(z) \end{pmatrix} = \begin{pmatrix} E_\alpha & 0 \end{pmatrix}\begin{pmatrix} E_\beta Q_\rho & E_\beta P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}$ and so $X_{\begin{pmatrix} E_\beta Q_\rho & E_\beta P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}}$ is an autonomous subbehavior.

Conversely, assume $\mathcal{B}_s \subset \mathcal{B}$ is an autonomous subbehavior. Then $\mathcal{B}_s = X_{\begin{pmatrix} A_{11}(\sigma) & A_{12}(\sigma) \\ A_{21}(\sigma) & A_{22}(\sigma) \end{pmatrix}}$ and there exists a factorization

$$\begin{pmatrix} Q & P \end{pmatrix} = \begin{pmatrix} L_1 & L_2 \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with the right factor nonsingular and $\begin{pmatrix} L_1 & L_2 \end{pmatrix}$ of full row rank. Using the Smith form and redefining the right factor, we can assume without loss of generality that $\begin{pmatrix} L_1 & L_2 \end{pmatrix} = \begin{pmatrix} E_\alpha & 0 \end{pmatrix}$ for some nonsingular $E_\alpha$. Equating both sides, we have

$$\begin{pmatrix} Q & P \end{pmatrix} = E_\rho \begin{pmatrix} Q_\rho & P_\rho \end{pmatrix} = E_\alpha \begin{pmatrix} A_{11}(\sigma) & A_{12}(\sigma) \end{pmatrix}.$$

Since $\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}$ has a polynomial right inverse, we conclude the existence of a factorization $E_\rho = E_\alpha E_\beta$ for some, necessarily nonsingular $E_\beta$. Thus $\begin{pmatrix} A_{11} & A_{12} \end{pmatrix} = \begin{pmatrix} E_\beta Q_\rho & E_\beta P_\rho \end{pmatrix}$ and writing $\begin{pmatrix} A_{11} & A_{12} \end{pmatrix} = \begin{pmatrix} -\overline{A} & \overline{B} \end{pmatrix}$, we have $\mathcal{B}_s = X^{\begin{pmatrix} E_\beta Q_\rho & E_\beta P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}}$. ∎

We present now the counterpart of Theorem 4.1. As in that case, for ease of reference, we present the following diagram.

**Diagram 4.2**

$$\begin{tikzcd}
E_\rho X_{\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}} & & & X^{Q_1} \arrow[r, "\begin{pmatrix} -\overline{P}(\sigma) \\ \overline{Q}(\sigma) \end{pmatrix}"] & X^{\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}} & \\
\bigoplus & = & X_{\begin{pmatrix} Q & P \end{pmatrix}} & X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \arrow[l, "M_{\begin{pmatrix} Q & P \end{pmatrix}}"'] & = & \bigoplus \\
\begin{pmatrix} 0 & I \end{pmatrix} X_{\begin{pmatrix} -\overline{A}_0 & \overline{B}_0 \\ E_\rho Q_\rho & E_\rho P_\rho \end{pmatrix}} & & & X^{E_\rho} \arrow[r, "\begin{pmatrix} B_0(\sigma) \\ A_0(\sigma) \end{pmatrix}"] & X^{\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}} &
\end{tikzcd}$$

**Theorem 4.3** *Given* $U = \begin{pmatrix} Q & P \end{pmatrix}$*, of full row rank, with* $Q$ *nonsingular and* $Q^{-1}P$ *proper. Let*

$$\begin{pmatrix} Q & P \end{pmatrix} = E_\rho \begin{pmatrix} Q_\rho & P_\rho \end{pmatrix} \tag{85}$$

*be a factorization with* $\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}$ *left prime, i.e. a maximal left skew prime factorization and let*

$$\begin{gathered}
\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix} \begin{pmatrix} B_0 & -\overline{P}_\rho \\ A_0 & \overline{Q}_\rho \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\
\begin{pmatrix} B_0 & -\overline{P}_\rho \\ A_0 & \overline{Q}_\rho \end{pmatrix} \begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.
\end{gathered} \tag{86}$$

*be a doubly coprime factorization. With $Q_1$ an arbitrary nonsingular polynomial matrix, we define*

$$\overline{A} = Q_1\overline{A}_0, \qquad \overline{B} = Q_1\overline{B}_0. \tag{87}$$

*Then*

1. *The factorizations*

$$\begin{pmatrix} Q & P \end{pmatrix} = E_\rho \begin{pmatrix} Q_\rho & P_\rho \end{pmatrix} = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} E_\rho Q_\rho & E_\rho P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix} \tag{88}$$

   *are, respectively, left and right skew prime factorizations.*

2. *The factorizations*

$$\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix} = \begin{pmatrix} E_\rho & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & Q_1 \end{pmatrix} \begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix} \tag{89}$$

   *are, respectively, left and right skew prime factorizations.*

3. *The following is a direct sum representation*

$$X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} = X^{\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}} \oplus X^{\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}}. \tag{90}$$

4. (a) *The map* $\begin{pmatrix} -\overline{P}_\rho(\sigma) \\ \overline{Q}_\rho(\sigma) \end{pmatrix} : X^{Q_1} \longrightarrow X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$ *is a continuous, injective behavior homomorphism with image* $X^{\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}}$.

   (b) *We have the following representation*

$$X^{\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}} = \left\{ \begin{pmatrix} B_0 f_1 - \overline{P}_\rho Q_1^{-1} f_2 \\ A_0 f_1 + \overline{Q}_\rho Q_1^{-1} f_2 \end{pmatrix} | f_1 \in \mathbb{F}^p[z], f_2 \in \mathbb{F}^{m-p}[z] \right\} \tag{91}$$

   *Moreover, in the representation*

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} B_0 f_1 - \overline{P}_\rho Q_1^{-1} f_2 \\ A_0 f_1 + \overline{Q}_\rho Q_1^{-1} f_2 \end{pmatrix}, \tag{92}$$

   *the polynomial vectors $f_1, f_2$ are uniquely determined.*

5. (a) *The map* $\begin{pmatrix} B_0(\sigma) \\ A_0(\sigma) \end{pmatrix} : X^{E_\rho} \longrightarrow X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$ *is a continuous, injective behavior homomorphism with image* $X^{\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}}$.

(b) *We have the following representation*

$$X^{\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}} = \left\{ \begin{pmatrix} B_0E_\rho^{-1}g_1 - \overline{P}_\rho g_2 \\ A_0E_\rho^{-1}g_1 + \overline{Q}_\rho g_2 \end{pmatrix} | g_1 \in \mathbb{F}^p[z], g_2 \in \mathbb{F}^{m-p}[z] \right\} \tag{93}$$

*Moreover, in the representation*

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} B_0E_\rho^{-1}g_1 - \overline{P}_\rho g_2 \\ A_0E_\rho^{-1}g_1 + \overline{Q}_\rho g_2 \end{pmatrix}, \tag{94}$$

*the polynomial vectors $g_1, g_2$ are uniquely determined.*

6. (a) *The multiplication map $M_{\begin{pmatrix} Q & P \end{pmatrix}} | X^{\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}}$ is injective and we have*

$$M_{\begin{pmatrix} Q & P \end{pmatrix}} X^{\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}} = \begin{pmatrix} I & 0 \end{pmatrix} X_{\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}} = X_{E_\rho}. \tag{95}$$

(b) *The multiplication map $M_{\begin{pmatrix} Q & P \end{pmatrix}} | X^{\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}}$ is injective if and only if all left Wiener-Hopf factorization indices of $\overline{Q}_\rho Q_1^{-1}$ are nonnegative.*
*We have the inclusion*

$$M_{\begin{pmatrix} Q & P \end{pmatrix}} X^{\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}} \subset E_\rho X_{\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}}$$

*and the equality*

$$M_{\begin{pmatrix} Q & P \end{pmatrix}} X^{\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}} = E_\rho X_{\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}} \tag{96}$$

*holds if and only if all left Wiener-Hopf factorization indices of $\overline{Q}_\rho Q_1^{-1}$ are nonpositive.*

(c) *The following is a direct sum representation*

$$X_{\begin{pmatrix} Q & P \end{pmatrix}} = X_{E_\rho} \oplus E_\rho X_{\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}}. \tag{97}$$

**Proof:**

1. Follows from the fact that $\begin{pmatrix} I & 0 & E_\rho \end{pmatrix}$ and $\begin{pmatrix} Q_\rho & P_\rho \\ E_\rho Q_\rho & E_\rho P_\rho \\ -oA_0 & \overline{B}_0 \end{pmatrix}$ are, respectively, left and right prime polynomial matrices.
2. Clearly $\begin{pmatrix} E_\rho & 0 & I & 0 \\ 0 & I & 0 & Q_1 \end{pmatrix}$ and $\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A} & \overline{B} \\ Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}$ are, respectively, left and right prime polynomial matrices.

3. Follows from the factorizations (101) and the coprimeness conditions. See Theorem 2.14 in Fuhrmann and Willems [1980].

4. (a) From the doubly coprime factorization (99), we conclude that

$$\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix} \begin{pmatrix} -\overline{P}_\rho \\ \overline{Q}_\rho \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} Q_1.$$

   By Theorem 3.4 in Fuhrmann [2003], it follows that the map $\begin{pmatrix} -\overline{P}_\rho(\sigma) \\ \overline{Q}_\rho(\sigma) \end{pmatrix} : X^{Q_1} \longrightarrow X^{\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}}$ is a continuous behavior homomorphism. The doubly coprime factorization (99) implies, applying Theorem 4.8 in Fuhrmann [2002], that this map is bijective. Since $X^{\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}}$ is a submodule of $X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$, then as a map from $X^{Q_1}$ into $X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$, $\begin{pmatrix} -\overline{P}_\rho(\sigma) \\ \overline{Q}_\rho(\sigma) \end{pmatrix}$ is injective with image $X^{\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}}$.

   (b) The proof of (91) follows the line of proof of the representation (66) and we omit the details.

5. (a) From the doubly coprime factorization (99), we conclude that

$$\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix} \begin{pmatrix} B_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix} E_\rho.$$

   By Theorem 3.4 in Fuhrmann [2003], it follows that the map $\begin{pmatrix} B_0(\sigma) \\ A_0(\sigma) \end{pmatrix}$ : $X^{E_\rho} \longrightarrow X^{\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}}$ is a continuous behavior homomorphism. The doubly coprime factorization (99) implies, applying Theorem 4.8 in Fuhrmann [2002], that this map is bijective. Since $X^{\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}}$ is a submodule of $X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$, then as a map from $X^{E_\rho}$ into $X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$, $\begin{pmatrix} B_0(\sigma) \\ A_0(\sigma) \end{pmatrix}$ is injective with image $X^{\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}}$.

   (b) The proof of (93) follows the line of proof of the representation (66) and we omit the details.

6. (a) We use the representation (93) of $X^{\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}}$. If $\begin{pmatrix} B_0 E_\rho^{-1} g_1 - \overline{P}_\rho g_2 \\ A_0 E_\rho^{-1} g_1 + \overline{Q}_\rho g_2 \end{pmatrix} \in \operatorname{Ker} M_{\begin{pmatrix} Q & P \end{pmatrix}}$ then $0 = \begin{pmatrix} Q & P \end{pmatrix} \begin{pmatrix} B_0 E_\rho^{-1} g_1 - \overline{P}_\rho g_2 \\ A_0 E_\rho^{-1} g_1 + \overline{Q}_\rho g_2 \end{pmatrix} = g_1$, i.e. $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} -\overline{P}_\rho g_2 \\ \overline{Q}_\rho g_2 \end{pmatrix}$ is necessarily strictly proper and hence vanishes. This proves the injectivity.

   Let $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} B_0 E_\rho^{-1} g_1 - \overline{P}_\rho g_2 \\ A_0 E_\rho^{-1} g_1 + \overline{Q}_\rho g_2 \end{pmatrix} \in X^{\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}}$, with $g_1$ necessarily in

$X_{E_\rho}$. We compute

$$\begin{pmatrix} I & 0 \end{pmatrix}\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}\begin{pmatrix} B_0E_\rho^{-1}g_1 - \overline{P}_\rho g_2 \\ A_0E_\rho^{-1}g_1 + \overline{Q}_\rho g_2 \end{pmatrix}$$

$$= \begin{pmatrix} Q & P \end{pmatrix}\begin{pmatrix} B_0E_\rho^{-1}g_1 - \overline{P}_\rho g_2 \\ A_0E_\rho^{-1}g_1 + \overline{Q}_\rho g_2 \end{pmatrix} = g_1,$$

and so (112) follows.

(b) We use the representation (91) of $X^{\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}}$.
Since
$\begin{pmatrix} Q & P \end{pmatrix}\begin{pmatrix} B_0f_1 - \overline{P}_\rho Q_1^{-1}f_2 \\ A_0f_1 + \overline{Q}_\rho Q_1^{-1}f_2 \end{pmatrix} = E_\rho f_1 \in E_\rho X_{\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}}$, then
$\begin{pmatrix} B_0f_1 - \overline{P}_\rho Q_1^{-1}f_2 \\ A_0f_1 + \overline{Q}_\rho Q_1^{-1}f_2 \end{pmatrix} \in \operatorname{Ker} M_{\begin{pmatrix} Q & P \end{pmatrix}}$ implies, noting the nonsingularity of $E_\rho$, that $f_1 = 0$. The rest of the proof is analogous to that of Theorem 4.1.7 and we omit the details.

(c) Clearly, both $E_\rho X_{\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}}$ and $X_{E_\rho} = \begin{pmatrix} I & 0 \end{pmatrix} X_{\begin{pmatrix} E_\rho Q_\rho & E_\rho P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}}$ are subspaces of $X_{\begin{pmatrix} Q & P \end{pmatrix}}$. We show first that

$$E_\rho X_{\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}} \cap \begin{pmatrix} I & 0 \end{pmatrix} X_{\begin{pmatrix} E_\rho Q_\rho & E_\rho P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}} = \{0\}.$$

Indeed, if $f$ belongs to this intersection, then necessarily $f = E_\rho g$, with $g \in X_{\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}} \subset \mathbb{F}^p[z]$, i.e. $f \in E_\rho \mathbb{F}^p[z]$.
On the other hand, $f \in \begin{pmatrix} I & 0 \end{pmatrix} X_{\begin{pmatrix} E_\rho Q_\rho & E_\rho P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}}$ implies that $f = Qh_1 + Ph_2$ for some $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X^{\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}}$. Using the representation (93), we have $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} B_0E_\rho^{-1}g_1 - \overline{P}_\rho g_2 \\ A_0E_\rho^{-1}g_1 + \overline{Q}_\rho g_2 \end{pmatrix}$ with $g_1 \in X_{E_\rho}$. Now $\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ implies that $\begin{pmatrix} I & 0 \end{pmatrix}\begin{pmatrix} Q & P \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = g_1 \in X_{E_\rho}$. So, necessarily, $f \in X_{E_\rho} \cap E_\rho \mathbb{F}^p[z] = \{0\}$.

∎

Next, we investigate the question of which submodules of $X_{\begin{pmatrix} Q & P \end{pmatrix}}$ are direct summands, with respect to a given shift mdule structure. An immediate consequence of Theorem 4.3 is that, corresponding to the factorization (85), $E_\rho X_{\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}}$ is a direct summand. In fact, we have $X_{\begin{pmatrix} Q & P \end{pmatrix}} = X_{E_\rho} \oplus E_\rho X_{\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}}$, and the direct sum is of submodules with respect to the module structure induced by the extension $\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}$. We note that the factorization (88), i.e. $E_\rho \begin{pmatrix} Q_\rho & P_\rho \end{pmatrix} = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} E_\rho Q_\rho & E_\rho P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}$ are, respectively, left and right skew prime factorizations. The next theorem extends this observation.

**Theorem 4.4** *Let* $U = \begin{pmatrix} Q & P \end{pmatrix}$ *be a full row rank polynomial matrix, with* $Q$ *nonsingular and* $Q^{-1}P$ *proper. Let*

$$\begin{pmatrix} Q & P \end{pmatrix} = E_\rho \begin{pmatrix} Q_\rho & P_\rho \end{pmatrix} \tag{98}$$

*be a factorization with* $\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}$ *left prime, i.e. a maximal left skew prime factorization and let*

$$\begin{aligned} \begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix} \begin{pmatrix} B_0 & -\overline{P}_\rho \\ A_0 & \overline{Q}_\rho \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\ \begin{pmatrix} B_0 & -\overline{P}_\rho \\ A_0 & \overline{Q}_\rho \end{pmatrix} \begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{aligned} \tag{99}$$

*be a doubly coprime factorization. With* $Q_1$ *an arbitrary nonsingular polynomial matrix, we define*

$$\overline{A} = Q_1 \overline{A}_0, \qquad \overline{B} = Q_1 \overline{B}_0. \tag{100}$$

*Let*

$$\begin{pmatrix} Q & P \end{pmatrix} = E_\alpha \begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix} \tag{101}$$

*be a factorization for which* $E_\alpha$ *is nonsingular. Then*

1. *There exists a factorization*

   $$E_\rho = E_\alpha E_\beta \tag{102}$$

   *and we have*

   $$\begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix} = E_\beta \begin{pmatrix} Q_\rho & P_\rho \end{pmatrix} \tag{103}$$

2. *The factorization (101) is a a left skew prime factorization if and only if (102) is a left skew prime factorization.*

3. *With respect to the shift module structure,* $E_\alpha X_{\begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix}}$ *is a submodule.*

4. $E_\alpha X_{\begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix}}$ *is a direct summand of* $X_{\begin{pmatrix} Q & P \end{pmatrix}}$ *if and only if the factorization (101) is a left skew prime factorization.*

**Proof:**

1. Note that as $\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}$ is left prime, it has a polynomial right inverse $\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}^\sharp$. We have

   $$\begin{pmatrix} Q & P \end{pmatrix} = E_\rho \begin{pmatrix} Q_\rho & P_\rho \end{pmatrix} = E_\alpha \begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix}. \tag{104}$$

   This implies $E_\rho = E_\alpha E_\beta$ with $E_\beta = \begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix} \begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}^\sharp$.

   Using the nonsingularity of $E_\alpha$, the factorization (103) follows from (102) and (104).

2. Assume first that $E_\alpha \begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix}$ is a left skew prime factorization. Thus, by Theorem 3.2, there exist polynomial matrices $X, Y$ for which $I = XE_\alpha - \begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix} Y = XE_\alpha - E_\beta [\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix} Y]$. This shows that $E_\alpha E_\beta$ is a left skew prime factorization.

   Conversely, assume that $E_\alpha E_\beta$ is a left skew prime factorization. Thus $I = XE_\alpha - E_\beta Y = XE_\alpha - E_\beta \begin{pmatrix} Q_\rho & P_\rho \end{pmatrix} (\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}^\sharp Y)$. This shows that (101) is a left skew prime factorization.

3. Assume now that the shift module structure on $X_{\begin{pmatrix} Q & P \end{pmatrix}}$ is induced by the extension $\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}$, with the assumptions in the theorem satisfied. We apply Theorem 4.3 twice in order to get the representations

$$\begin{array}{rcl} X_{\begin{pmatrix} Q & P \end{pmatrix}} & = & M_{\begin{pmatrix} Q & P \end{pmatrix}} X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \\ X_{\begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix}} & = & M_{\begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix}} X^{\begin{pmatrix} Q_\alpha & P_\alpha \\ -\overline{A} & \overline{B} \end{pmatrix}}. \end{array} \tag{105}$$

   The second equality implies

$$E_\alpha X_{\begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix}} = M_{\begin{pmatrix} Q & P \end{pmatrix}} X^{\begin{pmatrix} Q_\alpha & P_\alpha \\ -\overline{A} & \overline{B} \end{pmatrix}}.$$

   The factorization $\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix} = \begin{pmatrix} E_\alpha & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} Q_\alpha & P_\alpha \\ -\overline{A} & \overline{B} \end{pmatrix}$, implies that $X^{\begin{pmatrix} Q_\alpha & P_\alpha \\ -\overline{A} & \overline{B} \end{pmatrix}} \subset X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}}$ as a submodule. By our assumption, the multiplication map $M_{\begin{pmatrix} Q & P \end{pmatrix}} : X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \longrightarrow X_{\begin{pmatrix} Q & P \end{pmatrix}}$ is a module isomorphism, so necssarily $E_\alpha X_{\begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix}}$ is a submodule.

4. Assume the factorization (101) is left skew prime. By Part 2, the factorization $E_\alpha E_\beta$ is also left skew prime. Thus there exist nonsingular polynomial matrices $\hat{E}_\alpha, \hat{E}_\beta$ for which

$$E_\alpha E_\beta = \hat{E}_\beta \hat{E}_\alpha, \tag{106}$$

   with $E_\alpha, \hat{E}_\beta$ left coprime and $\hat{E}_\alpha, E_\beta$ right coprime. In turn this implies the factorizations

$$\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix} = \begin{pmatrix} E_\alpha & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} Q_\alpha & P_\alpha \\ -\overline{A} & \overline{B} \end{pmatrix} = \begin{pmatrix} \hat{E}_\beta & 0 \\ 0 & Q_1 \end{pmatrix} \begin{pmatrix} \hat{E}_\alpha Q_\rho & \hat{E}_\alpha P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}. \tag{107}$$

   We claim that these factorizations are skew prime.

   Indeed, the left coprimeness of $\begin{pmatrix} E_\alpha & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} \hat{E}_\beta & 0 \\ 0 & Q_1 \end{pmatrix}$ follows from the left coprimeness of $E_\alpha, \hat{E}_\beta$ and the right coprimeness of $\begin{pmatrix} Q_\alpha & P_\alpha \\ -\overline{A} & \overline{B} \end{pmatrix}, \begin{pmatrix} \hat{E}_\alpha Q_\rho & \hat{E}_\alpha P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}$ follows from the right coprimeness of $\hat{E}_\alpha, E_\beta$ and the unimodularity of $\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}$.

Equation (107) implies the direct sum representation

$$\begin{aligned}
X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} &= X^{\begin{pmatrix} Q_\alpha & P_\alpha \\ -\overline{A} & \overline{B} \end{pmatrix}} \oplus X^{\begin{pmatrix} \hat{E}_\alpha Q_\rho & \hat{E}_\alpha P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}} \\
&= X^{\begin{pmatrix} E_\beta & 0 \\ 0 & Q_1 \end{pmatrix}\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}} \oplus X^{\begin{pmatrix} \hat{E}_\alpha & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}} \\
&= X^{\begin{pmatrix} E_\beta & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}} \oplus X^{\begin{pmatrix} I & 0 \\ 0 & Q_1 \end{pmatrix}\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}} \oplus X^{\begin{pmatrix} \hat{E}_\alpha & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}}
\end{aligned} \tag{108}$$

With the equalities (105) and We compute

$$\begin{aligned}
M_{\begin{pmatrix} Q & P \end{pmatrix}} X^{\begin{pmatrix} E_\beta & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}} &= E_\alpha X_{E_\beta} \\
M_{\begin{pmatrix} Q & P \end{pmatrix}} X^{\begin{pmatrix} I & 0 \\ 0 & Q_1 \end{pmatrix}\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}} &= E_\rho X_{\begin{pmatrix} Q_\rho & P_\rho \end{pmatrix}} \\
M_{\begin{pmatrix} Q & P \end{pmatrix}} X^{\begin{pmatrix} \hat{E}_\alpha & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} Q_\rho & P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}} &= \hat{E}_\beta X_{\hat{E}_\alpha}
\end{aligned} \tag{109}$$

Thus $E_\alpha X_{\begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix}}$ is a direct summand and we have identified, for this particular shift module structure, its complementof $E_\alpha X_{\begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix}}$ with $\hat{E}_\beta X_{\hat{E}_\alpha}$.

To prove the converse, assume that, for some shift module structure on $X_{\begin{pmatrix} Q & P \end{pmatrix}}$, $E_\alpha X_{\begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix}}$ is a submodule and a direct summand. We apply Theorem 4.3, with $\overline{A}, \overline{B}$ defined by (101), to get the representations

$$\begin{aligned}
X_{\begin{pmatrix} Q & P \end{pmatrix}} &= M_{\begin{pmatrix} Q & P \end{pmatrix}} X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} \\
X_{\begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix}} &= M_{\begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix}} X^{\begin{pmatrix} Q_\alpha & P_\alpha \\ -\overline{A} & \overline{B} \end{pmatrix}}.
\end{aligned} \tag{110}$$

So, we obtain

$$E_\alpha X_{\begin{pmatrix} Q_\alpha & P_\alpha \end{pmatrix}} = M_{\begin{pmatrix} Q & P \end{pmatrix}} X^{\begin{pmatrix} Q_\alpha & P_\alpha \\ -\overline{A} & \overline{B} \end{pmatrix}} \tag{111}$$

By our assumption $X^{\begin{pmatrix} Q_\alpha & P_\alpha \\ -\overline{A} & \overline{B} \end{pmatrix}} \subset X^{\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}} = X^{\begin{pmatrix} E_\alpha E_\beta Q_\rho & E_\alpha E_\beta P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}}$ as a direct summand. Thus $\begin{pmatrix} Q_\alpha & P_\alpha \\ -\overline{A} & \overline{B} \end{pmatrix}$ is a right factor of $\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix}$ and it is trivial to check, using (104), that

$$\begin{pmatrix} Q & P \\ -\overline{A} & \overline{B} \end{pmatrix} = \begin{pmatrix} E_\alpha & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} E_\beta Q_\rho & E_\beta P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}. \tag{112}$$

Since $X^{\begin{pmatrix} Q_\alpha & P_\alpha \\ -\overline{A} & \overline{B} \end{pmatrix}}$ is a direct summand, the factorization (112) is a left skew prime factorization. Hence, there exist polynomial matrices $X_{ij}, Y_{ij}$ for which

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}\begin{pmatrix} E_\alpha & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} E_\beta Q_\rho & E_\beta P_\rho \\ -\overline{A} & \overline{B} \end{pmatrix}\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

In particular, this implies the equality $X_{11}E_\alpha - E_\beta(Q_\rho Y_{11} + P_\rho Y_{21}) = I$, which shows that $E_\alpha E_\beta$ is a left skew prime factorization. Therefore there exist nonsingular polynomial matrices $\hat{E}_\alpha, \hat{E}_\beta$ for which

$$E_\alpha E_\beta = \hat{E}_\beta \hat{E}_\alpha, \tag{113}$$

with $E_\alpha, \hat{E}_\beta$ left coprime and $\hat{E}_\alpha, E_\beta$ right coprime. Thus the factorizations (107) follow from which we obtain

$$\begin{pmatrix} E_\alpha & 0 \end{pmatrix} \begin{pmatrix} Q_\alpha & P_\alpha \\ -\overline{A} & \overline{B} \end{pmatrix} = \begin{pmatrix} \hat{E}_\beta & 0 \end{pmatrix} \begin{pmatrix} \hat{E}_\alpha Q_\rho & \hat{E}_\alpha P_\rho \\ -\overline{A}_0 & \overline{B}_0 \end{pmatrix}. \tag{114}$$

It is easily checked that these are skew prime factorizations.

∎

# 5 Output nulling subspaces

In this section we study the set of all out put nulling subspaces for a given linear system. These objects, a subclass of controlled invariant subspaces, arose first in the analysis of the disturbance ecoupling problem, see Wonham [1979]. For our analysis, we assume that a discrete time system is given in state space form, in the state space $X$, as

$$\begin{cases} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t + Du_t \end{cases} \tag{115}$$

and has the transfer function

$$G(z) = D + C(zI - A)^{-1}B = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{116}$$

Defining $x(z) = \sum_{j=1}^\infty x_j z^{-j} \in z^{-1}\mathbb{F}^n[[z^{-1}]]$, and similarly defining $u(z), y(z)$, then (115) can be rewritten in behavioral form as

$$\begin{cases} \sigma x &= Ax + Bu \\ y &= Cx + Du. \end{cases} \tag{117}$$

We say that a subspace $\mathcal{V} \subset X$ is **controlled invariant** if there exists a feedback map $K$ such that

$$(A + BK)\mathcal{V} \subset \mathcal{V} \tag{118}$$

and **output nulling** if there exists a feedback map $K$ such that

$$(A + BK)\mathcal{V} \subset \mathcal{V} \subset \operatorname{Ker}(C + DK). \tag{119}$$

Thus a subspace is output nulling if and only if, given any initial state in $\mathcal{V}$, we can find a state feedback controller that keeps the state in $\mathcal{V}$ while keeping the output zero.

A feedback map $K$ for which (118) holds is called a **friend** of $\mathcal{V}$. The set of all friends of $\mathcal{V}$ is denoted by $\mathcal{F}(\mathcal{V})$. For the case of output nulling subspaces we have to modify the notion of a friend. Thus, given an output nulling subspace $\mathcal{V}$ for the system (117), a feedback map $K$ for which (119) holds is called an **output nulling friend** of $\mathcal{V}$. By $\mathcal{F}_{ON}(\mathcal{V})$ we denote the set of all output nulling friends of $\mathcal{V}$. Obviously, we have $\mathcal{F}_{ON}(\mathcal{V}) \subset \mathcal{F}(\mathcal{V})$.

Given an output nulling subspace $\mathcal{V}$ and a friend $K \in \mathcal{F}_{ON}(\mathcal{V})$, there exists a natural $\mathbb{F}[z]$-module structure on $\mathcal{V}$, namely the module structure induced by $(A+BK)|\mathcal{V}$. The characterization of these module structures and their relation to systems interconnections will be one of our central aims in this paper. We will see later how this can be reduced to polynomial matrix completion problems.

It is easily seen that the assumption that the system (117) has a proper transfer function, rather than a strictly proper one does not add extra restrictions.

**Proposition 5.1** *Given a, not necessarily minimal, realization* $G = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *of a system* $\Sigma$. *We define the extended system* $\Sigma_e$ *by*

$$\begin{array}{ll} A_e = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, & B_e = \begin{pmatrix} B \\ D \end{pmatrix} \\ C_e = \begin{pmatrix} 0 & I \end{pmatrix} & D_e = 0. \end{array} \tag{120}$$

*Then*

1. *We have*
$$\frac{1}{z}G(z) = \left(\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array}\right) \tag{121}$$
2. *A subspace* $\mathcal{V} \subset X$ *is an output nulling subspace for* $\Sigma$ *if and only if* $\mathcal{V}_e = \mathcal{V} \oplus \{0\}$ *is an output nulling subspace for* $\Sigma_e$.

**Proof:**

1. Follows from an easy computation.
2. Assume $\mathcal{V}_e = \mathcal{V} \oplus \{0\}$ is an output nulling subspace for $\Sigma_e$. Then there exists a state feedback map $K_e = \begin{pmatrix} K & L \end{pmatrix}$ such that for every $v \in \mathcal{V}$ there exists $v' \in \mathcal{V}$ such that
$$\left[\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} + \begin{pmatrix} B \\ D \end{pmatrix}\begin{pmatrix} K & L \end{pmatrix}\right]\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} v' \\ 0 \end{pmatrix}.$$
This implies
$$\left\{\begin{array}{c} (A+BK)\mathcal{V} \subset \mathcal{V} \\ (C+DK)\mathcal{V} = \{0\}, \end{array}\right.$$
i.e. $\mathcal{V}$ is output nulling. This argument is clearly reversible.

∎

**Corollary 5.1** *Given the system (117), then there exists a maximal output nulling subspace subspace, which will be denoted by* $\mathcal{V}^*$.

**Proof:** This is trivial for the strictly proper case since the sum of output nulling subspaces is also output nulling and the zero subspace is clearly output nulling. Thus there exists a maximal output nulling subspace. The general case follows from this and Proposition 5.1. ∎

Our aim will be to study and characterize, output nulling subspaces in terms of external data of a system, namely its transfer function and representations of it in terms of matrix fractions. Thus we assume that a proper rational function $G$ with the not necessarily left coprime, matrix fraction representation $G = T^{-1}V$. With such a representation, we associate the shift realization given, in the state space $X_T$ by (18).

We can state

**Proposition 5.2** *Given a* $p \times m$ *proper rational matrix function* $G$ *with the, not necessarily left coprime, matrix fraction representation* $G = T^{-1}V$. *With* $X_V$ *defined by (10), we have*

$$X_V \subset X_T. \tag{122}$$

**Proof:** Let $f \in X_V$, then $f = Vh$ for some strictly proper $h$. Now $T^{-1}f = T^{-1}(Vh) = (T^{-1}V)h$. Since $T^{-1}V$ is proper and $h$ is strictly proper it follows that $T^{-1}f$ is strictly proper. This shows that $f \in X_T$. ∎

**Proposition 5.3** *Given a* $p \times m$ *proper rational matrix function with the, not necessarily left coprime, matrix fraction representation* $G = T^{-1}V$. *Assume* $V = E_1V_1$ *with* $E_1$ *a nonsingular polynomial matrix. Then, with respect to the shift realization (18) of* $G$, *the subspace* $\mathcal{V}$ *defined by*

$$\mathcal{V} = E_1X_{V_1} \tag{123}$$

*is an output nulling subspace. Moreover, we have*

$$E_1X_{V_1} \subset X_V \subset X_T. \tag{124}$$

**Proof:** We show first the inclusions (124). Assume $f \in E_1X_{V_1}$. Then $f = E_1g$ with $g = V_1h$, $g \in \mathbb{F}^p[z], h \in z^{-1}\mathbb{F}^m[[z^{-1}]]$. Clearly $f = E_1(V_1h) = (E_1V_1)h = Vh$. So $f \in X_V$. Now $T^{-1}f = T^{-1}(Vh) = (T^{-1}V)h$. Since $T^{-1}V$ is proper and $h$ is strictly proper, it follows that $T^{-1}f$ is strictly proper. This shows that $f \in X_T$.

Next we show that if $h \in z^{-1}\mathbb{F}^m[[z^{-1}]]$ is such that $g = V_1 h \in \mathbb{F}^p[z]$, then also $V_1(S_- h) \in \mathbb{F}^p[z]$. Denoting $\eta = (h)_{-1} \in \mathbb{F}^m$, we have $zh = \eta + S_- h$. This implies

$$V_1(S_- h) = V_1(zh) - V_1\eta = z(V_1 h) - V_1\eta = zg - V_1\eta \in \mathbb{F}^p[z].$$

This shows that $V_1(S_- h) \in \mathbb{F}^p[z]$.

Now, there is a unique way of writing $V(z) = T(z)D + U(z)$, with $T^{-1}U$ strictly proper. Simply, we define $D = \pi_+ T^{-1}V$. Hence, we can compute

$$B\eta = \pi_T V\eta = \pi_T(TD + U)\eta = U\eta.$$

In conclusion

$$Af = S_T f = E_1 V_1(S_- h) + U\eta.$$

As $E_1 V_1(S_- h) \in E_1 X_{V_1}$ and $U\eta \in \operatorname{Im} B$ we have proved that $\mathcal{V}$ is controlled invariant. Since an element $f \in X_V$ may have many preimages under the multiplication by $V$ map, in $z^{-1}\mathbb{F}^m[[z^{-1}]]$, in order to define an appropriate feedback map $K : X_T \longrightarrow \mathbb{F}^m$, we have to make choices. To this end, let $f_1, \dots, f_k$ be a basis for $X_V$ which we extend to a basis $f_1, \dots, f_n$ for $X_T$. We choose arbitrarily elements $h_1, \dots, h_k \in z^{-1}\mathbb{F}^m[[z^{-1}]]$ such that $Vh_i = f_i$ for $i = 1, \dots, k$ and let $\eta_i = (h_i)_{-1}$. We choose $\eta_{i+1}, \dots, \eta_n$ arbitrarily, define

$$Kf_i = \eta_i, \qquad i = 1, \dots, n \tag{125}$$

and extend $K$ linearly to all of $X_T$.
Now we show that $\mathcal{V} = E_1 X_{V_1}$ is controlled invariant. Let $f \in E_1 X_{V_1}$, i.e. $f = E_1 g$ and $g = V_1 h$. We compute, with $(h)_{-1} = \eta$,

$$\begin{aligned} S_T f &= \pi_T z f = \pi_T z E_1 V_1 h = \pi_T z V h \\ &= \pi_T V(zh) = \pi_T V(S_- h + \eta) \\ &= \pi_T V(S_- h) + \pi_T(TD + U)\eta \\ &= V(S_- h) + U\eta = V(S_- h) + BKf, \end{aligned}$$

i.e. $(A - BK)f = V(S_- h) \in E_1 X_{V_1} \subset X_V$. Next, we compute

$$\begin{aligned} Cf &= (T^{-1}f)_{-1} = (T^{-1}E_1 V_1 h)_{-1} \\ &= (T^{-1}Vh)_{-1} = (T^{-1}(TD + U)h)_{-1} \\ &= (Dh)_{-1} = D(h)_{-1} = D\eta = DKf, \end{aligned}$$

which can be rewritten as $(C - DK)|E_1 X_{V_1} = 0$. Thus we proved that $E_1 X_{V_1}$ is an output nulling subspace.

∎

Obviously, in general, there is a certain amount of freedom in the choice of the feedback map $K \in \mathcal{F}_{ON}(X_V)$, and hence in the module structure in $X_V$ induced by $(A + BK)|X_V$. The determination of the extent of this freedom will be pursued in Section **??**.

There is a natural order relation among the output nulling subspaces described in the previous proposition.

**Proposition 5.4** *Given a $p \times m$ proper rational matrix function with the, not necessarily left coprime, matrix fraction representation $G = T^{-1}V$. Assume $V_i = E_i V_i,\ i = 1, 2$ with $E_i$ nonsingular polynomial matrices. Then*

$$E_1 X_{V_1} \subset E_2 X_{V_2} \tag{126}$$

*if and only if, for some polynomial matrix $K$, we have*

$$E_1 = E_2 K. \tag{127}$$

**Proof:** Assume the factorization (127 holds, and hence also $V_2 = KV_1$. Then

$$E_1 X_{V_1} = E_2 K X_{V_1} \subset E_2 X_{KV_1} = E_2 X_{V_2}.$$

To prove the converse, we note that by left multiplying $V$ by a unimodular matrix we can reduce it to the form $V = \begin{pmatrix} V' \\ 0 \end{pmatrix}$, with $V'$ full row rank. Thus we can assume without loss of generality that $V$ has full row rank. We show now that this implies the equality

$$X_V + V\mathbb{F}^m[z] = \mathbb{F}^p[z].$$

Clearly we have the inclusion $X_V + V\mathbb{F}^m[z] \subset \mathbb{F}^p[z]$. Let $\Omega$ be any rational right inverse of $V$. Thus $V\Omega = I$. Given any $g \in \mathbb{F}^p[z]$, we have $g = V\pi_+(\Omega g) + V\pi_-(\Omega g)$. This shows that $V\pi_-(\Omega g) \in X_V$. Since $V\pi_+(\Omega g) \in V\mathbb{F}^m[z]$ the inverse inclusion holds.

Now (126) implies the inclusion

$$E_1 X_{V_1} + V\mathbb{F}^m[z] \subset E_2 X_{V_2} + V\mathbb{F}^m[z],$$

i.e.

$$E_1 X_{V_1} + E_1 V_1 \mathbb{F}^m[z] \subset E_2 X_{V_2} + E_2 V_2 \mathbb{F}^m[z].$$

Thus

$$E_1 \mathbb{F}^p[z] = E_1(X_{V_1} + V_1\mathbb{F}^m[z]) \subset E_2(X_{V_2} + V_2\mathbb{F}^m[z]) = E_2\mathbb{F}^p[z].$$

This in turn implies the existence of a, necessarily nonsingular polynomial matrix, $K$ for which (127) holds. ∎

Proposition 5.3 gave only sufficient conditions for a subspace of $X_T$ to be an output nulling subspace. The next theorem gives a full characterization.

**Theorem 5.1** *Given a $p \times m$ proper rational matrix function with the, not necessarily left coprime, matrix fraction representation $G = T^{-1}V$, and let $G(z) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ be the corresponding shift realization. A necessary and sufficient condition for a subspace $\mathcal{V} \subset X_T$ to be an output nulling subspace is that it has a representation of the form*

$$\mathcal{V} = V_0 X_{E_0} \tag{128}$$

*where*

$$V = V_0 E_0 \tag{129}$$

*is a factorization such that $E_0$ is nonsingular. Without loss of generality, we may assume that $V_0|X_{E_0}$ is injective.*

**Proof:** Assume $\mathcal{V}$ has such a representation. Clearly $\mathcal{V} \subset X_V \subset X_T$. If $f \in \mathcal{V}$ then $f = V_0 g$ and $g \in X_{E_0}$. Let $g = E_0 h$, then $zh = \eta + S_- h$ and

$$zg = zE_0 h = E_0(zh) = E_0(\eta + S_- h)$$

or $zg - E_0\eta = E_0(S_- h) \in X_{E_0}$. Next we compute

$$\begin{array}{rcl} S_T f & = & \pi_T z f = \pi_T z V_0 E_0 h = \pi_T V_0 E_0(\eta + S_- h) \\ & = & \pi_T V_0 E_0 \eta + V_0(E_0 S_- h) \\ & = & \pi_T(TD + U)\eta + V_0(E_0 S_- h) \\ & = & U\eta + V_0(E_0 S_- h) \end{array}$$

This shows that $\mathcal{V}$ is controlled invariant. We proceed to compute

$$\begin{array}{rcl} Cf & = & (T^{-1}f)_{-1} = (T^{-1}(V_0 E_0 h))_{-1} \\ & = & (T^{-1}Vh)_{-1} = (T^{-1}(TD + U)h)_{-1} \\ & = & (Dh)_{-1} = D\eta \end{array}$$

As in the proof of Proposition 5.3, there exists a linear map $K$ such that $\eta = Kf$ and so we have $(A - BK)f \in \mathcal{V}$ and $(C - DK)f = 0$.

To prove necessity, assume $\mathcal{V} \subset X_T$ is an output nulling subspace, with $\dim \mathcal{V} = k$. Thus, for some state feedback map we have $(A - BK)\mathcal{V} \subset \mathcal{V}$. Let

$$\overline{A} = (A - BK)|_{\mathcal{V}}.$$

Without loss of generality, we may assume $\mathcal{V} \simeq \mathbb{F}^k$. Clearly $X_{zI-\overline{A}}$ is a model space for $\overline{A}$. Let now $Z : X_{zI-\overline{A}} \longrightarrow X_T$ be the injective map satisfying

$$Z\overline{A} = (A - BK)Z.$$

Since $Z$ is linear, there exists a $p \times k$ polynomial matrix $\Psi(z)$, such that

$$(Z\xi)(z) = \Psi(z)\xi, \qquad \forall \xi \in \mathbb{F}^k.$$

Using the fact that $(S_T f)(z) = zf(z) - T(z)\eta_f$, we compute

$$\begin{array}{rcl} \Psi(z)\overline{A}\xi & = & Z\overline{A}\xi = (A - BK)Z\xi \\ & = & (S_T - U(z)K)\Psi(z)\xi \\ & = & z\Psi(z)\xi - T(z)\eta_1 - U(z)\eta_2 \end{array}$$

with $\eta_1, \eta_2$ depending linearly on $\xi$. Therefore there exist matrices $E, F$ for which

$$\Psi(z)\overline{A}\xi = z\Psi(z)\xi - T(z)E\xi - U(z)F\xi$$

or

$$\Psi(z)\overline{A} = z\Psi(z) - T(z)E - U(z)F.$$

which is equivalent to $\Psi(z)(zI - \overline{A}) = T(z)E + U(z)F$. Note that $K$ is an operator acting on $\mathcal{V}$ and $K\Psi(z) = F$, where $F$ is a $m \times k$ matrix. Taking into account that $V = TD + U$, we have

$$\begin{array}{rcl} \Psi(z) & = & T(z)E(zI - \overline{A})^{-1} + U(z)F(zI - \overline{A})^{-1} \\ & = & T(z)E(zI - \overline{A})^{-1} + (V(z) - T(z)D)F(zI - \overline{A})^{-1} \end{array}$$

Since $\operatorname{Ker}(C - DK) \supset \mathcal{V}$, we have

$$0 = (C - DK)\Psi(z)\xi = (T^{-1}\Psi\xi)_{-1} - DK\Psi(z)\xi.$$

But $BK\Psi(z)\xi = U(z)K\Psi(z)\xi = U(z)F\xi$. Thus

$$\begin{aligned} 0 &= (T^{-1}\Psi\xi)_{-1} - DF\xi \\ &= (E(zI - \overline{A})^{-1}\xi)_{-1} + (T^{-1}U(z)F(zI - \overline{A})^{-1}\xi)_{-1} - DF\xi \\ &= (E(zI - \overline{A})^{-1}\xi)_{-1} - DF\xi = (E - DF)\xi \end{aligned}$$

Hence

$$\begin{aligned} \Psi(z) &= T(z)DF(zI - \overline{A})^{-1} + U(z)F(zI - \overline{A})^{-1} \\ &= (T(z)D + U(z))F(zI - \overline{A})^{-1} \\ &= V(z)F(zI - \overline{A})^{-1} \end{aligned}$$

The injectivity of $\Psi(z)$ implies the observability of the pair $(F, \overline{A})$. Let now $E_0(z)^{-1}W_0(z)$ be a left coprime factorization of $F(zI - \overline{A})^{-1}$. Then

$$\Psi(z) = V(z)E_0(z)^{-1}W_0(z).$$

The left coprimeness of $E_0, W_0$ implies that, for some polynomial matrix $V_0(z)$ we have $V(z) = V_0(z)E_0(z)$ and $\Psi(z) = V_0(z)W_0(z)$. Now the columns of $W_0(z)$ span $X_{E_0}$ and hence we finally obtain $\mathcal{V} = V_0X_{E_0}$. Note that in our construction $\dim X_{E_0} = k$ and hence $V_0|X_{E_0}$ is an injective map. ∎

Theorem 5.1 provides an alternative approach to showing that a maximal output nulling subspace exists. First we note the following.

**Corollary 5.2** *The sum of two output nulling subspaces is output nulling.*

**Proof:** Let $\mathcal{V}_i = V_iX_{E_i} = VX^{E_i}$ be output nulling. Then

$$\mathcal{V}_0 = \mathcal{V}_1 + \mathcal{V}_2 = VX^{E_1} + VX^{E_2} = V(X^{E_1} + X^{E_2}) = VX^{E_0},$$

where $E_0$ is the least common left multiple of $E_1$ and $E_2$. We claim $VE_0^{-1}$ is a polynomial matrix. First, it is obvious that $VE_0^{-1}X_{E_0} = VX^{E_0}$ is a polynomial space. Secondly, it is trivial to observe that $VE_0^{-1}E_0\mathbb{F}^p[z] = V\mathbb{F}^p[z]$ is also a polynomial space. Together we have that $VE_0^{-1}\mathbb{F}^p[z] = VE_0^{-1}(X_{E_0} + E_0\mathbb{F}^p[z])$ is a polynomial space. In particular, this implies that $V_0 = VE_0^{-1}$ is a polynomial matrix. This factorization, using Theorem 5.1, shows that $\mathcal{V}_0$ is output nulling.

∎

The previous corollary proves the existence of a maximal output nulling subspace. The next proposition, due to Emre and Hautus [1980], gives a concrete characterization of it.

**Proposition 5.5** *With respect to the shift realization (18) of $T^{-1}V$ in $X_T$, there exists a maximal output nulling subspace and it is given by $\mathcal{V}^* = X_V$.*

**Proof:** That $X_V$ is output nulling follows from Proposition 5.3. If $\mathcal{V}$ is an arbitrary output nulling subspace, it has a representation $\mathcal{V} = V_\alpha X_{E_\alpha}$ for a factorization $V = V_\alpha E_\alpha$. Clearly,

$$\mathcal{V} = V_\alpha X_{E_\alpha} \subset X_{V_\alpha E_\alpha} = X_V.$$

Thus the maximality of $X_V$ follows. $\blacksquare$

Theorem 5.1 allows us to study arithmetically the partial order relation, given by inclusion, in the set of all output nulling subspaces associated with the shift realization of $T^{-1}V$.

**Proposition 5.6** *Let $\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$ be two output nulling subspaces of $X_V$. Then $\mathcal{V}_\alpha \subset \mathcal{V}_\beta$ if and only if there exist factorizations $V = \overline{V}_\alpha \overline{E}_\alpha = \overline{V}_\beta \overline{E}_\beta$ such that*

1. *$\overline{E}_\alpha, \overline{E}_\beta$ are nonsingular,*
2. *$\overline{E}_\alpha$ is a right factor of $\overline{E}_\beta$, i.e. $\overline{E}_\beta = Y\overline{E}_\alpha$ for some nonsingular polynomial matrix $Y$.*
3. *$\mathcal{V}_\alpha = \overline{V}_\alpha X_{\overline{E}_\alpha}$ and $\mathcal{V}_\beta = \overline{V}_\beta X_{\overline{E}_\beta}$.*

**Proof:** Assume there exists factorizations $V = \overline{V}_\alpha \overline{E}_\alpha = \overline{V}_\beta \overline{E}_\beta$ satisfying conditions (i)-(iii). By Theorem 5.1, $\overline{V}_\alpha X_{\overline{E}_\alpha}$ and $\mathcal{V}_\beta = \overline{V}_\beta X_{\overline{E}_\beta}$ are both output nulling subspaces. Moreover, since $\overline{E}_\beta = Y\overline{E}_\alpha$, it follows that $V = \overline{V}_\beta \overline{E}_\beta = \overline{V}_\beta Y \overline{E}_\alpha = \overline{V}_\alpha \overline{E}_\alpha$ and hence that $\overline{V}_\alpha = \overline{V}_\beta Y$. Therefore

$$\mathcal{V}_\alpha = \overline{V}_\alpha X_{\overline{E}_\alpha} = \overline{V}_\beta Y X_{\overline{E}_\alpha} \subset \overline{V}_\beta X_{Y\overline{E}_\alpha} = \overline{V}_\beta X_{\overline{E}_\beta} = \mathcal{V}_\beta.$$

Conversely, assume that $\mathcal{V}_\alpha \subset \mathcal{V}_\beta$ and both are output nulling. Since $\mathcal{V}_\alpha \subset \mathcal{V}_\beta \subset X_V$, all three subspaces are compatible. This implies the existence of a module structure on $X_V$, induced by $A + BK$, for which $\mathcal{V}_\alpha, \mathcal{V}_\beta$ are submodules. Now $\mathcal{V}^* = X_V$ has a representation $X_V = \overline{V}_\mu X_{\overline{E}_\mu}$ with $V = \overline{V}_\mu \overline{E}_\mu$ and the map $\overline{V}_\mu : X_{\overline{E}_\mu} \longrightarrow X_V = \overline{V}_\mu X_{\overline{E}_\mu}$ a bijection. Thus submodules of $X_V$ correspond bijectively to submodules of $X_{\overline{E}_\mu}$. These in turn are given by factorizations of $\overline{E}_\mu$. So we have factorizations $\overline{E}_\mu = \overline{F}_\beta \overline{E}_\beta = \overline{F}_\alpha \overline{E}_\alpha$ such that $\mathcal{V}_\alpha = \overline{V}_\mu \overline{F}_\alpha X_{\overline{E}_\alpha}$, $\mathcal{V}_\beta = \overline{V}_\mu \overline{F}_\beta X_{\overline{E}_\beta}$. Since $\overline{F}_\beta X_{\overline{E}_\beta} \supset \overline{F}_\alpha X_{\overline{E}_\alpha}$ if and only if $\overline{E}_\alpha$ is a right factor of $\overline{E}_\beta$, there exists a nonsingular polynomial matrix $Y$ such that $\overline{E}_\beta = Y\overline{E}_\alpha$. Setting $\overline{V}_\alpha = \overline{V}_\mu \overline{F}_\alpha$ and $\overline{V}_\beta = \overline{V}_\mu \overline{F}_\beta$, we have $\mathcal{V}_\alpha = \overline{V}_\mu \overline{F}_\alpha X_{\overline{E}_\alpha} = \overline{V}_\alpha X_{\overline{E}_\alpha}$ as well as $\mathcal{V}_\beta = \overline{V}_\mu \overline{F}_\beta X_{\overline{E}_\beta} = \overline{V}_\beta X_{\overline{E}_\beta}$.
$\blacksquare$

A comparison of Proposition 5.3 and Theorem 5.1 raises immediately the question of finding an intrinsic characterization of the output nulling subspaces appearing in Proposition 5.3. We will show that this characterization is related to output nulling reachability subspaces. The analysis is a modification of the method given in Fuhrmann [1981].

Given a system $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ in the state space $X$, we will say that a subspace $\mathcal{R}$ is an **output nulling reachability subspace** if, for some $K : X \longrightarrow \mathbb{F}^m$ and $L : \mathbb{F}^m \longrightarrow \mathbb{F}^m$, the following conditions are satisfied.

1. $\mathcal{R} =< A + BK|BL >$,
2. $(C + DK)|\mathcal{R} = 0$ and
3. $DL = 0$.

This means that all states in $\mathcal{R}$ are reachable from the origin, while the output of the system is zero.

Clearly, since the sum of controlled invariant subspaces is controlled invariant, there exists a maximal output nulling reachability subspace, which is denoted by $\mathcal{R}^*$. This is obtained by choosing $L$ to be the basis matrix of the subspace $\mathcal{L}$ defined by $\{\xi | B\xi \in \mathcal{V}, D\xi = 0\}$.

**Theorem 5.2** *Given a $p \times m$ proper, rational matrix function with the left coprime, matrix fraction representation $G = T^{-1}V$. Assume*

$$V = E_\rho V_\rho$$

*is a maximal, left skew-prime factorization. Then, with respect to the shift realization (18), we have*

$$\mathcal{R}^* = E_\rho X_{V_\rho}. \tag{130}$$

**Proof:** Note that as $G$ is proper, there exists a unique constant matrix $D$ for which $V(z) = T(z)D + U(z)$ and $T(z)^{-1}U(z)$ strictly proper.

Let us now define a subspace by $\mathcal{R} = E_\rho X_{V_\rho}$. By Proposition 5.3, $\mathcal{R}$ is an output nulling subspace. Let $L$ be a basis matrix of the subspace $\mathcal{V}$ defined in (**??**) and assume $f \in X_V \cap \operatorname{Im} BL$. Since $f \in X_V$ we have $f = Vh$ for some strictly proper rational function $h$. On the other hand $f \in \operatorname{Im} BL$ implies that, for some constant vector $\xi$, $f = UL\xi$. Now

$$UL\xi = (V - TD)L\xi = VL\xi - TDL\xi = VL\xi,$$

or $Vh = VL\xi$. Since $V = E_\rho V_\rho$ and $E_\rho$ is nonsingular, we have , multiplying by $E_\rho^{-1}$, that $V_\rho h = V_\rho L\xi \in \mathbb{F}^p[z]$. Thus $V_\rho h \in X_{V_\rho}$ and hence $f = E_\rho(V_\rho h) \in E_\rho X_{V_\rho}$. Thus we obtain the inclusion

$$\mathcal{R}^* \subset \mathcal{R}. \tag{131}$$

To prove the converse it suffices to show that $\mathcal{R}$ is a reachability subspace. Note that, since $V_\rho$ has constant invariant factors, there exists an $m \times p$ polynomial matrix $W_\rho$ with constant invariant factors for which

$$V_\rho W_\rho V_\rho = V_\rho. \tag{132}$$

To see this, assume $P$ and $Q$ are unimodular matrices for which

$$V_\rho = P \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} Q$$

and set

$$W_\rho = Q^{-1}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$

Let now $f \in E_\rho X_{V_\rho}$, i.e. $f = E_\rho(V_\rho h)$, with $V_\rho h$ polynomial. This implies

$$f = E_\rho V_\rho W_\rho V_\rho h = E_\rho V_\rho(W_\rho V_\rho h) = Vg,$$

where $g = W_\rho V_\rho h$ is polynomial. So we have $f = Vh = Vg$.

We prove now three lemmas.

**Lemma 5.1** *Given a $p \times m$ proper, rational matrix function with the left coprime, matrix fraction representation $G = T^{-1}V$. Then, with respect to the shift realization (18), we have*

$$\mathcal{L} = \{\xi \in \mathbb{F}^m | \exists h \in z^{-1}\mathbb{F}^m[[z^{-1}]], V\xi = Vh\} = \{\xi \in \mathbb{F}^m | B\xi \in X_V, D\xi = 0\}. \tag{133}$$

**Proof:** We have $V = U + TD$ with $T^{-1}U$ strictly proper. Clearly, $B\xi = U\xi$, so if $D\xi = 0$ then $B\xi = U\xi = (V - TD)\xi = V\xi$. Since $B\xi \in X_V$, we have $V\xi = Vh$ for some $h \in z^{-1}\mathbb{F}^m[[z^{-1}]]$. Thus $\mathcal{L} \subset \{\xi \in \mathbb{F}^m | \exists h \in z^{-1}\mathbb{F}^m[[z^{-1}]], V\xi = Vh\}$.

Conversely, assume $V\xi = Vh$ for some $h$. Writing $V = U + TD$, we obtain $TD\xi = Vh - U\xi$ or $D\xi = T^{-1}Vh - T^{-1}U\xi$. The right hand side is strictly proper whereas $D\xi$ is polynomial. Thus we have $D\xi = 0$ and $U\xi = (V - TD)\xi = Vh \in X_V$.

∎

**Lemma 5.2** *If $f(z) = V(z)(\gamma_0 + \cdots + \gamma_k z^k) \in X_V$ then $\gamma_k \in \mathcal{L}$.*

**Proof:** If $f \in X_V$ then $f = Vh$ with $h \in z^{-1}\mathbb{F}^m[[z^{-1}]]$. Assume $h(z) = \sum_{j=1}^{\infty} \frac{h_{-j}}{z^j}$. Therefore $V(z)\left[\gamma_k z^k + \cdots + \gamma_0 - \frac{h_{-1}}{z} - \cdots\right] = 0$, which can be rewritten as

$$V(s)\gamma_k = V(z)\left[-\frac{\gamma_{k-1}}{z} - \cdots - \frac{\gamma_0}{z^k} + \frac{h_{-1}}{z^{k+1}} + \cdots\right],$$

i.e. $\gamma_k \in \mathcal{L}$. ∎

**Lemma 5.3** *Let $K : X_T \longrightarrow \mathbb{F}^m$, $K \in \mathcal{F}_{ON}(X_V)$. Then, given $\gamma \in \mathcal{L}$, we have*

$$(S_T - BK)^k V\gamma = V(z)\left(\gamma_0 + \cdots + \gamma_k z^k\right), \tag{134}$$

*with $\gamma_k = \gamma$.*

**Proof:** We prove this by induction. For $k=0$, equality (134) is trivially satisfied.

Assume now that

$$(S_T - BK)^{k-1} V\gamma = V(z)\left(\gamma_0' + \cdots + \gamma_{k-1}' z^{k-1}\right) = f$$

with $\gamma_{k-1}' = \gamma$. Then

$$\begin{array}{rcl} (S_T - BK)^k V\gamma & = & (S_T - BK)V(z)\left(\gamma_0' + \cdots + \gamma_{k-1}' z^{k-1}\right) \\ & = & zV(z)\left(\gamma_0' + \cdots + \gamma_{k-1}' z^{k-1}\right) - T(z)\eta_f - U(z)Kf \\ & = & V(z)\left(z\gamma_0' + \cdots + \gamma_{k-1}' z^{k}\right) - T(z)\eta_f - U(z)Kf \end{array}$$

There exists a, not necessarily unique, strictly proper $h$ such that $V(z)\left(\gamma_0' + \cdots + \gamma_{k-1}' z^{k-1}\right) = Vh$. We compute now

$$\begin{array}{rcl} \eta_f & = & (T^{-1}f)_{-1} = (T^{-1}V\left(\gamma_0' + \cdots + \gamma_{k-1}' z^{k-1}\right))_{-1} \\ & = & (T^{-1}Vh)_{-1} = (T^{-1}(TD+U)h)_{-1} = Dh_{-1}. \end{array}$$

But we also have $Kf = h_{-1}$ and hence

$$\begin{array}{l} V(z)\left(z\gamma_0' + \cdots + \gamma_{k-1}' z^{k}\right) - T(z)\eta_f - U(z)Kf \\ = V(z)\left(z\gamma_0' + \cdots + \gamma_{k-1}' z^{k}\right) - (TD+U)h_{-1} \\ = V(z)\left(\gamma_0 + \cdots + \gamma_k z^k\right), \end{array}$$

where

$$\gamma_i = \begin{cases} \gamma_{i-1}' & i = 1, \ldots, k \\ -h_{-1} & i = 0. \end{cases}$$

∎

We complete now the proof of the Theorem 5.2. The proof is by induction. Let $K \in \mathcal{F}_{ON}(X_V)$. We will show that if $f \in \mathcal{R}$ then it has a representation of the form

$$f = \sum_{j=0}^{k} (S_T - BK)^j B\beta_j$$

with $\beta_j \in \mathcal{L}$.

Assume, by the induction hypothesis, that every vector $f = V(z)\left(\gamma_0 + \cdots + \gamma_{k-1} z^{k-1}\right) \in \mathcal{R}$ has such a representation. Let now $f = V(z)\left(\gamma_0 + \cdots + \gamma_k z^k\right) \in \mathcal{R}$. By Lemma 5.2 we have $\gamma_k \in \mathcal{L}$, and, applying Lemma 5.3, $(S_T - BK)^k V\gamma_k = V(z)\left(\beta_0 + \cdots + \beta_{k-1} z^{k-1} + \gamma_k z^k\right)$. So $f(z) - (S_T - BK)^k V\gamma_k = V\left(\gamma_0' + \cdots + \gamma_{k-1}' z^{k-1}\right)$, and we apply the induction hypothesis to conclude the proof. ∎

There is another way of proving Theorem 5.2, and this is by reduction to the strictly proper case. We feel it is important to explain the reduction. Let $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ be the shift realization associated with the left coprime factorization $T^{-1}V$. Since $T^{-1}V$ is proper, it follows that $(zT)^{-1}V$ is strictly proper. The subspace $TX_{zI} = \{T(z)\xi | \xi \in \mathbb{F}^p\}$ is an invariant subspace of the polynomial model $X_{zT}$. Moreover, we have $X_{zT} = X_T + TX_{zI}$, and the sum is a direct sum of linear subspaces. We consider now the shift realization $\left(\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array}\right)$ of $(zT)^{-1}V$. Obviously $D_e = 0$. For $f \in X_T$ we have $S_T f = zf - T(z)\xi_f$, where $\xi_f = (T^{-1}f)_{-1} = Cf$. So, since $T\xi \in X_{zT}$, we have $S_{zT} f = zf = S_T f + T(z)\xi_f$. On the other hand $S_{zT}T\xi = 0$. Next, $B_e\xi = V\xi = (U + TD)\xi = U\xi + TD\xi$. Finally, we compute, for $f \in X_T$ that $C_e f = ((zT)^{-1}f)_{-1} = 0$ and $C_e T\xi = ((zT)^{-1}T\xi)_{-1} = \xi$. Putting all this together, we obtain

$$A_e = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, \qquad B_e = \begin{pmatrix} B \\ D \end{pmatrix}$$
$$C_e = \begin{pmatrix} 0 & I \end{pmatrix}, \qquad D_e = 0,$$

and this is a minimal realization of $(zT)^{-1}V$.
Now $\mathcal{R} \subset X_e$ is an output nulling reachability subspace if and only if $\mathcal{R} \subset X$ and is there also an output nulling reachability subspace, see Wonham [1979, Ex. 5.9]. To see this, note that if $\mathcal{R} \subset X$ is an output nulling reachability subspace, then there exists an $L$ for which $DL = 0$ and $\mathcal{R} = < A + BK|BL >$, with $(C + DK)|\mathcal{R} = 0$. Since $X$ is naturally embedded in $X_e$, then $D_e = 0$ implies $D_e L = 0$. Let $K_e = \begin{pmatrix} K & 0 \end{pmatrix}$ which implies

$$A_e + B_e K_e = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} + \begin{pmatrix} B \\ D \end{pmatrix} \begin{pmatrix} K & 0 \end{pmatrix} = \begin{pmatrix} A + BK & 0 \\ C + DK & 0 \end{pmatrix},$$

which shows that

$$\mathcal{R}_e = < A_e + B_e K_e | B_e L > = \begin{pmatrix} < A + BK|BL > \\ 0 \end{pmatrix} = \mathcal{R}.$$

Conversely, assume $\mathcal{R}_e \subset X_e$ is an output nulling reachability subspace. Thus there exists an $L$ and a $K_e = \begin{pmatrix} K & K' \end{pmatrix}$, for which we have

$$(C_e + D_e K_e) \begin{pmatrix} BL \\ DL \end{pmatrix} = C_e \begin{pmatrix} BL \\ DL \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} BL \\ DL \end{pmatrix} = DL = 0.$$

Also,

$$\begin{aligned} < A_e + B_e K_e | B_e L > &= \left\langle \begin{pmatrix} A + BK & BK' \\ C + DK & DK' \end{pmatrix}, \begin{pmatrix} BL \\ 0 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} A + BK & 0 \\ C + DK & 0 \end{pmatrix}, \begin{pmatrix} BL \\ 0 \end{pmatrix} \right\rangle. \end{aligned}$$

This implies $(C + DK)BL = 0$ and, by induction, $(C + DK)(A + BK)^j BL = 0$. This shows that $\mathcal{R}_e = \mathcal{R} \subset X$ is also an output nulling reachability subspace for $T^{-1}V$.

We can use now the characterization of $\mathcal{R}^*$ given in Fuhrmann [1981] for the case of strictly proper functions and conclude that $\mathcal{R}^* = E_\rho X_{V_\rho}$ also for the case in which $T^{-1}V$ is proper.

We give now a characterization of $\mathcal{R}^*$ which is independent of factorizations.

**Proposition 5.7** *Given a* $p \times m$ *proper, rational matrix function with the left coprime, matrix fraction representation* $G = T^{-1}V$*. Then, with respect to the shift realization (18) of* $G$ *in the state space* $X_T$*, we have*

$$\mathcal{R}^* = V(z^{-1}\mathbb{F}^m[[z^{-1}]]) \cap V\mathbb{F}^m[z] = X_V \cap V\mathbb{F}^m[z]. \tag{135}$$

**Proof:** Let $f \in V(z^{-1}\mathbb{F}^m[[z^{-1}]]) \cap V\mathbb{F}^m[z]$, then $f = Vh = Vp$ with $h$ strictly proper and $p$ polynomial. We have $T^{-1}f = T^{-1}Vh \in z^{-1}\mathbb{F}^p[[z^{-1}]]$, which shows that $f \in X_T$, or that $V(z^{-1}\mathbb{F}^m[[z^{-1}]]) \cap V\mathbb{F}^m[z] \subset X_T$. Now $f = Vh = Vp$ implies, eliminating the regular left factor $E_\rho$, that $V_\rho h = V_\rho p$, i.e. $V_\rho p \in X_{V_\rho}$ and hence $Vp \in E_\rho X_{V_\rho} = \mathcal{R}^*$. This shows that $V(z^{-1}\mathbb{F}^m[[z^{-1}]] \cap V\mathbb{F}^m[z] \subset \mathcal{R}^*$.

Conversely, assume $f \in \mathcal{R}^* = E_\rho X_{V_\rho}$. Then $f = E_\rho g$ with $g \in X_{V_\rho}$. So $g = V_\rho h$ for some strictly proper $h$. Now let $W_\rho$ be a polynomial matrix for which $V_\rho W_\rho V_\rho = V_\rho$. For the existence of such a matrix, see the proof of Theorem 5.2. We can write now

$$g = V_\rho h = V_\rho W_\rho V_\rho h = V_\rho(W_\rho g),$$

i.e. $g \in V_\rho(z^{-1}\mathbb{F}^m[[z^{-1}]]) \cap V_\rho\mathbb{F}^m[z]$ and hence $f \in V(z^{-1}\mathbb{F}^m[[z^{-1}]]) \cap V\mathbb{F}^m[z]$. This shows the inclusion $\mathcal{R}^* \subset V(z^{-1}\mathbb{F}^m[[z^{-1}]]) \cap V\mathbb{F}^m[z]$ and the equality follows. ∎

The following is a direct corollary of Proposition 5.7.

**Corollary 5.3** *Let* $G$ *be a* $p \times m$ *proper, full column rank, rational matrix function. Then, with respect to an arbitrary minimal realization, we have* $\mathcal{R}^* = 0$*.*

**Proof:** By isomorphism, it suffices to show this for the shift realization. Assume the left coprime, matrix fraction representation $G = T^{-1}V$. By Proposition 5.7, $f \in \mathcal{R}^*$ if and only if $f = Vh = Vp$ for some strictly proper $h$ and polynomial $p$. By the full column rank assumption, $V$ is left invertible. Thus $h = p = 0$, and hence also $f = 0$, follows. ∎

**Proposition 5.8** *Let* $K : X_T \longrightarrow \mathbb{F}^m$ *and* $K \in \mathcal{F}_{ON}(X_V)$*. Then* $K \in \mathcal{F}_{ON}(E_\alpha X_{V_\alpha})$ *for any factorization* $V = E_\alpha V_\alpha$*, with* $E_\alpha$ *nonsingular.*

**Proof:** First, we show that $f \in E_\alpha X_{V_\alpha}$, implies $E_\alpha V_\alpha(S_- h) \in E_\alpha X_{V_\alpha}$. Indeed, assume $f = E_\alpha g$ with $g = V_\alpha h$. Since $V_\alpha sh = V_\alpha(S_- h + h_{-1})$, we have $V_\alpha S_- h = s(V_\alpha h) - V_\alpha h_{-1}$, i.e. $V_\alpha S_- h \in X_{V_\alpha}$. Assume now that $K \in \mathcal{F}_{ON}(X_V)$. Since $E_\alpha X_{V_\alpha} \subset X_V$, we have, with respect to the shift realization, $(C + DK)|E_\alpha X_{V_\alpha} = 0$. Now, for $f$ as above, and letting $Kf = \eta$, we compute

$$(C+DK)f = (T^{-1}f)_{-1} + D\eta = (T^{-1}Vh)_{-1} + D\eta = (T^{-1}(TD+U)h)_{-1} + D\eta = Dh_{-1} + D\eta.$$

Thus $\zeta = h_{-1} + \eta \in \operatorname{Ker} D$. Next we compute

$$(S_T + BK)f = (S_T + BK)Vh = V(S_- h) + Uh_{-1} + U\eta = V(S_- h) + U\zeta.$$

We saw that $V(S_-h) \in X_{V_\alpha}$, which implies that $U\zeta \in X_V$. But $D\zeta = 0$ implies $U\zeta = (V - TD)\zeta = V\zeta$. So $V(S_-h) + U\zeta = V(S_-h) + V\zeta = E_\alpha[V_\alpha(S_-h) + V_\alpha\zeta]$. This shows that $(S_T + BK)f \in E_\alpha X_{V_\alpha}$, i.e. $K \in \mathcal{F}_{ON}(E_\alpha X_{V_\alpha})$. ∎

**Corollary 5.4** *If* $K \in \mathcal{F}_{ON}(\mathcal{V}^*)$ *then* $K \in \mathcal{F}_{ON}(\mathcal{R}^*)$.

Given a system $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$, let $\mathcal{V}^*, \mathcal{R}^*$ be the maximal output nulling and output nulling reachability subspaces respectively. Since $\mathcal{V}^*$ is output nulling, there exists $K \in \mathcal{F}_{ON}(\mathcal{V}^*)$ such that $(A+BK)\mathcal{V}^* \subset \mathcal{V}^*$. Thus, we have an $\mathbb{F}[z]$-module structure on $\mathcal{V}^*$, namely the one induced by $A + BK$. Since $\mathcal{F}_{ON}(\mathcal{V}^*) \subset \mathcal{F}_{ON}(\mathcal{R}^*)$, it follows that $\mathcal{R}^*$ is a submodule. There exists also a constant matrix $L$ such that $(\operatorname{Im} B) \cap \mathcal{V}^* = \operatorname{Im} BL$ and $< A + BK|BL >= \mathcal{R}^*$. Choosing any subspace of $\mathcal{V}^*$ which is complementary to $\mathcal{R}^*$, we get the block matrix representation

$$A = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}.$$

This pair is reachable if and only if $\mathcal{V}^* = \mathcal{R}^*$. Applying the a general feedback transformation $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$, we get

$$A + BK = \begin{pmatrix} A_1 + B_1K_1 & A_3 + B_1K_2 \\ 0 & A_2 \end{pmatrix}.$$

Thus the quotient module structure $\mathcal{V}^*/\mathcal{R}^*$ is unaffected by feedback and is given by the module structure induced by the linear transformation $A_2$. Thus $A_2$ provides a measure of the extent that the module structure of $\mathcal{V}^*$ is independent of the choice of $K \in \mathcal{F}_{ON}(\mathcal{V}^*)$. In the next theorem we study this module structure from a polynomial point of view.

**Theorem 5.3** *Given any feedback induced module structure on* $X_V$*, i.e. given a choice* $K \in \mathcal{F}_{ON}(\mathcal{V}^*)$*, we have the* $\mathbb{F}[s]$*-module isomorphism*

$$\mathcal{V}^*/\mathcal{R}^* = X_V/E_\rho X_{V_\rho} \simeq X_{E_\rho}. \tag{136}$$

**Proof:** Choose $K \in \mathcal{F}_{ON}(\mathcal{V}^*)$ which, by Theorem 5.2 and Corollary 5.4, implies $K \in \mathcal{F}_{ON}(\mathcal{R}^*) = \mathcal{F}_{ON}(E_\rho X_{V_\rho})$. With respect to the module structure induced by $(S_T - BK)$, $E_\rho X_{V_\rho}$ is a submodule. Define a map $\tau : X_V \longrightarrow X_{E_\rho}$ by

$$\tau f = \pi_{E_\rho} f, \qquad f \in X_V. \tag{137}$$

We will show that $\tau$ is a surjective module homomorphism with $\operatorname{Ker} \tau = E_\rho X_{V_\rho}$. Let $f \in X_V$, then

$$\begin{aligned} \tau(S_T - BK)f &= \tau(zf(z) - V(z)\xi) = \pi_{E_\rho}(zf(z) - V(z)\xi) \\ &= \pi_{E_\rho} zf(z) - E_\rho \pi_- E_\rho^{-1} E_\rho V_\rho \xi \\ &= \pi_{E_\rho} z \pi_{E_\rho} f(z) = S_{E_\rho} \tau f. \end{aligned}$$

This shows that $\tau$ is indeed a module homomorphism.

Clearly, $f \in \operatorname{Ker} \tau$ if and only if $f = E_\rho g$ for some polynomial $g$. Since $f \in X_V$ we have $f = Vh = E_\rho(V_\rho h)$ and hence $g = V_\rho h$ or $g \in X_{V_\rho}$. This shows that $\operatorname{Ker} \tau = E_\rho X_{V_\rho}$.

To show that $\tau$ is surjective, we use the fact that $V = E_\rho V_\rho$ is a maximal, left skew-prime factorization. Thus there exists a factorization $V = \hat{V}_\rho \hat{E}_\rho$ such that $E_\rho, \hat{V}_\rho$ are left coprime and $\hat{E}_\rho, V_\rho$ are right coprime. Thus there exist polynomial matrices $P, Q$ for which the Bezout identity $\hat{V}_\rho P + E_\rho Q = I$ holds. Thus, for each $g \in \mathbb{F}^p[z]$, we have

$$\begin{aligned} g &= \hat{V}_\rho P g + E_\rho Q g = \hat{V}_\rho(\pi_{\hat{E}_\rho} + I - \pi_{\hat{E}_\rho}) P g + E_\rho Q g \\ &= \hat{V}_\rho g_1 + \hat{V}_\rho \hat{E}_\rho g_2 + E_\rho Q g \\ &= \hat{V}_\rho g_1 + E_\rho V_\rho g_2 + E_\rho Q g \\ &= \hat{V}_\rho g_1 + E_\rho(V_\rho g_2 + Q g), \end{aligned}$$

with $g_1 = \pi_{\hat{E}_\rho} P g \in X_{\hat{E}_\rho}$. Thus we conclude that

$$\mathbb{F}^p[z] = \hat{V}_\rho X_{\hat{E}_\rho} + E_\rho \mathbb{F}^p[z].$$

Now we claim that

$$X_V + E_\rho \mathbb{F}^p[z] = \mathbb{F}^p[z]. \tag{138}$$

Indeed, by skew primeness we have $X_V = \hat{V}_\rho X_{\hat{E}_\rho} + E_\rho X_{V_\rho}$, therefore

$$\begin{aligned} X_V + E_\rho \mathbb{F}^p[z] &= \hat{V}_\rho X_{\hat{E}_\rho} + E_\rho X_{V_\rho} + E_\rho \mathbb{F}^p[z] \\ &= \hat{V}_\rho X_{\hat{E}_\rho} + E_\rho \mathbb{F}^p[z] = \mathbb{F}^p[z]. \end{aligned}$$

From (138) it follows now that $\pi_{E_\rho}(X_V) = X_{E_\rho}$, i.e. $\tau$ is surjective. ∎

The invariant factors of the quotient module $\mathcal{V}^*/\mathcal{R}^*$ are called the **transmission polynomials** and their zeros are called the **transmission zeros**.

There is another way of looking at transmission zeros. By Theorem 5.1, given any $K \in \mathcal{F}_{ON}(\mathcal{V}^*)$, we have the representation $\mathcal{V}^* = \overline{V}_\mu X_{\overline{E}_\mu}$ for a (??) factorization $V = \overline{V}_\mu \overline{E}_\mu$ such that $\overline{V}_\mu | X_{\overline{E}_\mu}$ is injective. Thus the multiplication map $\overline{V}_\mu : X_{\overline{E}_\mu} \longrightarrow \overline{V}_\mu X_{\overline{E}_\mu} = \mathcal{V}^* = X_V$ provides an $\mathbb{F}[z]$-module isomorphism. $\mathcal{R}^* = E_\rho X_{V_\rho}$, being a submodule of $\mathcal{V}^*$, corresponds to the image, under multiplication by $\overline{V}_\mu$ of a submodule of $X_{\overline{E}_\mu}$. Such a submodule is necessarily given by $\overline{F} X_{\hat{E}_\rho}$ for a factorization $\overline{E}_\mu = \overline{F} \hat{E}_\rho$, and hence

$$\mathcal{R}^* = \overline{V}_\mu \overline{F} X_{\hat{E}_\rho} = \hat{V}_\rho X_{\hat{E}_\rho},$$

where $\hat{V}_\rho = \overline{V}_\mu \overline{F}$ and $\hat{V}_\rho \hat{E}_\rho = \overline{V}_\mu \overline{F} \hat{E}_\rho = \overline{V}_\mu \overline{E}_\mu = V$. In particular we have $\hat{V}_\rho \hat{E}_\rho = E_\rho V_\rho$.

Now $V = E_\rho V_\rho$ was a left skew-prime factorization, thus there exists a factorization $E_\rho V_\rho = \overline{V}_\rho \overline{E}_\rho$, with $E_\rho, \overline{V}_\rho$ left coprime and $V_\rho, \overline{E}_\rho$ right coprime. By the basic isomorphism theorem for polynomial models, see Fuhrmann [1976, Theorems 4.6 & 4.7], the polynomial models $X_{E_\rho}$ and $X_{\overline{E}_\rho}$ are isomorphic and the isomorphism is given by $f \mapsto \pi_{E_\rho} \overline{V}_\rho f$, for $f \in X_{\overline{E}_\rho}$. In particular, this implies that the multiplication map $\overline{V}_\rho|X_{\overline{E}_\rho}$ is injective. By skew-primeness we have

$$\begin{aligned} X_V &= \overline{V}_\mu X_{\overline{E}_\mu} = E_\rho X_{V_\rho} + \overline{V}_\rho X_{\overline{E}_\rho} \\ &= \hat{V}_\rho X_{\hat{E}_\rho} + \overline{V}_\rho X_{\overline{E}_\rho} \\ &= \overline{V}_\mu \hat{F} X_{\hat{E}_\rho} + \overline{V}_\rho X_{\overline{E}_\rho} \end{aligned}$$

So $\overline{V}_\rho X_{\overline{E}_\rho}$ is a complementary subspace to $\mathcal{R}^*$ in $\mathcal{V}^*$. We therefore have the isomorphisms

$$\mathcal{V}^*/\mathcal{R}^* \simeq \overline{V}_\rho X_{\overline{E}_\rho} \simeq X_{\overline{E}_\rho} \simeq X_{E_\rho}.$$

**Corollary 5.5** *Let $G = T^{-1}V$ and $\mathcal{R}^* = E_\rho X_{V_\rho}$ the maximal output nulling reachability subspace. A subspace $\mathcal{V} \subset X_T$ is an output nulling subspace that contains $\mathcal{R}^*$ if and only if*

$$\mathcal{V} = E_\alpha X_{V_\alpha} \tag{139}$$

*for some factorization $V = E_\alpha V_\alpha$ with $E_\alpha$ a nonsingular left factor of $E_\rho$.*

**Proof:** Assume $\mathcal{V} = E_\alpha X_{V_\alpha}$ and $E_\alpha H = E_\rho$. Then

$$\mathcal{R}^* = E_\rho X_{V_\rho} = E_\alpha H X_{V_\rho} \subset E_\alpha X_{HV_\rho} = E_\alpha X_{V_\alpha} = \mathcal{V}.$$

That $E_\alpha X_{V_\alpha}$ is output nulling has been proved in Proposition 5.3.

To prove the converse, let $\mathcal{V}$ be an output nulling subspace, satisfying $\mathcal{V} \supset \mathcal{R}^*$. Since $\mathcal{V} \subset X_V = \mathcal{V}^*$, the two subspaces are compatible. Thus there exists a $K \in \mathcal{F}_{ON}(X_V) \cap \mathcal{F}(\mathcal{V})$. By Corollary 5.4 we have $K \in \mathcal{F}_{ON}(E_\rho X_{V_\rho})$. This implies the module inclusions $E_\rho X_{V_\rho} \subset \mathcal{V} \subset X_V$. Let $\tau$ be defined by (137), then $\tau(\mathcal{V}) = \pi_{E_\rho}(\mathcal{V})$ is a submodule of $X_{E_\rho}$, and hence of the form $E_\alpha X_{E_\beta}$ where $E_\rho = E_\alpha E_\beta$. Now, for $f \in X_V$ we have $\pi_{E_\rho} f \in E_\alpha X_{E_\beta}$ if and only if $f \in E_\alpha X_{V_\alpha}$

∎

This corollary induces a partial order in the set of all output nulling subspaces that contain $\mathcal{R}^*$. We have $E_\alpha X_{V_\alpha} \subset E_\beta X_{V_\beta}$ if and only if $E_\beta$ is a left factor of $E_\alpha$.

**Corollary 5.6** *If $K \in \mathcal{F}_{ON}(\mathcal{V}^*)$ then $K \in \mathcal{F}_{ON}(\mathcal{V})$ for every output nulling subspace $\mathcal{V}$ that contains $\mathcal{R}^*$, i.e. $\mathcal{F}_{ON}(\mathcal{V}^*) \subset \mathcal{F}_{ON}(\mathcal{V})$.*

This can actually be strengthened as follows. In a different form this appears, without a proof, in Khargonekar, Georgiou and Özgüler [1983].

**Theorem 5.4** *Let $\mathcal{V}$ be an output nulling subspace. Then $\mathcal{V} \supset \mathcal{R}^*$ if and only if*

$$\mathcal{F}_{ON}(\mathcal{V}^*) \subset \mathcal{F}_{ON}(\mathcal{V}). \tag{140}$$

Given an output nulling subspace $\mathcal{V} = X_{\begin{pmatrix} Q & P \end{pmatrix}}$, the following theorem, stated without a proof, establishes the connection between the set $\mathcal{F}_{ON}(\mathcal{V})$ of all output nulling friends of $\mathcal{V}$ and the set of all nonsingular extensions of $\begin{pmatrix} Q & P \end{pmatrix}$.

**Theorem 5.5** *Given a controllable behavior $\mathcal{B}$ which has the representation*

$$\mathcal{B} = \operatorname{Ker} \begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix} = X^{\begin{pmatrix} Q & P \end{pmatrix}}, \tag{141}$$

*with $Q$ nonsingular, $Q, P$ left coprime and $Q^{-1}P$ proper. There exists a bijective correspondence between feedback maps $K \in \mathcal{F}_{ON}(X_{\begin{pmatrix} Q & P \end{pmatrix}})$ and nonsingular extensions $\begin{pmatrix} Q(z) & P(z) \\ -\overline{A}(z) & \overline{B}(z) \end{pmatrix}$ for which*

$$X_{\begin{pmatrix} Q & P \end{pmatrix}} = \begin{pmatrix} Q(\sigma) & P(\sigma) \end{pmatrix} X^{\begin{pmatrix} Q & P \\ -\overline{A}(z) & \overline{B}(z) \end{pmatrix}} \tag{142}$$

*The correspondence is as follows:*
*Given a nonsingular extension $\begin{pmatrix} Q(z) & P(z) \\ -\overline{A}(z) & \overline{B}(z) \end{pmatrix}$ such that*

$$\begin{pmatrix} Q(z) & P(z) \\ -\overline{A}(z) & \overline{B}(z) \end{pmatrix} = \begin{pmatrix} Q(z) & P(z) \\ -Q_1(z)\overline{A}_0(z) & Q_1(z)\overline{B}_0(z) \end{pmatrix},$$

*with $\overline{Q}Q_1^{-1}$ normalized biproper. Let $H \in \mathbb{F}^{n\times p}[z]$ be a basis matrix as in* **??**, *and let $(F, G)$ be the reachable pair defined by*

$$(zI - F)^{-1}G = H(z)\overline{Q}(z)^{-1}. \tag{143}$$

*Then, with $Q_1 - \overline{Q} = N$, there exists a unique $K \in \mathcal{F}_{ON}(X_{\begin{pmatrix} Q & P \end{pmatrix}})$ that satisfies $N(z) = KH(z)$.*

*Conversely, given $K \in \mathcal{F}_{ON}(X_{\begin{pmatrix} Q & P \end{pmatrix}})$, we define*

$$Q_1(z) = \overline{Q}(z) + KH(z) \tag{144}$$

*and*

$$\begin{pmatrix} -\overline{A}(z) & \overline{B}(z) \end{pmatrix} = \begin{pmatrix} -Q_1(z)\overline{A}_0(z) & Q_1(z)\overline{B}_0(z) \end{pmatrix} \tag{145}$$

*Then $\begin{pmatrix} Q(z) & P(z) \\ -\overline{A}(z) & \overline{B}(z) \end{pmatrix}$ is the corresponding nonsingular extension.*

Given the proper rational function $G = T^{-1}V$, the polynomial characterization of $\mathcal{V}^*$, with respect to the shift realization, was quite intricate. The dual object, namely $\mathcal{V}_*$, the minimal input containing conditioned invariant subspace, has a polynomial characterization, trivial to prove, namely $\mathcal{V}_* = X_T \cap V\mathbb{F}^m[z]$. This opens up the possibility of appying duality theory to simplify the analysis of output nulling subspaces. This analysis will be published elsewhere.

# 6 McMillan degree and Feedback interconnections

Given a behavior $\mathcal{B}$ in an AR, or kernel representation

$$\mathcal{B} = \operatorname{Ker} R(\sigma), \tag{146}$$

with $R \in \mathbb{F}^{p\times m}[z]$ having full row rank. We say that the representation (146) is **minimal** if $R$ has full row rank. We define the McMillan degree $\delta(\mathcal{B})$ of the behavior $\mathcal{B}$ by

$$\delta(\mathcal{B}) = \dim X_R. \tag{147}$$

This definition is not dependent on any i/o properties of the system, nor does it depend on any controllability assumptions. We can also define the McMillan degree in terms of minimal state space, i.e. $1^{st}$ order representations. These two definitions are equivalent.

To clarify the definition, we consider an autonomous behavior $\mathcal{B} = \operatorname{Ker} Q(\sigma) = X^Q$, with $Q$ nonsingular. Let $H(z)$ be any basis matrix for the polynomial model $X_Q$. $H$ is uniquely defined up to a constant, nonsingular right factor. Clearly, $Q^{-1}H$ is a left coprime factorization and, via the shift realization, see Fuhrmann [1975,1977], we have a right coprime factorization $Q^{-1}H = C(zI-A)^{-1}$ which is uniquely determined up to similarity. Thus it is clear that if the state space has dimension $n$, and since the coprimeness conditions imply that $\det Q(z) = \det(zI-A)$, we have

$$\delta(Q) = \dim X_Q = \deg\det Q = \deg\det(zI-A) = n. \tag{148}$$

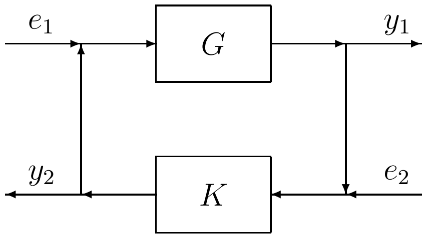
Clearly, this example has no i/o interpretation.

The analysis of interconnection of systems is a cornerstone of linear system theory. The most common and important types of system interconnections are parallel, series and feedback connections. In the i/o setting, the question of characterizing the minimality of the interconnected system, assuming the components were minimal, has been studied in the past, see Callier and Nahum [1975] for the series and feedback connections and Fuhrmann [1975] for the parallel connections.

We proceed now to analyze the minimality of the interconnection of systems in the behavioral setting. Moreover, we will characterize in all cases when the McMillan degree of the interconnected system is the sum of McMillan degree of its components. We study each case separately.

We begin our analysis from the following standard feedback configuration.

**Diagram 6.1**



The system equations are:

$$\begin{array}{rcl} G(e_1+y_2) & = & y_1 \\ K(e_2+y_1) & = & y_2 \end{array} \tag{149}$$

These can be written in matrix form as

$$\begin{pmatrix} I & -G \\ -K & I \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} G & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \tag{150}$$

Our standing assumption is that $(I-GK)$ is properly invertible. It is easily calculated that

$$\begin{pmatrix} I & -G \\ -K & I \end{pmatrix}^{-1} = \begin{pmatrix} (I-GK)^{-1} & G(I-KG)^{-1} \\ (I-KG)^{-1}K & (I-KG)^{-1} \end{pmatrix} \tag{151}$$

and so

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (I-GK)^{-1}G & G(I-KG)^{-1}K \\ (I-KG)^{-1}KG & (I-KG)^{-1}K \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \tag{152}$$

The closed loop transfer function $G_f$ from $e_1$ to $y_1$ is given by

$$G_f = (I-GK)^{-1}G = G(I-KG)^{-1}. \tag{153}$$

Assume now that the transfer functions of the system and the controller have the following coprime factorizations.

$$\begin{array}{rcl} G & = & Q^{-1}P = \overline{PQ}^{-1} \\ K & = & \overline{S}^{-1}\overline{R} = RS^{-1} \end{array} \tag{154}$$

Substituting into (153), we get for the closed loop transfer function $G_f$ the following representations

$$G_f = S(QS-PR)^{-1}P = \overline{P}(\overline{SQ}-\overline{RP})^{-1}\overline{S}. \tag{155}$$

Note that in terms of these coprime factorizations, we have

$$\begin{array}{rcl} \begin{pmatrix} I & -G \\ -K & I \end{pmatrix}^{-1}\begin{pmatrix} G & 0 \\ 0 & K \end{pmatrix} & = & \begin{pmatrix} I & -Q^{-1}P \\ -\overline{S}^{-1}\overline{R} & I \end{pmatrix}^{-1}\begin{pmatrix} Q^{-1}P & 0 \\ 0 & \overline{S}^{-1}\overline{R} \end{pmatrix} \\ & = & \begin{pmatrix} Q & -P \\ -\overline{R} & \overline{S} \end{pmatrix}^{-1}\begin{pmatrix} P & 0 \\ 0 & \overline{R} \end{pmatrix} \end{array}$$

The following proposition is due to Callier and Nahum [1975], where it is proved using the Rosenbrock theory of polynomial system matrices.

**Proposition 6.1** *Assume we are given proper rational functions* $G, K$ *having the coprime factorizations*

$$\begin{aligned} G &= Q^{-1}P = \overline{PQ}^{-1} \\ K &= \overline{S}^{-1}\overline{R} = RS^{-1}. \end{aligned} \tag{156}$$

*Then, under the assumption that the products* $GK$ *and* $KG$ *exist,*

1. *the following conditions are equivalent*
   - *(a)* $P, \overline{S}$ *are right coprime.*
   - *(b)* $\overline{SQ}, \overline{P}$ *are right coprime.*
   - *(c)* $PR, S$ *are right coprime.*
   - *(d) The series connection of minimal realizations of* $G$ *and* $K$ *is an observable realization of* $GK$*.*
2. *the following conditions are equivalent*
   - *(a)* $\overline{P}, S$ *are left coprime.*
   - *(b)* $QS, P$ *are left coprime.*
   - *(c)* $\overline{RP}, \overline{S}$ *are left coprime.*
   - *(d) The series connection of minimal realizations of* $K$ *and* $G$ *is a controllable realization of* $KG$*.*
3. *the following conditions are equivalent*
   - *(a)* $\overline{Q}, R$ *are left coprime.*
   - *(b)* $PR, Q$ *are left coprime.*
   - *(c)* $\overline{SQ}, \overline{R}$ *are left coprime.*
   - *(d) The series connection of minimal realizations of* $G$ *and* $K$ *is a controllable realization of* $GK$*.*
4. *the following conditions are equivalent*
   - *(a)* $\overline{R}, Q$ *are right coprime.*
   - *(b)* $\overline{RP}, Q$ *are right coprime.*
   - *(c)* $QS, R$ *are right coprime.*
   - *(d) The series connection of minimal realizations of* $K$ *and* $G$ *is an observable realization of* $KG$*.*

**Proof:** $(1a) \Rightarrow (1b)$
By the assumed right coprimeness of $P, \overline{S}$, there exist polynomial matrices $X_0, Y_0$ such that $X_0 P + Y_0 \overline{S} = I$ and hence also $X_0 P\overline{Q} + Y_0 \overline{SQ} = \overline{Q}$. Also, by the right coprimeness of $\overline{P}, \overline{Q}$, there exist polynomial matrices $A_0, B_0$ such that $A_0\overline{P} + B_0\overline{Q} = I$. We compute, using the equality $P\overline{Q} = Q\overline{P}$,

$$\begin{aligned} I &= A_0\overline{P} + B_0\overline{Q} = A_0\overline{P} + B_0(X_0 P\overline{Q} + Y_0\overline{SQ}) \\ &= (A_0 + B_0 X_0 Q)\overline{P} + Y_0(\overline{SQ}), \end{aligned}$$

i.e. $\overline{SQ}, \overline{P}$ are right coprime.

$(1b) \Rightarrow (1a)$
By assumption, there exist polynomial matrices $X, Y$ such that $X\overline{SQ} + Y\overline{P} = I$. By multiplying this equality with $\overline{Q}^{-1}$ from the right and by $\overline{Q}$ from the left, we obtain $I = \overline{Q}X\overline{S} + \overline{Q}Y\overline{PQ}^{-1} = \overline{Q}X\overline{S} + \overline{Q}YQ^{-1}P$. Since $Q, P$ are left coprime, it follows that there exists a polynomial matrix $\overline{Y}$ such that $\overline{Q}Y = \overline{Y}Q$. Substituting back, we obtain $(\overline{Q}X)\overline{S} + \overline{Y}P = I$, i.e. $P, \overline{S}$ are right coprime.

$(1a) \Rightarrow (1c)$
By the assumed right coprimeness of $R, S$, there exist polynomial matrices $X_0, Y_0$ such that $X_0 R + Y_0 S = I$. Also, the assumed right coprimeness of $P, \overline{S}$ implies the existence of polynomial matrices $X, Y$ such that $XP + Y\overline{S} = I$ and hence that $X_0 XPR + X_0 Y\overline{S}R = X_0 R$. Using the equality $\overline{S}R = \overline{R}S$ and adding the term $\overline{R}S$ to both sides, we get

$$(X_0 X)PR + (X_0 Y\overline{R} + Y_0)S = X_0 R + Y\overline{S} = I,$$

i.e. $PR, S$ are right coprime.

$(1c) \Rightarrow (1a)$
From the equality

$$PRS^{-1} = P\overline{S}^{-1}\overline{R},$$

it is clear, using the shift realization, that necessarily $P, \overline{S}$ are right coprime.

The other statements are proved analogously or, alternatively, can be derived by duality considerations.

∎

**Proposition 6.2** *Assume a rectangular polynomial matrix is given in the form* $\begin{pmatrix} Q & -P \end{pmatrix}$ *with* $Q$ *nonsingular and* $Q^{-1}P$ *proper and the factorization left coprime. Let* $\overline{PQ}^{-1}$ *be a right coprime factorization of* $Q^{-1}P$. *Let* $\begin{pmatrix} Q & -P \\ -\overline{R} & \overline{S} \end{pmatrix}$ *be an extension to a nonsingular polynomial matrix, assuming that* $\overline{S}, \overline{R}$ *are left coprime and that* $RS^{-1}$ *is a right coprime factorization of* $\overline{S}^{-1}\overline{R}$. *Then we have the equality*

$$\det \begin{pmatrix} Q & -P \\ -\overline{R} & \overline{S} \end{pmatrix} = \det(QS - PR) = \det(\overline{SQ} - \overline{RP}). \tag{157}$$

**Proof:** We compute

$$\begin{aligned}\begin{pmatrix} I & 0 \\ \overline{R}Q^{-1} & I \end{pmatrix}\begin{pmatrix} Q & -P \\ -\overline{R} & \overline{S} \end{pmatrix} &= \begin{pmatrix} Q & -P \\ 0 & \overline{S}-\overline{R}Q^{-1}P \end{pmatrix} = \begin{pmatrix} Q & -P \\ 0 & \overline{S}-\overline{RPQ}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} Q & -P\overline{Q} \\ 0 & \overline{SQ}-\overline{RP} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & \overline{Q}^{-1} \end{pmatrix}\end{aligned}$$

Thus we have, using the equality $\det(\overline{Q}^{-1}) = (\det \overline{Q})^{-1}$,

$$\det \begin{pmatrix} Q & -P \\ -\overline{R} & \overline{S} \end{pmatrix} = \det Q \cdot \det(\overline{SQ}-\overline{RP})(\det \overline{Q})^{-1}.$$

Now, the equality $P\overline{Q} = Q\overline{P}$, taken together with the coprimeness conditions, implies $\det \overline{Q} = \det Q$. Thus we have $\det \begin{pmatrix} Q & -P \\ -\overline{R} & \overline{S} \end{pmatrix} = \det(\overline{SQ}-\overline{RP})$.

In a similar way, we compute

$$\begin{aligned}\begin{pmatrix} Q & -P \\ -\overline{R} & \overline{S} \end{pmatrix}\begin{pmatrix} I & 0 \\ \overline{S}^{-1}\overline{R} & I \end{pmatrix} &= \begin{pmatrix} Q-P\overline{S}^{-1}\overline{R} & -P \\ 0 & \overline{S} \end{pmatrix} = \begin{pmatrix} Q-PRS^{-1} & -P \\ 0 & \overline{S} \end{pmatrix} \\ &= \begin{pmatrix} QS-PR & -P \\ 0 & \overline{S} \end{pmatrix}\begin{pmatrix} S^{-1} & 0 \\ 0 & I \end{pmatrix}.\end{aligned}$$

This implies, by the same reasoning as before, that $\det \begin{pmatrix} Q & -P \\ -\overline{R} & \overline{S} \end{pmatrix} = \det(QS-PR)$.

∎

**Example 1: State feedback:**
We show how the analysis of state feedback fit into the previous analysis. To this end, let $(A, B)$ be a reachable pair and let $(zI-A)^{-1}B = H(z)D(z)^{-1}$ be coprime factorizations. The left coprimeness of $(zI-A), B$ is of course equivalent to the reachability assumption. The behavioral equation of the system is given by $\mathcal{B} = \mathrm{Ker} \begin{pmatrix} \sigma I - A & -B \end{pmatrix}$. Assume we have a state feedback map $u = Kx$. The closed loop behavior equations are given by $\mathcal{B} = \mathrm{Ker} \begin{pmatrix} \sigma I - A & -B \\ K & I \end{pmatrix}$. A simple calculation yields the following identity.

$$\begin{pmatrix} zI-A & -B \\ K & I \end{pmatrix}\begin{pmatrix} H(z) \\ D(z) \end{pmatrix} = \begin{pmatrix} 0 \\ D(z)+KH(z) \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}(D(z)+KH(z)).$$

We get an intertwining relation that can be interpreted in terms of behavior homomorphisms. We note that the there exists a doubly unimodular embedding, see Fuhrmann [2002] of $\begin{pmatrix} 0 & zI-A & -B \\ -I & K & I \end{pmatrix}, \begin{pmatrix} D(z)+KH(z) \\ H(z) \\ D(z) \end{pmatrix}$. Thus the map $\begin{pmatrix} H(\sigma) \\ D(\sigma) \end{pmatrix} : X^{(D+KH)} \longrightarrow X^{\begin{pmatrix} zI-A & -B \\ K & I \end{pmatrix}}$ is a behavior isomorphism. It is a simple check that $\det \begin{pmatrix} zI-A & -B \\ K & I \end{pmatrix} = \det(zI-A+BK) = \det(D(z)+KH(z))$. In fact, more is true. The invariant factors of $A-BK$ are equal to the invariant factors of $D(z)+KH(z)$. Now as $H(z)D(z)^{-1}$ is a transfer function of a reachable input to state map, a polynomial matrix $Q(z)$ is such that $QD^{-1}$ is strictly proper if and only if $Q(z) = KH(z)$ for some constant matrix $K$. This describes all the transfer functions achievable by state feedback, recovering a result of Hautus and Heymann [1978]. In this connection see also Fuhrmann [1979].

From Diagram 6.1, it follows that the McMillan degree of $(I-GK)^{-1}G$ is majorized by the sum of the McMillan degrees of $G$ and $K$, i.e. we have

$$\delta((I-GK)^{-1}G) \leq \delta(G)+\delta(K). \tag{158}$$

We characterize the case where equality occurs.

**Lemma 6.1** *Given rational matrix functions $G, K$, with $G$ stricly proper and $K$ proper. Then we have*

$$\delta((I-GK)^{-1}G) = \delta(G)+\delta(K) \tag{159}$$

*if and only if the following conditions are satisfied.*

1. *The series connection of any minimal realizations of $K$ and $G$ is a controllable realization of $KG$.*
2. *The series connection of any minimal realizations of $G$ and $K$ is an observable realization of $KG$.*

**Proof:** Assume $G, K$ have the coprime factorizations (154). We always have the inequality (158). By equation (155) we have $\delta((I-GK)^{-1}G) = \delta(G)+\delta(K)$ if and only if $QS, P$ are left coprime and $S, PR$ are right coprime. By Proposition 6.1, these coprimeness conditions are equivalent to the stated conditions.

∎

The assumption that $G$ is strictly proper is not a restriction. It suffices to assume only that $I-GK$ is properly invertible.

# References

[1973] G. Basile and G. Marro, "Controlled and conditioned invariant subspaces in linear system theory", *J. Optim. Theory and Appl.* 3, 306-316.

[1975] F.M. Callier and C.D. Nahum, "Necessary and sufficient conditions for the complete controllability and observability of systems in series using the coprime decomposition of a rational matrix", *IEEE Trans. Aut. Control*, AC-22, 90-95.

[1980] E. Emre and M.L.J. Hautus, "A polynomial characterization of $(A,B)$-invariant and reachability subspaces", *SIAM J. Contr. & Optim.* 18, 420-436.

[1975] P.A. Fuhrmann, "On controllability and observability of systems connected in parallel", *IEEE Trans. Circuits and Systems*, CAS-22, 57.

[1976] P.A. Fuhrmann, "Algebraic system theory: An analyst's point of view", *J. Franklin Inst.* 301, 521-540.

[1979] P.A. Fuhrmann, "Linear feedback via polynomial models", *Int. J. Contr.* 30 (1979), 363-377.

[1981] P.A. Fuhrmann, "Duality in polynomial models with some applications to geometric control theory", *IEEE Trans. Aut. Control*, AC-26, 284-295.

[1994] P.A. Fuhrmann, "On skew primeness of inner functions", *Lin. Alg. Appl.*, vols. 208-209, 539-551.

[2001] P.A. Fuhrmann, "On behavior homomorphisms and system equivalence", *Systems and Control Letters*, 44, 127-134.

[2002] P.A. Fuhrmann, "A study of behaviors", *Lin. Alg. Appl.*, vols. 351-352, 303-380.

[2003] P.A. Fuhrmann, "A note on continuous behavior homomorphisms", *Systems and Control Letters*, 49, (2003), 359-363.

[2001] P.A. Fuhrmann and U. Helmke, "On the parametrization of conditioned invariant subspaces and observer theory", *Lin. Alg. Appl.*, vols. 332-334, 265-353.

[1980] P.A. Fuhrmann and J.C. Willems, "A study of (A,B)-invariant subspaces via polynomial models", *Int. J. Contr.*, 31, 467-494.

[1978] M.L.J. Hautus and M. Heymann, "Feedback - an algebraic approach", *SIAM J. Contr.*, 16, 83-105.

[1982] P.P. Khargonekar and E. Emre, "Further results on polynomial characterizations of $(F, G)$-invariant and reachability subspaces", *IEEE Trans. Autom. Contr.* AC-27, 352-366.

[1983] P.P. Khargonekar, T.T. Georgiou and A.B. Özgüler, "Skew-prime polynomial matrices: the polynomial model approach", *Lin. Alg. Appl*, 50, 403-435.

[2001] V. Lomadze, "On interconnections and control", preprint.

[1994] A.B. Özgüler, *Linear Multichannel Control*, Prentice Hall.

[1986] A.B. Özgüler, "Polynomial characterizations of $(H, F)$-invariant subspaces with applications", *Lin. Alg. Appl*, 73, 1-31.

[1997] J.W. Polderman and J.C. Willems, *Introduction to Mathematical System Theory*, Springer, New York.

[1970] H.H. Rosenbrock, *State-space and Multivariable Theory*, John Wiley, New York.

[2002] H.L. Trentelman, "Regular feedback implementability of linear differential behaviors", MTNS-2002 Open Problem Book, 9-12.

[1986] J.C. Willems, "From time series to linear systems. Part I: Finite-dimensional linear time invariant systems", *Automatica*, 22, 561-580.

[1989] J.C. Willems, "Models for dynamics", *Dynamics Reported*, vol. 2, 171-269, U. Kirchgraber and H.O. Walther (eds.), Wiley-Teubner.

[1991] J.C. Willems, "Paradigms and puzzles in the theory of dynamical systems", *IEEE Trans. Autom. Contr.* AC-36, 259-294.

[1997] J.C. Willems, "On interconnections, control and feedback", *IEEE Trans. Autom. Contr.*, AC- 42, 326-337.

[1978] W.A. Wolovich, "Skew prime polynomial matrices", *IEEE Trans. Autom. Contr.* AC-23, 880-887.

[1979] W.M. Wonham, *Linear Multivariable Control*, 2nd ed., Springer Verlag, New York.