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LINEAR SYSTEM**

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REPORT No. 41, 2002/2003, spring

ISSN 1103-467X

ISRN IML-R- -41-02/03- -SE+spring



**INSTITUT MITTAG-LEFFLER**  
THE ROYAL SWEDISH ACADEMY OF SCIENCES

# **Bounds on the optimal $l_2$ gain in adaptive control of a first order linear system**

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October 2003

## **Abstract**

The following feedback design problem is considered: given a family of first order LTI discrete time plants, design, when possible, a single nonlinear feedback law which, when applied to each of the plants, satisfies a given (plant-dependent) closed loop  $l_2$  gain bound. Sufficient conditions of feasibility and infeasibility of the adaptive control problem are given.

**Key Words:** nonlinear feedback, robust control, adaptive control,  $l_2$  gain bounds.

# 1 Introduction

Considering the wealth of results available for adaptive control design for uncertain linear systems, it appears that little is known about the fundamental limitations a problem setup imposes on the quality of transient response in a closed loop adaptive system. In particular the problem of minimizing  $l_2$  gains of adaptive systems appears to be far from being solved.

In this paper, a special case of the following general setup is considered. Let  $\{P_S\}_{S \in \mathcal{S}}$  be a family of discrete time systems with fixed dimensions of control input  $u = u(t)$ , disturbance input  $w = w(t)$ , cost output  $z = z(t)$ , sensor output  $y = y(t)$ , and state vector  $x = x(t)$ , governed by equations

$$\begin{bmatrix} x(t+1) \\ z(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix}, \quad t = 0, 1, 2, \dots, \quad (1)$$

and parameterized by matrices  $S$  ranging over a given set  $\mathcal{S}$ . The objective is to construct a single nonlinear feedback law

$$u(t) = K_t(y(t), y(t-1), \dots, y(0)), \quad (2)$$

such that for every  $S \in \mathcal{S}$  there exists a constant  $c$  such that the inequality

$$\sum_{t=0}^T |z(t)|^2 \leq c|x(0)|^2 + \gamma(S)2 \sum_{t=0}^T |w(t)|^2 \quad (3)$$

is satisfied for all  $T \geq 0$ ,  $S \in \mathcal{S}$ , and for all solutions of (1),(2), where  $\gamma : \mathcal{S} \mapsto \mathbf{R}_+$  is a given “desired  $l_2$  gain profile” of the closed loop system.

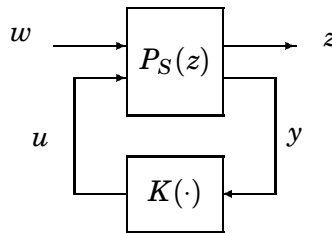


Figure 1: Feedback design setup

When  $S$  is a compact set of matrices each of which defines a *stabilizable* plant one would expect that, for a sufficiently small positive constant  $\gamma$ , the design objective can be satisfied. However, even for a basic statement like this, the authors were not able to find a proof in the literature. The conference paper [1] contains the claim without a proof, referencing instead an unpublished manuscript. The more recent paper [2] proves a number of results about induced gains for a particular adaptive control algorithm.

In this paper, first order plants are considered, and sufficient conditions for feasibility and infeasibility of the adaptive control setup are given. More precisely, for the family of plants described by equations

$$x(t+1) = ax(t) + u(t) + w(t), \quad (4)$$

$$x(0) = 0, \quad z(t) = y(t) = x(t),$$

where  $a \in \mathcal{A} \subset \mathbf{R}$  is an unknown parameter it is shown that a simple controller produces closed loop  $l_2$  gain  $\gamma = \gamma(a)$  which is not larger than  $(2 + 32M^2 + \varepsilon)^{1/2}$  for every  $a \in \mathcal{A} = [-M, M]$ , and it is shown that a controller which makes  $\gamma(a) < \infty$  for all  $a$  from an interval of length  $L$ , where  $L \geq 100$ , must have a gain larger than  $L/20$  for  $a = 0$ . In particular, this shows that it is impossible to find a controller which makes  $\gamma(a) < \infty$  for all  $a \in (0, \infty)$ . On the other hand, it also shows that the ratio of the upper and lower performance bounds for  $a$  ranging over  $[-M, M]$  remains bounded while the bounds grow linearly with  $M$  as  $M \rightarrow \infty$ .

## 2 A lower bound for adaptive $l_2$ gain

**Theorem 1** *Let  $\gamma_0 \geq 5$  be a real number. Let  $I$  be a closed interval in  $\mathbf{R}$  such that the length of its intersection with  $\mathbf{R}_+$  or  $\mathbf{R}_-$  is at least  $10\gamma_0$ . For a given family  $K = \{K_t\}_{t=0}^\infty$  of functions*

$$K_t : \mathbf{R}^{t+1} \mapsto \mathbf{R}, \quad t = 0, 1, 2, \dots,$$

let  $\Gamma = \Gamma(a, K)$  be the infimum of those  $\gamma$  for which

$$\gamma^2 \sum_{t=1}^T |x(t) - ax(t-1) - u(t-1)|^2 \quad (5)$$

$$\geq \sum_{t=1}^T |x(t)|^2 \quad \forall T, x(\cdot) : x(0) = 0,$$

whenever

$$u(t) = K_t(x(t), x(t-1), \dots, x(1), x(0)) \quad \forall t.$$

If  $\Gamma(0) \leq \gamma_0$  then there exists at least one  $a \in I$  such that  $\Gamma(a) = \infty$ .

**Proof** Let

$$\begin{aligned} X(T) &= \sum_{t=1}^T |x_{t-1}|^2 \\ Y(T) &= \sum_{t=1}^T x(t-1)[u(t-1) - x(t)] \\ Z(T) &= \sum_{t=1}^T |u_{t-1} - x(t)|^2 \end{aligned}$$

Assuming, without loss of generality, that  $x(1) \neq 0$ , define

$$\begin{aligned} \theta(T) &= -\frac{Y(T)}{X(T)} \\ \kappa(T) &= \frac{Z(T)X(T) - Y(T)^2}{X(T)^2} \\ h(T) &= \frac{x(T)}{X(T)^{1/2}} \\ q(T) &= \frac{u(T) + \theta(T)x(T)}{X(T)^{1/2}(1 + h(T)^2)^{1/2}} \\ p(T) &= \frac{x(T+1) + \theta(T)x(T)}{X(T)^{1/2}(1 + h(T)^2)^{1/2}} \end{aligned}$$

Then

$$\begin{aligned} \kappa(t+1) &= \frac{\kappa(t) + p(t)^2}{1 + h(t)^2} \\ \theta(t+1) &= \theta(t) - \frac{h(t)p(t)}{(1 + h(t)^2)^{1/2}} \\ h(t+1) &= q(t) - p(t) \end{aligned}$$

and the inequality in (5) can be re-written as

$$\gamma^2[\kappa(T) + (a - \theta(T))^2] \geq (1 + h(T)^2).$$

For a given  $\alpha$ , the setup describes a game in which  $q(t)$  is the control action decided at time  $t$  with the knowledge of  $h(t)$ ,  $\kappa(t)$ ,  $\theta(t)$  with the aim of satisfying (5) perpetually, and  $p(t)$  is the antagonist disturbance action decided at time  $t$  with the knowledge of  $h(t)$ ,  $\kappa(t)$ ,  $\theta(t)$ , and  $q(t)$ , with the aim of making (5) fail eventually.

Let us show that if the control action guarantees that (5) is satisfied with  $\gamma = \gamma_0$  for  $\alpha = 0$  then (5) cannot be guaranteed (no matter how large  $\gamma$  is) for some  $\alpha \in I$ . Let  $a_c$  be the closest to zero point of  $I$ . Without loss of generality, assume that  $a_c \geq 0$ . Then by assumption  $[a_c, a_c + 10\gamma_0] \in I$ . Define  $p = p(t)$  according to the following law:

(a)  $p(t) = 0$  if

$$\begin{aligned} |q(t)| &\geq 2\gamma_0, & \text{and} \\ \theta(t) - a_c &\in [5\gamma_0, 9.02\gamma_0], & \text{and} \\ \kappa(t) &\leq 4.03, \end{aligned}$$

(b)  $p(t) = 0.01\gamma_0 \text{sgn}(h(t)(\theta(t) - 7.01\gamma_0))$  if

$$\begin{aligned} |q(t)| &\geq 2.01\gamma_0, & \text{but} \\ \theta(t) - a_c &\notin [5\gamma_0, 9.02\gamma_0], & \text{or} \\ \kappa(t) &\geq 4.03, \end{aligned}$$

(c)  $q(t) = 4.01\gamma_0 \text{sgn}(h(t)(\theta(t) - 7.01\gamma_0))$  otherwise.

Note first that, by construction,

$$|h(t)| \geq 2\gamma_0 \text{ for all } t > 1.$$

Indeed, if  $|q(t)| \geq 2.01\gamma_0$  then (a) or (b) are used to define  $p(t)$ , in which case  $|p(t)| \leq |q(t)| - 2\gamma_0$  and

$$|h(t+1)| = |q(t) - p(t)| \geq 2\gamma_0.$$

Otherwise  $|q(t)| \leq 2.01\gamma_0$  and  $|p(t)| \geq 4.01\gamma_0$ , which also implies  $|h(t+1)| \geq 2\gamma_0$ .

Now let us show that  $\kappa(t) \leq 4.03$  for all sufficiently large  $t$ . Indeed,  $|p(t)| \leq 4.01\gamma_0$  for all  $t > 0$ . Hence

$$\kappa(t+1) - 4.025 = \frac{\kappa(t) + p(t)2}{1 + h(t)2} \leq \frac{\kappa(t) + 16.1\gamma_02}{1 + 4\gamma_02} - 4.025 \leq \frac{\kappa(t) - 4.025}{1 + 4\gamma_02}$$

for all  $t > 1$ . Hence  $\max\{0, \kappa(t) - 4.025\}$  is exponentially decreasing. Hence there exists  $t_*$  such that  $\kappa(t) \leq 4.03$  for  $t \geq t_*$ .

Now for  $t \geq t_*$  the definition of  $p(t)$  is such that

$$|\theta(t+1) - a_c - 7.01\gamma_0| \leq |\theta(t) - a_c - 7.01\gamma_0| - 0.01\gamma_0$$

whenever  $\theta(t) - a_c \notin [5\gamma_0, 9.02\gamma_0]$ . Hence  $\theta(t) - a_c \in [5\gamma_0, 9.2\gamma_0]$  for sufficiently large  $t$ .

Note that, in order to maintain (5) for  $a = 0, \gamma = \gamma_0$ , the control action must satisfy

$$\gamma_0 2 \left[ \frac{\kappa(t)}{1+h(t)2} + \frac{\theta(t)}{h(t)2} \right] - 1 \geq \left| q(t) - \frac{\theta(t)(1+h(t)2)^{1/2}}{h(t)} \right|$$

whenever  $h(t) \neq 0$ . Indeed, otherwise using

$$p(t) = \frac{\theta(t)(1+h(t)2)^{1/2}}{h(t)}$$

yields  $\theta(t+1) = 0$  and

$$\kappa(t+1) = \frac{\kappa(t) + p(t)2}{1+h(t)2} = \frac{\kappa(t)}{1+h(t)2} + \frac{\theta(t)}{h(t)2}.$$

In order to maintain the gain bound of  $\gamma_0$  at  $a = 0$ , the inequality

$$\begin{aligned} \gamma_0 2 \kappa(t+1) &\geq 1 + h(t+1)2 \\ &= 1 + \left( q(t) - \frac{\theta(t)(1+h(t)2)^{1/2}}{h(t)} \right) \end{aligned}$$

must hold, which implies the desired constraint on  $q(t)$ .

Combining the constraint with the lower bound  $\theta(t) \geq 5\gamma_0$  implies that at every sufficiently large time instance

$$\begin{aligned} |q| &\geq \frac{|\theta|(1+h2)^{1/2}}{|h|} - \frac{\gamma_0 \kappa^{1/2}}{(1+h2)^{1/2}} - \frac{\gamma_0 |\theta|}{|h|} \\ &\geq \frac{|\theta|}{2} - \frac{\gamma_0 4.03^{1/2}}{1+4\gamma_0 2} \geq 2\gamma_0. \end{aligned}$$

Hence  $p(t) = 0$  and  $\theta(t) = \theta_0 \in I$  is constant for sufficiently large  $t$ , and therefore  $\kappa(t) \rightarrow 0$  exponentially as  $t \rightarrow \infty$ . Hence the  $l_2$  gain is infinite for  $a = \theta_0$ . ■

### 3 An upper bound for adaptive $l_2$ gain

**Theorem 2** Let  $0 \leq \lambda < 1$ . For the linear system

$$x(k+1) = ax(k) + u(k) + w(k) \quad x(0) = 0$$

with  $|a| \leq A$  and  $w(0) \neq 0$ , the non-linear control law

$$u(k) = -\hat{a}(k)x(k),$$

where

$$\hat{a}(k) = \text{sat}_A \left( \frac{\sum_{j=0}^{k-1} \lambda^{k-j} x(j) [x(j+1) - u(j)]}{\sum_{j=0}^{k-1} \lambda^{k-j} x(j)^2} \right),$$

$$\text{sat}_A(y) = \begin{cases} y, & |y| \leq A, \\ Ay/|y|, & |y| > A, \end{cases}$$

gives the stable closed loop performance

$$\sum_{k=0}^n x(k)^2 \leq \left( \frac{32A^2}{1-\lambda} + 2 \right) \sum_{k=0}^n w(k)^2$$

**Proof** For brief notation, we will write  $x$  and  $x_+$  instead of  $x(k)$  and  $x(k+1)$ . The system equation then reads

$$x_+ = ax + u + w$$

Define  $S$ ,  $T$ ,  $W$  and  $X$  through the difference equations

$$\begin{aligned} X_+ &= \lambda X + x^2 & X(0) &= 0 \\ T_+ &= \lambda T + xw & T(0) &= 0 \\ W_+ &= \lambda W + w^2 & W(0) &= 0 \\ S_+ &= \lambda S + x(x_+ - u) & & \\ &= \lambda S + x(ax + w) & S(0) &= 0 \end{aligned}$$

and note that  $S = aX + T$ . Let

$$\hat{a} = A \text{sat} \left( \frac{S}{AX} \right) \quad \tilde{a} = a - \hat{a}$$

Then the control law can be written  $u = -\hat{a}x$  and the closed loop system

$$x_+ = \tilde{a}x + w$$

The proof is completed by verifying that the Lyapunov function

$$V = x^2 + dW$$

with  $d = 32A^2/(1 - \lambda)$  satisfies

$$V_+ - V \leq (d + 2)w^2 - x^2$$

To see this, some calculations are needed

$$|\tilde{a}| = |a - \hat{a}| = \left| a - A \operatorname{sat}\left(\frac{S}{AX}\right) \right| = \left| a - A \operatorname{sat}\left(a + \frac{T}{AX}\right) \right| \leq \left| \frac{T}{X} \right|$$

The definition of  $T$ ,  $W$  and  $X$  shows that

$$0 \leq \begin{bmatrix} X & T \\ T & W \end{bmatrix}$$

Consequently  $T^2 \leq XW$  and

$$\tilde{a}^2 \leq \frac{T^2}{X^2} \leq \frac{W}{X}$$

At the same time

$$\tilde{a}^2 \leq (|a| + |\hat{a}|)^2 \leq 4A^2$$

Together, the two bounds give

$$\begin{aligned} x_+^2 &= (\tilde{a}x + w)^2 \leq 2\tilde{a}^2x^2 + 2w^2 \\ &\leq 8A^2X_+ + 2W_+ \end{aligned}$$

$$x^2 \leq 8A^2X + 2W$$

$$\begin{aligned} \tilde{a}^2x^2 &\leq 8\tilde{a}^2A^2X + 2\tilde{a}^2W \\ &\leq 8A^2W + 2(4A^2)W = 16A^2W \end{aligned}$$

$$\begin{aligned} V_+ - V &= (x_+^2 + dW_+) - (x^2 + dW) \\ &\leq (\tilde{a}x + w)^2 - x^2 + d(\lambda W + w^2 - W) \\ &\leq 2\tilde{a}^2x^2 + 2w^2 - x^2 + d(\lambda W + w^2 - W) \\ &\leq 32A^2W + 2w^2 - x^2 + d(\lambda W + w^2 - W) \\ &= (d + 2)w^2 - x^2 \end{aligned}$$

Adding over  $k$  gives

$$\begin{aligned} V(n+1) &= \sum_{k=0}^n [V(k+1) - V(k)] \\ &\leq \sum_{k=0}^n [(d+2)w(k)^2 - x(k)^2] \end{aligned}$$

and the desired performance bound is proved. ■

## 4 Acknowledgment

This work was prepared during the spring semester 2003 at the Mittag-Leffler Institute. The authors are grateful for the generous support and atmosphere of the institute.

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