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Omega-limit Sets of Nonlinear Systems that Are Semiglobally Practically Stabilized *

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Abstract

In nonlinear control theory, the equilibrium of a system is semiglobally practically stabilizable if, given two balls centered at the equilibrium, one of arbitrarily large radius and one of arbitrarily small radius, we are able to design a feedback so that the resulting closed-loop system has the following property: all the trajectories originating in the large ball enter, within a fixed finite time, into the small ball and stay inside thereafter.

In this work, given a nonlinear system that is semiglobally practically stabilized, we focus on the problem of characterizing the asymptotic behavior of its trajectories that start inside the large ball. It turns out that inside the small ball where these trajectories enter, there is a compact, invariant, connected, stable set that attracts them; such set is the omega-limit set of the large ball. Here, we address the problem of studying the structure of this omega-limit set. Specifically, we carry

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out the study for closed-loop systems obtained applying a semiglobally practically stabilizing feedback law to a nonlinear minimum-phase system belonging to a certain class. The study is done first when a memory-less state feedback is employed and then when a dynamic output feedback is employed. It is then found that using output feedback rather than state feedback does not affect the structure of the omega-limit set.

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1 Introduction

Using large feedback gains is a powerful design technique in the stabilization of a nonlinear system. For instance, in the fundamental work by Teel and Praly [11], it is shown that, in the stabilization of the equilibrium at the origin of a minimum-phase nonlinear system having relative degree d, a high-gain memory-less linear feedback of the output and its first $d-1$ derivatives can be used to steer trajectories starting from any arbitrarily large ball centered at the origin into an arbitrary small ball centered at the origin; the latter property is known as semiglobal practical stabilizability. In general the trajectories of the semiglobally practically stabilized system are not guaranteed to converge asymptotically to the equilibrium at the origin. As a matter of fact, in some cases the origin is made unstable by such feedbacks.

If only the output is available, a powerful technique to cope with partial measurements consists in using a high-gain coarse observer, introduced by Khalil and Esfandiari [10], that estimates the output and its first $d-1$ derivatives, and then in plugging those estimates into the high-gain feedback law. By this certainty-equivalence type of design, semiglobal practical stabilizability is still achieved, but again in general we cannot guarantee asymptotic convergence to the origin.

In both cases of partial state and output feedback it is interesting to investigate the asymptotic behavior of the trajectories of the closed-loop system inside the small ball that captures them.

When the relative degree $d = 1$ such investigation has been carried out in the work by Byrnes and Isidori [2]. Byrnes and Isidori point out that, when the zero dynamics are critically asymptotically stable, the closed-loop system may possess nontrivial compact attractors. Moreover, they characterize the structure of such attractors, and derive conditions under which the attractors degenerate to the origin, and semiglobal stabilizability is obtained.

The goal of the present work is to extend the study in [2] to the case of minimum-phase systems having relative degree $d \geq 2$ that are semiglobally practically stabilized using the method in [11]. We consider both the partial-state and the output feedback stabilization schemes.

We point out that, regardless of the specific design method, achieving semiglobal practical stabilization by itself guarantees that the closed-loop system possesses a compact, invariant, connected, stable set that is contained in the arbitrary small ball centered at the origin and that uniformly attracts all trajectories starting inside the arbitrary large ball centered at the origin; such set is the ω-limit set of the large ball. As a result, our goal shifts to characterizing the structure of this ω-limit set.

We perform our study using two main tools. First, we use iteratively a reduction principle for ω-limit sets that allows us to lower significantly the dimension of the system to be investigated; second, we perform a bifurcation analysis considering some large feedback gains as bifurcation parameters in the reduced system thus obtained.

2 Notations and Definitions

- $\mathbb{R}^+$ denotes $(0, \infty)$
- Given $A \subseteq \mathbb{R}^n$ we denote by $int(A)$ its interior.
- B^n_r denotes the *closed* ball $\{x \in \mathbb{R}^n : \|x\| \leq r\}$.
- Given a C^1 function $g : A \to \mathbb{R}^m$, with A open subset of $\mathbb{R}^n$, we denote by $Dg : A \to \mathbb{R}^{m \times n}$ its Jacobian matrix.

Let (2.1) denote the number of the differential equation

$$\dot{x} = f(x) \tag{2.1}$$

where $f : U \to \mathbb{R}^n$ is a locally Lipschitz function defined on an open subset U of $\mathbb{R}^n$.

- We denote by $\phi_{(2.1)} : E \to \mathbb{R}^n,\ \ (t, x_0) \to \phi_{(2.1)}(t, x_0)$ the flow generated by the differential equation (2.1); $\phi_{(2.1)}$ is defined on $E = \{(t, x_0) :\ x_0 \in U,\ t \in (-t^-_{(2.1)}(x_0), t^+_{(2.1)}(x_0))\}$ where $t^-_{(2.1)}, t^+_{(2.1)} : U \to (0, \infty]$; $\phi_{(2.1)}$ is continuous and differentiable with respect to t in E and is such that

$$\frac{\partial \phi_{(2.1)}}{\partial t}(t, x_0) = f(\phi_{(2.1)}(t, x_0))\ , \quad \phi_{(2.1)}(0, x_0) = x_0 \ \text{ for all } (t, x_0) \in E\ ,$$

and $(-t^-_{(2.1)}(x_0), t^+_{(2.1)}(x_0))$ is the maximal interval of existence of the solution $\phi_{(2.1)}(\cdot, x_0)$.

- Given a set $B \subseteq U$ such that $t^+_{(2.1)}(x_0) = \infty$ for all $x_0 \in B$, and given $t \geq 0$, we denote by $\phi_{(2.1)}(t, B)$ the set

$$\phi_{(2.1)}(t, B) = \bigcup_{x_0 \in B} \phi_{(2.1)}(t, x_0)\ .$$

- Given a set $B \subseteq U$ such that $t^+_{(2.1)}(x_0) = \infty$ for all $x_0 \in B$, we denote by $\omega(B)$ the *ω-limit set of B* (under the flow $\phi_{(2.1)}$) defined as follows: $y \in \omega(B)$ if there are sequences $t_k \to \infty$ as $k \to \infty$, $x_k \in B$, such that $\phi_{(2.1)}(t_k, x_k) \to y$ as $k \to \infty$. Note that $\omega(B) \supseteq \cup_{x_0 \in B}\omega(\{x_0\})$, but equality may not hold (see [7, pp. 7–8]).

- A set $A \subseteq U$ is *positively invariant* (under the flow $\phi_{(2.1)}$) if $t^+_{(2.1)}(x_0) = \infty$ for all $x_0 \in A$ and, $\phi_{(2.1)}(t, x_0) \in A$ for all $t \in [0, \infty)$ and for all $x_0 \in A$.

- A set $A \subseteq U$ is *invariant* (under the flow $\phi_{(2.1)}$) if $t^-_{(2.1)}(x_0) = t^+_{(2.1)}(x_0) = \infty$ for all $x_0 \in A$ and, $\phi_{(2.1)}(t, x_0) \in A$ for all $t \in \mathbb{R}$ and for all $x_0 \in A$.

- Given two sets $A, B \subseteq U$, it is said that A *attracts* B (under the flow $\phi_{(2.1)}$) if $t^+_{(2.1)}(x_0) = \infty$ for all $x_0 \in B$, and $\lim_{t \to \infty} \operatorname{dist}(\phi_{(2.1)}(t, B), A) = 0$, where

$$\operatorname{dist}(B, A) = \sup_{x \in B} \operatorname{dist}(x, A) = \sup_{x \in B} \inf_{y \in A} \|x - y\|$$

(see [7, p. 8]). Note that $A \subseteq U$ attracts $B \subseteq U$ if and only if $t^+_{(2.1)}(x_0) = \infty$ for all $x_0 \in B$, and for any $\varepsilon > 0$, there is a $\bar{t} > 0$ such that $\operatorname{dist}(\phi_{(2.1)}(t, x_0), A) \leq \varepsilon$ for all $t \geq \bar{t}$ and for all $x_0 \in B$.

- Given a set $V \subseteq U$, an invariant set $A \subseteq V$ is said to be the *maximal invariant set contained in* V (under the flow $\phi_{(2.1)}$) if every set contained in V that is invariant (under the same flow) is also contained in A.

- An invariant set $J \subseteq U$ is *stable* (under the flow $\phi_{(2.1)}$) if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $x_0 \in U$ and $\text{dist}(x_0, J) < \delta$, then $t^+_{(2.1)}(x_0) = \infty$ and $\text{dist}(\phi_{(2.1)}(t, x_0), J)) < \varepsilon$ for all $t \geq 0$.

- Given a system of differential equations

$$\dot{x}_i = f_i(x_1, x_2, \cdots, x_m) \quad i = 1, \cdots, m \;, \tag{2.2}$$

where $f_i : U \to \mathbb{R}^{n_i}$ $i = 1, \cdots, m$ are locally Lipschitz functions defined on an open subset U of $\mathbb{R}^{n_1+n_2+\cdots+n_m}$, we denote by $\phi^{x_i}_{(2.2)}$ the x_i component of $\phi_{(2.2)}$; that is

$$\phi_{(2.2)}(t, (x_1^0, x_2^0, \cdots, x_m^0)) = \begin{pmatrix} \phi^{x_1}_{(2.2)}(t, (x_1^0, x_2^0, \cdots, x_m^0) \\ \vdots \\ \phi^{x_n}_{(2.2)}(t, (x_1^0, x_2^0, \cdots, x_m^0)) \end{pmatrix}$$

- Given a C^1 function $V : A \to \mathbb{R}$, with A open subset of U, we denote by $\dot{V}_{(2.1)}$ the function $\dot{V}_{(2.1)} : A \to \mathbb{R}$ defined by

$$\dot{V}_{(2.1)}(x) = DV(x)f(x) \;. \tag{2.3}$$

3 Some Useful Lemmas

Given the differential equation

$$\dot{x} = f(x) \;, \tag{3.1}$$

where $f : U \to \mathbb{R}^n$ is a locally Lipschitz function defined on an open subset U of $\mathbb{R}^n$, the following lemmas hold.

Lemma 3.2. *(see [7, Lemma 2.0.1]) If $B \subseteq U$ is nonempty, connected, and bounded, and there exists a compact set $J \subseteq U$ that attracts B under $\phi_{(3.1)}$, then $\omega(B)$ under the same flow is nonempty, compact, invariant, connected, and attracts B.*

Lemma 3.3. *If $B \subseteq U$ is open, and $A \subset B$ is an invariant compact set that attracts B, then A is stable.*

Proof. The lemma can be proved generalizing the proof of Theorem 38.1 of [6] where A is assumed to be an equilibrium. □

Remark 3.4. It is well known that in general attractivity of an invariant set does not imply its stability. However, in this case there are two critical assumptions that render the implication true. First A is a subset of B, and second A attracts B in the sense defined in section 2; that is, given $\varepsilon > 0$ a trajectory originating from $x_0 \in B$ is guaranteed to enter into the neighborhood of A of radius ε by a time $\bar{t}$ which *does not depend on* x_0; that is $\bar{t}$ is uniform with respect to B.

4 Semiglobal Practical Stabilizability

In this paper we are interested in characterizing the asymptotic behavior of the trajectories of systems that are *semiglobally practically stabilized.*

Definition 4.1. *(semiglobal practical stabilizability). Given the single-input single-output system*

$$\dot{x} = f(x,u), \;\; y = h(x) \tag{4.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, f and h are locally Lipschitz, we say that its origin is semiglobally practically stabilizable by memory-less state feedback *if, for each pair of real numbers r and R, such that $0 < r < R$, there exists a memory-less state feedback*

$$u = \alpha(x) \tag{4.3}$$

such that the flow $\phi_{(4.4)}$ of the resulting closed-loop system,

$$\dot{x} = f(x, \alpha(x)) \tag{4.4}$$

has the following property: there exists $T > 0$ such that $\phi_{(4.4)}(t, B^n_R) \subseteq B^n_r$ for all $t \geq T$.

The definition of semiglobal practical stabilizability of the origin of (4.2) by *dynamic output feedback* is analogous, except that the feedback is of the type

$$\dot{\varphi} = \theta(\varphi, y), \;\; u = \kappa(\varphi)$$

where $\varphi \in \mathbb{R}^\nu$ and θ and κ are locally Lipschitz.

In the remainder of this section we refer to semiglobal practical stabilizability *by memory-less state feedback*; however, clearly all what follows applies also to semiglobal practical stabilizability *by dynamic output feedback*.

Remark 4.5. If in the definition above we had *only* required that each trajectory of the closed-loop system (4.4) originating in $x_0 \in B_R^n$ is captured in a finite time by B_r^n, then a *finite* $T > 0$ such that $\phi_{(4.4)}(t, B_R^n) \subseteq B_r^n$ for all $t \geq T$ might not exist. In fact, the minimum time $\tau(x_0)$ such that $\phi_{(4.4)}(t, x_0) \in B_r^n$ for all $t \geq \tau(x_0)$ can grow unboundedly even if x_0 ranges over a compact set. This occurs for instance if system (4.4) has a homoclinic orbit that originates and ends at the origin. An example of a system of this type is described in [6, section 40].

In this paper we consider closed-loop systems (4.4) obtained from semiglobal practical stabilization of a system of the type (4.2); that is, we assume what follows: the origin of (4.2) is semiglobally practically stabilizable; $0 < r < R$ have been assigned; a memory-less state feedback (4.3) has been designed so that the resulting closed-loop system (4.4) has the following property: there exist $T > 0$ such that $\phi_{(4.4)}(t, B_R^n) \subseteq B_r^n$ for all $t \geq T$. Our goal is to study the *asymptotic behavior of the trajectories of (4.4) that originate in* B_R^n. A first characterization of such behavior is given by the following theorems.

Theorem 4.6. *The ω-limit set of B_R^n under $\phi_{(4.4)}$ is a nonempty, compact, invariant, connected, stable set that attracts B_R^n; $\omega(B_R^n)$ is contained in B_r^n, and it is the maximal invariant set contained in B_R^n.*

Proof. Note that B_r^n attracts B_R^n since $\mathrm{dist}(\phi_{(4.4)}(t, B_R^n), B_r^n) = 0$ for $t \geq T$. Then, by lemma 3.2, $\omega(B_R^n)$ is nonempty, compact, invariant, connected, and it attracts B_R^n. To prove that $\omega(B_R^n) \subseteq B_r^n$, let $y \in \omega(B_R^n)$ and let $t_k \in \mathbb{R}$ and $x_k \in B_R^n$ sequences such that $t_k \to \infty$ and $\phi_{(4.4)}(t_k, x_k) \to y$ as $k \to \infty$. For all k's such that $t_k \geq T$ we have $\phi_{(4.4)}(t_k, x_k) \in B_r^n$; then, $y \in B_r^n$. Stability of $\omega(B_R^n)$ comes from lemma 3.3 since $\omega(B_R^n) \subseteq B_R^n \subset int(B_R^n)$. To prove that $\omega(B_R^n)$ is maximal, let A be an invariant set contained in B_R^n, and let $y \in A$. Pick any sequence $t_k \to -\infty$ as $k \to \infty$, and define $x_k = \phi_{(4.4)}(t_k, y)$. Note that $x_k \in A \subseteq B_R^n$ and $\phi_{(4.4)}(-t_k, x_k) = \phi_{(4.4)}(0, y) = y$ for all k's; consequently $y \in \omega(B_R^n)$. ∎

Theorem 4.7. *Let $\Omega \subseteq B_R^n$ have the following property: there exists $\hat{T}$ such that $\phi_{(4.4)}(t, B_R^n) \subseteq \Omega$ for all $t \geq \hat{T}$; then, $\omega(B_R^n)$ under $\phi_{(4.4)}$ is equal to $\omega(\Omega)$ under the same flow.*

Proof. $\omega(\Omega) \subseteq \omega(B_R^n)$ since $\Omega \subseteq B_R^n$. To prove the opposite inclusion, let $y \in \omega(B_R^n)$; then, there exist sequences $t_k \to \infty$ as $k \to \infty$ and $x_k \in B_R^n$, such that $\phi_{(4.4)}(t_k, x_k) \to y$ as $k \to \infty$. Let $\tilde{t}_k = t_k - T$ and $\tilde{x}_k = \phi_{(4.4)}(T, x_k)$. Note that $\tilde{t}_k \to \infty$ as $k \to \infty$, $\tilde{x}_k \in \Omega$, and $\phi_{(4.4)}(\tilde{t}_k, \tilde{x}_k) = \phi_{(4.4)}(t_k, x_k)$; consequently, $\phi_{(4.4)}(\tilde{t}_k, \tilde{x}_k) \to y$ as $k \to \infty$ and $y \in \omega(\Omega)$. □

From the previous theorem it follows that $\omega(B_R^n) = \omega(B_r^n)$. Then, to characterize the asymptotic behavior of the trajectories of (4.4) originating in B_R^n, *it suffices to determine the structure of* $\omega(B_r^n)$ *under* $\phi_{(4.4)}$.

In what follows we will consider some classes of closed-loop systems obtained using a semiglobally practically stabilizing feedback, and we will characterize the structure of $\omega(B_r^n)$.

5 Semiglobal Practical Stabilization of Minimum-phase Systems

Consider a single-input single-output smooth nonlinear system, having uniform relative degree d, that can be transformed via a global smooth change of coordinates into the *global normal form* (see [8, section 9.1])

$$\begin{aligned}
\dot{z} &= f(z) + p(z, \xi_1) \\
\dot{\xi}_1 &= \xi_2 \\
\vdots &\quad \vdots \\
\dot{\xi}_{d-1} &= \xi_d \\
\dot{\xi}_d &= q(z, \xi_1, \ldots, \xi_d) + b(z, \xi_1, \cdots, \xi_d)u \\
\\
y &= \xi_1 \,.
\end{aligned} \tag{5.1}$$

In (5.1) $z \in \mathbb{R}^n$, $\xi_i \in \mathbb{R}$ for $i = 1, \ldots, d$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, and

$$\begin{aligned}
f(0) &= 0 \\
p(z, 0) &= 0 \\
q(0, 0, \ldots, 0) &= 0 \\
b(z, \xi_1, \ldots, \xi_d) &> 0 \,.
\end{aligned}$$

We assume that the system is *minimum-phase*; that is, its zero dynamics

$$\dot{z} = f(z) \tag{5.2}$$

have a globally asymptotically stable equilibrium at $z = 0$.

In [11] (see also [9, section 12.1]) it is shown that the origin of system (5.1) is semiglobally practically stabilizable by a memory-less partial-state feedback of the form

$$u = -\bar{k}(k^{d-1}a_0\xi_1 + k^{d-2}a_1\xi_2 + \ldots + ka_{d-2}\xi_{d-1} + \xi_d) \tag{5.3}$$

in which a_i's are fixed coefficients of a Hurwitz polynomial, and k and $\bar{k}$ are high-gains. More precisely, for every $0 < r < R$, there exists $k^* > 0$ which depends on R, such that fixed $k > k^*$, there exists $\bar{k}^* > 0$ which depends on R, r, and the k fixed, such that if $\bar{k} > \bar{k}^*$, then the following property holds: there exists $T > 0$ such that $\phi_{(5.1)(5.3)}(t, B_R^{n+d}) \subseteq B_r^{n+d}$ for all $t \geq T$.

Moreover, in [11] (see also [9, section 12.4]) it is shown that the origin of system (5.1) is semiglobally practically stabilizable by a dynamic output feedback of the form

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \vdots \\ \dot{\eta}_{d-1} \\ \dot{\eta}_d \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \eta_3 \\ \vdots \\ \eta_d \\ 0 \end{pmatrix} + \begin{pmatrix} gc_{d-1} \\ g^2c_{d-2} \\ \vdots \\ g^{d-1}c_1 \\ g^dc_0 \end{pmatrix} (y - \eta_1) \tag{5.4}$$

$$u = -\sigma_l(\bar{k}(k^{d-1}a_0\eta_1 + k^{d-2}a_1\eta_2 + \cdots + ka_{d-2}\eta_{d-1} + \eta_d)) \,,$$

in which σ_l is a saturation function defined as

$$\sigma_l(r) = \begin{cases} r & \text{if } |r| \leq l \\ \text{sgn(r)}l & \text{if } |r| > l \,, \end{cases}$$

a_i's and c_j's are fixed coefficients of Hurwitz polynomials, and k, $\bar{k}$, and g are high-gains. More precisely, for every $0 < r < R$, there exists $k^* > 0$ which depends on R, such that fixed $k > k^*$, there exists $\bar{k}^* > 0$ which depends on R, r, and the k fixed, such that fixed $\bar{k} > \bar{k}^*$, there exist $0 < l^* < l^{**}$ which depend on R and the k and $\bar{k}$ fixed, such that fixed $l \in [l^*, l^{**}]$ there exists $g^* > 0$ which depends on R, r, and the k, $\bar{k}$, and l fixed such that, if $g > g^*$, then the following property hold: there exists $T > 0$ such that $\phi_{(5.1)(5.4)}(t, B_R^{n+2d}) \subseteq B_r^{n+2d}$ for all $t \geq T$.

In addition, in [11] it is proved that if the zero dynamics (5.2) have an *exponentially stable* equilibrium at the origin, then, given $R > 0$ arbitrarily large, if $\bar{k}$ is large enough, the partial state feedback (5.3) is such that the origin of the resulting closed-loop system is exponentially stable and its region of attraction contains B_R^{n+d}. As a result, $\omega(B_r^{n+d}) = \{0\}$. Moreover, if $\bar{k}$ and g are large enough, the same type of result is achieved using the output feedback (5.4). Consequently, if the origin of (5.2) is exponentially stable, and $\bar{k}$ is large enough (and in the case of output feedback g is large enough, too) , no further investigation on the asymptotic behavior of the trajectories of the closed-loop system is needed, and we will assume throughout that *the origin of (5.2) is critically asymptotically stable.* In this hypothesis the Jacobian matrix of f at $z = 0$ has at least one eigenvalue on the imaginary axis, and all the remaining spectrum is in $\mathbb{C}^-$. Then, using a linear change of coordinates in the z variables, (5.1) can be put in the form

$$\begin{array}{rcl} \dot{z}_1 & = & Sz_1 + g_1(z_1, z_2) + p_1(z_1, z_2, \xi_1) \\ \dot{z}_2 & = & Hz_2 + g_2(z_1, z_2) + p_2(z_1, z_2, \xi_1) \\ \dot{\xi}_1 & = & \xi_2 \\ \vdots & & \vdots \\ \dot{\xi}_{d-1} & = & \xi_d \\ \dot{\xi}_d & = & q(z_1, z_2, \xi_1, \ldots, \xi_d) + b(z_1, z_2, \xi_1, \ldots, \xi_d)u \\ & & \\ y & = & \xi_1 \end{array} \tag{5.5}$$

in which $z_1 \in \mathbb{R}^m$, $z_2 \in \mathbb{R}^{n-m}$, the matrix S has all eigenvalues on the imaginary axis, H is a Hurwitz matrix, $g_i(0,0) = 0$, $Dg_i(0,0) = 0$, and $p_i(z_1, z_2, 0) = 0$ for $i = 1, 2$.

6 A Reduction Principle for ω-limit Sets

The main objective of this paper is to determine the structure of $\omega(B_r^{n+d})$ under $\phi_{(5.5)(5.3)}$ or $\omega(B_r^{n+2d})$ under $\phi_{(5.5)(5.4)}$.

It can easily be seen that, setting $\bar{k} = 1/\varepsilon$, and in the case of output feedback also $g = 1/(\varepsilon\delta)$, and scaling the time properly, the closed-loop systems (5.5)(5.3) and (5.5)(5.4) can be put in the form

$$\begin{aligned} \dot{x} &= Ax + f(x, y, \varepsilon, \delta) \\ \dot{y} &= By + g(x, y, \varepsilon, \delta) \ , \end{aligned} \tag{6.1}$$

where A has all eigenvalues on the imaginary axis, B has all eigenvalues with negative real part, and f and g vanish at $(x, y, \varepsilon, \delta) = (0, 0, 0, 0)$ together with their first order partial derivatives.

We augment (6.1) with the equations

$$\begin{aligned} \dot{\varepsilon} &= 0 \\ \dot{\delta} &= 0 \ . \end{aligned} \tag{6.2}$$

Since, as said in section 5, ε and δ can be chosen arbitrarily small, the analysis of system (6.1)(6.2) can be done using the Center Manifold Theory (see [3]). Here we will establish an extension to ω-limit sets of the reduction principle of the Center Manifold Theory. First we need the following

Definition 6.3. *Given a system*

$$\dot{x} = F(x, \varepsilon, \delta) \tag{6.4}$$

where $F : \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ *is such that* $F(\cdot, \varepsilon, \delta)$ *is locally Lipschitz for each* $(\varepsilon, \delta) \in \mathbb{R}^+ \times \mathbb{R}^+$*, we say that its origin is* practically stabilizable *in the* parameters (ε,δ) *with* radial upper bound R *if for any positive real number* $r < R$ *there exist*

(i) a closed subset Ω_r *of* $\mathbb{R}^n$ *satisfying*

$$\{0\} \subset \Omega_r \subseteq B^n_r \ ,$$

(ii) a time $T_r > 0$ *,*

(iii) two positive numbers ε_r *and* δ_r*,*

such that, for all $0 < \varepsilon < \varepsilon_r$ *and* $0 < \delta < \delta_r$*, the set* Ω_r *is positively invariant under* $\phi_{(6.4)}$*, and* $\phi_{(6.4)}(t, B^n_R) \subseteq \Omega_r$ *for all* $t \geq T_r$*.*

Assume that (6.1) is C^k (with $k \geq 2$), and that its *origin is practically stabilizable in the parameters* (ε, δ) *with radial upper bound* R. Let m be the dimension of x and $n - m$ the dimension of y.

System (6.2) (6.1) possesses a *center manifold* at $(x, y, \varepsilon, \delta) = (0, 0, 0, 0)$, that can be expressed as the graph of a C^k mapping $y = h(x, \varepsilon, \delta)$, defined for all $||x|| < r_0$, $|\varepsilon| < e_0$, and $|\delta| < d_0$ (for some positive r_0, e_0, d_0), and satisfying

$$h(0, 0, 0) = 0,\ Dh(0, 0, 0) = 0\ ,$$

and

$$\frac{\partial h}{\partial x}(Ax + f(x, h(x, \varepsilon, \delta), \varepsilon, \delta)) = Bh(x, \varepsilon, \delta) + g(x, h(x, \varepsilon, \delta), \varepsilon, \delta)\ . \qquad (6.5)$$

Let P be the solution of the Lyapunov equation $PB + B^{\mathrm{T}}P = -I$. Without loss of generality we can assume that r_0, e_0, and d_0 are small enough so that $r_0 < R$ and

$$\left\| g(x, y, \varepsilon, \delta) - g(x, h(x, \varepsilon, \delta), \varepsilon, \delta) - \frac{\partial h}{\partial x}(f(x, y, \varepsilon, \delta) - f(x, h(x, \varepsilon, \delta), \varepsilon, \delta)) \right\|$$
$$\leq \frac{1}{4\|P\|}\|y - h(x, \varepsilon, \delta)\| \quad \text{for all } \|x\| < r_0,\ \|y\| < r_0,\ |\varepsilon| < e_0,\ |\delta| < d_0\ .$$

Furthermore, there is no loss of generality in assuming that the quantities ε_r and δ_r from definition 6.3 satisfy $\varepsilon_r < e_0$ and $\delta_r < d_0$ for all $0 < r < R$. Note also that for each $|\varepsilon| < e_0$ and $|\delta| < d_0$

$$\mathcal{C}_{\varepsilon\delta} = \{(x, y) : y = h(x, \varepsilon, \delta), \|x\| < r_0\}$$

is a locally invariant manifold under $\phi_{(6.1)}$.

We assume that we have assigned $0 < r < r_0$ and fixed $0 < \varepsilon < \varepsilon_r$ and $0 < \delta < \delta_r$. Note that because of the previous assumptions we have $r < R$, $\varepsilon < e_0$, and $\delta < d_0$. Let Ω_r be a closed set with the properties described in definition 6.3.

Lemma 6.6. *(reduction principle for ω-limit sets). $\omega(\Omega_r)$ under $\phi_{(6.1)}$ is equal to $\omega(\mathcal{C}_{\varepsilon\delta} \cap \Omega_r)$ under the same flow.*

Proof. Note that since Ω_r is positively invariant and closed, we have that $\omega(\Omega_r) \subseteq \Omega_r$. We will show that $\mathcal{C}_{\varepsilon\delta} \cap \Omega_r$ attracts Ω_r. Change the coordinate y into

$$w = y - h(x, \varepsilon, \delta) \qquad (6.7)$$

to transform system (6.1) into

$$\dot{x} = f_0(x, \varepsilon, \delta) + f_1(x, w, \varepsilon, \delta) \qquad (6.8)$$
$$\dot{w} = Hw + g_1(x, w, \varepsilon, \delta)\ , \qquad (6.9)$$

in which

$$
\begin{aligned}
f_0(x,\varepsilon,\delta) &= Ax + f(x,h(\xi,\varepsilon,\delta),\varepsilon,\delta) \\
f_1(x,w,\varepsilon,\delta) &= f(x,w+h(x,\varepsilon,\delta),\varepsilon,\delta) - f(x,h(x,\varepsilon,\delta),\varepsilon,\delta)\ ,
\end{aligned} \tag{6.10}
$$

and

$$
\begin{aligned}
g_1(x,w,\varepsilon,\delta) &= g(x,w+h(x,\varepsilon,\delta),\varepsilon,\delta) - g(x,h(x,\varepsilon,\delta),\varepsilon,\delta) \\
&\quad - \frac{\partial h}{\partial x}(x,\varepsilon,\delta)[f(x,w+h(x,\varepsilon,\delta),\varepsilon,\delta) - f(x,h(x,\varepsilon,\delta),\varepsilon,\delta)]\ .
\end{aligned}
$$

For all $(x,w) \in \Omega_r$ we have

$$
\|g_1(x,w,\varepsilon,\delta)\| \leq \frac{1}{4\,\|P\|}\|w\|\ .
$$

Note also that the quadratic form $Q(w) = w^{\mathrm{T}}Pw$ satisfies

$$
\dot{Q}_{(6.9)}(w) \leq -\|w\|^2 + 2\|w\|\cdot\|P\|\cdot\|g_1(x,w,\varepsilon)\| \leq -\frac{1}{2}\|w\|^2 \tag{6.11}
$$

for all $(x,w) \in \Omega_r$. From (6.11) it can be concluded that there exist $\alpha > 0$ and $M > 0$ such that

$$
\|\phi^w_{(6.8)(6.9)}(t,(x(0),w(0)))\| \leq Me^{-\alpha t} \tag{6.12}
$$

for all $t \geq 0$ and all initial conditions $(x(0),w(0)) \in \Omega_r$; this shows that $\mathcal{C}_{\varepsilon\delta} \cap \Omega_r$ attracts Ω_r. Moreover, this implies that all points of $\omega(\Omega_r)$ under $\phi_{(6.8)(6.9)}$ have the w component equal to zero. In fact, suppose by contradiction that $(\hat{x},\hat{w}) \in \omega(\Omega_r)$ with $\|\hat{w}\| = \gamma > 0$. Then there exist sequences $\hat{t}_k \to \infty$ as $k \to \infty$ and $(\hat{x}_k,\hat{w}_k) \in \Omega_r$, such that $\phi^x_{(6.8)(6.9)}(\hat{t}_k,(\hat{x}_k,\hat{w}_k)) \to \hat{x}$ and

$$
\phi^w_{(6.8)(6.9)}(\hat{t}_k,(\hat{x}_k,\hat{w}_k)) \to \hat{w} \tag{6.13}
$$

as $k \to \infty$. However, from (6.12), we conclude that there exists $\bar{t} > 0$ such that

$$
\|\phi^w_{(6.8)(6.9)}(t,(x,w))\| < \gamma/2
$$

for all $t > \bar{t}$ and all $(x,w) \in \Omega_r$. This clearly contradicts (6.13) since $\|\hat{w}\| = \gamma$. As a result, $\omega(\Omega_r) \subseteq \Omega_r \cap \mathcal{C}_{\varepsilon\delta}$. Now take $z \in \omega(\Omega_r)$ and $t_k \to \infty$ as $k \to \infty$. Let $u_k = \phi_{(6.1)}(-t_k,z)$. Note that $u_k \in \omega(\Omega_r) \subseteq \Omega_r \cap \mathcal{C}_{\varepsilon\delta}$ since $\omega(\Omega_r)$ is invariant. Then, $z \in \omega(\Omega_r \cap \mathcal{C}_{\varepsilon\delta})$ since $\phi_{(6.1)}(t_k,u_k) = z$ for all k's. $\square$

Remark 6.14. If $z^* \in B_R^n$ is an equilibrium of (6.1) then $z^* \in \mathcal{C}_{\varepsilon\delta} \cap \Omega_r$.

Remark 6.15. The restriction of (6.1) to $\mathcal{C}_{\varepsilon\delta}$ is described by the *reduced system*

$$\dot{\xi} = f_0(\xi, \varepsilon, \delta) \tag{6.16}$$

defined for $\|\xi\| < r_0$, where f_0 is defined in (6.10). Then, lemma 6.6 implies what follows. Let $\mathcal{B} = \{\xi : (\xi, h(\xi, \varepsilon, \delta)) \in \Omega_r\} \subseteq B_{r_0}^m$ and let $\mathcal{A}_0$ denote the ω-limit set of $\mathcal{B}$ under $\phi_{(6.16)}$; then, $\omega(\Omega_r)$ under $\phi_{(6.1)}$ is equal to $\{(x, y) : x \in \mathcal{A}_0, y = h(x, \varepsilon, \delta)\}$.

7 Memory-less Partial State Feedback

The closed-loop system consisting of (5.3) and (5.5) can be seen as a system of the form

$$\begin{aligned} \dot{\tilde{z}}_1 &= S\tilde{z}_1 + \tilde{g}_1(\tilde{z}_1, \tilde{z}_2) + \tilde{p}_1(\tilde{z}_1, \tilde{z}_2, \zeta) \\ \dot{\tilde{z}}_2 &= \tilde{H}\tilde{z}_2 + \tilde{g}_2(\tilde{z}_1, \tilde{z}_2) + \tilde{p}_2(\tilde{z}_1, \tilde{z}_2, \zeta) \\ \dot{\zeta} &= \tilde{q}(\tilde{z}_1, \tilde{z}_2, \zeta) + \tilde{b}(\tilde{z}_1, \tilde{z}_2, \zeta)u \\ \\ y &= \zeta \; , \end{aligned} \tag{7.1}$$

controlled by

$$u = -\bar{k}y \; . \tag{7.2}$$

In fact, if (5.5) has relative degree $d = 1$, then it is already in the form (7.1), and it is semiglobally practically stabilized by the (output) feedback (7.2). If $d \geq 2$ and we apply the partial state feedback (5.3), then, using a linear change of coordinates, the resulting closed-loop system can be put in the form (7.1) (7.2) as follows. Change the coordinate ξ_d into

$$\zeta = k^{d-1}a_0\xi_1 + k^{d-2}a_1\xi_2 + \ldots + ka_{d-2}\xi_{d-1} + \xi_d \; , \tag{7.3}$$

and set

$$\tilde{z}_2 = \mathrm{col}(z_2, \xi_1, \cdots, \xi_{d-1}) \; . \tag{7.4}$$

The resulting system is the feedback interconnection of (7.2) and of a system of the type

$$\begin{aligned} \dot{z}_1 &= Sz_1 + \hat{g}_1(z_1, \tilde{z}_2) + \hat{p}_1(z_1, \tilde{z}_2) \\ \dot{\tilde{z}}_2 &= \tilde{H}\tilde{z}_2 + \hat{g}_2(z_1, \tilde{z}_2) + \hat{p}_2(z_1, \tilde{z}_2, \zeta) \\ \dot{\zeta} &= \hat{q}(z_1, \tilde{z}_2, \zeta) + \hat{b}(z_1, \tilde{z}_2, \zeta)u \\ \\ y &= \zeta \; . \end{aligned} \tag{7.5}$$

If $\tilde{P} = \dfrac{\partial \hat{p}_1}{\partial \tilde{z}_2}(0,0) \neq 0$, change again the coordinate z_1 into

$$\tilde{z}_1 = z_1 + X\tilde{z}_2 \tag{7.6}$$

where X is the unique solution of the Sylvester equation $\tilde{P} + X\tilde{H} = SX$, to obtain a system of the type (7.1) with $\tilde{p}_i$ $i = 1, 2$ independent of $(\tilde{z}_1, \tilde{z}_2)$, and linear in ζ.

Let $\mathcal{A} = \omega(B_r^{n+d})$ under $\phi_{(7.1)(7.2)}$; then, from theorem 4.6 it follows that $\mathcal{A}$ is a nonempty, compact, invariant, connected, stable set that attracts B_R^{n+d}; $\mathcal{A}$ is contained in B_r^{n+d}, and it is the maximal invariant set contained in B_R^{n+d}. In this section we will characterize the structure of $\mathcal{A}$ in two cases that correspond to zero dynamics of (5.5) having a critically asymptotically stable equilibrium at the origin that is critical in the mildest way.

Case I. *The linearization of the zero dynamics at the origin has a simple zero eigenvalue and all remaining spectrum is in* $\mathbb{C}^-$. *Moreover, the (1-dimensional) restriction of the zero dynamics to its center manifold at the origin can be expanded as*

$$\dot{x} = ax^3 + \mathcal{O}(x^4)\ , \qquad a < 0\ .$$

Case II. *The linearization of the zero dynamics at the origin has two purely imaginary eigenvalues and all remaining spectrum is in* $\mathbb{C}^-$. *Moreover, the normal form (see [12, section 2.2] and [5, section 3.3]) of the (2-dimensional) restriction of the zero dynamics to its center manifold at the origin can be expanded, in polar coordinates, as*

$$\begin{aligned} \dot{r} &= ar^3 + \mathcal{O}(r^5)\ , \qquad a < 0 \\ \dot{\theta} &= \omega + br^2 + \mathcal{O}(r^4)\ . \end{aligned} \tag{7.7}$$

For both Case I and Case II, let

$$P = \frac{\partial \tilde{p}_1}{\partial \zeta}(0,0,0), \quad Q = \frac{\partial \tilde{q}}{\partial z_1}(0,0,0)\ . \tag{7.8}$$

Only for Case I, let

$$
\begin{aligned}
N &= \tilde{H}^{-1}\frac{\partial \tilde{p}_2}{\partial \zeta}(0,0,0)\ ,\\
\alpha &= P\left(\frac{\partial \tilde{q}}{\partial \tilde{z}_2}(0,0,0)\tilde{H}^{-1}\frac{\partial^2 \tilde{g}_2}{\partial \tilde{z}_1^2}(0,0) - \frac{\partial^2 \tilde{q}}{\partial \tilde{z}_1^2}(0,0,0)\right)\ ,\\
\beta &= Q\left(N^{\mathrm{T}}\frac{\partial^2 \tilde{g}_1}{\partial \tilde{z}_2^2}(0,0)N - 2\frac{\partial^2 \tilde{p}_1}{\partial \tilde{z}_2 \partial \zeta}(0,0,0)N + \frac{\partial^2 \tilde{p}_1}{\partial^2 \zeta^2}(0,0,0)\right)\ ,\\
\gamma &= Q\left(\frac{\partial^2 \tilde{g}_1}{\partial \tilde{z}_1 \partial \tilde{z}_2}(0,0)N - \frac{\partial^2 \tilde{p}_1}{\partial \tilde{z}_1 \partial \zeta}(0,0,0)\right)\ .
\end{aligned}
\tag{7.9}
$$

For $\bar{k}$ *large enough*, the following theorems, characterizing the structure of $\mathcal{A}$, hold.

Theorem 7.10. *Consider Case I;*

(i) if $QP < 0$
$\mathcal{A}$ consists of the origin that is an exponentially *stable equilibrium.*

(ii) if $QP > 0$
$\mathcal{A}$ is a 1-manifold with boundary; the boundary consists of two exponentially stable equilibria; the (relative) interior of $\mathcal{A}$ consists of the origin, which is an unstable hyperbolic equilibrium, and its 1-dimensional unstable manifold.

(iii) if ($Q = 0$ and $\alpha = 0$) or ($P = 0$ and $\beta = 0$ and $\gamma = 0$)
$\mathcal{A}$ consists of the origin that is a critically asymptotically *stable equilibrium.*

(iv) if ($Q = 0$ and $\alpha \neq 0$) or ($P = 0$ and ($\beta \neq 0$ or $\gamma \neq 0$))
$\mathcal{A}$ is a 1-manifold with boundary; the boundary consists of an exponentially stable equilibrium and of the origin, which is a unstable non-hyperbolic equilibrium; the (relative) interior of $\mathcal{A}$ consists of the unique heteroclinic orbit that connects the origin to the exponentially stable equilibrium.

Proof. Here we give only a sketch of the proof; more details can be found in [4].

In (7.1) (7.2) set $\bar{k} = 1/\varepsilon$, and scale the time t to $s = t/\varepsilon$. The origin of the resulting system is practically stabilizable in the parameter ε (there is no parameter δ in this case) with radial upper bound R; then, we can use the reduction principle for ω-limit sets of lemma 6.6. Denoting by " $''$ " the derivative with respect to s, add the extra equation $\varepsilon'' = 0$. The system thus obtained possesses an $(n+1)$-dimensional center manifold at $(\tilde{z}_1, \tilde{z}_2, \zeta, \varepsilon) = 0$, the graph of a C^k mapping $\zeta = \mu(\tilde{z}_1, \tilde{z}_2, \varepsilon)$ that is defined for all $\|z_1\| < \alpha$, $\|z_2\| < \alpha$, $|\varepsilon| < \alpha$.

The resulting reduced system after scaling back the time variable from s to t is

$$\begin{aligned} \dot{\tilde{z}}_1 &= S\tilde{z}_1 + \tilde{g}_1(\tilde{z}_1, \tilde{z}_2) + \tilde{p}_1(\tilde{z}_1, \tilde{z}_2, \mu(\tilde{z}_1, \tilde{z}_2, \varepsilon)) \\ \dot{\tilde{z}}_2 &= \tilde{H}\tilde{z}_2 + \tilde{g}_2(\tilde{z}_1, \tilde{z}_2) + \tilde{p}_2(\tilde{z}_1, \tilde{z}_2, \mu(\tilde{z}_1, \tilde{z}_2, \varepsilon)) \; . \end{aligned} \tag{7.11}$$

By assumption the zero dynamics of (7.1) have a locally asymptotically stable equilibrium at $(\tilde{z}_1, \tilde{z}_2) = (0, 0)$. Let $\mathcal{D} \supset B^n_R$ be its region of attraction; then, there exists $\hat{V} : \mathcal{D} \to \mathbb{R}$ smooth, positive definite, and proper such that $\dot{\hat{V}}_{(7.1)} < 0$ for all $(\tilde{z}_1, \tilde{z}_2) \in \mathcal{D}$, $(\tilde{z}_1, \tilde{z}_2) \neq (0, 0)$. Let $g > 0$ such that

$$int(\hat{\Omega}_g) = \{(\tilde{z}_1, \tilde{z}_2) : \hat{V}(\tilde{z}_1, \tilde{z}_2) < g\} \subseteq \mathcal{D} \cap \{(\tilde{z}_1, \tilde{z}_2) : \|\tilde{z}_1\| < \alpha, \; \|\tilde{z}_2\| < \alpha\} \; ,$$

and let $0 < \check{R} < R$ such that $B^n_{\check{R}} \subseteq \hat{\Omega}_g$. Again, it can be proved that the origin of (7.11) is practically stabilizable in the parameter ε with radial upper bound $\check{R}$. Thus, we can use once more the reduction principle for ω-limit sets. System (7.11) augmented with $\dot{\varepsilon} = 0$ possesses a $n + d$-dimensional center manifold at $(\tilde{z}_1, \tilde{z}_2, \varepsilon) = 0$, the graph of a C^k mapping $\tilde{z}_2 = \nu(\tilde{z}_1, \varepsilon)$. The reduced system we obtain is

$$\dot{\tilde{z}}_1 = S\tilde{z}_1 + \tilde{g}_1(\tilde{z}_1, \nu(\tilde{z}_1, \varepsilon)) + \tilde{p}_1(\tilde{z}_1, \nu(\tilde{z}_1, \varepsilon), \mu(\tilde{z}_1, \nu(\tilde{z}_1, \varepsilon), \varepsilon)) \; . \tag{7.12}$$

In summary, two applications of the reduction principle have reduced our problem to determine the ω-limit set of a certain connected closed subset of $\mathbb{R}$ containing the origin, under the flow $\phi_{(7.12)}$.

We call (7.12) the *perturbed reduced zero-dynamics*. In fact, (7.12) can be interpreted as a perturbation of the 1-dimensional system that represents the restriction of the zero-dynamics of (7.1) to the graph of $\tilde{z}_2 = \nu(\tilde{z}_1, 0)$, and such graph is a center manifold at the origin of the zero-dynamics of (7.1). In (7.12), the positive parameter ε can be seen as a perturbation parameter, and the structure of the ω-limit set of (7.12) can be determined performing a bifurcation analysis with respect to the parameter ε. Such analysis is feasible since (7.12) is 1-dimensional. □

Theorem 7.13. *Consider Case II;*

(i) if $QP < 0$
$\mathcal{A}$ *consists of the origin that is an* exponentially *stable equilibrium.*

(ii) if $QP > 0$
$\mathcal{A}$ *is a 2-manifold with boundary; the boundary consists of a periodic orbit; the (relative) interior of* $\mathcal{A}$ *consists of the origin, which is an unstable hyperbolic equilibrium, and its 2-dimensional unstable manifold.*

(iii) if $Q = 0$ *or* $P = 0$
$\mathcal{A}$ *consists of the origin that is a* critically asymptotically *stable equilibrium.*

(iv) if ($QP = 0$ *and* $Q \neq 0$ *and* $P \neq 0$*)*
$\mathcal{A}$ *is either the origin, an* exponentially *stable equilibrium, or a 2-manifold with boundary. In the latter case, the boundary of* $\mathcal{A}$ *consists of a periodic orbit; the (relative) interior of* $\mathcal{A}$ *consists of the origin, which is an unstable hyperbolic equilibrium, and its 2-dimensional unstable manifold.*

Proof. The proof is identical to that of theorem 7.10 in the part related to the reductions. However, the two proofs differ in the bifurcation analysis of (7.12) since the assumptions on the reduced zero-dynamics of (7.1) are different in the two cases. More details can be found in [4]. □

8 Dynamic Output Feedback

Here we consider the case in which in (5.5) $d \geq 2$ and the output feedback (5.4) has been applied. Having assigned $0 < r < R$, the controller designed according to the method presented in [11] is such that there exist $\mathcal{O} \subseteq B_r^{n+2d}$ and $T' > 0$ with the following properties: $\mathcal{O}$ is open and positively invariant under the flow $\phi_{(5.5)(5.4)}$, and $\phi_{(5.5)(5.4)}(t, B_R^{n+2d}) \subseteq \mathcal{O}$ for all $t \geq T'$. We have already shown that $\omega(B_R^{n+2d}) = \omega(B_r^{n+2d})$; then, by theorem 4.6 $\omega(B_r^{n+2d}) = \omega(\mathcal{O})$. Consequently, to characterize $\omega(B_r^{n+2d})$, it is enough to consider the restriction of (5.5) (5.4) to $\mathcal{O}$. It turns out that if g is large enough, then

$$|\bar{k}(k^{d-1}a_0\eta_1 + k^{d-2}a_1\eta_2 + \cdots + ka_{d-2}\eta_{d-1} + \eta_d| \leq l \quad \text{for all } (z, \xi, \eta) \in \mathcal{O} \,.$$

As a result, when we consider the restriction of (5.5) (5.4) to $\mathcal{O}$, in (5.4) we can replace σ_l with the identity function . Let the coordinate ξ_d be changed into ζ defined in (7.3), and $\eta_1, \eta_2, \ldots, \eta_d$ into

$$e_i = g^{d-i}(\xi_i - \eta_i) \qquad i = 1, \ldots, d \,. \tag{8.1}$$

Setting

$$K = \begin{pmatrix} k^{d-1}a_0 & k^{d-2}a_1 & \dots & ka_{d-2} & 1 \end{pmatrix} ,$$

and

$$D_g = \mathrm{diag}\{g^{d-1}, \dots, g, 1\} ,$$

equations (7.3) and (8.1) can be expressed in a more compact notation as

$$\begin{array}{rcl} \zeta & = & K\xi \\ e & = & D_g(\xi - \eta) \ . \end{array} \tag{8.2}$$

In addition, set $\bar{k} = 1/\varepsilon$ and $g = 1/(\varepsilon\delta)$ where ε and δ are positive numbers, and scale the time t to $\tau = t/(\varepsilon\delta)$; then, denoting by " $'$ " the derivative with respect to τ the system we are interested in becomes

$$\begin{pmatrix} z_1' \\ z_2' \\ \xi_1' \\ \vdots \\ \xi_{r-2}' \\ \xi_{r-1}' \end{pmatrix} = \varepsilon\delta \begin{pmatrix} Sz_1 + g_1(z_1, z_2) + p_1(z_1, z_2, \xi_1) \\ Hz_2 + g_2(z_1, z_2) + p_2(z_1, z_2, \xi_1) \\ \xi_2 \\ \vdots \\ \xi_{r-1} \\ -k^{r-1}a_0\xi_1 - k^{r-2}a_1\xi_2 - \cdots - ka_{r-2}\xi_{r-1} + \zeta \end{pmatrix}$$

$$\begin{array}{rcl} \zeta' & = & \varepsilon\delta\bar{q}(z_1, z_2, \xi_1, \cdots, \xi_{r-1}, \zeta) - \delta\bar{b}(z_1, z_2, \xi_1, \cdots, \xi_{r-1}, \zeta)(\zeta - KD_{\varepsilon\delta}e) \\ e' & = & Ae + \delta B[\varepsilon\hat{q}(z_1, z_2, \xi_1, \cdots, \xi_{r-1}, \zeta) - \bar{b}(z_1, z_2, \xi_1, \cdots, \xi_{r-1}, \zeta)(\zeta - KD_{\varepsilon\delta}e)] \ ; \end{array} \tag{8.3}$$

in which $\bar{q}, \bar{b}, \hat{q}$ are appropriate functions and

$$A = \begin{pmatrix} -c_{d-1} & 1 & 0 & \cdots & 0 \\ -c_{d-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_1 & 0 & 0 & \cdots & 1 \\ -c_0 & 0 & 0 & \cdots & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} .$$

Our goal is to characterize the structure of $\omega(\mathcal{O})$ under $\phi_{(8.3)}$.

An important property of (8.3) that can be easily checked is as follows

Property 8.4. $(z_1^0, z_2^0, \xi_1^0, \cdots, \xi_{d-1}^0, \zeta^0, e^0)$ *is an equilibrium of (8.3) if and only if* $e^0 = 0$ *and* $(z_1^0, z_2^0, \xi_1^0, \cdots, \xi_{d-1}^0, \zeta^0)$ *is an equilibrium of (5.5) (5.3) with* $\bar{k} = 1/\varepsilon$.

8.1 Practical Stabilizability of (8.3)

In order to apply the reduction principle of section 6, we need to show first that if the coefficients a_i's, c_j's, and k are fixed as in the design method of [11], then the origin of (8.3) is practically stabilizable in the parameters (ε,δ) with radial upper bound R according to definition 6.3. Note that this is not a direct consequence of the fact that the origin of (5.5) is semiglobally practically stabilized by the feedback (5.4). In fact, (8.3) is obtained from (5.5) (5.4) changing coordinates, scaling time, and replacing in (5.4) the saturation function σ_l with the identity function; the latter replacement might render (5.4) unable to semiglobally practically stabilize the origin of (5.5) because some of the trajectories of the resulting closed-loop system that start inside B_R^{n+2d} might have a finite escape time.

Rewrite (8.3) as

$$\begin{array}{rcl} x' & = & \varepsilon\delta(F(x) + GHw) \\ w' & = & \delta(\varepsilon Q(x,w,\varepsilon,\delta) + P(x,w,\delta)w) \end{array} \tag{8.5}$$

where

$$x = \mathrm{col}(z_1, z_2, \xi_1, \ldots, \xi_{d-1}) \,, \qquad w = \mathrm{col}(\zeta, e_1, \ldots, e_d) \,,$$

$$F(x) = \begin{pmatrix} Sz_1 + g_1(z_1,z_2) + p_1(z_1,z_2,\xi_1) \\ Hz_2 + g_2(z_1,z_2) + p_2(z_1,z_2,\xi_1) \\ \xi_2 \\ \vdots \\ \xi_{d-1} \\ -k^{d-1}a_0\xi_1 - k^{d-2}a_1\xi_2 - \ldots - ka_{d-2}\xi_{d-1} \end{pmatrix} , \quad G = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} ,$$

$$H = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} ,$$

$$Q(x,w,\varepsilon,\delta) =$$
$$= \begin{pmatrix} \bar{q}(z_1,z_2,\xi_1,\cdots,\xi_{d-1},\zeta) + \\ +\bar{b}(z_1,z_2,\xi_1,\cdots,\xi_{d-1},\zeta)(k^{d-1}a_0\varepsilon^{d-2}\delta^{d-1}e_1 + \cdots + ka_{d-2}\delta e_{d-1}) \\ 0 \\ \vdots \\ 0 \\ \hat{q}(z_1,z_2,\xi_1,\cdots,\xi_{d-1},\zeta) + \\ +\bar{b}(z_1,z_2,\xi_1,\cdots,\xi_{d-1},\zeta)(k^{d-1}a_0\varepsilon^{d-2}\delta^{d-1}e_1 + \cdots + ka_{d-2}\delta e_{d-1}) \end{pmatrix} ,$$

and

$$P(x, w, \delta) =$$

$$\begin{pmatrix} -\bar{b}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta) & 0 & 0 & \cdots & 0 & \bar{b}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta) \\ 0 & -\frac{1}{\delta}c_{d-1} & \frac{1}{\delta} & \cdots & 0 & 0 \\ 0 & -\frac{1}{\delta}c_{d-2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & -\frac{1}{\delta}c_1 & 0 & \cdots & 0 & \frac{1}{\delta} \\ -\bar{b}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta) & -\frac{1}{\delta}c_0 & 0 & \cdots & 0 & \bar{b}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta) \end{pmatrix} . \tag{8.6}$$

The practical stabilizability of the origin of (8.5) in the parameters (ε,δ) with radial upper bound R can be proved by means of an appropriate adaptation of the classical argument in [1, section 3]. In fact, following the design method of [11] for the output feedback (5.4), we have that $\dot{x} = F(x)$ has an asymptotically stable equilibrium at the origin and B_R^{n+d-1} is contained in its region of attraction $\mathcal{D}$. Then there exists $V : \mathcal{D} \to \mathbb{R}$ smooth, positive definite, and proper such that $DV(x)F(x) < 0$, for all $x \in \mathcal{D}$, $x \neq 0$. Let $\bar{U}$ be the solution of the Lyapunov equation $\bar{U}A + A^T\bar{U} = -I$, and define $U = \begin{pmatrix} 1 & 0 \\ 0 & \bar{U} \end{pmatrix}$; let $W : \mathcal{D} \times \mathbb{R}^{d+1} \to \mathbb{R}$ defined by $W(x, w) = V(x) + w^TUw$; note that W is smooth, positive definite, and proper. Let $\Omega_\beta = \{(x, w) : W(x, w) \leq \beta\}$; given r such that $0 < r < R$, let $\rho > 0$ such that $\Omega_\rho \subset B_r^{n+2d}$, and let $c > 0$ such that $B_R^{n+2d} \subset \Omega_c$. Define $S = \{(x, w) : \rho \leq W(x, w) \leq c\}$, and $b_0 = \min_{(x,w)\in S} \bar{b}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta) > 0$. Fix $\gamma \in (0, 2b_0)$, then we have

Lemma 8.7. *There exists $\delta^* > 0$ such that*

$$UP(x, w, \delta) + P(x, w, \delta)^TU < -\gamma I \qquad \text{for all } (x, w) \in S,\ \delta \in (0, \delta^*) . \tag{8.8}$$

Proof. See [4]. ∎

Let $W'_{(8.5)}$ denote the quantity in the time scale τ corresponding to (2.3), we have that for all $\delta \in (0, \delta^*)$ and $(x, w) \in S$

$$W'_{(8.5)}(x,w) = \delta \left[\varepsilon \left(\frac{\partial V}{\partial x}(x)F(x) + \frac{\partial V}{\partial x}(x)GHw + 2w^T UQ(x,w,\varepsilon,\delta) \right) \right.$$

$$\left. + w^T[UP(x,w,\delta) + P(x,w,\delta)^T U]w \right]$$

$$\leq \delta \left[\varepsilon \left(\frac{\partial V}{\partial x}(x)F(x) + \frac{\partial V}{\partial x}(x)GHw + 2\|w\|\|U\|\|Q(x,w,\varepsilon,\delta)\| \right) - \gamma \|w\|^2 \right] .$$

Note that, without loss of generality, we can assume that $\varepsilon \leq 1$. Let $S' \supset S$ be a bounded open set, and $N : S' \to \mathbb{R}$ be a continuous function such that

$$N(x,w) \geq \max_{\substack{\varepsilon \in [0,1] \\ \delta \in [0,\delta^*]}} \|Q(x,w,\varepsilon,\delta)\| \qquad \text{for all } (x,w) \in S .$$

Then, for all $\delta \in (0, \delta^*)$, $\varepsilon \in (0,1)$, and $(x, w) \in S$ we have

$$W'_{(8.5)}(x,w) \leq \delta \left[\varepsilon \left(\frac{\partial V}{\partial x}(x)F(x) + \frac{\partial V}{\partial x}(x)GHw + 2\|w\|\|U\|N(x,w) \right) - \gamma \|w\|^2 \right] .$$

Let $S_0 = S \bigcap \{(x,w) : w = 0\}$, and note that

$$\frac{\partial V}{\partial x}(x)F(x) + \frac{\partial V}{\partial x}(x)GHw + 2\|w\|\|U\|N(x,w) < 0 \qquad \text{for all } (x,w) \in S_0 \; ; \tag{8.9}$$

then, by continuity, the left hand side of (8.9) is negative at each point of some open set $T \supset S_0$. Consider now the compact set $\tilde{S} = S \setminus T$; since $w \neq 0$ at each point of $\tilde{S}$, we have $m = \min_{(x,w)\in\tilde{S}} \|w\|^2 > 0$. Let

$$M = \max_{(x,w)\in\tilde{S}} \left| \frac{\partial V}{\partial x}(x)F(x) + \frac{\partial V}{\partial x}(x)GHw + 2\|w\|\|U\|N(x,w) \right|$$

and set $\varepsilon_r = \begin{cases} \frac{\gamma m}{M} & \text{if } M \neq 0 \\ 1 & \text{if } M = 0 \end{cases}$; then, for all $\delta \in (0, \delta^*)$, $\varepsilon \in (0, \varepsilon_r)$, and $(x, w) \in S$ we have $W'_{(8.5)}(x, w) < 0$; as a result, for all such ε and δ, $\Omega_\rho \subseteq B_r^{n+2d}$ is positively invariant, and there exists $T_r > 0$ such that $\phi_{(8.5)}(\tau, B_R^{n+2d}) \subseteq \Omega_\rho$ for all $\tau \geq T_r$.

Remark 8.10. Note that, even though the origin of a system such as (8.3) is practically stabilizable in the parameters $(\varepsilon,\ \delta)$ with radial upper bound R as just shown, this *does not mean* at all that an output feedback law of the form (5.4) *without saturation* (i.e. with $\sigma_l(\cdot)$ replaced by the identity function) semiglobally practically stabilizes system (5.5). The key difference is that if the initial conditions for (5.5) (5.4) are ranging over a fixed bounded set, the corresponding initial conditions for (8.3) range over a set which grows unbounded as $g = 1/(\varepsilon\delta)$ increases (because so does $e(0)$). On the contrary, in the analysis of practical stabilizability in the parameters $(\varepsilon,\ \delta)$ of the origin of system (8.3), $e(0)$ by definition is supposed to range over a fixed bounded set.

8.2 First Reduction

Since the origin of (8.3) is practically stabilizable in the parameters $(\varepsilon,\ \delta)$ with radial upper bound R as just shown, we can apply the reduction principle of section 6. Augmenting (8.3) with the extra equations $\varepsilon' = 0$, $\delta' = 0$, we obtain

$$\varepsilon' = 0 \quad \delta' = 0 \tag{8.11}$$

$$\begin{array}{rcl} x' & = & \varepsilon\delta(F(x) + G\zeta) \\ \zeta' & = & \delta[\varepsilon\bar{q}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta) - \bar{b}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta)(\zeta - KD_{\varepsilon\delta}e)] \\ e' & = & Ae + \delta B[\varepsilon\hat{q}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta) \\ & & -\delta\bar{b}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta)(\zeta - KD_{\varepsilon\delta}e)]\ . \end{array} \tag{8.12}$$

The Jacobian matrix of system (8.11) (8.12) at the origin is $\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$; hence, (8.11) (8.12) possesses a $n + d + 2$-dimensional center manifold given by the graph of a C^k mapping, with k arbitrarily large, $e = \theta(x, \zeta, \varepsilon, \delta)$, defined for all $\|x\| < \rho'$, $|\zeta| < \rho'$, $|\varepsilon| < \rho'$, and $|\delta| < \rho'$, satisfying $\theta(0, 0, 0, 0) = 0$, $D\theta(0, 0, 0, 0) = 0$, and

$$\begin{array}{l} \dfrac{\partial\theta}{\partial x}(x, \zeta, \varepsilon, \delta)\varepsilon\delta(F(x) + G\zeta) + \dfrac{\partial\theta}{\partial\zeta}(x, \zeta, \varepsilon, \delta)\delta\{\varepsilon\bar{q}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta) \\ -\bar{b}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta)[\zeta - KD_{\varepsilon\delta}\theta(x, \zeta, \varepsilon, \delta)]\} = \\ = A\theta(x, \zeta, \varepsilon, \delta) + \\ \delta B[\varepsilon\hat{q}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta) - \bar{b}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta)(\zeta - KD_{\varepsilon\delta}\theta(x, \zeta, \varepsilon, \delta))]\ . \end{array} \tag{8.13}$$

Plugging $\delta = 0$ in (8.13) we obtain

$$\theta(x, \zeta, \varepsilon, 0) = 0 \; ; \tag{8.14}$$

moreover, since for any ε and δ the point $(x, \zeta, e, \varepsilon, \delta) = (0, 0, 0, \varepsilon, \delta)$ is an equilibrium of (8.11) (8.12), we can conclude that

$$\theta(0, 0, \varepsilon, \delta) = 0 \; . \tag{8.15}$$

Plugging $\zeta = 0$ and $\varepsilon = 0$ in (8.13) we obtain that

$$\theta(x, 0, 0, \delta) = 0 \; . \tag{8.16}$$

The resulting reduced system is

$$\begin{array}{rcl} x' &=& \varepsilon\delta(F(x) + G\zeta) \\ \zeta' &=& \delta[\varepsilon\bar{q}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta) \\ && -\bar{b}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta)(\zeta - KD_{\varepsilon\delta}\theta(x, \zeta, \varepsilon, \delta))] \; . \end{array} \tag{8.17}$$

Note that

$$\begin{array}{l} KD_{\varepsilon\delta}\theta(x, \zeta, \varepsilon, \delta) = (k\varepsilon\delta)^{d-1}a_0\theta_1(x, \zeta, \varepsilon, \delta) + \ldots \\ \qquad\qquad + (k\varepsilon\delta)a_{d-2}\theta_{d-1}(x, \zeta, \varepsilon, \delta) + \theta_d(x, \zeta, \varepsilon, \delta) \; ; \end{array}$$

from (8.14) and (8.16) we can write θ_d as

$$\theta_d(x, \zeta, \varepsilon, \delta) = \delta[\varepsilon\alpha(x, \zeta, \varepsilon, \delta) + \zeta\beta(x, \zeta, \varepsilon, \delta)] \; . \tag{8.18}$$

Using (8.15), we obtain that

$$\alpha(0, 0, \varepsilon, \delta) = 0 \; . \tag{8.19}$$

Then, scaling time back to t, the reduced system (8.17) becomes

$$\begin{array}{rcl} \dot{x} &=& F(x) + G\zeta \\ \dot{\zeta} &=& \bar{\bar{q}}(x, \zeta, \varepsilon, \delta) - \dfrac{1}{\varepsilon}\bar{\bar{b}}(x, \zeta, \varepsilon, \delta)\zeta \; , \end{array} \tag{8.20}$$

where

$$\begin{array}{rcl} \bar{\bar{q}}(x, \zeta, \varepsilon, \delta) &=& \bar{q}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta) + \\ && +\delta[\bar{b}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta)(k^{d-1}(\varepsilon\delta)^{d-2}a_0\theta_1(x, \zeta, \varepsilon, \delta) + \ldots + \\ && +ka_{d-2}\theta_{d-1}(x, \zeta, \varepsilon, \delta) + \alpha(x, \zeta, \varepsilon, \delta))] \; , \\ \\ \bar{\bar{b}}(x, \zeta, \varepsilon, \delta) &=& \bar{b}(z_1, z_2, \xi_1, \cdots, \xi_{d-1}, \zeta) - \delta\beta(x, \zeta, \varepsilon, \delta) \; , \end{array}$$

and both are defined for all $\|x\| < \rho'$, $|\zeta| < \rho'$, $|\varepsilon| < \rho'$, and $|\delta| < \rho'$.

8.3 A Relative Degree One System Controlled by Memory-less Output Feedback

Note that $\bar{\bar{q}}(0,0,\varepsilon,\delta) = 0$ and $\bar{\bar{b}}(0,0,0,0) = b(0,0,\cdots,0) > 0$; then, there exists $0 < \rho'' < \rho'$ such that $\bar{\bar{b}}(x,\zeta,\varepsilon,\delta) > 0$ for all $\|x\| < \rho''$, $|\zeta| < \rho''$, $|\varepsilon| < \rho''$, $|\delta| < \rho''$. Consequently, for each $|\varepsilon| < \rho''$ $|\delta| < \rho''$, (8.20) can be interpreted as the relative degree one system

$$\begin{aligned} \dot{x} &= F(x) + G\zeta \\ \dot{\zeta} &= \bar{\bar{q}}(x,\zeta,\varepsilon,\delta) + \bar{\bar{b}}(x,\zeta,\varepsilon,\delta)u \\ \\ y &= \zeta \end{aligned} \tag{8.21}$$

controlled by the "high-gain" output feedback

$$u = -\frac{1}{\varepsilon}y\ . \tag{8.22}$$

Note that the zero-dynamics of (8.21) have a locally asymptotically stable equilibrium at the origin. Performing the change of coordinates (7.6) (7.4), we transform system (8.21) into

$$\begin{aligned} \dot{\tilde{z}}_1 &= S\tilde{z}_1 + \tilde{g}_1(\tilde{z}_1,\tilde{z}_2) + \tilde{P}_1\zeta \\ \dot{\tilde{z}}_2 &= \tilde{H}\tilde{z}_2 + \tilde{g}_2(\tilde{z}_1,\tilde{z}_2) + \tilde{P}_2\zeta \\ \dot{\zeta} &= \check{q}(\tilde{z}_1,\tilde{z}_2,\zeta,\varepsilon,\delta) + \check{b}(\tilde{z}_1,\tilde{z}_2,\zeta,\varepsilon,\delta)u \\ \\ y &= \zeta\ , \end{aligned} \tag{8.23}$$

where $\tilde{g}_i(0,0) = 0$, $D\tilde{g}_i(0,0) = 0$ for $i = 1,2$, $\check{q}(0,0,0,\varepsilon,\delta) = 0$, and there exists ρ such that $\check{b}(\tilde{z}_1,\tilde{z}_2,\zeta,\varepsilon,\delta) > 0$ for all $\|\tilde{z}_1\| < \rho$, $\|\tilde{z}_2\| < \rho$, $|\zeta| < \rho$, $|\varepsilon| < \rho$, and $|\delta| < \rho$.

It is worth to observe that system (8.23) has a structure similar, but not identical, to that of system (7.1). Both systems are controlled by the "high-gain" feedback law (8.22); in system (8.23), though, the parameters ε and δ appear, which do not show up in system (7.1). In principle, the presence of those parameters might suggest the occurrence of more complicated ω-limit sets than those described in the case of system (7.1), but this is not the case, as it will be demonstrated in the sequel.

8.4 Further Reductions

Proceeding similarly to the proof of theorem 7.10 we can apply the reduction principle for ω-limit sets two more times.

In fact, in (8.23) (8.22) scale the time t to $s = t/\varepsilon$. The origin of the resulting system is practically stabilizable in the parameters (ε, δ) with a certain radial upper bound. Denote by $\zeta = \pi(\tilde{z}_1, \tilde{z}_2, \varepsilon, \delta)$ a C^k map whose graph is an attractive invariant manifold that comes from the reduction principle. The resulting reduced system after scaling back the time variable from s to t is

$$\begin{array}{rcl} \dot{\tilde{z}}_1 & = & S\tilde{z}_1 + \tilde{g}_1(\tilde{z}_1, \tilde{z}_2) + \tilde{P}_1\pi(\tilde{z}_1, \tilde{z}_2, \varepsilon, \delta) \\ \dot{\tilde{z}}_2 & = & \tilde{H}\tilde{z}_2 + \tilde{g}_2(\tilde{z}_1, \tilde{z}_2) + \tilde{P}_2\pi(\tilde{z}_1, \tilde{z}_2, \varepsilon, \delta)\ . \end{array} \tag{8.24}$$

The origin of (8.24) is practically stabilizable in the parameters (ε, δ) with a certain radial upper bound. Thus, we can apply once more the reduction principle. Denote by $\tilde{z}_2 = \gamma(\tilde{z}_1, \varepsilon, \delta)$ a C^k map whose graph is an attractive invariant manifold that comes from the reduction principle. The reduced system we obtain is

$$\dot{\tilde{z}}_1 = S\tilde{z}_1 + \tilde{g}_1(\tilde{z}_1, \gamma(\tilde{z}_1, \varepsilon, \delta)) + \tilde{P}_1\pi(\tilde{z}_1, \gamma(\tilde{z}_1, \varepsilon, \delta), \varepsilon, \delta) \tag{8.25}$$

In summary, iterative applications of the reduction principle have simplified our problem to determine the ω-limit set of a certain connected closed subset of $\mathbb{R}^m$, containing the origin, under the flow $\phi_{(8.25)}$. System (8.25) plays, in the present context, a role totally similar to the role played by system (7.12) in the analysis of the attractors of (7.5). As before, system (8.25) can be seen as a *perturbed reduced zero dynamics* and the structure of the ω-limit set can be determined by performing a bifurcation analysis with respect to the pair of parameters (ε, δ). In fact, at $(\varepsilon, \delta) = (0, 0)$, the eigenvalues of the Jacobian matrix of the right-hand side of (8.25) at $\tilde{z}_1 = 0$, are on the imaginary axis.

In the next subsections we will investigate the types of bifurcations that can occur in (8.25). However, we will analyze only systems (8.25) that are obtained when (5.5) satisfies the assumptions of either Case I or Case II of section 7. In those two cases the bifurcation analysis can be carried out generalizing the results in [2] where the functions $\check{q}$ and $\check{b}$ did not depend on (ε, δ).

8.5 Bifurcation Analysis in Case I

Here we present only the results of the analysis; their proofs can be found in [4].

The occurrence of a bifurcation and its nature depend on the values of the following parameters.

$$P = \tilde{P}_1, \quad Q = \frac{\partial q}{\partial z_1}(0,0,0) \ , \tag{8.26}$$

$$\begin{aligned} N &= \tilde{H}^{-1}\frac{\partial \tilde{p}_2}{\partial \zeta}(0,0,0) \ , \\ \alpha &= P\left(\frac{\partial \tilde{q}}{\partial \tilde{z}_2}(0,0,0)\tilde{H}^{-1}\frac{\partial^2 \tilde{g}_2}{\partial \tilde{z}_1^2}(0,0) - \frac{\partial^2 q}{\partial z_1^2}(0,0,0)\right) \ , \\ \beta &= QN^{\mathrm{T}}\frac{\partial^2 \tilde{g}_1}{\partial \tilde{z}_2^2}(0,0)N \ , \\ \gamma &= Q\frac{\partial^2 \tilde{g}_1}{\partial \tilde{z}_1 \partial \tilde{z}_2}(0,0)N \ . \end{aligned} \tag{8.27}$$

If $QP < 0$ there exist $\psi > 0$, $\varepsilon^* > 0$, and $\delta^* > 0$, such that, for all $0 < \varepsilon < \varepsilon^*$ and $|\delta| < \delta^*$, $\tilde{z}_1 = 0$ is a *locally exponentially stable* equilibrium of (8.25), and its region of attraction contains $(-\psi, \psi)$.

If $QP > 0$ there exist $\psi > 0$, $\varepsilon^* > 0$, and $\delta^* > 0$, such that, for all $0 < \varepsilon < \varepsilon^*$ and $|\delta| < \delta^*$, system (8.25) has three equilibria in the interval $(-\psi, \psi)$:

$$\tilde{z}_1^-(\varepsilon,\delta) < 0, \ 0, \ 0 < \tilde{z}_1^+(\varepsilon,\delta) \ .$$

$\tilde{z}_1^-(\varepsilon,\delta) \to 0$ and $\tilde{z}_1^+(\varepsilon,\delta) \to 0$ as $\varepsilon \to 0$. Moreover, $\tilde{z}_1 = 0$ is a hyperbolic and unstable equilibrium, whereas $\tilde{z}_1^-(\varepsilon,\delta)$ and $\tilde{z}_1^+(\varepsilon,\delta)$ are exponentially stable equilibria. This generalizes to our case the classical pitchfork bifurcation (see [5] and [12]).

If $QP = 0$ there exist $\psi > 0$, $\varepsilon^* > 0$, and $\delta^* > 0$, such that, for all $0 < \varepsilon < \varepsilon^*$ and $|\delta| < \delta^*$, only two situations can occur

(i) $\tilde{z}_1 = 0$ is a critically asymptotically stable equilibrium of (8.25), and its region of attraction contains $(-\psi, \psi)$;

(ii) system (8.25) has two equilibria in the interval $(-\psi, \psi)$:

$$0, \ \tilde{z}_1^*(\varepsilon,\delta) \ .$$

$\tilde{z}_1^*(\varepsilon,\delta) \to 0$ as $\varepsilon \to 0$. Moreover, $\tilde{z}_1 = 0$ is a non-hyperbolic and unstable equilibrium, whereas $\tilde{z}_1^*(\varepsilon,\delta)$ is an exponentially stable equilibrium. This generalizes to our case the critical-transcritical bifurcation (see [2, section 5.3]). We can distinguish the occurring of situation (i) rather than (ii) using the results established for the partial-state feedback case, in view of remark 6.14 and property 8.4.

8.6 Bifurcation Analysis in Case II

In this case, after coordinate transformation, system (8.25) can be expressed in polar coordinates as

$$\dot{r} = \alpha(\varepsilon,\delta)r + a(\varepsilon,\delta)r^3 + \mathcal{O}(r^5) \ , \tag{8.28}$$

$$\dot{\theta} = \beta(\varepsilon,\delta) + b(\varepsilon,\delta)r^2 + \mathcal{O}(r^4) \ , \tag{8.29}$$

with $a(0,0) < 0$ (see [12, section 3.1B]). Again, here we present only the results of the bifurcation analysis; their proofs can be found in [4].

The occurrence of a bifurcation and its nature depend on the values of the parameters Q and P defined in (8.26).

If $QP < 0$ there exist $\psi > 0$, $\varepsilon^* > 0$, and $\delta^* > 0$, such that, for all $0 < \varepsilon < \varepsilon^*$ and $|\delta| < \delta^*$, $r = 0$ is a locally exponentially stable equilibrium of (8.28), and its region of attraction contains $[0,\psi)$.

If $QP > 0$ there exist $\psi > 0$, $\varepsilon^* > 0$, and $\delta^* > 0$, such that, for all $0 < \varepsilon < \varepsilon^*$ and $|\delta| < \delta^*$, system (8.28) has two equilibria in the interval $[0,\psi)$:

$$0, \ r^*(\varepsilon,\delta) \ .$$

$r^*(\varepsilon,\delta) \to 0$ as $\varepsilon \to 0$. Moreover, $r = 0$ is a hyperbolic unstable equilibrium, whereas $r^*(\varepsilon,\delta)$ is an exponentially stable equilibrium. This represents a generalization to our case of the classical supercritical Hopf bifurcation (see [5] and [12]).

If $QP = 0$, we consider two sub-cases.

(a) $Q = 0$ or $P = 0$. In this sub-case there exist $\psi > 0$, $\varepsilon^* > 0$, and $\delta^* > 0$, such that, for all $0 < \varepsilon < \varepsilon^*$ and $|\delta| < \delta^*$, $r = 0$ is a critically asymptotically stable equilibrium of (8.28), and its region of attraction contains $[0,\psi)$.

(b) $Q \neq 0$ and $P \neq 0$. In this sub-case either the same situation as $QP < 0$ or the the same situation as $QP > 0$ occurs.

8.7 Final result

The entire analysis carried out in the previous subsections can be summarized as follows. Assume that system (5.5) satisfies the assumptions of either Case I or Case II of section 7. Given $0 < r < R$, let (5.4) be an output feedback designed using the method in [11]. Then, *if* $\bar{k}$ *and* g *are large enough*, the trajectories of the closed-loop system (5.5)–(5.4) are attracted by a set, $\mathcal{A} = \omega(B_R^{n+2d})$. In Case I the structure of $\mathcal{A}$ is determined by the values of the certain quantities P, Q, N, α, β, and γ, defined in (8.26)(8.27); in Case II, it is determined by the values of P and Q only. The occurrence of each particular structure of attractor is analyzed in subsections 8.5 and 8.6. Comparing the results in these subsections with those presented in theorems 7.10 and 7.13, it is seen that the relation between the attractors of (5.5)–(5.4) and the parameters P, Q, N, α, β, and γ defined in (8.26)(8.27) is identical to the the relation between the attractors of (5.5)–(5.3) and the parameters P, Q, N, α, β, and γ defined in (7.8)(7.9). Hence, in the case of output feedback, results totally analogous to those in theorems 7.10 and 7.13 hold.

Remark 8.30. It is well known that, in the semiglobal practical stabilization scheme obtained by means of the dynamic feedback law (5.4), the subset of the state space where $e = 0$ *is not an invariant subset*, as opposite to the classical situation occurring when using an observer-based dynamic controller. As a matter of fact, the dynamics of (5.4) is not that of an observer for the partial state $\xi_1, \ldots, \xi_d$ of (5.5). Semiglobal practical stabilizability *only* guarantees that the state of the closed loop system (and thus in particular $e(t)$) converges to an arbitrarily small neighborhood of the origin. However, we have seen in property 8.4 that all equilibria of (8.3) belong to the subset of the state space where $e = 0$. Consequently, whenever the trajectory of the closed-loop system converges to an equilibrium, $e(t)$ actually *converges to zero* as $t \to \infty$. However, it can be proved that there may exist invariant subsets that are not equilibria (e.g. periodic orbits) that are *not* contained in the subset of the state space where $e = 0$ and that attract some trajectories of the closed-loop system; consequently, for those trajectories $e(t)$ does *not* go to 0 as $t \to \infty$.

In concluding the section, it is useful to compare the results of this analysis with the one of the case of partial-state feedback. Consider system (5.5) and assume that it satisfies the assumptions of either Case I ore Case II . Then, (5.5) can be semiglobally practically stabilized by either partial-state or dynamic output feedback using methods presented in [11], and in both cases we have been able to characterize the structure of the ω-limit set of the resulting closed-loop system. From the previous analysis it follows that for $\bar{k}$ and g large enough, *the structure of the* ω*-limit set is the same for both types of feedbacks.* In fact, assume that the coefficients a_i's and k are chosen identical in the two designs; then, the quantities P, Q, α, T, β, and γ are identical for both resulting closed-loop systems, and this implies that for $\bar{k}$ and g large enough the structure of $\mathcal{A}$ is the same in both cases of partial-state feedback and dynamic output feedback. Consequently, in the situations examined, if we use the coarse estimates of the output and its derivatives, that are provided by the observer included in the output feedback, rather than their direct measures, the performance of the controller *is not deteriorated.* In particular, if partial-state feedback is able to asymptotic stabilize (exponentially or even *critically*) the origin, then the output feedback which uses only a course estimation of the partial state does the same job.

The "conservation" of the structure of the ω-limit set is surprising to a certain extent since the output feedback is based on an approximate observer that does not guarantee the invariance of the subspace on which the observation error is zero. Nevertheless, we have found that the use of the approximate observer does not induce any additional "dimension" in the ω-limit set. This provides a new insight into these high-gain output stabilization schemes and reveals an interesting property of the observer introduced in [10].

9 Conclusions

In [11] Teel and Praly showed that a nonlinear minimum-phase system can be semiglobally practically stabilized by either memory-less partial-state feedback or dynamic output feedback.

In this paper we have described the asymptotic behavior of the resulting closed-loop system inside the arbitrarily small neighborhood of the origin where the trajectories are guaranteed to enter.

First, we have pointed out that in general semiglobally practically stabilization implies that in that neighborhood of the origin there exists a compact, invariant, and stable set that attracts the trajectories that enter the neighborhood; moreover, such set can be characterized as an ω-limit set of a set.

Then, we have investigated the structure of this ω-limit set when the zero dynamics of the nonlinear minimum-phase system that is semiglobally practically stabilized have a linearization at the origin that has either a simple eigenvalue at 0 or a simple pair of purely imaginary eigenvalues. In the first case the ω-limit set can be either just the origin or a 1-dimensional subspace whose boundary consists of equilibria; in the second case the ω-limit set can be either just the origin or a 2-dimensional manifold with a boundary consisting of a periodic orbit. Moreover, we have found that the structure of the ω-limit set does not change if the nonlinear minimum-phase system is semiglobally practically stabilized using observer-based dynamic output feedback rather than memory-less partial-state feedback or vice versa.

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