THE FUNDAMENTAL CORRESPONDENCE IN SUPER-COMMUTATIVE HOPF ALGEBRAS

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REPORT No. 11, 2003/2004

ISSN 1103-467X
ISRN IML-R- -11-03/04- -SE

INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES

The Fundamental Correspondence in Super-Commutative Hopf Algebras [⋆]

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Abstract

The 1-1 correspondence (due to Takeuchi [23]) between the Hopf subalgebras B in a fixed commutative Hopf algebra A and the conormal quotients $A \to H$ is generalized to super-commutative Hopf algebras in arbitrary characteristic $\neq 2$. Some results on Hopf algebras in braided monoidal categories as well as super-hyperalgebras are proved, and they together with the original correspondence are applied.

Introduction

Throughout this paper we work over a field k. An affine group (over k) is by definition such a group-functor on the category of commutative algebras that is represented, then necessarily by a uniquely determined commutative Hopf algebra: the two algebraic systems thus form mutually anti-isomorphic categories. The fundamental theorem [3, Chap. III, Sect. 3, Thm. 7.2] on affine groups was purely algebraically proved by Takeuchi [23], by introducing the notion of (relative) Hopf modules. The theorem was reformulated as follows: (1) A commutative Hopf algebra is a faithfully flat module over every Hopf subalgebra; (2) For a commutative Hopf algebra A, there is a natural 1-1 correspondence between the Hopf subalgebras B in A and the normal Hopf ideal I in A (or equivalently, the conormal quotients $A \to A/I$). Part 2 is translated into a 1-1 correspondence between affine quotient groups and affine normal subgroups in an affine group. The advantage of his Hopf-algebraic approach was realized by Takeuchi [26], himself. He freely discussed there correspondences between wider classes of subalgebras and quotients, in not necessarily commutative Hopf algebras, as well as stronger module-properties, such as projectivity or freeness, rather than faithful flatness; it was especially proved

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[⋆] Dedicated to Professor Hans-Jürgen Schneider on his 60th birthday.

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Preprint submitted to Elsevier Science *25 April 2004*

that Part 1 above holds true with 'faithful flat' replaced by 'projective generator'. These topics on Hopf algebras, including Hopf modules, are sometimes called 'quotient theory of Hopf algebras'; see [4], [19], [11], [13], [21], [27], and especially the new [18] by Schauenburg and Schneider.

Our objective of this paper is to extend the results (1) (strengthened), (2) above to super-commutative Hopf algebras, by using the original results and applying results from 'quotient theory' as well as (super-)hyperalgebra theory; see Theorem 2.8. It follows that the super-commutative and super-cocommutative Hopf algebras form an abelian category; see Corollary 2.9. If the characteristic $\operatorname{ch} k \neq 2$, the vector spaces graded by $\mathbb{Z}_2 = \{0, 1\}$ form a symmetric monoidal category, $\mathbf{S}$, and [(co)commutative] Hopf algebra objects in $\mathbf{S}$ are called super-[(co)commutative] Hopf algebras.

Recently Deligne [2] proved the strong result that if k is an algebraically closed field of characteristic zero, a k-linear abelian symmetric rigid monoidal category with some mild assumptions is realized as the category of finite-dimensional A-comodules in $\mathbf{S}$, where A is some super-commutative Hopf algebra. This made it possible for Etingof and Gelaki [5] to complete their successful classification of finite-dimensional triangular Hopf algebras over k as above. Nevertheless, our knowledge on super-commutative Hopf algebras seems rather restricted, especially in positive characteristic. So, it might be emphasized that we will prove our results in arbitrary characteristic $\neq 2$. In place of super-Lie algebras in characteristic zero, super-hyperalgebras, a direct generalization of Takeuchi's hyperalgebras [24], will play a rôle, just as hyperalgebras do in his theory of affine groups in positive characteristic.

Super-Hopf algebras are a special example of Hopf algebras in the braided monoidal category of Yetter-Drinfeld modules. In Section 1, some basic results from [26], [10] are reformulated for those generalized Hopf algebras. The results will be frequently used in the following sections.

In Section 2, we prove especially that a super-commutative Hopf algebra A is a projective generator as a left and right module over every super-Hopf subalgebra B; see Corollary 2.4. This immediately implies that our correspondence

$$B \mapsto H = A/AB^{+}$$

from the super-Hopf subalgebras B in A to the conormal quotient super-Hopf algebras $A \to H$ is injective. The surjectivity will be proved in Section 5. For this, affine (i.e., finitely generated super-commutative) super-Hopf algebras and super-hyperalgebras are discussed in Sections 3, 4, respectively. We will see that splitting results are brought by smooth commutative Hopf algebras as well as smooth hyperalgebras; see Propositions 3.1 and 4.2.

As an officious remark to deal with Hopf algebras in braided monoidal categories, including super-Hopf algebras, the diagrammatics used in [7], [9], for example, would make the arguments clear and enjoyable, although we will dispense with them in this paper.

1 Hopf algebras in the category of Yetter-Drinfeld modules

Let J be a Hopf algebra with bijective antipode.

Let $\mathbf{YD}_J^J$ denote the category of the Yetter-Drinfeld modules with right J-module action and right J-comodule coaction. It is known (see [9]) that this naturally forms a braided monoidal category with the braiding

$$c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V, \quad v \otimes w \mapsto \sum w_0 \otimes v \cdot w_1,$$

where $w \mapsto \sum w_0 \otimes w_1$, $W \to W \otimes J$ denotes the comodule structure, as usual. Let A be a Hopf algebra in $\mathbf{YD}_J^J$. In particular, A is a right J-module algebra and right J-comodule coalgebra. The Radford-Majid bosonization ([16], [9]) gives rise to an ordinary Hopf algebra, $J \mathbin{\triangleright\!\!\!<} A$; this is the smash product $J \ltimes A$ as an algebra, and is the smash coproduct $J \mathbin{\blacktriangleright\!\!\!<} A$ as a coalgebra.

As mentioned above, A is an algebra and coalgebra in the monoidal category $\mathbf{M}^J$ of right J-comodules. Let $B \subset A$ be a right coideal subalgebra in $\mathbf{M}^J$, or a subalgebra in the monoidal category $(\mathbf{M}^J)^A$ of right A-comodules in $\mathbf{M}^J$. Define

$$(\mathbf{M}^J)^A_B := [(\mathbf{M}^J)^A]_B,$$

the category of right B-modules in $(\mathbf{M}^J)^A$. This can be also defined in the same way as the ordinary Hopf modules [26, p. 454], but replacing the trivial flip with $c_{A,-}$; see Remark 1.2 (2) below. Let us write, as usual,

$$B^+ = \mathrm{Ker}(\varepsilon|_B : B \to k),$$

the kernel of the counit of A restricted to B. One sees that $Q := A/AB^+$ is the quotient left A-module coalgebra of A in $\mathbf{M}^J$.

In the same way as in [26, p. 455], we have the functor

$$M \mapsto M/MB^+, \quad (\mathbf{M}^J)^A_B \to (\mathbf{M}^J)^Q \tag{1}$$

which is left adjoint to

$$V \mapsto V \square_Q A, \quad (\mathbf{M}^J)^Q \to (\mathbf{M}^J)^A_B. \tag{2}$$

Moreover, if $M \in (\mathbf{M}^J)^A$, $N \in {}_A(\mathbf{M}^J)$, then

$$M \otimes N \to M \otimes N, \quad m \otimes n \mapsto \sum m_0 \otimes m_1 n \tag{3}$$

gives an isomorphism in $\mathbf{M}^J$. If $M \in (\mathbf{M}^J)^A_B$, this induces an isomorphism

$$M \otimes_B N \simeq M/MB^+ \otimes N \tag{4}$$

in $\mathbf{M}^J$, and in particular an isomorphism

$$A \otimes_B A \simeq Q \otimes A \tag{5}$$

in ${}^Q(\mathbf{M}^J)$. The proof of [26, Thm. 1] works well to prove:

Proposition 1.1 *Suppose that there is an object N in ${}_A(\mathbf{M}^J)$ such that $- \otimes_B N : (\mathbf{M}^J)_B \to \mathbf{M}^J$ is faithfully exact. Then the functors given by (1), (2) are equivalences, and B coincides with the subalgebra*

$${}^{\mathrm{co}\,Q}A = \{a \in A \mid \sum \pi(a_1) \otimes a_2 = \pi(1) \otimes a\}$$

of left Q-coinvariants. Here, $\pi : A \to Q$ denotes the quotient, and $\Delta(a) = \sum a_1 \otimes a_2$ denotes the comultiplication, as usual.

Remark 1.2 (1) The proposition can be reduced to a large extent to [26, Thm. 1], by applying the theorem to the right coideal subalgebra $B = k \otimes B$ in $\tilde{A} := J \rhd\!\!< A$. We see that

$$\tilde{A}/\tilde{A}B^+ \simeq \tilde{Q} := J \blacktriangleright\!\!< Q, \quad V \square_{\tilde{Q}} \tilde{A} \simeq V \square_Q A,$$

where $V \in \mathbf{M}^{\tilde{Q}}$. Moreover, $(\mathbf{M}^J)^A_B$ is naturally identified with the ordinary Hopf module category $\mathbf{M}^{\tilde{A}}_B$, and $(\mathbf{M}^J)^Q$ with $\mathbf{M}^{\tilde{Q}}$. But, one then needs to suppose the N above to be a left $\tilde{A}$-module; A will do since it is a left $\tilde{A}$-module with respect to the left multiplication by A, together with the twisted J-action through the composite-inverse of the antipode.

(2) On the other hand the proposition can be proved in such a generalized situation that is suggested by Takeuchi [29]. A *k-linear monoidal category over the category $\mathbf{M}_k$ of vector spaces* [29] is a k-linear abelian, monoidal category $\mathbf{C}$ together with such a faithfully exact k-linear functor $F : \mathbf{C} \to \mathbf{M}_k$ that satisfies

$$F(X) \otimes F(Y) = F(X \otimes Y), \quad F(I) = k,$$

where $X, Y \in \mathbf{C}$ and I is the unit object in $\mathbf{C}$, and preserves the associativity and the unit constraints. Let $\mathcal{Z}(\mathbf{C})$ denote the *left* center of $\mathbf{C}$; see [7, Chap. XIII, Sect. 4]. Thus an object in $\mathcal{Z}(\mathbf{C})$ is a pair $(X, c_{X,-})$ of $X \in \mathbf{C}$ and a natural isomorphism $c_{X,-} : X \otimes - \xrightarrow{\sim} - \otimes X$. This is again a k-linear monoidal category over $\mathbf{M}_k$, and hence a *k-linear braided category over* $\mathbf{M}_k$ in the sense of [28]. The proposition is generalized to $\mathbf{C}$ (instead of $\mathbf{M}^J$), a Hopf algebra A in $\mathcal{Z}(\mathbf{C})$, and a subalgebra $B \subset A$ in $\mathbf{C}^A$; notice that the crossings in the diagrams which might have been drawn for the proposition are all of the form $c_{A,-}$. For example, $\mathbf{C} = \mathbf{YD}^J_J$, a Hopf algebra A in $\mathbf{YD}^J_J$, and a right coideal subalgebra $B \subset A$ in $\mathbf{YD}^J_J$ will do.

Let ${}^J_J\mathbf{YD}$ denote the braided monoidal category of the Yetter-Drinfeld modules with left J-module action and left J-comodule coaction. This is the left center of the monoidal category ${}_J\mathbf{M}$ of left J-modules. Let A be a Hopf algebra in ${}^J_J\mathbf{YD}$, and let $\pi : A \to Q$ be a quotient left A-module coalgebra in ${}_J\mathbf{M}$. Define

$$ {}^Q_A({}_J\mathbf{M}) := {}^Q[{}_A({}_J\mathbf{M})]. $$

Let $B = {}^{\mathrm{co}\,Q}A$; this is a right coideal subalgebra of A in ${}_J\mathbf{M}$. In the same way as in [26, p. 457], we have the functor

$$ T \mapsto A \otimes_B T, \quad {}_B({}_J\mathbf{M}) \to {}^Q_A({}_J\mathbf{M}) \tag{6} $$

which is left adjoint to

$$ N \mapsto {}^{\mathrm{co}\,Q}N, \quad {}^Q_A({}_J\mathbf{M}) \to {}_B({}_J\mathbf{M}). \tag{7} $$

The isomorphism in ${}_J\mathbf{M}$ which is given by (3), but this time for $M \in ({}_J\mathbf{M})^A$, $N \in {}_A({}_J\mathbf{M})$, especially induces an isomorphism

$$ A \otimes B \simeq A \Box_Q A \tag{8} $$

in $({}_J\mathbf{M})_B$. In the same way as [26, Thm. 2], we obtain:

Proposition 1.3 *Suppose that there is an object M in $({}_J\mathbf{M})^A$ such that $M \Box_Q - : {}^Q({}_J\mathbf{M}) \to {}_J\mathbf{M}$ is faithfully exact. Then the functors given by (6), (7) are equivalences, and $A/AB^+ \simeq Q$.*

With the notation of Remark 1.2 (2), the proposition is generalized to $\mathbf{C}$ (instead of ${}_J\mathbf{M}$), a Hopf algebra A in $\mathscr{Z}(\mathbf{C})$, and a quotient coalgebra $A \to Q$ in ${}_A\mathbf{C}$.

We shall say that a coalgebra is *connected*, if it is pointed and irreducible, or in other words if its coradical is 1-dimensional.

Let Γ be a group. We will sometimes write simply Γ for the group algebra $k\Gamma$, for example $\mathbf{YD}^\Gamma_\Gamma$ for $\mathbf{YD}^{k\Gamma}_{k\Gamma}$.

Proposition 1.4 *Let A be such a Hopf algebra in $\mathbf{YD}^\Gamma_\Gamma$ that is connected as a coalgebra.*

(1) *If $B \subset A$ is a right coideal subalgebra in $\mathbf{YD}^\Gamma_\Gamma$, A is a projective generator as a left and right B-module.*

(2) *If $A \to Q$ is a quotient left A-module coalgebra in $\mathbf{YD}^\Gamma_\Gamma$, A is an injective cogenerator as a left and right Q-comodule.*

(3) *The right coideal subalgebras $B \subset A$ in $\mathbf{YD}^\Gamma_\Gamma$ and the quotient left A-module coalgebras $A \to Q$ in $\mathbf{YD}^\Gamma_\Gamma$ are in 1-1 correspondence, under $B \mapsto A/AB^+$,*

$Q \mapsto {}^{\mathrm{co}\,Q}A$.

(4) *If* $B \leftrightarrow Q$ *under the correspondence, there is a left* Q*-colinear and right* B*-linear isomorphism* $A \simeq Q \otimes B$.

PROOF. We see that $\tilde{A} := \Gamma \triangleright\!\!< A$ is a pointed Hopf algebra.

(1) $B = k \otimes B$ is a right coideal subalgebra in $\tilde{A}$, which contains no non-trivial grouplike. It follows by [10, Thm. 1.3 (1)] (see also [19, Thm. 4.15]) and [13, Thm. 2.1] that $\tilde{A}$ is a left and right B-projective generator. Part 1 follows since $\tilde{A}$ is free as a left and right A-module.

(2) Recall from [25, Prop. A.2.1] that a comodule over a coalgebra is (faithfully) coflat if and only if it is (an) injective (cogenerator).

$\tilde{Q} := \Gamma \blacktriangleright\!\!< Q$ is a quotient left $\tilde{A}$-module coalgebra of $\tilde{A}$. It follows by [10, Thm. 1.3 (2)] that $\tilde{A}$ is a right (and left) $\tilde{Q}$-injective cogenerator. Since $\tilde{A}$ and $\tilde{Q}$ is right cofree over A and Q, respectively, it follows that A is a right Q-injective cogenerator. We apply Proposition 1.3 and the isomorphism (5) in the generalized situation in which ${}_J\mathbf{M}$ and $\mathbf{M}^J$ are replaced by $\mathbf{YD}_{\Gamma}^{\Gamma}$. We can take A as the M in the proposition, to see that $A/AB^+ = Q$, where $B = {}^{\mathrm{co}\,Q}A$. Since A is a left B-projective generator by Part 1, the isomorphism implies that A is a left Q-injective cogenerator.

(3) The one equality $A/AB^+ = Q$ with $B = {}^{\mathrm{co}\,Q}A$ was just proved. The other follows from Part 1 by applying Proposition 1.1 in the generalized situation as above.

(4) This is proved in the same way as [10, Thm. 1.3 (4)], based on the argument of [19, Thm. 2.1]. $\square$

2 Projectivity over super-Hopf subalgebras

From now on until the end we suppose that the characteristic $\mathrm{ch}\, k \neq 2$. Let $\mathbf{S} = \mathbf{M}^{\mathbb{Z}_2}$ denote the category of vector spaces $V = V_0 \oplus V_1$ graded by $\mathbb{Z}_2 = \{0, 1\}$. This forms a symmetric monoidal category, that is, a braided monoidal category with the involutory braiding

$$t_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V, \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v,$$

where $|v|, |w|$ denote the degrees of homogeneous elements v, w. The unique Hopf-algebra isomorphism $(k\mathbb{Z}_2) \simeq (k\mathbb{Z}_2)^*$ gives an identification ${}_{\mathbb{Z}_2}\mathbf{M} = \mathbf{M}^{\mathbb{Z}_2}$ $(= \mathbf{S})$ of monoidal categories. Given V in ${}_{\mathbb{Z}_2}\mathbf{M}$ or $\mathbf{M}^{\mathbb{Z}_2}$, it together with the corresponding $\mathbb{Z}_2$-gradation or $\mathbb{Z}_2$-action give rise to a Yetter-Drinfeld module, so that we have a faithfully exact, braided monoidal functor

$$\mathbf{S} \to \mathbf{YD}^{\mathbb{Z}_2}_{\mathbb{Z}_2} = {}^{\mathbb{Z}_2}_{\mathbb{Z}_2}\mathbf{YD}.$$

Algebraic systems in $\mathbf{S}$ are called with 'super' prefixed. A *super-commutative algebra* is thus a $\mathbb{Z}_2$-graded algebra $R = R_0 \oplus R_1$ such that

$$xy = (-1)^{|x||y|}yx \quad (x, y \in R_0 \cup R_1).$$

Hence, R_0 is a central subalgebra, every element in R_1 is square-zero, and every homogeneous element x in R is normalized by every graded subspace $V \subset R$ in the sense $Vx = xV$. It follows that the ideal (R_1) generated by R_1 is nil, and equals $R_1^2 \oplus R_1$.

A super-Hopf algebra $A = A_0 \oplus A_1$ is regarded as a Hopf algebra in $\mathbf{YD}^{\mathbb{Z}_2}_{\mathbb{Z}_2}$, and hence the results in Section 1 can apply. The ideal $I = (A_1)$ generated by A_1 is the smallest super-Hopf ideal such that A/I is an ungraded (or ordinary) Hopf algebra. Dually, the inverse image $\Delta^{-1}(A_0 \otimes A_0)$ is the largest ungraded Hopf subalgebra.

Our main interest is in super-commutative Hopf algebras. A super-commutative Hopf algebra A represents the group-functor

$$\operatorname{Sp} A : R \mapsto \{\text{super-algebra maps } A \to R\}$$

on the category of super-commutative algebras R, which has $\otimes$ as finite direct sum. This can be called a *super-affine group*. We say that A is *affine* if it is finitely generated as an algebra. Notice that the antipode of A is involutory. Since in addition, every super-coalgebra is a directed union of finite-dimensional super-subcoalgebras (or $\mathbb{Z}_2$-stable subcoalgebras), we see that a super-commutative Hopf algebra is a directed union of affine super-Hopf subalgebras.

Let B be a super-commutative algebra. Suppose $M \in \mathbf{S}_B$ with respect to $\leftharpoonup: M \otimes B \to M$. Then,

$$B \otimes M \xrightarrow{t_{B,M}} M \otimes B \xrightarrow{\leftharpoonup} M$$

makes M an object in ${}_B\mathbf{S}$; this will be denoted by ${}_tM$.

Lemma 2.1 (1) *${}_tM$ together with $\leftharpoonup$ is a (B, B)-bimodule in* $\mathbf{S}$.

(2) *$M \mapsto {}_tM$ gives an equivalence $\mathbf{S}_B \approx {}_B\mathbf{S}$.*

(3) *M is flat (resp., projective, resp., a generator) as a right B-module if and only if ${}_tM$ is so as a left B-module.*

PROOF. (1), (2) These are directly seen.

(3) For flatness, it suffices by Part 2 to see that M is right B-flat if and only if the functor $M \otimes_B - : {}_B\mathbf{S} \to \mathbf{S}$ is exact (and that the symmetric statement for ${}_tM$ holds). The 'only if' part is trivial. For the 'if' part, let $N \in {}_B\mathbf{S} = {}_{B \ltimes \mathbb{Z}_2}\mathbf{M}$. Then

$$M \otimes_{B \ltimes \mathbb{Z}_2} N = e(M \otimes_B N),$$

where $e \in k\mathbb{Z}_2$ is the idempotent integral. Therefore, if the functor above is exact, M is right $B \ltimes \mathbb{Z}_2$-flat, and hence is right B-flat since $B \ltimes \mathbb{Z}_2$ is right B-free.

The statement on projectivity follows similarly by using the fact that $B \subset B \rtimes \mathbb{Z}_2$ is a separable extension of rings.

The remaining follows if one notices (up to symmetry): (i) ${}_t(\mathbb{Z}_2 \otimes M) \simeq {}_tM \otimes \mathbb{Z}_2$ via $t_{\mathbb{Z}_2, M}$; (ii) M is a right B-generator if and only if $\mathbb{Z}_2 \otimes M$ is a generator in $\mathbf{S}_B$. It is easy to see (i). Here we supposed that the structure of $\mathbb{Z}_2 \otimes M$ is such that B acts on M. But, there is another, isomorphic structure such that $(\mathbb{Z}_2 \otimes M)_i = (\mathbb{Z}_2)_i \otimes M$ $(i = 0, 1)$, and $(a \otimes m)b = a|b| \otimes mb$ $(a \in \mathbb{Z}_2$, $m \in M$, $b \in B)$. Hence a right B-linear epimorphism $\bigoplus M \to B$ gives rise to an epimorphism $\bigoplus(\mathbb{Z}_2 \otimes M) \to \mathbb{Z}_2 \otimes B$ in $\mathbf{S}_B$. Since $\mathbb{Z}_2 \otimes B$ is a generator in S_B (for $k\mathbb{Z}_2$ is Frobenius), Part (ii) follows. □

Lemma 2.2 *Let A be a super-commutative bialgebra, and let $B \subset A$ be a super-right coideal subalgebra.*

(1) *$M \mapsto {}_tM$ gives an equivalence $\mathbf{S}_B^A \approx {}_B\mathbf{S}^A$.*

(2) *A is flat (resp., projective, resp., a generator) as a left B-module if and only if it is so as a right B-module.*

PROOF. (1) This is directly seen.

(2) This follows by Lemma 2.1 (3), since ${}_tA$ in ${}_B\mathbf{S}$ equals A with the left multiplication by B. □

Theorem 2.3 *Let A be a super-commutative Hopf algebra, and let $B \subset A$ be a super-right coideal subalgebra. Then A is flat as a left and right B-module.*

PROOF. We modify the proof of the corresponding result [13, Thm. 3.4] in the ungraded context; see also [21, Thm. 3.3].

We may suppose by base extension that k is algebraically closed. We may suppose that B is finitely generated, since it is a directed union of finitely generated super-right coideal subalgebras. Since A is then a directed union of affine super-Hopf subalgebras including B, we may suppose that A is affine.

For a super-commutative algebra $R = R_0 \oplus R_1$ in general, $p \mapsto P = p \oplus R_1$ gives a 1-1 correspondence from the maximal ideals p in R_0 to the left or right (and necessarily two-sided) maximal ideals P in R.

By Lemma 2.2 (2), it suffices to prove that A is right B-flat; we will often omit to say 'right'. First we write $p = B_0^+$, and prove that for each $M \in \mathbf{S}_B^A$, the localization M_p by the central multiplicative set $B_0 \setminus p$ is B-flat. Since M is a directed union of sub-objects which are finitely generated as B-modules, we may suppose that M is finitely generated, so that the k-dimension $r := \dim M/MB^+$ is finite. The isomorphism (4) with $N = A$ implies that $M_p \otimes_{B_p} A_p \simeq A_p^r$ as right A_p-modules. By lifting a k-basis of M/MB^+ up to M, we obtain such a B-linear map $f : B^r \to M$ that is an isomorphism modulo the maximal ideal $P = p \oplus B_1$. Now, B_p is local with the maximal ideal PB_p. It follows by the Nakayama lemma that the localization $f_p : B_p^r \to M_p$ is a B_p-linear epimorphism, since the f_p modulo PB_p is identified with the isomorphism, the f modulo P. Hence we have an A_p-linear epimorphism

$$f_p \otimes_{B_p} A_p : A_p^r \to M_p \otimes_{B_p} A_p \cong A_p^r. \tag{9}$$

This is necessarily an isomorphism, since in general, a super-commutative algebra R will become commutative if divided by the Jacobson radical ($\supset (R_1)$), and hence is weakly finite in the sense that any R-linear epimorphism $R^s \to R^s$ ($s > 0$) is necessarily an isomorphism. Since f_p is embedded into the map (9), it is an isomorphism and in particular, M_p is B-flat.

We have seen that A_p ($p = B_0^+$) is B-flat. Let m be a maximal ideal in A_0. By applying the Hilbert Nullstellensatz to $A_0/A_1^2 = A/(A_1)$, we see $A_0/m = k$. Let $g : A \to A/(m \oplus A_1) = k$ denote the natural super-algebra map. The right translation

$$T_g : A \to A, \quad T_g(a) = \sum a_1 g(a_2)$$

gives a super-algebra automorphism, with inverse $T_{g^{-1}}$, such that $T_g(B) = B$, $T(q) = B_0^+$, where $q = m \cap B_0$. It follows that A_q is B-flat. Since

$$A_m = (A_0)_m \otimes_{A_0} A = (A_0)_m \otimes_{(A_0)_q} A_q$$

and the localization $(A_0)_q = (B_0)_q \otimes_{B_0} A_0 \to (A_0)_m$ is flat, A_m is B-flat for every maximal ideal m in A_0. Hence A is B-flat. □

Corollary 2.4 *Let A, B as above. The following are equivalent:*

(a) *B is a simple object in $\mathbf{S}_B^A$;*
(b) *A is faithfully flat as a left or equivalently right B-module;*
(c) *A is a projective generator as a left or right B-module;*

(d) *Every non-zero object in* $\mathbf{S}^A_B$ *is a projective generator as a right* B*-module.*

If B *is a super-Hopf subalgebra, it satisfies these conditions.*

PROOF. $\tilde{A} := \mathbb{Z}_2 \ltimes A$ is a Hopf algebra with bijective antipode, $B = k \otimes B$ is a right coideal subalgebra, and $\tilde{A}$ is left B-flat by Theorem 2.3. Hence by [13, Thm. 2.1], Condition (a) and the conditions (b)–(c) with A replaced by $\tilde{A}$ are all equivalent. Since $\tilde{A}$ is left and right A-free, Conditions (a)–(c) are equivalent. Trivially, (d) implies (c).

To prove (c) $\Rightarrow$ (d), let $0 \neq M \in \mathbf{S}^A_B$. We see from Lemma 2.1 (1) that the isomorphism

$${}_tM \otimes_B A \simeq \bar{M} \otimes A \quad (\bar{M} = M/MB^+) \tag{10}$$

given by (4) is now in ${}_B\mathbf{S}$, where B acts on A in $\bar{M} \otimes A$ from the left, across $\bar{M}$ through $t_{B,\bar{M}}$. Since by (c), $\bar{M} \otimes A$ is (a) left B-projective (generator), and A is a left B-(projective) generator, it follows by (10) that ${}_tM$ is (a) left B-projective (generator). By Lemma 2.1 (3), M is a right B-projective generator.

If B is a super-Hopf subalgebra, then it is simple in $\mathbf{S}^B_B$ ($\approx \mathbf{S}$), and hence is so in $\mathbf{S}^A_B$. Thus it satisfies (a) and hence (b)–(d). □

Proposition 2.5 *Let* A *be a super-commutative Hopf algebra. Those super-right coideal subalgebras* $B \subset A$ *over which* A *is a left or equivalently right projective generator and those quotient super-Hopf algebras* $A \to H$ *along which* A *is a left or equivalently right injective cogenerator are in 1-1 correspondence, under* $B \mapsto A/AB^+$, $H \mapsto {}^{\mathrm{co}\,H}A$.

PROOF. Let B be as above, and write $H = A/AB^+$; this is a quotient super-Hopf algebra, as is easily seen. By Proposition 1.1 (A can be taken as the N), $B = {}^{\mathrm{co}\,H}A$. By (5), A is a left H-injective cogenerator. Regard $\pi : A \to H$ as a quotient right A-module coalgebra, and apply the opposite version of Proposition 1.3, in which A can play the rôle of the M. Then one sees that $A/C^+A = H$, where C denotes the super-left coideal subalgebra

$$A^{\mathrm{co}\,H} = \{a \in A \mid \sum a_1 \otimes \pi(a_2) = a \otimes \pi(1)\}$$

of right H-coinvariants. Moreover, A is a left and hence right C-projective generator by the opposite versions of (8) and Corollary 2.4. By the version of (5), A is a right H-injective cogenerator. These arguments also prove that if A is a left or right injective cogenerator along a quotient $A \to H$, then it is a left and right injective cogenerator, is a left and right projective generator over $B = {}^{\mathrm{co}\,H}A$ (and $A^{\mathrm{co}\,H}$), and $A/AB^+ = H$. □

Definition 2.6 A super-Hopf ideal I in a super-commutative Hopf algebra A is said to be *normal*, if for every $a \in I$,

$$\sum(-1)^{|a_2||a_3|} a_1 S(a_3) \otimes a_2 \in A \otimes I$$

or equivalently

$$\sum(-1)^{|a_1||a_2|} a_2 \otimes S(a_1) a_3 \in I \otimes A,$$

where S denotes the antipode of A. In this case we shall say that the quotient $A \to A/I$ is *conormal*. The condition is equivalent to that for each super-commutative algebra R, the subgroup $\mathrm{Sp}(A/I)(R)$ in $\mathrm{Sp}\,A(R)$ is normal.

Lemma 2.7 *Let $A \to H$ be a conormal quotient super-Hopf algebra of a super-commutative Hopf algebra A. Then*

$$^{\mathrm{co}\,H}A = A^{\mathrm{co}\,H},$$

and this is a super-Hopf subalgebra of A.

PROOF. This is proved in the same way as [23, Lemma 4.4], or by translating the proof diagrammatically. □

Theorem 2.8 *Let A be a super-commutative Hopf algebra.*

(1) *If $B \subset A$ is a super-Hopf subalgebra, A is a projective generator as a left and right B-module.*

(2) *If $A \to H$ is a conormal quotient super-Hopf algebra, A is a injective cogenerator as a left and right H-comodule.*

(3) *The super-Hopf subalgebras $B \subset A$ and the conormal quotient super-Hopf algebras $A \to H$ are in 1-1 correspondence, under $B \mapsto A/AB^+$, $H \mapsto {}^{\mathrm{co}\,H}A$ $(= A^{\mathrm{co}\,A})$.*

Part 1 was proved in Corollary 2.4. If B is as in Part 1, AB^+ $(= B^+A)$ is a normal super-Hopf ideal, as is easily seen; see [23, Lemma 4.2]. By Proposition 2.5, $B = {}^{\mathrm{co}\,H}A$, where $H = A/AB^+$. Given a conormal $A \to H$, let $B = {}^{\mathrm{co}\,H}A$; this is a super-Hopf subalgebra by Lemma 2.7. By Part 1 and Proposition 2.5, to prove that A is a left or equivalently right H-injective cogenerator is equivalent to proving $A/AB^+ = H$. This will be proved in Theorem 5.1, and the proof of Theorem 2.8 will then complete.

The corresponding theorem in the ungraded context was proved by Takeuchi [23, Thm. 4.3]. The next corollary follows from Theorem 2.8, just as in the ungraded context, [23, Cor. 4.16] followed from [23, Thm. 4.3].

Corollary 2.9 *The super-commutative and super-cocommutative Hopf algebras form an abelian category.*

3 Affine super-Hopf algebras

Throughout this section we suppose that k is algebraically closed. Let A be an affine super-Hopf algebra. We have an ungraded affine Hopf algebra $A_{\mathrm{un}} := A/(A_1)$. The reduced algebra

$$\bar{A} := (A_{\mathrm{un}})_{\mathrm{red}} \tag{11}$$

divided by the nil radical is again an affine Hopf algebra. If $\operatorname{ch} k = 0$, every affine Hopf algebra is reduced, so that $A_{\mathrm{un}} = \bar{A}$. Define

$$K = {}^{\operatorname{co} \bar{A}}A, \tag{12}$$

the super-right coideal subalgebra of left $\bar{A}$-coinvariants along the quotient $A \to \bar{A}$. Notice that A_0 is a left $\bar{A}$-comodule commutative algebra such that the natural map $A_0 \to \bar{A}$ is an epimorphism of left $\bar{A}$-comodule algebras.

Proposition 3.1 (1) *There is a splitting $\sigma : \bar{A} \to A_0$ of left $\bar{A}$-comodule algebras, so that*

$$\bar{A} \otimes K \to A, \quad x \otimes a \mapsto \sigma(x)a \tag{13}$$

gives a counit-preserving isomorphism of super-left $\bar{A}$-comodule algebras; cf. [12, Prop. 2.4 (2)].

(2) *We have $AK^+ = \mathrm{Ker}(A \to \bar{A})$. Moreover, K is finite-dimensional, and the ideal K^+ is nilpotent.*

PROOF. (1) Since A is left (and right) Noetherian by the generalized Hilbert basis theorem [17, Prop. 3.5.2], (A_1) as a left ideal has finitely many nilpotent homogeneous generators, and hence is nilpotent. It follows that $\mathrm{Ker}(A \to A_{\mathrm{un}})$, and hence $\mathrm{Ker}(A \to \bar{A})$, $\mathrm{Ker}(A_0 \to \bar{A})$ are all nilpotent.

Since $\bar{A}$ is smooth as a commutative algebra, any symmetric Hochschild 2-cocycle $\bar{A} \otimes \bar{A} \to k$ with values in the trivial $\bar{A}$-module k is a coboundary. By [12, Prop. 1.10], we have a splitting σ as above, so that by [15, Prop. 7.2.3] (due to Doi and Takeuchi), the map given by (13) is such an isomorphism as described (notice that σ necessarily preserves the counit).

(2) The equality follows from the isomorphism just obtained. It implies that K^+ is nilpotent. Since the isomorphism also proves that K is finitely generated,

each $(K^+)^i$ is finitely generated as a left, say, K-module, and so $(K^+)^i/(K^+)^{i+1}$ is finite-dimensional. Hence K is finite-dimensional. $\square$

It follows from Part 1 above that A_0 is a finitely generated commutative algebra. Hence it includes the largest separable subalgebra $\pi_0(A_0)$ (see [30, Sect. 6.5]), which we will denote by $\pi_0(A)$.

Definition 3.2 A is said to be *connected* if $\pi_0(A) = k$.

Proposition 3.3 (1) *$\pi_0(A)$ is a Hopf subalgebra of A, which is naturally isomorphic to $\pi_0(\bar{A})$. It follows that*

$$\pi_0(A) \simeq \pi_0(\bar{A}) \simeq k^\Gamma,$$

where k^Γ $(= (k\Gamma)^*)$ is the function algebra of some finite group Γ.

(2) *Define $A_\mathrm{c} = A/\pi_0(A)^+A$, a quotient super-Hopf algebra of A. There is a counit preserving isomorphism*

$$A \simeq \pi_0(A) \otimes A_\mathrm{c}$$

of super-right A_c-comodule algebras. It follows that A_c is connected.

PROOF. (1) If a map $R \to T$ of finitely generated commutative algebras induces an isomorphism $R_\mathrm{red} \simeq T_\mathrm{red}$, then $\pi_0(R) \simeq \pi_0(T)$, since in general, $\pi_0(R) \simeq \pi_0(R_\mathrm{red})$. Apply this fact to $A_0 \otimes A_0 \subset (A \otimes A)_0$. Since $A_1 \otimes A_1$ in $(A \otimes A)_0$ consists of nilpotent elements, $\pi_0(A_0 \otimes A_0) = \pi_0((A \otimes A)_0)$, and so

$$\Delta(\pi_0(A_0)) \subset \pi_0((A \otimes A)_0) = \pi_0(A_0) \otimes \pi_0(A_0).$$

Therefore, $\pi_0(A)$ $(= \pi_0(A_0))$ is a Hopf subalgebra. By applying the fact to $A_0 \to \bar{A}$, we see $\pi_0(A) \simeq \pi_0(\bar{A})$.

(2) Suppose that Γ consists of $g_1 = 1, \ldots, g_s$; these give the projections $k^\Gamma \to k$. Write $I_i = \operatorname{Ker} g_i$. For each $1 \leq i \leq s$, choose a super-algebra map $p_i : A/I_iA \to k$, and let q_i denote the composite of p_i with the projection $A \to A/I_iA$. We can choose p_1 so as $q_1 = \varepsilon$, the counit of A. Then,

$$\lambda : A \simeq \prod_{i=1}^{s} A/I_iA \xrightarrow{(p_i)_i} \prod_{i=1}^{s} k \simeq k^\Gamma$$

gives a counit-preserving super-algebra map, such that $\lambda|_{k^\Gamma} = \mathrm{id}$. We see that

$$\phi : A \to k^\Gamma \otimes A_\mathrm{c}, \quad \phi(a) = \sum \lambda(a_1) \otimes \bar{a}_2$$

is a counit-preserving map of super-right A_{c}-comodule algebras. The left translation

$$T_{q_i} : A \xrightarrow{\sim} A, \quad T_{q_i}(a) = \sum q_i(a_1)a_2$$

induces an isomorphism $A/I_iA \simeq A/I_1A = A_{\mathrm{c}}$. The commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "\simeq"'] & k^{\Gamma} \otimes A_{\mathrm{c}} \arrow[d, "\simeq"', "(g_i\otimes\mathrm{id})_i"] \\
\prod_{i=1}^{s} A/I_iA \arrow[r, "\sim", "(T_{q_i})_i"'] & \prod_{i=1}^{s} A_{\mathrm{c}}
\end{tikzcd}$$

proves that ϕ is a desired isomorphism. □

Let $\pi : A \to H$ be an epimorphism of affine super-Hopf algebras. Let $\bar{A}, K$ be as in (11), (12). Define

$$H_{\mathrm{un}} = H/(H_1), \quad \bar{H} = (H_{\mathrm{un}})_{\mathrm{red}}, \quad L = {}^{\mathrm{co}\,\bar{H}}H. \tag{14}$$

Proposition 3.4 *π induces a map $K \to L$ of super-right H-comodule algebras. This is surjective if π is conormal.*

PROOF. If π is conormal, the induced $A_{\mathrm{un}} \to H_{\mathrm{un}}$, $\bar{A} \to \bar{H}$ are both conormal, as is seen by considering the corresponding closed embedding of affine groups and linear algebraic groups, respectively.

We claim that the commutative diagrams

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & H \arrow[d] \\
A_{\mathrm{un}} \arrow[r] & H_{\mathrm{un}}
\end{tikzcd}, \quad
\begin{tikzcd}
A_{\mathrm{un}} \arrow[r] \arrow[d] & H_{\mathrm{un}} \arrow[d] \\
\bar{A} \arrow[r] & \bar{H}
\end{tikzcd}, \quad
\begin{tikzcd}
A \arrow[r] \arrow[d] & H \arrow[d] \\
\bar{A} \arrow[r] & \bar{H}
\end{tikzcd}$$

are all pushout diagrams of algebra epimorphisms. The claim for the last diagram means that the kernel $\mathrm{Ker}(A \to \bar{A})$ is mapped via $A \to H$ onto $\mathrm{Ker}(H \to \bar{H})$; this follows from the claim for the first two. The first diagram is pushout since $(A_1) = A_1^2 \oplus A_1$ is mapped onto (H_1). For the second we apply the two results by Takeuchi [23] cited in the Introduction, one of which we have generalized in Corollary 2.4. Define

$$B_{\mathrm{un}} = A_{\mathrm{un}}^{\mathrm{co}\,H_{\mathrm{un}}}, \quad \bar{B} = \bar{A}^{\mathrm{co}\,\bar{H}}.$$

By [23, Thm. 4.3], these are Hopf subalgebras such that

$$A_{\mathrm{un}}/B_{\mathrm{un}}^{+}A_{\mathrm{un}} = H_{\mathrm{un}}, \quad \bar{A}/\bar{B}^{+}\bar{A} = \bar{H}. \tag{15}$$

We see by [23, Thm. 3.1] that $(B_{\mathrm{un}})_{\mathrm{red}} = \bar{B}$ as the coordinate Hopf algebra of the quotient linear algebraic group G/N, where $G = \mathrm{Sp}\, A_{\mathrm{un}}(k) = \mathrm{Sp}\, \bar{A}(k)$, $N = \mathrm{Sp}\, H_{\mathrm{un}}(k) = \mathrm{Sp}\, \bar{H}(k)$. Hence B_{un}^{+} is mapped onto $\bar{B}^{+}$ via $A_{\mathrm{un}} \to \bar{A}$. This together with (15) prove the claim.

Since the last diagram is thus pushout, one sees by Proposition 3.1 (2) that AK^{+} is mapped onto HL^{+}, and so that the image of K^{+} in H generates the left H-module H. Since $H \simeq \bar{H} \otimes L$ similarly as in (13), we have $LK^{+} = L^{+}$. This means that the right K-linear map $K \to L$ is surjective modulo the nilpotent ideal K^{+}. Hence the map itself is surjective. $\square$

We see from the proof that if $\mathrm{ch}\, k = 0$, $K \to L$ is surjective even when π is not conormal. In fact this immediately follows through super-Lie algebras; see (23) below.

4 Super-hyperalgebras

A *hyperalgebra* [24, 1.3.5] is a connected cocommutative Hopf algebra. It is said to be *smooth* [24, 1.9.5] if it is isomorphic as a coalgebra to the cofree connected cocommutative coalgebra $B(V)$ of some vector space V [22, Sect. 12.2].

Definition 4.1 A *super-hyperalgebra* is a super-cocommutative Hopf algebra which is connected as a coalgebra.

A super-hyperalgebra $\mathcal{A}$ represents the contravariant group-functor

$$\mathrm{Sp}^{*} \mathcal{A} : C \mapsto \{\text{super-coalgebra maps } C \to \mathcal{A}\}$$

on the category of connected super-cocommutative coalgebras C, which has $\otimes$ as finite direct product. If $\mathrm{ch}\, k = 0$, it is known [8, Prop. 3.2] that $\mathcal{A}$ is isomorphic to the universal envelope $U(\mathfrak{g})$ of the super-Lie algebra $\mathfrak{g}$ of primitives in $\mathcal{A}$.

In general the component C_1 in a suer-coalgebra C is a coideal. Hence the identification $C_0 = C/C_1$ gives to C_0 the quotient coalgebra structure of C.

Proposition 4.2 *Let $\mathcal{A}$ be a super-hyperalgebra, and let $\bar{\mathcal{A}} \subset \mathcal{A}$ be a smooth sub-hyperalgebra. The monomorphism $\bar{\mathcal{A}} \hookrightarrow \mathcal{A}_0$ $(= \mathcal{A}/\mathcal{A}_1)$ of left $\bar{\mathcal{A}}$-module coalgebras has a splitting, say, $\gamma : \mathcal{A}_0 \to \bar{\mathcal{A}}$, so that*

$$\mathcal{A} \to \bar{\mathcal{A}} \otimes \mathcal{A}/\bar{\mathcal{A}}^{+}\mathcal{A}, \quad a \mapsto \sum \tilde{\gamma}(a_1) \otimes \bar{a}_2 \tag{16}$$

gives a unit-preserving isomorphism of super-left $\bar{\mathcal{A}}$-module coalgebras, where $\tilde{\gamma}$ denotes the composite of γ with the projection $\mathcal{A} \to \mathcal{A}/\mathcal{A}_1 = \mathcal{A}_0$.

PROOF. By the cofreeness of $\bar{\mathcal{A}}$, any coalgebra monomorphism $\bar{\mathcal{A}} \hookrightarrow C$ into a connected cocommutative coalgebra C splits. This means that any symmetric co-Hochschild 2-cocycle $k \to \bar{\mathcal{A}} \otimes \bar{\mathcal{A}}$, where k is the trivial $\bar{\mathcal{A}}$-comodule, is a coboundary. Notice in addition that trivially, $\bar{\mathcal{A}}$ includes coradical of C. Then the dual argument of the proof of Proposition 3.1 (1) proves the proposition. □

A super-Hopf subalgebra $\mathcal{H}$ of a super-hyperalgebra $\mathcal{A}$ is said to be *normal*, if

$$\sum (-1)^{|h||a_2|} a_1 h S(a_2) \in \mathcal{H}$$

for all $h \in \mathcal{H}$, $a \in \mathcal{A}$. The following will be a key to prove Theorem 5.1.

Proposition 4.3 *Let*

$$\begin{tikzcd}
\mathcal{H} \arrow[r, hook] & \mathcal{A} \\
\bar{\mathcal{H}} \arrow[u, hook] \arrow[r, hook] & \bar{\mathcal{A}} \arrow[u, hook]
\end{tikzcd}$$

be a pullback diagram of monomorphisms of super-hyperalgebras. Suppose that $\mathcal{H}$ is normal in $\mathcal{A}$, and $\bar{\mathcal{H}}$ and $\bar{\mathcal{A}}$ are both smooth hyperalgebras. If

$$\mathcal{A}/\bar{\mathcal{A}}^{+}\mathcal{A} \otimes_{\mathcal{H}} \mathcal{H}/\mathcal{H}^{+} \simeq k, \tag{17}$$

then the induced map $\mathcal{H}/\bar{\mathcal{H}}^{+}\mathcal{H} \to \mathcal{A}/\bar{\mathcal{A}}^{+}\mathcal{A}$ of super-right $\mathcal{H}$-module coalgebras is an isomorphism.

PROOF. Since $\mathcal{H}$ is normal in $\mathcal{A}$, we have a quotient super-hyperalgebra, $\mathcal{B} := \mathcal{A}/\mathcal{H}^{+}\mathcal{A}$. Since the diagram is pullback, $\bar{\mathcal{H}}$ is normal in $\bar{\mathcal{A}}$, so that we have a quotient hyperalgebra of $\bar{\mathcal{A}}$, $\bar{\mathcal{B}} := \bar{\mathcal{A}}/\bar{\mathcal{H}}^{+}\bar{\mathcal{A}}$.

First we claim that the induced map $\bar{\mathcal{B}} \to \mathcal{B}$ of super-hyperalgebras is injective. Let $\mathcal{D}$ denote the image of $\bar{\mathcal{B}}$ in $\mathcal{B}$. We apply Proposition 1.4 (3) to have $\mathcal{H} = \mathcal{A}^{\mathrm{co}\,\mathcal{B}}$; this is possible since hypersubalgebras are in particular connected Hopf algebras in $\mathbf{YD}_{\mathbb{Z}_2}^{\mathbb{Z}_2}$. Since we thus have

$$\bar{\mathcal{H}} = \mathcal{H} \cap \bar{\mathcal{A}} = \mathcal{A}^{\mathrm{co}\,\mathcal{B}} \cap \bar{\mathcal{A}} = \bar{\mathcal{A}}^{\mathrm{co}\,\mathcal{D}},$$

it follows again by Proposition 1.4 (3) (or [10, Thm. 1.3 (3)] in the ungraded context) that $\mathcal{B} = \mathcal{D}$, which proves the claim.

Next, we claim that the quotient $\mathcal{A} \to \mathcal{B}$ induces an isomorphism

$$\mathcal{A}/\bar{\mathcal{A}}^+\mathcal{A} \otimes_{\mathcal{H}} \mathcal{H}/\mathcal{H}^+ \simeq \mathcal{B}/\bar{\mathcal{B}}^+\mathcal{B} \tag{18}$$

of super-right $\mathcal{B}$-module coalgebras. Since $\mathcal{H}$ is normal in $\mathcal{A}$, $\bar{\mathcal{H}}^+\mathcal{A} \subset \mathcal{H}^+\mathcal{A} = \mathcal{A}\mathcal{H}^+$, and hence

$$\mathcal{B} = \mathcal{A}/\mathcal{A}\mathcal{H}^+ = k \otimes_{\bar{\mathcal{H}}} \mathcal{A}/\mathcal{A}\mathcal{H}^+ = \mathcal{A}/\bar{\mathcal{H}}^+\mathcal{A} \otimes_{\mathcal{H}} k. \tag{19}$$

Here and in the following, k is regarded as trivial modules along the counits. Since $\mathcal{A} \simeq \bar{\mathcal{A}} \otimes \mathcal{A}/\bar{\mathcal{A}}^+\mathcal{A}$ as in (16), we have $\mathcal{A}/\bar{\mathcal{H}}^+\mathcal{A} \simeq \bar{\mathcal{B}} \otimes \mathcal{A}/\bar{\mathcal{A}}^+\mathcal{A}$ in ${}_{\bar{\mathcal{B}}}\mathbf{S}$. Hence the quotient $\mathcal{A}/\bar{\mathcal{H}}^+\mathcal{A} \to \mathcal{A}/\bar{\mathcal{A}}^+\mathcal{A}$ induces an isomorphism $k \otimes_{\bar{\mathcal{B}}} \mathcal{A}/\bar{\mathcal{H}}^+\mathcal{A} \simeq \mathcal{A}/\bar{\mathcal{A}}^+\mathcal{A}$. This together with (19) prove the claim.

We have the following commutative diagram of connected super-cocommutative coalgebras:

$$\begin{tikzcd}
\mathcal{H}/\bar{\mathcal{H}}^+\mathcal{H} \arrow[r] & \mathcal{A}/\bar{\mathcal{A}}^+\mathcal{A} & \\
\mathcal{H} \arrow[u, two heads] \arrow[r, hook] & \mathcal{A} \arrow[u, two heads] \arrow[r, two heads] & \mathcal{B} \\
\bar{\mathcal{H}} \arrow[u, hook] \arrow[r, hook] & \bar{\mathcal{A}} \arrow[u, hook] \arrow[r, two heads] & \bar{\mathcal{B}}. \arrow[u, hook]
\end{tikzcd}$$

Apply Sp^* to this, and take values in a connected super-cocommutative coalgebra C. Then we have a commutative diagram of groups and pointed sets,

$$\begin{tikzcd}
\bar{N}\backslash N \arrow[r] & \bar{G}\backslash G & \\
N \arrow[u, two heads] \arrow[r, hook] & G \arrow[u, two heads] \arrow[r] & F \\
\bar{N} \arrow[u, hook] \arrow[r, hook] & \bar{G} \arrow[u, hook] \arrow[r, two heads] & \bar{G}/\bar{N}, \arrow[u, hook]
\end{tikzcd}$$

with exact rows and columns. Since $\mathcal{A} \simeq \bar{\mathcal{A}} \otimes \mathcal{A}/\bar{\mathcal{A}}^+\mathcal{A}$ especially as super-coalgebras, $\mathrm{Sp}^*\,\mathcal{A}(C) \to \mathrm{Sp}^*(\mathcal{A}/\bar{\mathcal{A}}^+\mathcal{A})(C)$ is surjective, and so $\mathrm{Sp}^*(\mathcal{A}/\bar{\mathcal{A}}^+\mathcal{A})(C) = \bar{G}\backslash G$ with $G = \mathrm{Sp}^*\,\mathcal{A}(C)$, $\bar{G} = \mathrm{Sp}^*\,\bar{\mathcal{A}}(C)$. Similarly we have $\bar{N}\backslash N$ and $\bar{G}/\bar{N}$. By (18), the assumption (17) implies that $\bar{\mathcal{B}} \hookrightarrow \mathcal{B}$ and hence $\bar{G}/\bar{N} \hookrightarrow F$ are isomorphisms. The snake lemma proves that $\bar{N}\backslash N \to \bar{G}\backslash G$ is bijective, which proves the proposition. $\square$

Let A be an affine super-Hopf algebra. Since A is finitely generated, each $A/(A^+)^n$ $(n > 0)$ is finite-dimensional, so that the dual vector space $(A/(A^+)^n)^*$ is a connected super-cocommutative coalgebra, and

$$A' := \bigcup_{n>0} (A/(A^+)^n)^* \quad [15, \text{Sect. } 9.2]$$

forms a super-hyperalgebra; see [12, Sect. 2]. We suppose until the end of this section that k is algebraically closed. Then we have such an isomorphism $A \simeq \bar{A} \otimes K$ as given in (13). Let us write

$$\mathcal{A} = A', \quad \bar{\mathcal{A}} = \bar{A}'. \tag{20}$$

Since K^+ is nilpotent, $K' = K^*$. By [24, Prop. 3.3.5], $\bar{\mathcal{A}}$ is a smooth hyperalgebra. By applying $(\ )'$ to the last isomorphism, we obtain such an isomorphism $\mathcal{A} \simeq \bar{\mathcal{A}} \otimes K^*$ as given in (16). In particular we have a canonical isomorphism

$$\mathcal{A}/\bar{\mathcal{A}}^+\mathcal{A} \simeq K^* \tag{21}$$

of super-right $\mathcal{A}$-module coalgebras.

If $\operatorname{ch} k = 0$, $\mathcal{A} = U(\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the super-Lie algebra of primitives in $\mathcal{A}$. This is the super-Lie algebra of the algebraic super-affine group $\operatorname{Sp} A$, while $\mathfrak{g}_0$ is the Lie algebra of the algebraic affine group $\operatorname{Sp} A_{\mathrm{un}} = \operatorname{Sp} \bar{A}$. We have $\bar{\mathcal{A}} = U(\mathfrak{g}_0)$, and

$$\mathcal{A} = U(\mathfrak{g}) \simeq U(\mathfrak{g}_0) \otimes \bigwedge(\mathfrak{g}_1) \tag{22}$$

as super-left $U(\mathfrak{g}_0)$-module coalgebras [14, Thm. 5.15]. The natural map $K \to \mathcal{A}^*$ induces a counit-preserving isomorphism

$$K \simeq (\bigwedge(\mathfrak{g}_1))^* = \bigwedge(\mathfrak{g}_1^*) \tag{23}$$

of super-algebras; see [1, Thm. 1], [12, Prop. 2.4].

Let $\pi : A \to H$ be an epimorphism of affine super-Hopf algebras in arbitrary characteristic $\neq 2$. Let $\bar{A}, \bar{H}, K, L, \mathcal{A}, \bar{\mathcal{A}}$ be as in (11), (12), (14), (20), and define $\mathcal{H} = H'$, $\bar{\mathcal{H}} = \bar{H}'$.

Lemma 4.4 *Suppose that π is conormal. If*

$$K^*/K^*\mathcal{H}^+ \simeq k, \tag{24}$$

then the natural epimorphism $K \to L$ of super-right H-comodule algebras (see Proposition 3.4) is an isomorphism.

PROOF. The dual of $K \to L$ is identified with

$$\mathcal{L} := \mathcal{H}/\bar{\mathcal{H}}^{+}\mathcal{H} \to \mathcal{K} := \mathcal{A}/\bar{\mathcal{A}}^{+}\mathcal{A},$$

which is hence injective; see (21). The lemma follows from Proposition 4.3, if we prove that $\mathcal{H}$ is normal in $\mathcal{A}$, and $\bar{\mathcal{H}} = \mathcal{H} \cap \bar{\mathcal{A}}$. The last equation follows, since $\mathcal{L} \to \mathcal{K}$ is injective, and hence

$$\bar{\mathcal{H}} = \mathcal{H}^{\mathrm{co}\,\mathcal{L}} = \mathcal{H}^{\mathrm{co}\,\mathcal{K}} = \mathcal{H} \cap \bar{\mathcal{A}}.$$

For each connected super-cocommutative coalgebra C, we have

$$\mathrm{Sp}^{*}\,\mathcal{A}(C) = \mathrm{Ker}(\mathrm{Sp}\,A(C^{*}) \to \mathrm{Sp}\,A(k)),$$

where the group homomorphism arises from the dual $C^{*} \to k$ of the unique coalgebra map $k \to C$. To see this we suppose $\dim C < \infty$, and see that the super-coalgebra maps $C \to \mathcal{A}$ are in 1-1 correspondence with the super-algebra maps $A \to C^{*}$ annihilating some $(A^{+})^{n}$, which are precisely the counit-preserving super-algebra maps $A \to C^{*}$. It now follows that

$$\begin{array}{ccc} \mathrm{Sp}^{*}\,\mathcal{A}(C) & \subset & \mathrm{Sp}\,A(C^{*}) \\ \cup & & \cup \\ \mathrm{Sp}^{*}\,\mathcal{H}(C) & \subset & \mathrm{Sp}\,H(C^{*}) \end{array}$$

is a pullback diagram of groups. Since the right vertical inclusion is normal, the left one is, too. This proves that $\mathcal{H}$ is normal in $\mathcal{A}$. $\square$

Remark 4.5 If $\mathrm{ch}\,k = 0$, the lemma follows more easily. Let $\mathfrak{g}, \mathfrak{h}$ be the super-Lie algebras of primitives in $\mathcal{A}, \mathcal{H}$, respectively. Since $\mathcal{H}$ is normal in $\mathcal{A}$, $\mathfrak{h}$ is a super-Lie ideal in $\mathfrak{g}$, so that $\mathfrak{b} := \mathfrak{g}/\mathfrak{h}$ is a quotient super-Lie algebra of $\mathfrak{g}$. By (22), (23),

$$K^{*}/K^{*}\mathcal{H}^{+} = k \otimes_{U(\mathfrak{g}_0)} U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} k = k \otimes_{U(\mathfrak{b}_0)} U(\mathfrak{b}) = \bigwedge(\mathfrak{b}_1).$$

Therefore, $\mathfrak{b}_1 = 0$ under the assumption (24). This means $\mathfrak{h}_1 = \mathfrak{g}_1$, or $K = \bigwedge(\mathfrak{g}_1)^{*} \simeq \bigwedge(\mathfrak{h}_1)^{*} = L$.

5 Injectivity along conormal quotients

To complete the proof of Theorem 2.8, we prove:

Theorem 5.1 *Let A be a super-commutative Hopf algebra, and let $\pi : A \to H$ be a conormal quotient super-Hopf algebra. Set $B = {}^{\mathrm{co}\,H}A$ $(= A^{\mathrm{co}\,H})$, the super-Hopf subalgebra of H-coinvariants. Then, $A/AB^{+} = H$.*

Recall from the paragraph following Theorem 2.8 that the theorem above is equivalent to say that A is an injective cogenerator as a left or right H-comodule.

To prove the theorem we may suppose by base extension that k is algebraically closed. We may also suppose that A and hence H are affine, since π is a directed union of conormal quotients of affine super-Hopf algebras. Obviously, π induces a conormal quotient $A/AB^+ \to H$. The equalizer diagram

$$0 \to B \to A \rightrightarrows H \otimes A$$

is in $\mathbf{S}_B^A$, where the structure of $H \otimes A$ arises from the factor A. Since every object in $\mathbf{S}_B^A$ is right B-projective by Corollary 2.4, the diagram

$$0 \to k \to A/AB^+ \rightrightarrows H \otimes A/AB^+$$

obtained by applying $-\otimes_B B/B^+$ still gives an equalizer. Thus, $J := A/AB^+ \to H$ is a conormal quotient of an affine super-Hopf algebra J with ${}^{\mathrm{co}\,H}J = k$. Therefore the theorem follows from the next lemma.

Lemma 5.2 *Suppose that k is algebraically closed. Let $\pi : A \to H$ be a conormal quotient super-Hopf algebra of an affine super-Hopf algebra A such that ${}^{\mathrm{co}\,H}A = k$. Then π is an isomorphism.*

PROOF. We first prove that the lemma reduces to the case when H is connected.

Recall from Proposition 3.3 (2) that $H \simeq \pi_0(H) \otimes H_{\mathrm{c}}$. We will write $R = \pi_0(H)$, so that $R = k^\Gamma$ with Γ a finite group. We claim that the composite $A \xrightarrow{\pi} H \to H_{\mathrm{c}}$ is conormal. Since π is conormal, $a \mapsto \sum(-1)^{|a_1||a_2|}a_2 \otimes S(a_1)a_3$ induces a map, say, $\alpha : H \to H \otimes A$ of super-commutative algebras, we see as in the proof of Proposition 3.3 (1) that $\alpha(\pi_0(H)) \subset \pi_0(H) \otimes \pi_0(A)$, and $\alpha(\pi_0(H)^+) \subset \pi_0(H)^+ \otimes A$. Therefore, α induces $H_{\mathrm{c}} \to H_{\mathrm{c}} \otimes A$; this proves the claim.

Suppose that the lemma holds true when H is connected. Then Theorem 5.1 holds true for $A \to H_{\mathrm{c}}$, and hence A is left (and right) H_{c}-injective. Set $D = A^{\mathrm{co}\,H_{\mathrm{c}}}$. By the opposite version of Proposition 1.3 that we have an equivalence $\mathbf{S}_A^{H_{\mathrm{c}}} \approx \mathbf{S}_D$. Hence the lemma in the general case will follow if we prove that the quotient $D \to R = H^{\mathrm{co}\,H_{\mathrm{c}}}$ arising from π is an isomorphism, or that $\dim D = \dim R$. We see that D is an ordinary right R-comodule algebra such that the so-called Galois map $D \otimes D \to D \otimes R$ is surjective. It follows by the Kreimer-Takeuchi theorem [15, Thm. 8.3.1] that $D \supset D^{\mathrm{co}\,R}$ is an R-Galois extension, and D is a left (and right) finitely generated (projective) module over $D^{\mathrm{co}\,R}$. It remains to prove that $D^{\mathrm{co}\,R} = k$, since this implies that the Galois map above is bijective, and so $\dim D = \dim R$.

We claim that for each $V \in \mathbf{S}^H$, the obvious inclusion $V^{\mathrm{co}\,H} \subset (V^{\mathrm{co}\,H_c})^{\mathrm{co}\,R}$ is the identity. In fact we have an exact sequence

$$0 \to V \to V_1 \to V_2$$

in $\mathbf{S}^H$, where V_i $(i = 1, 2)$ are the direct sums of some copies of the cogenerator $\mathbb{Z}_2 \blacktriangleright\!\!< H$. Since the claim is obviously true for V_i, the left exactness of the coinvariant functors proves the claim. In particular, $D^{\mathrm{co}\,R} = A^{\mathrm{co}\,H} = k$, as desired.

To prove the lemma when H is connected, we prove:

Lemma 5.3 *With the notation of Lemma 5.2, suppose in addition that H is connected. Let $\bar{A}, \bar{H}$ be as in (11), (14). Then the natural epimorphism $K = {}^{\mathrm{co}\,\bar{A}}A \to L = {}^{\mathrm{co}\,\bar{H}}H$ of super-right H-comodule algebras (see Proposition 3.4) is an isomorphism.*

PROOF. By Lemma 4.4, it suffices to prove $K^*/K^*\mathcal{H}^+ \simeq k$, where $\mathcal{H} = H'$.

Since H is connected, $\bar{H}$ is a Noetherian integral domain, so that $\bigcap_{n>0}(\bar{H}^+)^n = 0$ by the Krull intersection theorem. Since L^+ is nilpotent, $\bigcap_{n>0}(H^+)^n = 0$. Since $A^{\mathrm{co}\,H} = k$ by assumption, $K^{\mathrm{co}\,H} = k$. Since K is finite-dimensional, its image under the map $K \to K \otimes H$ given by

$$x \mapsto x \otimes \pi(1) - \sum x_1 \otimes \pi(x_2)$$

trivially intersects with $K \otimes (H^+)^m$ for $m \gg 0$. This implies that the super-subalgebra $\subset K$ of coinvariants under the induced $K \to K \otimes H/(H^+)^m$ still equals k. Dualizing this we see $K^*/K^*\mathcal{H}^+ \simeq k$, as desired. $\square$

Proof of Lemma 5.2 (continued). Suppose that H is connected. By Lemma 5.3, $K \simeq L$. Hence, A is naturally an object in $\mathbf{S}^H_L$.

For each object M in $\mathbf{S}^H_L$, we have a natural map

$$\chi : M^{\mathrm{co}\,H} \to (M/ML^+)^{\mathrm{co}\,\bar{H}}.$$

We claim that this is an isomorphism. Since H is right L-free, it is left L-faithfully flat by Corollary 2.4. By Proposition 1.1 we have the equivalence $\mathbf{S}^H_L \approx \mathbf{S}^{\bar{H}}$. Since $\mathbb{Z}_2 \otimes \bar{H}$ is a cogenerator in $\mathbf{S}^{\bar{H}}$, $\mathbb{Z}_2 \otimes H$ is a cogenerator in $\mathbf{S}^H_L$. Hence we have an exact sequence

$$0 \to M \to M_1 \to M_2 \tag{25}$$

in $\mathbf{S}^H_L$, where M_i $(i = 1, 2)$ are the direct sums of some copies of $\mathbb{Z}_2 \otimes H$. The maps χ for M_i are obviously isomorphisms. By the equivalence above the sequence (25) remains exact after applying $-\otimes_L L/L^+$, and then $(\ )^{\mathrm{co}\,\bar{H}}$. This implies that the map χ for M is an isomorphism, too.

When $M = A$, we have $k = A^{\mathrm{co}\,H} \simeq \bar{A}^{\mathrm{co}\,\bar{H}}$. Thus, $\bar{A} \to \bar{H}$ is a conormal quotient Hopf algebra of an affine Hopf algebra $\bar{A}$, such that $\bar{A}^{\mathrm{co}\,\bar{H}} = k$. It follows by Takeuchi's correspondence [23, Thm. 4.3] cited in the Introduction that $\bar{A} = \bar{H}$. The equivalence above proves that π is an isomorphism. $\square$

Acknowledgements

This work was done while I was visiting the Institut Mittag-Leffler, February 15-April 20, 2004. I am very grateful to the Royal Swedish Academy of Sciences for the financial support, and to all the staffs of the institute for their warm hospitality. My special thanks to Hans-Jürgen Schneider for his longstanding encouragement to me, including during this work.

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