2-categorical K-theories

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2-categorical K-theories.

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Abstract

For bicategories we define the associated charted 2-bundles and hence $2K$-theories. For the bicategory of 2-vector spaces in the sense of Kapranov and Voevodsky this gives the $2K$-theory of Baas, Dundas and Rognes.

Restricting to 2-categories, we provide assumptions sufficient to show that the 2-nerve is a classifying space. This applies to examples based on the 2-category of 2-vector spaces in the sense of Baez, and our calculations show that their $2K$-theory is just two copies of ordinary first order K-theory.

1 Introduction

In this paper we extend the point of view on 2-vector bundles introduced in [BDR04]. Let $2\mathcal{C}$ be a discrete, topological or simplicial bicategory. We introduce a general notion of a $2\mathcal{C}$-bundle, and functorialize the definition. Restricting to 2-categories, we present a general setting for studying these bundles, and give a sufficient condition on $2\mathcal{C}$ such that is 2-nerve (geometric nerve) is a classifying space for $2\mathcal{C}$-bundles. This relates charted and represented $2\mathcal{C}$-bundles.

To the various notion of $2\mathcal{C}$-bundles there are associated K-type cohomology theories. For $2\mathcal{C} =$ 2-Vect in the sense of Kapranov and Voevodsky we know that the $2K$-theory of [BDR04] is represented by the (Ω^∞)-spectrum $K(ku)$ - which is essentially of chromatic filtration two and hence represents a homotopy theoretic model of elliptic cohomology.

However, the structure of these theories vary according to whether $2\mathcal{C}$ is a 2-groupoid (the strict version of Duskin's bigroupoid) or not. In the

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examples just mentioned this is not the case, and this is what leads to group-completions and the use of algebraic K-theory.

If $2\mathcal{C}$ is a sufficiently fibrant (to be defined) 2-groupoid we obtain second order K-theories which can be calculated using standard techniques similar to those applied to ordinary K-theories. In particular if $2\mathcal{C} = 2$-Vect in the sense of Baez (objects defined as a category in Vect), then the corresponding "second-order" K-theory is just equivalent to two copies of ordinary K-theory - which is of chromatic filtration one, and hence not a model of elliptic cohomology.

Clearly there may be other interesting 2-categories (or bicategories) to consider together with their associated $2K$-theories in the above sense. Therefore we think that the general setting developed here may be useful in such studies. In particular, this article does not rule out the possibility that there might exist a 2-groupoid giving a $2K$-theory of chromatic filtration 2.

Recall that bicategories, as defined first in [Bén67], are a weakened form of 2-categories (also known as strict 2-categories, i.e. a category enriched in categories), given by relaxing the associativity of the composition.

2 Charted 2-bundles

In this section we formulate equivalent ways to define a charted $2\mathcal{C}$-bundle. Here $2\mathcal{C}$ will be a discrete, topological or simplicial bicategory. This includes the cases where the objects form topological spaces or simplicial sets.

We start with an explicit definition which specializes to the chartered 2-vector bundles of [BDR04] for a suitably chosen $2\mathcal{C}$. An *ordered open cover* $(\mathcal{U}, \mathcal{I})$ of X, as defined in [BDR04], is a cover $\mathcal{U}$ of X indexed by a partial ordered set $\mathcal{I}$ that restricts to a total ordering on each finite subset $\{\alpha_1, \ldots, \alpha_k\}$ of $\mathcal{I}$ whenever $U_{\alpha_1 \cdots \alpha_k} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$ is nonempty. We use the notation $\mathcal{U}^k$ for the collection of k-fold intersections of sets in $\mathcal{U}$. A cover $\mathcal{U}$ is *good* if each finite intersection $U_{\alpha_1 \cdots \alpha_k}$ is either empty or contractible.

Definition 2.1 *Let X be a topological space, with an ordered open cover $(\mathcal{U}, \mathcal{I})$, and let $2\mathcal{C}$ be a discrete, topological, or simplicial bicategory. A charted $2\mathcal{C}$-bundle $\mathcal{E}$ over X consists of*

(1) for each U_α in $\mathcal{U}$ a bundle V^α over U_α consisting of objects in $2\mathcal{C}$, and

(2) for each $U_{\alpha\beta}$ in $\mathcal{U}^2$, $\alpha < \beta$, a bundle of 1-morphisms in $2\mathcal{C}$

$$E^{\alpha\beta} : V^\alpha \to V^\beta \quad ,$$

and

(3) for each $U_{\alpha\beta\gamma}$ in $\mathcal{U}^3$, $\alpha < \beta < \gamma$, a bundle of 2-morphisms in $2\mathcal{C}$

$$\phi^{\alpha\beta\gamma} : E^{\alpha\beta} \cdot E^{\beta\gamma} \to E^{\alpha\gamma} \quad ,$$

such that

(4) the diagram

$$\begin{array}{ccccc} E^{\alpha\beta}\cdot(E^{\beta\gamma}\cdot E^{\gamma\delta}) & & \xrightarrow{\underline{\alpha}} & & (E^{\alpha\beta}\cdot E^{\beta\gamma})\cdot E^{\gamma\delta} \\ \downarrow{\scriptstyle id\cdot\phi^{\beta\gamma\delta}} & & & & \downarrow{\scriptstyle \phi^{\alpha\beta\gamma}\cdot id} \\ E^{\alpha\beta}\cdot E^{\beta\delta} & \xrightarrow[\phi^{\alpha\beta\delta}]{} & E^{\alpha\delta} & \xleftarrow[\phi^{\alpha\gamma\delta}]{} & E^{\alpha\gamma}\cdot E^{\gamma\delta} \end{array}$$

commutes over $U_{\alpha\beta\gamma\delta}$, for each chain $\alpha < \beta < \gamma < \delta$ in $\mathcal{I}$. The map $\underline{\alpha}$ is the natural associativity isomorphism of the bicategory $2\mathcal{C}$.

Here the term "bundle" needs both an explanation and a justification. Consider the case where $2\mathcal{C}$ is topological. A bundle of objects (1-morphisms, or 2-morphisms respectively) in $2\mathcal{C}$ should be a locally trivial family of objects (1-morphisms, or 2-morphisms respectively). However, for general $2\mathcal{C}$ it is unclear how to give the local triviality condition. Therefore we view the space of objects (1-morphisms, or 2-morphisms respectively) as a classifying space for the appropriate kind of bundle. Thus an object bundle over over U_α is precisely a continuous map $U_\alpha \to \mathrm{Obj}(2\mathcal{C})$. If $2\mathcal{C}$ is simplicial we interpret object and morphism bundles as maps into the geometric realizations of $\mathrm{Obj}(2\mathcal{C})$ and $\mathrm{Mor}(2\mathcal{C})$ respectively. For discrete bicategories $2\mathcal{C}$, we regard the object and morphism sets as discrete topological spaces.

Example 2.2 *We recover gerbes as the charted $2\mathcal{C}$-bundles of the following topological bicategory: Let the objects of $2\mathcal{C}$ be the space having a single point $*$. Define the 1-morphisms to be $\mathbb{CP}^\infty$, let 2-morphisms be linear isomorphisms, and use the tensor product to define composition.*

Remark 2.3 *When $2\mathcal{C}$ is a 2-category the cocycle condition in definition above simplifies since the natural associativity isomorphism $\underline{\alpha}$ in this case is an identity. Most of this article concerns the case of 2-categories. Hence we will from now on assume that $2\mathcal{C}$ is a 2-category, unless we explicitly say otherwise.*

To an ordered cover $(\mathcal{U}, \mathcal{I})$ of X we associate the Čech complex $\check{C}_\bullet$, the ordered Čech complex $U_\bullet$ and Segal's topological category $X_\mathcal{U}$. These are defined as follows: $\check{C}_\bullet$ and $U_\bullet$ are 1-coskeletal simplicial spaces (and thus determined by their 0- and 1-simplices, see [Bek04, proposition 3.1]). We define $\check{C}_0$ and $\check{C}_1$ to be

$$\coprod_\alpha U_\alpha \quad \text{and} \quad \coprod_{\alpha,\beta} U_{\alpha\beta}$$

respectively. $U_\bullet$ is the sub-simplicial space of $\check{C}_\bullet$ having the same 0-simplices, but the 1-simplices being the disjoint union of all $U_{\alpha\beta}$'s with $\alpha \leq \beta$. The space of objects of $X_{\mathcal{U}}$ is the disjoint union of all U_S where S is a finite set of indices. The space of morphisms is the disjoint union of U_S over all pairs $R \subseteq S$, with S as above. The source map forgets R, and the target map is induced from the inclusions $U_S \subseteq U_R$.

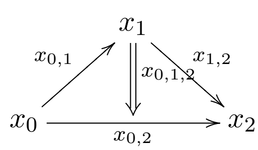
From theorem 2.1, proposition 2.6 and proposition 2.7 in [DI04] we have that the spaces

$$|\check{C}_\bullet| \quad , \quad |U_\bullet| \quad \text{and} \quad |N_\bullet X_{\mathcal{U}}|$$

all are weakly equivalent to X via the natural maps.

Definition 2.4 *To an ordered cover $(\mathcal{U}, \mathcal{I})$ of X we can also associate a topological 2-category $2X_{\mathcal{U}}$. We shall denote a finite nonempty increasing chain $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ in $\mathcal{I}$ by α_*. Define now the space of objects of the 2-category as pairs $(\alpha; x)$ where α lies in $\mathcal{I}$ and $x \in U_\alpha$. Let the 1-morphisms be pairs $(\alpha_*; x)$ where α_* is as above and $x \in U_{\alpha_1\alpha_2\cdots\alpha_k}$, and finally let the 2-morphisms be triples $(\alpha_*, \beta_*; x)$ where β_* is a subchain of α_* with $\beta_1 = \alpha_1$ and $\beta_l = \alpha_k$, and $x \in U_{\alpha_1\alpha_2\cdots\alpha_k}$. Source and target of 1-morphisms are restriction to α_1 and α_k respectively. Identity 1-morphisms have chains of length 1, and composition is given by joining the chains. Source and target of 2-morphisms is restricting to the α_* or β_* chain respectively, 2-identities have $\alpha_* = \beta_*$, vertical composition is removal of the middle chain, and horizontal composition is joining of chains.*

Recall from [Str87], [BC03], or [Dus02], the notion of the geometric nerve $\Delta 2\mathcal{C}$ of $2\mathcal{C}$. It is well-defined for all bicategories, but for our purposes we only consider strict 2-categories. Moreover, we use a modified version $\Delta' 2\mathcal{C}$ with opposite 2-morphisms. It has 0-simplices the objects x_0 of $2\mathcal{C}$. The 1-simplices are the 1-morphisms $x_0 \xrightarrow{x_{0,1}} x_1$ of $2\mathcal{C}$, and the 2-simplices are triangles



where $x_{0,1,2}$ is a 2-morphism $x_{1,2} \circ x_{0,1} \to x_{0,2}$. For $n \geq 3$ the n-simplices are build from 2-simplices, such that for each subtetrahedron the obvious coherence condition for 2-morphisms is satisfied. The geometric nerve is 3-coskeletal. If $2\mathcal{C}$ is topological or simplicial, then $\Delta' 2\mathcal{C}$ becomes a simplicial space or a bisimplicial set respectively.

Theorem 2.5 *Let $(\mathcal{U}, \mathcal{I})$ be an ordered open cover over X. Then there are one-to-one correspondences between*

charted $2\mathcal{C}$-bundles over X subordinate to the given ordered open cover,

continuous strict 2-functors $2X_{\mathcal{U}} \to 2\mathcal{C}$, and *simplicial maps $U_\bullet \to \Delta' 2\mathcal{C}$.*

Proof: The proof is to inspect the definitions. Assume that $\alpha < \beta < \gamma < \delta$ in $\mathcal{I}$. From a strict 2-functor F we recover V^α, $E^{\alpha\beta}$ and $\phi^{\alpha\beta\gamma}$ of the charted $2\mathcal{C}$-bundle at a point x in U_α, $U_{\alpha\beta}$, or $U_{\alpha\beta\gamma}$ by restricting F to the object, 1-morphism, or 2-morphism $(\alpha; x)$, $(\alpha < \beta; x)$, and $(\alpha < \beta < \gamma, \alpha < \gamma; x)$ respectively. To get the cocycle condition at a point x in $U_{\alpha\beta\gamma\delta}$ apply F to the following commutative diagram of 2-morphisms in $2X_{\mathcal{U}}$:

$$\begin{array}{ccc} (\alpha < \beta < \gamma < \delta; x) & \xrightarrow{\text{forget } \beta} & (\alpha < \gamma < \delta; x) \\ {\scriptstyle \text{forget } \gamma}\downarrow & & \downarrow{\scriptstyle \text{forget } \gamma} \\ (\alpha < \beta < \delta; x) & \xrightarrow{\text{forget } \beta} & (\alpha < \delta; x) \end{array}.$$

From a charted $2\mathcal{C}$-bundle one can construct F by first defining it on objects, 1-morphisms, and 2-morphisms of the form $(\alpha; x)$, $(\alpha < \beta; x)$, and $(\alpha < \beta < \gamma, \alpha < \gamma; x)$. The construction is completed by using strictness of F and the fact that $2X_{\mathcal{U}}$ is generated by these 1- and 2-morphisms.

A map $U_\bullet \to \Delta' 2\mathcal{C}$ is uniquely determined by the maps of k-simplices for $k \leq 3$. For the correspondence to charted $2\mathcal{C}$-bundles the map on 0-simplices, 1-simplices, and 2-simplices give the V^α's, the $E^{\alpha\beta}$'s, and the $\phi^{\alpha\beta\gamma}$'s respectively. The coherence condition on tetrahedrons in $\Delta' 2\mathcal{C}$ corresponds to the cocycle condition of the charted $2\mathcal{C}$-bundle. $\square$

Remark 2.6 *It is probably possible to generalize definition 2.4 to a topological bicategory $2X_{\mathcal{U}}^w$. The idea is letting the 1-morphisms be nonassociatively freely generated by pairs $(\alpha_*; x)$ where the chain has length 2. Thus it should also be possible to extend the theorem above to all bicategories.*

Given a charted $2\mathcal{C}$-bundle $\mathcal{E}$ over X and a subspace A of X it is clear that we may restrict $\mathcal{E}$ to A. We denote the restriction by $\mathcal{E}|_A$. More generally, there is a pullback $f^*\mathcal{E}$ over Y, for every continuous map $f : Y \to X$.

We define equivalence of charted $2\mathcal{C}$-bundles by concordance:

Definition 2.7 *Two charted $2\mathcal{C}$-bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ over X are equivalent (or concordant) if there exists a charted $2\mathcal{C}$-bundle $\mathcal{E}$ over $X \times I$ such that $\mathcal{E}|_{X\times\{i\}} = \mathcal{E}_i$, $i = 0, 1$.*

3 Classifying spaces

An important tool for the study of bundles is the classifying space. Restricting to the cases where $2\mathcal{C}$ is a 2-groupoid the 2-nerves gives classifying spaces in the discrete case. For the topological and simplicial case we need in addition a fibrancy condition on $2\mathcal{C}$. Let us begin by defining what a classifying space is:

Definition 3.1 *If there exists a space B such that for all finite CW-complexes X there is a one-to-one correspondence between equivalence classes of charted $2\mathcal{C}$-bundles over X and homotopy classes of maps $X \to B$, then we say that B is a classifying space for charted $2\mathcal{C}$-bundles. We denote such classifying spaces by $B2\mathcal{C}$.*

Now restrict to the case where $2\mathcal{C}$ is a 2-groupoid, i.e a 2-category where all 1-morphisms are weak equivalences and all 2-morphisms are isomorphisms. In **the discrete case** results by Duskin, Dugger and Isaksen, and Bullejos and Cegarra yields a theory for the classifying spaces of charted $2\mathcal{C}$-bundles. We now review this, and summarizing their results we get the following theorem:

Theorem 3.2 *Let $2\mathcal{C}$ be a discrete 2-groupoid, then $|\Delta' 2\mathcal{C}| \simeq |N_{\bullet\bullet} 2\mathcal{C}|$ is a classifying space for charted $2\mathcal{C}$-bundles.*

Observe the following facts:

- The realization of the ordered Čech complex $U_\bullet$ is weakly equivalent to X by [DI04, theorem 2.1 and proposition 2.6].
- The main theorem of [BC03] states that $|\Delta' 2\mathcal{C}| \simeq |N_{\bullet\bullet} 2\mathcal{C}|$ for arbitrary 2-categories $2\mathcal{C}$.
- The geometric nerve $\Delta' 2\mathcal{C}$ of a 2-groupoid is a Kan complex by theorem 8.6 in [Dus02].
- A *refinement* $(\mathcal{U}', \mathcal{I}')$ of an ordered open cover $(\mathcal{U}, \mathcal{I})$ is an order preserving function $f : \mathcal{I}' \to \mathcal{I}$ such that $U'_\alpha \subseteq U_{f(\alpha)}$. Given a charted $2\mathcal{C}$-bundle $\mathcal{E}$ subordinate to $(\mathcal{U}, \mathcal{I})$ the restriction $\mathcal{E}|_{\mathcal{U}'}$ to a refinement is well-defined and concordant to $\mathcal{E}$.
- At least in the case where X is a finite CW-complex, it is clear that every ordered open cover has a good refinement. Hence, every concordance class of charted $2\mathcal{C}$-bundles can be charted subordinate to a good ordered open cover.
- Suppose that two charted $2\mathcal{C}$-bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ are subordinate to the same ordered open cover $(\mathcal{U}, \mathcal{I})$. If their representing simplicial maps $U_\bullet \to \Delta' 2\mathcal{C}$ are homotopic by a simplicial homotopy, then it is obvious that $\mathcal{E}_0$ and $\mathcal{E}_1$ are concordant.

Before completing the proof we need to discuss covers relative to a subspace. Assuming that A is a subspace of X and that $(\mathcal{U}, \mathcal{I})$ an ordered open cover of X, we can restrict the cover to A by defining $(\mathcal{V}, \mathcal{I})$ by the formula $V_\alpha = U_\alpha \cap A$, and using the same indexing set. We say that $(\mathcal{U}, \mathcal{I})$ is *good relative to* A if $(\mathcal{U}, \mathcal{I})$ is good as a cover of X, and each nonempty finite intersection $V_{\alpha_1 \cdots \alpha_k}$ is a deformation retract of $U_{\alpha_1 \cdots \alpha_k}$. Clearly, at least in the case where X is a finite CW-complex and A a subcomplex, all ordered open covers of X have refinements which are good relative to the subspace.

This enables us to formulate the following lemma:

Lemma 3.3 *If $(\mathcal{U}, \mathcal{I})$ is a ordered cover good relative to a neighborhood deformation retract A of X, then there is a bijection between simplicial homotopy classes $[U_\bullet, \Delta' 2\mathcal{C}]_{rel\ A}$ and topological homotopy classes $[X, |\Delta' 2\mathcal{C}|]_{rel\ A}$.*

Proof: Define the simplicial space $Y_\bullet$ to be the quotient of the ordered Čech complex $U_\bullet$ where Y_n is given by collapsing each path component of U_n to a point. The natural quotient map $U_\bullet \to Y_\bullet$ induces a bijection

$$[U_\bullet, \Delta' 2\mathcal{C}]_{\text{rel } A} \xleftarrow{\cong} [Y_\bullet, \Delta' 2\mathcal{C}]_{\text{rel } A}$$

since $\Delta' 2\mathcal{C}$ is a simplicial set.

Observe that both $U_\bullet$ and $Y_\bullet$ have free degeneracies, see [DI04, definition A.4]. Moreover, each $U_n \to Y_n$ is a weak equivalence since $(\mathcal{U}, \mathcal{I})$ is good. By [DI04, corollary A.6] it follows that $|U_\bullet| \simeq |Y_\bullet|$. Hence also $\simeq X$.

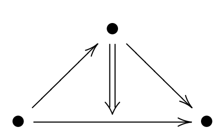
Using that $\Delta' 2\mathcal{C}$ is a fibrant simplicial set, we have that

$$[Y_\bullet, \Delta' 2\mathcal{C}]_{\text{rel } A} \cong [|Y_\bullet|, |\Delta' 2\mathcal{C}|]_{\text{rel } A} \quad .$$

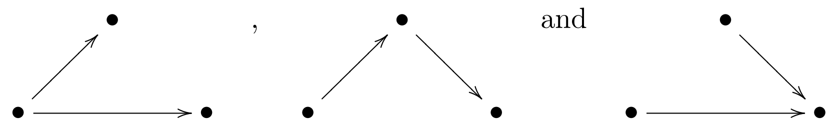
The lemma follows from these facts. □

Without loss of generality we may assume that all ordered open covers are good. A charted $2\mathcal{C}$-bundle corresponds to some map $U_\bullet \to \Delta' 2\mathcal{C}$, and geometric realization yields a topological homotopy class in $[X, |\Delta' 2\mathcal{C}|]$. By the lemma, with $A = \emptyset$, any such class actually comes from a charted $2\mathcal{C}$-bundle. To see that two such bundles mapping to the same class in $[X, |\Delta' 2\mathcal{C}|]$ are concordant, apply the lemma in the case $X \times I$ relative to $X \times \{0, 1\}$. This concludes the proof of theorem 3.2 in the discrete case.

We begin our treatment of **the topological case** by introducing our fibrancy condition on topological 2-groupoids. Let D be the diagram (small 2-category) given by the following picture:



Let H_0, H_1 and H_2 be the subdiagrams



respectively.

Definition 3.4 *A topological* 2*-groupoid* $2\mathcal{C}$ *is called sufficiently Serre fibrant if the source and target maps from* 1*-morphisms to objects are Serre fibrations, and the maps*

$$2\,\mathrm{Funct}_{str}(D, 2\mathcal{C}) \to 2\,\mathrm{Funct}_{str}(H_i, 2\mathcal{C}) \quad \textit{for } i = 0, 1, 2$$

also are Serre fibrations. Similarly, we can define sufficiently Hurewicz fibrant topological 2*-groupiods.*

The consequence, whose proof we have postponed until the appendix, see theorem A.3 and lemma A.4, is the following:

Corollary 3.5 *If* $2\mathcal{C}$ *is a sufficiently Serre fibrant topological* 2*-groupoid, then the bisimplicial set* $\mathrm{Sing}_\bullet\,\Delta' 2\mathcal{C}$ *is Moerdijk fibrant.*

Using this fact, we are able to show the following theorem. The proof is long and technical, and has unfortunately not been written down yet.

Theorem 3.6 *Let* $2\mathcal{C}$ *be a sufficiently Hurewicz fibrant topological* 2*-groupoid, then* $|\Delta' 2\mathcal{C}| \simeq |N_{\bullet\bullet} 2\mathcal{C}|$ *is a classifying space for charted* $2\mathcal{C}$*-bundles.*

Let us call a simplicial 2-groupoid *sufficiently fibrant* if the simplicial maps analogous to those in definition 3.4 are surjective Kan fibrations. Since the geometric realization of a Kan fibration is a Serre fibration, we get classifying spaces in **the simplicial case**:

Corollary 3.7 *Let* $2\mathcal{C}$ *be a sufficiently fibrant simplicial* 2*-groupoid, then* $|\Delta' 2\mathcal{C}| \simeq |N_{\bullet\bullet} 2\mathcal{C}|$ *is a classifying space for charted* $2\mathcal{C}$*-bundles.*

This follows immediately from theorem 3.6.

There is another approach to the classifying space of a charted $\mathcal{C}$-bundle, namely via 2-functors $2X_{\mathcal{U}} \to 2\mathcal{C}$. We now develop this:

Lemma 3.8 *Let* $(\mathcal{U}, \mathcal{I})$ *be an ordered open cover of* X*, then the natural map* $|N_{\bullet\bullet}(2X_{\mathcal{U}})| \to X$ *is a weak equivalence. In particular if* X *is a finite CW-complex, then* $|N_{\bullet\bullet}(2X_{\mathcal{U}})|$ *and* X *are homotopy equivalent.*

We first need a Quillen's Theorem A adopted to the setting of simplicial (or topological) 2-categories. Suitable for our purposes we have the following.

Theorem 3.9 *Let $F_\bullet : 2\mathcal{C}_\bullet \to 2\mathcal{C}'_\bullet$ be a simplicial strict 2-functor between simplicial strict 2-categories. Let F_q be the restriction of $F_\bullet$ to simplicial degree q. If for every q and object y in $2\mathcal{C}'_q$ the classifying space $B(y//F_q)$ is contractible, then*

$$BF : B2\mathcal{C} \to B2\mathcal{C}'$$

is a homotopy equivalence.

It is conceivable that a stronger version can be proved, however that is beyond the scope of this article.

Proof: By Bullejos and Cegarra's theorem 1.2, [BC03], we have that $BF_q : B2\mathcal{C}_q \to B2\mathcal{C}'_q$ is a homotopy equivalence for each q. Realizing the last simplicial direction we get that BF is also a homotopy equivalence. □

If $2\mathcal{C}$ is a strict 2-category, we can form its classifying space $B2\mathcal{C}$ by applying the double nerve, and then realizing the resulting bisimplicial set.

Proof: Recall Segal's topological poset $X_\mathcal{U}$. Its space of objects is the disjoint union

$$\coprod_S U_S \quad ,$$

where S runs through all nonempty finite subsets of $\mathcal{I}$. The space of morphisms is

$$\coprod_{R\subseteq S} U_S \quad ,$$

where R and S are finite nonempty subsets of $\mathcal{I}$. The source map forgets R, while the target map is induced by the inclusions $U_S \hookrightarrow U_R$.

There is a natural map $BX_\mathcal{U} \to X$, and Segal shows that this map is a homotopy equivalence. Our strategy for proving the lemma is to introduce a hybrid between $X_\mathcal{U}$ and $2X_\mathcal{U}$, a topological 2-category denoted by $2\mathcal{C}$. It comes with forgetful topological 2-functors $X_\mathcal{U} \leftarrow 2\mathcal{C} \to 2X_\mathcal{U}$, and we get a commutative diagram

$$\begin{array}{ccc} B2\mathcal{C} & \longrightarrow & B(2X_\mathcal{U}) \\ \downarrow & & \downarrow \\ BX_\mathcal{U} & \xrightarrow{\simeq} & X \end{array} \quad .$$

The singular simplicial complex functor replaces the topological 2-categories with corresponding simplicial 2-categories. In this setting we may apply the weak Quillen's Theorem A to show that $B2\mathcal{C} \to BX_U$ and $B2\mathcal{C} \to B(2X_U)$ are homotopy equivalences. This will complete the proof.

Let us define $2\mathcal{C}$. The space of objects is

$$\{(\alpha, S, x) \in \mathcal{I} \times \mathcal{P}(\mathcal{I}) \times X \quad , \text{ where } \alpha \in S \text{ and } x \in U_S.\}$$

The space of 1-morphisms is

$$\{(\alpha_1 < \cdots < \alpha_k, S, R, x) \quad , \text{ where } R \subseteq S,\ \alpha_1, \dots, \alpha_{k-1} \in S,\ \alpha_k \in R, \text{ and } x \in U_S.\}$$

The 2-morphisms are tuples $(\alpha_1 < \cdots < \alpha_k, \beta_1 < \cdots < \beta_l, S, R, x)$ such that $(\alpha_1 < \cdots < \alpha_k, S, R, x)$ and $(\beta_1 < \cdots < \beta_l, S, R, x)$ are 1-morphisms (the source and target respectively), and β_* is a refinement of α_* with $\beta_1 = \alpha_1$ and $\beta_l = \alpha_k$. It is obvious how to define the two forgetful topological 2-functors into $X_{\mathcal{U}}$ and $2X_U$.

□

4 Calculations

In this section we will investigate possible definitions of 2-vector bundles based on Baez' 2-vector space. However, the result is quite disappointing; the topological K-theory of these 2-vector bundles splits as two copies of the K-theory of ordinary vector bundles. The main reason for this behavior is that Baez' 2-vector spaces, which can be defined as 2 term chain complexes, admits a 2-retraction into the direct sum of two general linear groups. This 2-retraction is given by taking the homology of the 2 term chain complex. Moreover, the 2-morphisms in Baez' 2-vector spaces, i.e. the chain homotopies, between fixed 1-morphisms form affine spaces.

We now list abstract properties giving a sufficient condition for equivalence between the classifying space of a topological bigroupiod $2\mathcal{C}$ and the classifying space of a topological group G: Suppose that $2\mathcal{C}$ is sufficiently fibrant and contains G as a retract. This means that there are continuous 2-functors $G \xrightarrow{i} 2\mathcal{C} \xrightarrow{r} G$ such that ri is the identity on G. Suppose further that 2-morphisms in $2\mathcal{C}$ from f to g is an affine space, and thus 2-automorphisms of f is a vector space. Also assume that whenever $f, g : y \to y'$ are two 1-morphisms in $2\mathcal{C}$ such that $r(f) = r(g)$, then there exists a 2-morphism $\phi : f \to g$ in $2\mathcal{C}$.

Theorem 4.1 *Under the assumptions above the classifying space of $2\mathcal{C}$ is weakly equivalent to BG.*

Proof: Recall the notion of the homotopy fiber 2-category $y//F$ for a 2-functor $F : 2\mathcal{C}' \to 2\mathcal{C}$ at an object $y \in 2\mathcal{C}$. For our purposes we look at $y//i$, where i is the inclusion of G in $2\mathcal{C}$. In this case the homotopy fiber 2-category simplifies. Let $y_0 = i(*)$. The objects of $y//i$ are morphisms $u : y \to y_0$ in $2\mathcal{C}$, the 1-morphisms from u_1 to u_2 are pairs (g, ϕ) where $g \in G$ and ϕ is a 2-morphism in $2\mathcal{C}$ from $i(g) \circ u_1$ to u_2, and $y//i$ has only trivial 2-morphisms since G has no non-trivial 2-morphism.

We now observe that $y//i$ is a groupoid. The inverse of (g, ϕ) is (g^{-1}, ψ), where ψ is the horizontal composition of $i(g^{-1})$ and $-\phi$.

We also check that $y//i$ is connected. Let u_1 and u_2 be objects. Choose weak inverse $v : y_0 \to y$ to u_1 in $2\mathcal{C}$. Let $g = r(u_2 \circ v)$. Notice that $r(i(g) \circ u_1) = r(u_2) \circ r(v) \circ r(u_1) = r(u_2)$. By assumption there then exists a 2-morphism $\phi : i(g) \circ u_1 \to u_2$ in $2\mathcal{C}$, and we get a 1-morphism (g, ϕ) in $y//i$ from u_1 to u_2.

Hence, the nerve of $y//i$ is weakly equivalent to the nerve of the automorphism group of some u in $y//i$. Let (g, ϕ) be an automorphism of u. By definition ϕ is a 2-morphism from $i(g) \circ u$ to u. The retraction r sends every 2-morphism in $2\mathcal{C}$ to an equality in G. Hence, $g \circ r(u) = r(u)$ and consequently $g = 1$ since $r(u)$ is invertible. We see that the automorphisms of u in $y//i$ is equal to the 2-automorphisms of u in $2\mathcal{C}$, and these are contractible by assumption.

What we now have shown is that $|N_\bullet(y//i)|$ is contractible for every object y in $2\mathcal{C}$. The theorem follows. $\square$

Let us now discuss choices of $2\mathcal{C}$ based on Baez' 2-vector spaces. The numerous variations depend on how strictly the 1-morphisms should be equivalences, how much structure the 2-morphisms should contain, and how to choose the objects from 2-vector spaces within the same weak equivalence class. Let n, b_0 and b_1 be positive integers and let C_* be a 2-term chain complex of vector spaces.

> Let $\mathcal{B}_{\text{strict}}(C_*)$ be the 2-category with a single object C_*, the 1-morphisms are the chain isomorphisms $f : C_* \to C_*$, and the 2-morphisms $f \to g$ are chain homotopies ϕ from f to g.
>
> Let $\mathcal{B}_{\text{weak}}(C_*)$ be the 2-category with a single object C_*, the 1-morphisms are the chain equivalences $f : C_* \to C_*$, and the 2-morphisms $f \to g$ are chain homotopies ϕ from f to g.
>
> Let $\mathcal{B}_{\text{eq}}(C_*)$ be the 2-category with a single object C_*, the 1-morphisms are tuples $(f, \bar{f}, \iota_f, \epsilon_f)$, where f and $\bar{f}$ are chain maps $C_* \to C_*$, and $\iota_f : 1 \to f\bar{f}$ and $\epsilon_f : \bar{f}f \to 1$ are chain homotopies, and the 2-morphisms $(f, \bar{f}, \iota_f, \epsilon_f) \to (g, \bar{g}, \iota_g, \epsilon_g)$ are pairs of chain homotopies $\phi : f \to g$ and $\bar{\phi} : \bar{f} \to \bar{g}$ such that the following identities hold:
>
> $$\iota_g - \iota_f = f\bar{\phi} + \phi\bar{g} \quad \text{and} \quad \epsilon_f - \epsilon_g = f\bar{\phi} + \phi\bar{g} \quad .$$
>
> Let $\mathcal{B}_{\text{ad}}(C_*)$ be the 2-category with a single object C_*, the 1-morphisms are tuples $(f, \bar{f}, \iota_f, \epsilon_f)$, where f and $\bar{f}$ are chain maps $C_* \to C_*$, and $\iota_f : 1 \to f\bar{f}$ and $\epsilon_f : \bar{f}f \to 1$ are chain homotopies such that the zigzag identities
>
> $$f\epsilon_f + \iota_f f = 0 \quad \text{and} \quad \epsilon_f\bar{f} + \bar{f}\epsilon_f = 0$$

hold, and the 2-morphisms $(f, \bar{f}, \iota_f, \epsilon_f) \to (g, \bar{g}, \iota_g, \epsilon_g)$ are pairs of chain homotopies $\phi : f \to g$ and $\bar{\phi} : \bar{f} \to \bar{g}$ such that the following identities hold:

$$\iota_g - \iota_f = f\bar{\phi} + \phi\bar{g} \quad \text{and} \quad \epsilon_f - \epsilon_g = f\bar{\phi} + \phi\bar{g} \quad .$$

Let $\mathcal{B}^H_{\text{weak}}(b_0, b_1)$ be the 2-category with objects all 2 term chain complexes C_* with Betti-numbers b_0, b_1, the 1-morphisms are the chain equivalences $f : C_* \to C'_*$, and the 2-morphisms $f \to g$ are chain homotopies ϕ from f to g.

Let $\mathcal{B}^H_{\text{eq}}(b_0, b_1)$ be the 2-category with objects all 2 term chain complexes C_* with Betti-numbers b_0, b_1, the 1-morphisms are tuples $(f, \bar{f}, \iota_f, \epsilon_f)$, where $f : C_* \to C'_*$ and $\bar{f} : C'_* \to C_*$ are chain maps, and $\iota_f : 1 \to f\bar{f}$ and $\epsilon_f : \bar{f}f \to 1$ are chain homotopies, and the 2-morphisms $(f, \bar{f}, \iota_f, \epsilon_f) \to (g, \bar{g}, \iota_g, \epsilon_g)$ are pairs of chain homotopies $\phi : f \to g$ and $\bar{\phi} : \bar{f} \to \bar{g}$ such that the following identities hold:

$$\iota_g - \iota_f = f\bar{\phi} + \phi\bar{g} \quad \text{and} \quad \epsilon_f - \epsilon_g = f\bar{\phi} + \phi\bar{g} \quad .$$

Let $\mathcal{B}^H_{\text{ad}}(b_0, b_1)$ be the 2-category with objects all 2 term chain complexes C_* with Betti-numbers b_0, b_1, the 1-morphisms are tuples $(f, \bar{f}, \iota_f, \epsilon_f)$, where $f : C_* \to C'_*$ and $\bar{f} : C'_* \to C_*$ are chain maps, and $\iota_f : 1 \to f\bar{f}$ and $\epsilon_f : \bar{f}f \to 1$ are chain homotopies such that the zigzag identities

$$f\epsilon_f + \iota_f f = 0 \quad \text{and} \quad \epsilon_f\bar{f} + \bar{f}\epsilon_f = 0$$

hold, and the 2-morphisms $(f, \bar{f}, \iota_f, \epsilon_f) \to (g, \bar{g}, \iota_g, \epsilon_g)$ are pairs of chain homotopies $\phi : f \to g$ and $\bar{\phi} : \bar{f} \to \bar{g}$ such that the following identities hold:

$$\iota_g - \iota_f = f\bar{\phi} + \phi\bar{g} \quad \text{and} \quad \epsilon_f - \epsilon_g = f\bar{\phi} + \phi\bar{g} \quad .$$

Observe that in all cases there is an inclusion 2-functor

$$GL(b_1) \oplus GL(b_1) \to \mathcal{B} \quad ,$$

where $\mathcal{B}$ is one of the 2-categories above. There is a retraction to this inclusion given as the homology of the 2-term chain complex C_*. Furthermore, this retraction satisfies the assumptions of theorem 4.1. Thus:

Corollary 4.2 *Let X be a finite CW-complex and $\mathcal{B}$ be one of the 2-categories above. There is a bijection between concordance classes of $\mathcal{B}$-bundles over X and a pair of concordance classes of ordinary vector bundles over X of fiberdimension b_0 and b_1 respectively. Consequently the 2-K-theory associated to Baez' 2-vector spaces splits as two copies of ordinary K-theory.*

Thus we may write

$$2K_{\text{Baez}} \simeq K \oplus K \quad ,$$

where K is ordinary topological K-theory of vector bundles.

However, based on Kapranov and Voevodsky's 2-vector spaces we could define $2\mathcal{C}$ to be one of the following:

> Let $\mathcal{KV}(n)$ be the 2-category with n-tuples $V = (V_1, V_2, \ldots, V_n)$ of complex vector spaces as objects, the 1-morphisms $V \to V'$ are $n \times n$ matrices $E = (E_{ij})$ with $\det \dim E = \pm 1$ and $E \cdot V = V'$, and the 2-morphisms $E \to E'$ are $n \times n$ matrices $\phi = (\phi_{ij})$ of isomorphisms $\phi_{ij} : E_{ij} \xrightarrow{\cong} E'_{ij}$.
>
> Let $\mathcal{KV}_{\text{pt}}(n)$ be the 2-category with a single object n, $n \times n$ matrices $E = (E_{ij})$ with $\det \dim E = \pm 1$ as 1-morphisms, and the 2-morphisms $E \to E'$ are $n \times n$ matrices $\phi = (\phi_{ij})$ of isomorphisms $\phi_{ij} : E_{ij} \xrightarrow{\cong} E'_{ij}$.

In this case the 2-category is not a bigroupoid, so theorem 4.1 does not apply. Furthermore, Baas, Dundas and Rognes has calculated the associated 2-K-theory in [BDR04] and shown that this generalized cohomology theory is $K_{\text{alg}}(ku)$, and essentially of chromatic filtration 2, thus resembling something elliptic. We have:

$$2K_{\text{BDR}} \simeq K_{\text{alg}}(ku)$$

If $\mathcal{V} = \text{FinSet}$ (the category of finite sets and maps) and $2\mathcal{C} = \mathcal{V} - \text{mod}$, then the representing spectrum is $K(\mathbb{S}) = A(*)$, relating this to Waldhausens A-theory (as pointed out by John Rognes).

A The topological Kan condition.

We consider a simplicial space $X_\bullet$ with a topological Kan property. To be precise, in the notation of Duskin, we define the space of k-horns in dimension n to be

$$\begin{gathered}\Lambda^k_n(X_\bullet) = \\ \{(x_0, \ldots, x_{k-1}, -, x_{k+1}, \ldots, x_n) \mid d_i(x_j) = d_{j-1}(x_i), i < j, i \neq k \neq j\} \\ \subset \underbrace{X_{n-1} \times X_{n-1} \times \cdots \times X_{n-1}}_{n\text{factors}}\end{gathered}$$

For our purposes, it seems more natural to associate this horn to the complementary set $I(k) = \{0, 1, \ldots, k-1, k+1, \ldots n\}$. So we think of a k-horn as an $I(k)$-cohorn. If $I \subset [n] = \{0, 1, \ldots, n\}$, we define the space of I-co-horns to be

$$C\Lambda_n^I(X_\bullet) =$$
$$\{(x_i)_{\in I} \mid d_i(x_j) = d_{j-1}(x_i), i < j, i \in I \ni j\}$$
$$\subset \underbrace{X_{n-1} \times X_{n-1} \times \cdots \times X_{n-1}}_{|I|\text{factors}}$$

Note that $\Lambda_n^k = C\Lambda_n^{I(k)}$ There is a canonical map $c_n^{I(k)}, \colon X_n \to \Lambda_n^k(X_\bullet) = C\Lambda_n^{I(k)}(X_\bullet)$, and more generally, for any $I \subset [n]$ we have a canonical map $c_n^I \colon X_n \to C\Lambda_n^I(X_\bullet)$.

Definition A.1 *We say that $X_\bullet$ satisfies the discrete Kan condition in dimension n if c_k is surjective for all $0 \leq k \leq n$. We say that $X_\bullet$ satisfies the topological Kan condition in dimension n if the maps c_k are surjective Serre fibrations for all $0 \leq k \leq n$.*

Lemma A.2 *If the simplicial space $X_\bullet$ satisfies the topological Kan condition both in dimension n and in $n-1$, then for every non-empty $I \subset [n]$ the map c_n^I is a surjective Serre fibration.*

Proof: If $|I| = n-1$, then $I = I(k)$ for some $0 \leq k \leq n$, and the lemma follows directly from the definitions. Suppose inductively that the lemma is true for all $J \subset [n]$ with $n - |J| < n - |I| = m > 1$. Assume that $k \notin I$ and $J = I \cup \{k\}$, so that by the induction hypothesis c_n^J is a surjective Serre fibration. Consider the canonical map which forgets x_k

$$r_J^I \colon C\Lambda_n^J(X_\bullet) \to C\Lambda_n^I(X_\bullet).$$

Then $c_n^I = c_n^J \circ r_J^I$. So by the induction assumption, it is enough to show that the map r_I^J is a surjective Serre fibration.

Recall that $\delta_k \colon [n-1] \to [n]$ is the injective map which omits k, so that the face map $d_k \colon X_n \to X_{n-1}$ is the map induced by δ_k. Let the sequence $x = (x_i)_{i \in I}$ be a point in $C\Lambda^I(X_\bullet)$. We can apply the face map d_k to each entry. We obtain a sequence of points in X_{n-1}, namely $(d_k(x_i))_{i \in I}$. Let $I' \subset [n-1]$ be the subset determined by that $I = \partial_k(I')$. There is such a subset, since $k \notin I$.Moreover

$$|I'| = |I|$$

and the map

$$\delta_k \colon I' \to I$$

is a bijection. We define elements

$$y_i = \begin{cases} d_{k-1}(x_{\delta_k(i)}) = d_{k-1}(x_i) & \text{if } i \in I', \text{ and } i < k. \\ d_k(x_{\delta_k(i)}) = d_k(x_{i+1}) & \text{if } i \in I', \text{ and } k \leq i. \end{cases}$$

We claim that the sequence $(y_i)_{i\in I'}$ forms an I'-co-horn in X_{n-1}, which we can fill by induction. This can be seen either by meditating on the geometry of the simplex, or more computationally in the following way (for $i, j \in I'$ and $i < j$):

$$d_i y_j = \begin{cases} d_i d_{k-1}(x_j) = d_{k-2} d_i(x_j) & i < j < k \\ d_i d_k(x_{j+1}) = d_{k-1} d_i(x_{j+1}) & i < k \leq j \\ d_i d_k(x_{j+1}) = d_k d_{i+1}(x_{j+1}) & k \leq i < j \end{cases}$$

and also (using that x is in the k^{th} I-co-horn):

$$d_{j-1} y_i = \begin{cases} d_{j-1} d_{k-1}(x_i) = d_{k-2} d_{j-1}(x_i) = d_{k-2} d_i x_j & i < j \leq k \\ d_{j-1} d_{k-1}(x_i) = d_{k-1} d_j(x_i) = d_{k-1} d_i x_{j+1} & i < k < j \\ d_{j-1} d_k(x_{i+1}) = d_k d_j(x_{i+1}) = d_{i+1} x_{j+1} & k \leq i < j \end{cases}$$

The association

$$x = (x_i)_{i\in I} \mapsto P^{I,k}(x) = (y_i)_{i\in I'}$$

defines a continuous map of horns $P^{I,k}\colon C\Lambda^I_n(X_\bullet) \to C\Lambda^{I'}_{n-1}(X_\bullet)$. Since $k \in J$ there is also a continuous map $p^k\colon C\Lambda^J_n(X_\bullet) \to X_n$ defined by picking out the appropriate component: $p^k((x_i)_{i\in J}) = x_k$.

These maps fit into a diagram of spaces

$$\begin{array}{ccc} C\Lambda^J_n(X_\bullet) & \xrightarrow{r^I_J} & C\Lambda^I_n(X_\bullet) \\ \downarrow{\scriptstyle p^k} & & \downarrow{\scriptstyle P^{I,k}} \\ X_n & \xrightarrow{c^{I'}_{n-1}} & C\Lambda^{I'}_{n-1}(X_\bullet) \end{array}$$

This is actually pull back diagram. The bottom map is a surjective Serre fibration by assumption, so it follows that the top map is also a surjective Serre fibration. □

We can now prove our main theorem.

Theorem A.3 *Let $X_\bullet$ be a simplicial space.*

- *Assume that $X_\bullet$ satisfies the topological Kan condition in dimensions n and $n-1$. Then the diagonal simplicial space $T_\bullet = \delta_\bullet(Sing(X))$ satisfies the Kan condition in dimension n.*

- *Assume that the three boundary maps $d_i\colon X_2 \to X_1$ are Serre fibrations, and assume that $X_\bullet$ satisfies the discrete Kan extension condition in dimension 2. Then the diagonal simplicial space $T_\bullet = \delta_\bullet(Sing(X))$ satisfies the Kan condition in dimension 2.*

Proof: We need to show that any horn in $T_\bullet$ can be filled. The Kan condition is always satisfied for $n = 1$, since a 1-horn is filled by the degeneracy map $s_0\colon X_0 \to X_1$. So we assume without restriction that $n \geq 2$. A k-horn in dimension n in $T_\bullet$ consists of a sequence of continuous maps

$$x_i\colon \Delta_{n-1} \to X_{n-1} \quad \text{defined for } 0 \leq i \leq n-1, i \neq k.$$

with the property that if x_i and x_j are defined and $i < j$ then the following two continuous maps agree:

$$\Delta_{n-2} \xrightarrow{\delta_i} \Delta_{n-1} \xrightarrow{x_j} X_{n-1} \xrightarrow{d_i} X_{n-2}$$
$$\Delta_{n-2} \xrightarrow{\delta_{j-1}} \Delta_{n-1} \xrightarrow{x_i} X_{n-1} \xrightarrow{d_{j-1}} X_{n-2}$$

Here, and in the rest of the proof, we abuse notation by not distinguishing between a map $[m] \to [n]$ and its affine extension to a continuous map $\Delta_m \to \Delta_n$. The way to visualize this, is to consider the topological horn

$$\Lambda_n^k = \cup_{i \neq k} \delta_i(\Delta_{n-1}) \subset \Delta_n$$

The map x_i can be identified with a map from the subset $\delta_i(\Delta_{n-1})$ of Λ_n^k. The maps corresponding to x_i and x_j do not necessarily agree on the intersection $\delta_i(\Delta_{n-1}) \cap \delta_j(\Delta_{n-1})$ of these subsets, but satisfy a more complicated compatibility condition.

Main step. We define a continuous map

$$y\colon \Lambda_n^k \to X_n$$

such that for every $i \neq k$ we have that $x_i = d_i \circ y \circ \delta_i$. We define this map inductively. The induction is done over a sequence of subspaces of the skeletons of Λ_n^k, which we define now.. Let Z_r be the union of all r-dimensional subsimplices of Δ_n which contain the vertex k. Then

$$\{k\} = Z_0 \subset Z_1 \subset \dots Z_{n-1} = \Lambda_n^k$$

Start of induction: Z_0 is included in every subspace $\partial_i(\Delta_n)$, so we obtain a sequence of elements $a_i = x_i \partial_i(k) \in X_n$. These element satisfy that if $i < j$, $i \neq k \neq j$, then

$$d_i a_j = d_i x_j(\partial_j(k)) = d_{j-1} x_i(\partial_j(k)) = d_{j-1} a_i$$

By assumption the map c_k is surjective, so we can find an element $b \in X_{n+1}$ such that $d_i(b) = a_i$. We define y on $Z_0 = \{k\}$ by $y(k) = b \in X_{n+1}$.

Induction step: Assume that we have defined $y\colon Z_{r-1} \to X_{n+1}$ such that for each $i \neq k$ we have a commutative diagram:

$$\begin{array}{ccccc} Z_{r-1} \cap \partial_i(\Delta_{n-1}) & \hookrightarrow & Z_{r-1} & \xrightarrow{y} & X_n \\ \big\downarrow & & & & \big\downarrow{\scriptstyle d_i} \\ \Delta_{n-1} & & \xrightarrow{\quad x_i \quad} & & X_{n-1} \end{array}$$

We want to extend y to a map from Z_r with the same property. To define this extension, it is sufficient to define it on each r-simplices in Z_r. Such a simplex is determined by a subset $I \subset [n]$ with the two properties $k \in I$ and $|I| = r$. There is a unique order preserving injection with image I. $\alpha\colon [r] \to I \subset [n+1]$, and some k', $0 \leq k' \leq r$ so that $\alpha(k') = k$. Using α, we can translate the problem of extending the map y to the simplex determined by I into a problem of constructing a map $y'\colon \Delta_r \to X_n$ with certain properties. In order to construct the extension of y to Z_r, we have to solve this transformed problem for all injective $\alpha\colon [r] \to [n]$ such that the image $I = \alpha([r])$ contains k. We have to be careful about what the transformed problem is.

First, note that $\alpha(\Delta_r) \cap Z_{r-1} = \alpha(\Lambda_r^{k'})$, so y' is already defined on $\Lambda_r^{k'}$.

There are two cases for the maps x_i. If $i \notin I$, then x_i is defined on a sub space $\alpha(\Lambda_r^k)$, and by induction, there it agrees with $d_i y$. So these maps do not give extra conditions on y', beyond that it has to be an extension of the already defined map $y'\colon \Lambda_r^k$. On the other hand, if $i \in I$, say $i = \alpha(i')$, we do get an extra condition. First note that we have a canonical factorization

$$\begin{array}{ccc} [r-1] & \xrightarrow{\alpha'_{i'}} & [n-1] \\ \big\downarrow{\scriptstyle \delta_{i'}} & & \big\downarrow{\scriptstyle \delta_i} \\ [r] & \xrightarrow{\alpha} & [n] \end{array}$$

where

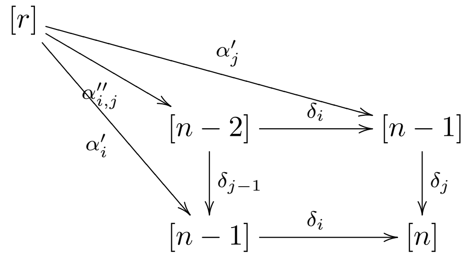
$$\alpha'_{i'}(j') = \begin{cases} \alpha(j') & \text{if } j' < i'. \\ \alpha(j'+1) - 1 & \text{if } j' \geq i'. \end{cases}$$

For $i = \alpha(i')$ and $i \neq k$ we define

$$x_{\alpha,i}\colon \Delta_{r-1} \xrightarrow{\alpha'_{i'}} \Delta_{n-1} \xrightarrow{x_i} X_{n-1}$$

We claim, that these maps fit together to a map $x_\alpha\colon \Delta_r \to C\Lambda_n^I(X_\bullet)$. To see this, assume that $0 \leq i' < j' \leq r$, $i = \alpha(i')$,$j = \alpha(j')$ and $i \neq k \neq j$.

There is a commutative diagram (whose diagonal composite is $\alpha\colon [r] \to [n]$).



Taking this diagram into account, we compute:

$$\begin{aligned} d_i x_{\alpha,j} &= d_i x_j \alpha'_{j'} \\ &= d_{j-1} x_i \delta_{j-1} \alpha''_{i,j} \end{aligned}$$

and

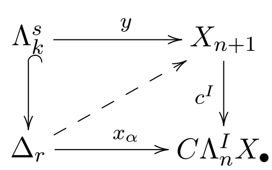
$$\begin{aligned} d_{j-1} x_{\alpha,i} &= d_{j-1} x_i \alpha'_{i'} \\ &= d_{j-1} x_i \delta_{j-1} \alpha''_{i,j} \end{aligned}$$

End of argument for $n \geq 3$.

So the maps $x_{\alpha,i}$ collectively define a map

$$x_\alpha\colon \Delta^r \to C\Lambda^I_n(X_\bullet)$$

The extension problem we have to solve is the following:



However, this is solvable, since the map c_I is a Serre fibration by lemma A.2.

The case $n = 2$. Let $0 \leq k \leq 2$, and consider a k-horn in $\delta_1(Sing(X_\bullet)$. Use the Kan extension property at the vertex k to obtain a point in X_2. Then use that $d_i\colon X_2 \to X_1$ is a Serre fibration to obtain a continuous map $y\colon \Lambda^k_2 \to X_2$. Finally extend this to the 2-simplex.

In this case, we do not need lemma A.2. □

We deduce:

Lemma A.4 *If $2\mathcal{C}$ is a sufficiently fibrant topological 2-groupoid, then for all n and k, $0 \leq k \leq n$ the map from the n-simplexes of $\Delta' 2\mathcal{C}$ into the space of k horns in dimension n in $\Delta' 2\mathcal{C}$ are surjective Serre fibrations.*

Proof: Surjectivity follows from Duskin's result that the geometric nerve of a discrete 2-groupoid is a Kan complex.

For $n = 1$ and $n = 2$ the definition of sufficiently fibrant corresponds precisely to the geometric nerve having the topological Kan condition in that dimension.

For $n > 3$ there is nothing to prove since in this case any n-simplex in the geometric nerve is uniquely determined by its restriction to the 3-skeleton.

For $n = 3$ consider an extension problem

$$c_k : (\Delta' 2\mathcal{C})_3 \to \Lambda_3^k(\Delta' 2\mathcal{C}) \quad .$$

This amounts to finding a 2-morphism in $2\mathcal{C}$ satisfying a certain coherence condition. Since every 2-morphism in a 2-groupoid is an isomorphism, the map c_k is injective. But we know from Duskin that c_k is also surjective. Thus c_k is a homeomorphism, and in particular a surjective Serre fibration. □

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