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Minimal γ-sheaves

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MINIMAL γ–SHEAVES

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Abstract. In this note we show that finitely generated unit $\mathcal{O}_X[\sigma]$–modules for X regular and F–finite have a minimal root (in the sense of [Lyu97] Definition 3.6). This question was asked by Lyubeznik and answered by himself in the complete case. In fact, we construct a minimal subcategory of the category of coherent γ–sheaves (in the sense of [BB06]) which is equivalent to the category of γ–crystals. Some applications to tight closure are included at the end of the paper.

1. Introduction

In [Lyu97] introduces the category of finitely generated unit $R[\sigma]$–modules and applies the resulting theory successfully to study finiteness properties of local cohomology modules. One of the main tools in proving results about unit $R[\sigma]$–modules is the concept of a generator or root. In short, a generator (later on called γ–sheaf) is a finitely generated module M together with a map $\gamma : M \longrightarrow \sigma^* M$. By repeated application of σ^* to this map one obtains a direct limit system, whose limit we call $\mathsf{Gen} M$. One checks easily that γ induces an map $\mathsf{Gen} M \longrightarrow \sigma^* \mathsf{Gen} M$ which is an isomorphism. A finitely generated unit $R[\sigma]$–module $\mathcal{M}$ is precisely a module which is isomorphic to $\mathsf{Gen} M$ for some γ–sheaf (M, γ), it hence comes equipped with an isomorphism $\mathcal{M} \cong \sigma^* \mathcal{M}$. Of course, different γ–sheaves may generate isomorphic unit $R[\sigma]$–modules so the question arises if there is a unique minimal (in an appropriate sense) γ–sheaf that generates a given unit $R[\sigma]$–module. In the case that R is complete, this is shown to be the case in [Lyu97] Theorem 3.5. In [Bli04] this is extended to the case that R is local (at least if R is F-finite). The purpose of this note is to prove this in general, i.e for any F–finite regular ring R (see Theorem 2.22). A notable point in the proof is that it does not rely on the hard finiteness result [Lyu97] Theorem 4.2, but only on the (easier) local case of it which is in some sense proven here *en passant* (see Remark 2.13).

The approach in this note is not the most direct one imaginable since we essentially develop a theory of minimal γ–sheaves from scratch (section 2). However, with this theory at hand, the results on minimal generators are merely a corollary.

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The ideas in this paper have two sources. Firstly, the ongoing project [BB06] of the author with Gebhard Böckle lead to a systematic study of γ–sheaves (the notation γ–sheaf is chosen to remind of the notion of a **g**enerator introduced in [Lyu97]). Secondly, insight gained from the D–module theoretic viewpoint on generalized test ideals developed in [BMS] lead to the observation that these techniques can be successfully applied to study γ–sheaves.

In the final section 3 we give some applications of the result on the existence of minimal γ–sheaves. First, we show that the category of minimal γ–sheaves is equivalent to the category γ–crystals of [BB06]. We show that a notion from tight closure theory, namely the parameter test module, is a global object (Proposition 3.3). Statements of this type are notoriously hard in the theory of tight closure. Furthermore, we give a concrete description of minimal γ–sheaves in a very simple case (Proposition 3.5), relating it to the generalized test ideals studied in [BMS]. This viewpoint also recovers (and slightly generalizes, with new proofs) the main results of [BMS] and [AMBL05]. A similar generalization, however using different (but related) methods, was recently obtained independently by Lyubeznik, Katzman and Zheng in [KLZ].

Notation. Throughout we fix a *regular* scheme X over a field $k \supseteq \mathbb{F}_q$ of characteristic $p > 0$ (with $q = p^e$ fixed). We further assume that X is F–finite, i.e. the Frobenius morphism $\sigma : X \longrightarrow X$, which is given by sending $f \in \mathcal{O}_X$ to f^q, is a finite morphism[1]. In particular, σ is affine. This allows to reduce in many arguments below to the case that X itself is affine and I will do so if convenient. We will use without further mention that because X is regular, the Frobenius morphism $\sigma\colon X \longrightarrow X$ is flat such that σ^* is an exact functor (see [Kun69]).

2. Minimal γ–sheaves

We begin with recalling the notion of γ–sheaves and nilpotence.

Definition 2.1. A *γ–sheaf* on X is a pair (M, γ_M) consisting of a quasi-coherent $\mathcal{O}_X$–module M and a $\mathcal{O}_X$–linear map $\gamma : M \longrightarrow \sigma^* M$. A γ sheaf is called *coherent* if its underlying sheaf of $\mathcal{O}_X$–modules is coherent.

A γ–sheaf (M, γ) is called *nilpotent* (of order n) if $\gamma^n \overset{\text{def}}{=} \sigma^{n*}\gamma \circ \sigma^{(n-1)*}\gamma \circ \ldots \circ \sigma^*\gamma \circ \gamma = 0$ for some $n > 0$. A γ–sheaf is called *locally nilpotent* if it is the union of nilpotent γ subsheaves.

Maps of γ–sheaves are maps of the underlying $\mathcal{O}_X$–modules such that the obvious diagram commutes. The following proposition summarizes some properties of γ–sheaves, for proofs and more details see [BB06].

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[1] It should be possible to replace the assumption of F–finiteness to saying that if X is a k–scheme with k a filed that the relative Frobenius $\sigma_{X/k}$ is finite. This would extend the results given here to desirable situations such as X of finite type over a field k with $[k : k^q] = \infty$. The interested reader should have no trouble to adjust our treatment to this case.

Proposition 2.2. (a) *The set of γ–sheaves forms an abelian category which is closed under extensions.*

(b) *The coherent, nilpotent and locally nilpotent γ-sheaves are abelian subcategories, also closed under extension.*

Proof. The point in the first statement is that the O_X–module kernel, co-kernel and extension of (maps of) γ–sheaves naturally carrys the structure of a γ–sheaf. This is really easy to verify such that we only give the construction of the γ–structure on the kernel as an illustration. Recall that we assume that X is regular such that σ is flat, hence σ^* is an exact functor. If $\varphi : M \longrightarrow N$ is a homomorphism of γ-sheaves, i.e. a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\gamma_M"'] & N \arrow[d, "\gamma_N"] \\ \sigma^* M \arrow[r, "\sigma^*\varphi"] & \sigma^* N \end{tikzcd}$$

which induces a map $\ker\varphi \longrightarrow \ker(\sigma^*\varphi)$. Since σ^* is exact, the natural map $\sigma^*(\ker\varphi) \longrightarrow \ker(\sigma^*\varphi)$ is an isomorphism. Hence the composition

$$\ker\varphi \longrightarrow \ker(\sigma^*\varphi) \xrightarrow{\cong} \sigma^*(\ker\varphi)$$

equips $\ker\varphi$ with a natural structure of a γ–sheaf.

The second part of Proposition 2.2 is also easy to verify such that we leave it to the reader, cf. the proof of Lemma 2.3 below. ☐

Lemma 2.3. *A morphism $\varphi : M \longrightarrow N$ of γ-sheaves is called* nil-injective *(resp.* nil-surjective*,* nil-isomorphism*) if its kernel (resp. cokernel, both) is locally nilpotent.*

(a) *If N is coherent and φ is* nil-injective *(resp.* nil-surjective*) then* $\ker\varphi$ *(resp.* $\operatorname{coker}\varphi$*) is nilpotent.*

(b) *Kernel and cokernel of φ are nilpotent (of order n and m resp.) if and only if there is, for some $k \geq 0$ ($k = n+m$), a map $\psi : N \longrightarrow \sigma^{k*}M$ such that $\gamma_M^k = \psi \circ \varphi$.*

(c) *If N is nilpotent of degree $\leq n$ (i.e. $\gamma_N^n = 0$) and $N' \subseteq N$ contains the kernel of γ_N^i for $1 \leq i \leq n$, then N' is nilpotent of degree $\leq i$ and N/N' is nilpotent of degree $\leq n - i$.*

Proof. The first statement is clear since X is noetherian. For the second statement consider the diagram obtained from the exact sequence $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow$

$C \longrightarrow 0$.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] \arrow[dl, "\psi"'] \arrow[ddl, dotted] & C \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \sigma^{n*}K \arrow[r] \arrow[d, "0"] & \sigma^{n*}M \arrow[r] \arrow[d] & \sigma^{n*}N \arrow[r] \arrow[d] & \sigma^{n*}C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \sigma^{(n+m)*}K \arrow[r] & \sigma^{(n+m)*}M \arrow[r] & \sigma^{(n+m)*}N \arrow[r] & \sigma^{(n+m)*}C \arrow[r] & 0
\end{tikzcd}$$

If there is ψ as indicated, then clearly the leftmost and rightmost vertical arrows of the first row are zero, i.e. K and C are nilpotent. Conversely, let $K = \ker\varphi$ be nilpotent of degree n and $C = \operatorname{coker}\varphi$ be nilpotent of degree m. Then the top right vertical arrow and the bottom left vertical arrow are zero. This easily implies that there is a dotted arrow as indicated, which will be the sought after ψ.

For the last part, the statement about the nilpotency of N' is trivial. Consider the short exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N/N' \longrightarrow 0$ and the diagram one obtains by considering $\sigma^{(n-i)*}$ and σ^{n*} of this sequence.

$$\begin{tikzcd}
0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, "\gamma^{n-i}"] & N/N' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \sigma^{(n-i)*}N' \arrow[r] \arrow[d, "0"] & \sigma^{(n-i)*}N \arrow[r] \arrow[d, "\sigma^{(n-i)*}\gamma^{i}"] & \sigma^{(n-i)*}(N/N') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \sigma^{n*}N' \arrow[r] & \sigma^{n*}N \arrow[r] & \sigma^{n*}(N/N') \arrow[r] & 0
\end{tikzcd}$$

The composition of the middle vertical map is γ_N^n which is zero by assumption. To conclude that the top right vertical arrow is zero one uses the fact that $\sigma^{(n-i)^*}N' \supseteq \sigma^{(n-i)*}\ker\gamma^i = \ker(\sigma^{(n-i)^*}\gamma^i)$. With this it is an easy diagram chase to conclude that the top right vertical map is zero. $\square$

Lemma 2.4. *Let $M \xrightarrow{\varphi} N$ be a map of γ–sheaves. Let $N' \subseteq N$ be such that N/N' is nilpotent (hence $N' \subseteq N$ is a nil-isomorphism). Then $M/(\varphi^{-1}N')$ is also nilpotent.*

Proof. If φ is injective/surjective, the Snake Lemma shows that $M/(\varphi^{-1}N')$ injects/surjects to N/N'. Now split φ into $M \twoheadrightarrow \operatorname{image}\varphi \hookrightarrow N$. $\square$

If (M, γ) is a γ–sheaf, then σ^*M is naturally a γ–sheaf with structural map $\sigma^*\gamma$. Furthermore, the map $\gamma : M \longrightarrow \sigma^*M$ is then a map of γ–sheaves which is a *nil-isomorphism*, i.e. kernel and cokernel are nilpotent. We can iterate this process to obtain a directed system

$$M \xrightarrow{\gamma} \sigma^*M \xrightarrow{\sigma^*\gamma} \sigma^{2*}M \xrightarrow{\sigma^{2*}\gamma} \dots \tag{2.1}$$

whose limit we denote by $\mathsf{Gen}M$. Clearly $\mathsf{Gen}M$ is a γ–sheaf whose structural map $\gamma_{\mathsf{Gen}M}$ is injective. In fact, it is an isomorphism since clearly $\sigma^*\mathsf{Gen}M \cong \mathsf{Gen}M$. Note that even if M is coherent, $\mathsf{Gen}M$ is generally not coherent. Furthermore, let $\overline{M}$ be the image of M under the natural map $M \longrightarrow \mathsf{Gen}M$. Then, if M is coherent, so is $\overline{M}$ and the map $M \twoheadrightarrow \overline{M}$ is a nil-isomorphism. Since $\overline{M}$ is a γ–submodule of $\mathsf{Gen}M$ whose structural map is injective, the structural map $\overline{\gamma}$ of $\overline{M}$ is injective as well.

Proposition 2.5. *The operation that assigns to each γ–sheaf M its image $\overline{M}$ in $\mathsf{Gen}M$ is an end-exact functor (preserves exactness only at the end of sequences) from $\mathbf{Coh}_\gamma(X)$ to $\mathbf{Coh}_\gamma(X)$. The kernel $M^\circ = \bigcup \ker \gamma^i_M$ of the natural map $M \longrightarrow \overline{M}$ is the maximal (locally) nilpotent subsheaf of M.*

Proof. The point is that one has a functorial map between the exact functors $\mathrm{id} \longrightarrow \mathsf{Gen}$. An easy diagram chase shows that the image of such a functorial map is an end-exact functor (see for example [Kat96, 2.17 Appendix 1]). The verification of the statement about M° left to the reader. ☐

Such γ–submodules with injective structural map enjoy a certain minimality property with respect to nilpotent subsheaves:

Lemma 2.6. *Let (M,γ) be a γ–sheaf. The structural map γ_M is injective if and only if M does not have a non-trivial nilpotent subsheaf.*

Proof. Assume that the structural map of M is injective. This implies that the structural map of any γ-subsheaf of M is injective. But a γ–sheaf with injective structural map is nilpotent if and only it is zero.

Conversely, $\ker \gamma_M$ is a nil-potent subsheaf of M. If γ_M is not injective it is nontrivial. ☐

2.1. Definition of minimal γ–sheaves.

Definition 2.7. A coherent γ–sheaf M is called *minimal* if the following two conditions hold.

(a) M does not have nontrivial nilpotent subsheaves.
(b) M does not have nontrivial nilpotent quotients.

A simple consequence of the definition is

Lemma 2.8. *Let M be a γ-sheaf. If M satisfies (a) then any γ–subsheaf of M also satisfies (a). If M satisfies (b) the so does any quotient.*

Proof. Immediate from the definition. ☐

As the preceding Lemma 2.6 shows, (a) is equivalent to the condition that the structural map γ_M is injective. We give a concrete description of the second condition.

Proposition 2.9. *For a coherent γ–sheaf M, the following conditions are equivalent.*

(a) *M does not have nontrivial nilpotent quotients.*
(b) *For any map of γ-sheaves $\varphi : N \longrightarrow M$, if $\gamma_M(M) \subseteq \varphi(\sigma^* N)$ (as subsets of $\sigma^* M$) then φ is surjective.*

Proof. I begin with showing the easy direction that (a) implies (b): Note that the condition $\gamma_M(M) \subseteq \varphi(\sigma^* N)$ in (b) precisely says that the induced structural map on the cokernel of $N \longrightarrow M$ is the zero map, thus in particular $M/\varphi(N)$ is a nilpotent quotient of M. By assumption on M, $M/\varphi(N) = 0$ and hence $\varphi(N) = M$.

Let $M \twoheadrightarrow C$ be such that C is nilpotent. Let $N \subseteq M$ be its kernel. We have to show that $N = M$. The proof is by induction on the the order of nilpotency of C (simultaneously for all C). If $C = M/N$ is nilpotent of order 1 this means precisely that $\gamma(M) \subseteq \sigma^* N$, hence by (b) we have $N = M$ as claimed. Now let N be such that the nilpotency order of $C \overset{\text{def}}{=} M/N$ is equal to $n \geq 2$. Consider the γ-submodule $N' = \pi^{-1}(\ker \gamma_C)$ of M. This N' clearly contains N and we have that $M/N' \cong C/(\ker \gamma_C)$. By the previous Lemma 2.3 we conclude that the nilpotency order of M/N' is $\leq n-1$. Thus by induction $N' = M$. Hence $M/N = N'/N \cong \ker \gamma_C$ is of nilpotency order 1. Again by the base case of the induction we conclude that $M = N$. □

These observations immediately lead to the following corollary.

Corollary 2.10. *A coherent γ–sheaf M is minimal if and only if the following two conditions hold.*

(a) *The structural map of M is injective.*
(b) *If $N \subseteq M$ is a subsheaf such that $\gamma(M) \subseteq \sigma^* N$ then $N = M$.*

The conditions in the Corollary are essentially the definition of a *minimal root* of a finitely generated unit $R[\sigma]$–module in [Lyu97]. The finitely generated unit $R[\sigma]$–module generated by (M, γ) is of course $\mathsf{Gen} M$. Lyubeznik shows in the case that R is a complete regular ring, that minimal roots exist. In [Bli04, Theorem 2.10] I showed how to reduce the local case to the complete case if R is F–finite. For convenience we give a streamlined argument of the result in the local case in the language of γ–sheaves.

2.2. **Minimal γ–sheaves over local rings.** The difficult part in establishing the existence of a minimal root is to satisfy condition (b) of Definition 2.7. The point is to bound the order of nilpotency of any nilpotent quotient of a fixed γ–sheaf M.

Proposition 2.11. *Let $(R, \mathfrak{m})$ be regular, local and F–finite. Let M be a coherent γ–sheaf and N_i be a collection of γ–sub-sheaves which is closed under finite intersections and such that M/N_i is nilpotent for all i. Then $M/\bigcap N_i$ is nilpotent.*

Proof. Since R is regular, local and F–finite, R is via σ a free R–module of finite rank. Hence σ^* is nothing but tensorisation with a free module of finite rank. Such an operation commutes with the formation of inverse limits such that $\sigma^* \bigcap N_i = \bigcap(\sigma^* N_i)$ and hence $\bigcap N_i$ is a γ–subsheaf of M. Clearly we may replace M by $M/\bigcap N_i$ such that we have $\bigcap N_i = 0$. By faithfully flatness of completion M is nilpotent if and only if $\hat{R} \otimes_R M$ is a nilpotent γ–sheaf over $\hat{R}$ (and similar for all M/N_i). Hence we may assume that $(R, \mathfrak{m})$ is complete. We may further replace M by its image $\overline{M}$ in $\mathsf{Gen} M$. Thus we may assume that M has injective structural map $\gamma : M \subseteq \sigma^* M$. We have to show that $M = 0$.

By the Artin-Rees Lemma (applied to $M \subseteq \sigma^* M$) there exists $t \geq 0$ such that for all $s > t$

$$M \cap \mathfrak{m}^s \sigma^* M \subseteq \mathfrak{m}^{s-t}(M \cap \mathfrak{m}^t \sigma^* M) \subseteq \mathfrak{m}^{s-t} M \ .$$

By Chevalleys's Theorem in the version of [Lyu97, Lemma 3.3], for some $s \gg 0$ (in fact $s \geq t+1$ will suffice) we find N_i with $N_i \subseteq \mathfrak{m}^s M$. Possibly increasing s we may assume that $N_i \not\subseteq \mathfrak{m}^{s+1} M$ (unless, of course $N_i = 0$ in which case $M/N_i = M$ is nilpotent $\Rightarrow M = 0$ since γ_M is injective, and we are done). Combining these inclusions we get

$$\begin{aligned} N_i \subseteq \sigma^* N_i \cap M &\subseteq \sigma^*(\mathfrak{m}^s M) \cap M \\ &\subseteq (\mathfrak{m}^s)^{[q]} \sigma^* M \cap M \subseteq \mathfrak{m}^{sq} \sigma^* M \cap M \\ &\subseteq \mathfrak{m}^{sq-t} M \ . \end{aligned}$$

But since $sq - t \geq s + 1$ for our choice of $s \geq t + 1$ this is a contradiction (to the assumption $N_i \neq 0$) and the result follows. □

Corollary 2.12. *Let R be regular, local and F–finite and M a coherent γ–sheaf. Then M has a nil-isomorphic subsheaf without non-zero nilpotent quotients (i.e. satisfying (b) of the definition of minimality). In particular, M is nil-isomorphic to a minimal γ–sheaf.*

Proof. Let N_i be the collection of all nil-isomorphic subsheaves of M. This collection is closed under finite intersection: If N and N' are two such, then Lemma 2.4 shows that $N \cap N'$ is a nil-isomorphic subsheaf of N. Since composition of nil-isomorphisms are nil-isomorphisms it follows that $N \cap N' \subseteq M$ is a nil-isomorphism as well.

Since M is coherent each M/N_i is indeed nilpotent such that we can apply Proposition 2.11 to conclude that $M/\bigcap N_i$ is nilpotent. Hence $N \stackrel{\text{def}}{=} \bigcap N_i$ is the unique smallest nil-isomorphic subsheaf of M. It is clear that N cannot have non-zero nilpotent quotients (since the kernel would be a strict subsheaf of N, nil-isomorphic to M, by Proposition 2.2 (b)).

By first replacing M by $\overline{M}$ we can also achieve that condition (a) of the definition of minimality holds. As condition (a) passes to subsheaves, the smallest nil-isomorphic subsheaf of $\overline{M}$ is the sought after minimal γ–sheaf which is nil-isomorphic to M. $\square$

Remark 2.13. Essentially the same argument as in the proof of Proposition 2.11 shows the following: If R is local and M is a coherent γ–sheaf over R with injective structural map, then any descending chain of γ–submodules of M stabilizes. This was shown (with essentially the same argument) in [Lyu97] and implies immediately that γ–sheaves with injective structural map satisfy DCC.

If one tries to reduce the general case of Corollary 2.12 (i.e. R not local) to the local case just proven one encounters the problem of having to deal with the behavior of the infinite intersection $\bigcap N_i$ under localization. This is a source of troubles I do not know how to deal with directly. The solution to this is to take a detour and realize this intersection in a fashion such that each term functorially depends on M and furthermore that this functorial construction commutes with localization. This is explained in the following section.

2.3. $D_X^{(1)}$–modules and Frobenius descent.

Let D_X denote the sheaf of differential operators on X. This is a sheaf of rings on X which locally, on each affine subvariety $\operatorname{Spec} R$ is described as follows.

$$D_R = \bigcup_{i=0}^{\infty} D_R^{(i)}$$

where $D_R^{(i)}$ is the subset of $\operatorname{End}_{\mathbb{F}_q}(R)$ consisting of the operators which are linear over R^{q^i}, the subring of $(q^i)^{\text{th}}$ powers of elements of R. In particular $D_R^{(0)} \cong R$ and $D_R^{(1)} = \operatorname{End}_{R^q}(R)$. Clearly, R itself becomes naturally a left $D_R^{(i)}$–module. Now denote by $R^{(1)}$ the $D_R^{(1)}$–R–bi-module which has this left $D_R^{(1)}$–module structure and the right R–module structure via Frobenius. i.e. for $r \in R^{(1)}$ and $x \in R$ we have $r \cdot x = rx^q$. With this notation we may view $D_R^{(1)} = \operatorname{End}^{\mathrm{r}}_R(R^{(1)})$ as the right R-linear endomorphisms of $R^{(1)}$. Thus we have

$$\sigma^*(\underline{}) = R^{(1)} \otimes_R \underline{} : R - \text{mod} \longrightarrow D_R^{(1)} - \text{mod}$$

which makes σ^* into an equivalence of categories from R–modules to $D_R^{(1)}$–modules (because, since σ is flat and R is F–finite, $R^{(1)}$ is a locally free right R–module of finite rank). Its inverse functor is given by

$$\sigma^{-1}(\underline{}) = \operatorname{Hom}^{\mathrm{r}}_R(R^{(1)}, R) \otimes_{D_R^{(1)}} \underline{} : D_R^{(1)} - \text{mod} \longrightarrow R - \text{mod} \tag{2.2}$$

For details see [AMBL05, Section 2.2]. I want to point out that these constructions commute with localization at arbitrary multiplicative sets. Let S be a multiplicative set of R.[2]. We have

$$\begin{aligned} S^{-1}D_R^{(1)} &= S^{-1}\operatorname{End}^{\mathrm{r}}_R(R^{(1)}) \\ &= \operatorname{End}^{\mathrm{r}}_{S^{-1}R}((S^{[q]})^{-1}R^{(1)}) = \operatorname{End}^{\mathrm{r}}_{S^{-1}R}((S^{-1}R)^{(1)}) \\ &= D^{(1)}_{S^{-1}R} \end{aligned} \tag{2.3}$$

Furthermore we have for an $D_R^{(1)}$–module M:

$$\begin{aligned} S^{-1}(\sigma^{-1}M) &= S^{-1}(\operatorname{Hom}^{\mathrm{r}}_R(R^{(1)},R)\otimes_{D_R^{(1)}} M) \\ &= S^{-1}\operatorname{Hom}^{\mathrm{r}}_R(R^{(1)},R)\otimes_{S^{-1}D_R^{(1)}} S^{-1}M \\ &= \operatorname{Hom}^{\mathrm{r}}_{S^{-1}R}((S^{-1}R)^{(1)},S^{-1}R)\otimes_{D^{(1)}_{S^{-1}R}} S^{-1}M \\ &= \sigma^{-1}(S^{-1}M) \end{aligned}$$

These observations are summarized in the following Proposition

Proposition 2.14. *Let X be F–finite and regular. Let U be an open subset (more generally, U is locally given on* $\operatorname{Spec} R$ *as* $\operatorname{Spec} S^{-1}R$ *for some (sheaf of) multiplicative sets on X). Then*

$$(D_X^{(1)})|_U = D_U^{(1)}$$

and for any sheaf of $D_X^{(1)}$–modules M one has that

$$(\sigma^{-1}M)|_U = (\operatorname{Hom}^{\mathrm{r}}(\mathcal{O}_X^{(1)},\mathcal{O}_X)\otimes_{D_X^{(1)}}M)|_U \cong \operatorname{Hom}^{\mathrm{r}}(\mathcal{O}_U^{(1)},\mathcal{O}_U)\otimes_{D_U^{(1)}}M|_U = \sigma^{-1}(M|_U)$$

as $\mathcal{O}_U$–modules.

2.4. **A criterion for minimality.**

The Frobenius descent functor σ^{-1} can be used to define an operation on γ–sheaves which assigns to a γ–sheaf M its smallest γ–subsheaf N with the property that M/N has the trivial (=0) γ–structure. This is the opposite of what the functor σ^* does: $\gamma : M \longrightarrow \sigma^*M$ is a map of γ sheaves such that $\sigma^*M/\gamma(M)$ has trivial γ–structure.

We define the functor σ_γ^{-1} from γ–sheaves to γ–sheaves as follows. Let $M \xrightarrow{\gamma} \sigma^*M$ be a γ sheaf. Then $\gamma(M)$ is an $\mathcal{O}_X$–submodule of the $D_X^{(1)}$–module σ^*M. Denote by $D_X^{(1)}\gamma(M)$ the $D_X^{(1)}$–submodule of σ^*M generated by $\gamma(M)$. To this inclusion of $D_X^{(1)}$–modules

$$D_X^{(1)}\gamma(M) \subseteq \sigma^*M$$

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[2] Since $S^{-1}R = (S^{[q]})^{-1}R$ we may assume that $S \subseteq R^q$. This implies that S is in the center of $D_R^{(1)}$ such that localization in this non-commutative ring along S is harmless. With this I mean that we may view the localization of the left R–module $D_R^{(1)}$ at S^{-1} in fact as the localization of $D_R^{(1)}$ at the central multiplicative set $(S^{[q]})^{-1}$

we apply the Frobenius descent functor $\sigma^{-1} : D_X^{(1)}$–mod $\longrightarrow \mathcal{O}_X$–mod defined above in Equation 2.2 and use that $\sigma^{-1} \circ \sigma^* = \mathrm{id}$ to define

$$\sigma_\gamma^{-1} M \overset{\text{def}}{=} \sigma^{-1}(D_X^{(1)}\gamma(M)) \subseteq \sigma^{-1}\sigma^* M = M$$

In general one has $\sigma_\gamma^{-1}(\sigma^* M) = \sigma^{-1} D_X^{(1)} \sigma^*(\gamma)(\sigma^* M) = \gamma(M)$ since $\sigma^*(\gamma)(\sigma^* M)$ already is a $D_X^{(1)}$–subsheaf of the $D_X^{(2)}$–module $\sigma^*(\sigma^* M) = \sigma^{2*} M$.

By construction $\sigma_\gamma^{-1} M \subseteq M \xrightarrow{\gamma} \gamma(M) \subseteq D_X^{(1)}\gamma(M) = \sigma^*\sigma^{-1} D_X^{(1)}\gamma(M) = \sigma^*\sigma_\gamma^{-1} M$ such that $\sigma_\gamma^{-1} M$ is a γ–subsheaf of M.

Furthermore, the quotient $M/\sigma_\gamma^{-1} M$ has zero structural map. One makes the following observation

Lemma 2.15. *Let M be a γ sheaf. Then $\sigma_\gamma^{-1} M$ is the smallest subsheaf N of M such that $\sigma^* N \supseteq \gamma(M)$.*

Proof. Clearly $\sigma^{-1} M$ satisfies this condition. Let N be as in the statement of the Lemma. Then $\sigma^* N$ is a $D_X^{(1)}$–subsheaf of $\sigma^* M$ containing $\gamma(M)$. Hence $D_X^{(1)}\gamma(M) \subseteq \sigma^* N$. Applying σ^{-1} we see that $\sigma^{-1} M \subseteq N$. ☐

Therefore, the result of the lemma could serve as an alternative definition of σ_γ^{-1} (one would have to show that the intersection of all such N has again the property that $\gamma(M) \subseteq \sigma^* \bigcap N$ but this follows since σ^* commutes with inverse limits). The following lemma is the key point in our reduction to the local case. It is an immediate consequence of Proposition 2.14. Nevertheless we include here a proof using only the characterization of Lemma 2.15. Hence one may avoid the appearance of $D^{(1)}$–modules in this paper altogether but I believe it to be important to explain where the ideas for the arguments originated, hence $D^{(1)}$–modules are still there.

Lemma 2.16. *Let M be a γ sheaf and let $S \subseteq \mathcal{O}_X$ be multiplicative set. Then $S^{-1}(\sigma_\gamma^{-1} M) = \sigma_\gamma^{-1}(S^{-1} M)$.*

Proof. This follows from Proposition 2.14. However, this can also be proven using only the characterization in Lemma 2.15: By this we have

$$\sigma^*(S^{-1}(\sigma_\gamma^{-1} M)) = S^{-1}(\sigma^*(\sigma_\gamma^{-1})) \supseteq S^{-1}\gamma(M) = \gamma(S^{-1} M) \tag{2.4}$$

which implies that $\sigma_\gamma^{-1}(S^{-1} M) \subseteq S^{-1}(\sigma_\gamma^{-1} M)$ because $\sigma_\gamma^{-1}(S^{-1} M)$ is smallest (by Lemma 2.15) with respect to the inclusion shown in the displayed equation Equation 2.4. On the other hand one has the chain of inclusions

$$\begin{aligned} \sigma^*(M \cap S^{-1}\sigma_\gamma^{-1}(M)) &= \sigma^* M \cap \sigma^*\sigma_\gamma^{-1}(S^{-1} M) \\ &\supseteq \sigma^* M \cap \gamma(S^{-1} M) \supseteq \gamma(M) \end{aligned}$$

and hence Lemma 2.15 applied to M yields

$$\sigma_\gamma^{-1} M \subseteq M \cap S^{-1}\sigma_\gamma^{-1}(M)\,.$$

Therefore $S^{-1}\sigma_\gamma^{-1} M \subseteq S^{-1} M \cap S^{-1}(\sigma_\gamma^{-1} S^{-1} M) = \sigma_\gamma^{-1} S^{-1} M$ which finishes the argument. ☐

Proposition 2.17. *Let M be a γ-sheaf. Then $\sigma_\gamma^{-1}M = M$ if and only if M has no proper nilpotent quotients (i.e. satisfies condition (b) of the definition of minimality)*

If M is coherent. The condition on $x \in X$ that the inclusion $\sigma_\gamma^{-1}(M_x) \subseteq M_x$ is equality is an open condition on X.

Proof. One direction is clear since $M/\sigma_\gamma^{-1}M$ is a nilpotent quotient of M. We use the characterization in Proposition 2.9. For this let $N \subseteq M$ be such that $\gamma(M) \subseteq \sigma^*N$. $\sigma_\gamma^{-1}M$ was the smallest subsheaf with this property, hence $\sigma_\gamma^{-1}M \subseteq N \subseteq M$. Since $M = \sigma_\gamma^{-1}M$ by assumption it follows that $N = M$. Hence, by Proposition 2.9, M does not have non-trivial nilpotent quotients.

By Lemma 2.16 σ_γ^{-1} commutes with localization which means that $\sigma_\gamma^{-1}(M_x) = (\sigma_\gamma^{-1}M)_x$. Hence the second statement follows simply since both M and $\sigma_\gamma^{-1}M$ are coherent (and equality of two coherent modules via a given map is an open condition). □

Lemma 2.18. *The assignment $M \mapsto \sigma_\gamma^{-1}M$ is an end-exact functor on γ–sheaves.*

Proof. Formation of the image of the functorial map id $\xrightarrow{\gamma} \sigma^*$ of exact functors is end-exact (see for example [Kat96, 2.17 Appendix 1]). If M is a $D_X^{(1)}$–module and $A \subseteq B$ are $\mathcal{O}_X$–submodules of M then $D_X^{(1)}A \subseteq D_X^{(1)}B$. If $M \twoheadrightarrow N$ is a surjection of $D^{(1)}$–modules with induces a surjection on $\mathcal{O}_X$–submodules $A \twoheadrightarrow B$ then, clearly, $D_X^{(1)}A$ surjects onto $D_X^{(1)}B$. Now one concludes by observing that σ^{-1} is an exact functor. □

Lemma 2.19. *Let $N \subseteq M$ be an inclusion of γ–sheaves such that $\sigma^{n*}N \supseteq \gamma^n(M)$ (i.e. the quotient is nilpotent of order $\leq n$). Then $\sigma^{(n-1)*}(N \cap \sigma_\gamma^{-1}M) \supseteq \gamma^{n-1}(\sigma_\gamma^{-1}M)$.*

Proof. Consider the γ–subsheaf $M' = (\gamma^{n-1})^{-1}(\sigma^{(n-1)*}N)$ of M. One has

$$\sigma^*M' = (\sigma^*\gamma^{n-1})^{-1}(\sigma^{n*}N) \supseteq \gamma(M)$$

by the assumption that $\gamma^n(M) \subseteq \sigma^{n*}N$. Since $\sigma_\gamma^{-1}M$ is minimal with respect to this property we have $\sigma_\gamma^{-1}M \subseteq (\gamma^{n-1})^{-1}(\sigma^{(n-1)*}N)$. Applying γ^{n-1} we conclude that $\gamma^{n-1}(\sigma_\gamma^{-1}M) \subseteq \sigma^{(n-1)*}N$. Since $\sigma_\gamma^{-1}M$ is a γ–sheaf we have $\gamma(\sigma_\gamma^{-1}M) \subseteq \sigma^{(n-1)*}(\sigma_\gamma^{-1}M)$ such that the claim follows. □

2.5. **Existence of minimal γ–sheaves.** For a given γ–sheaf M we can iterate the functor σ_γ^{-1} to obtain a decreasing sequence of γ–subsheaves

$$\ldots \subseteq M_3 \subseteq M_2 \subseteq M_1 \subseteq M(\xrightarrow{\gamma} \sigma^*M \longrightarrow \ldots)$$

where $M_i = \sigma_\gamma^{-1}M_{i-1}$. Note that each inclusion $M_i \subseteq M_{i-1}$ is a nil-isomorphism.

Proposition 2.20. *Let M be a coherent γ–sheaf. Then the following conditions are equivalent.*

(a) *M has a nil-isomorphic γ–subsheaf $\underline{M}$ which does not have non-trivial nilpotent quotients (i.e. $\underline{M}$ satisfies condition (b) in the definition of minimal γ–sheaf).*
(b) *M has a* unique *smallest nil-isomorphic subsheaf (equiv. M has a (unique) maximal nilpotent quotient).*
(c) *For some $n \geq 0$, $M_n = M_{n+1}$.*
(d) *There is $n \geq 0$ such that for all $m \geq n$, $M_m = M_{m+1}$.*

Proof. (a) $\Rightarrow$ (b): Let $\underline{M} \subseteq M$ be the nil-isomorphic subsheaf of part (a) and let $N \subseteq M$ be another nil-isomorphic subsheaf of M. By Lemma 2.4 it follows that $\underline{M} \cap N$ is also nil-isomorphic to M. In particular $\underline{M}/(\underline{M} \cap N)$ is a nilpotent quotient of $\underline{M}$ and hence must be trivial. Thus $N \subseteq \underline{M}$ which shows that $\underline{M}$ is the smallest nil-isomorphic subsheaf of M.

(b) $\Rightarrow$ (c): Let N be this smallest subsheaf as in (b). Since each M_i is nil-isomorphic to M, it follows that $N \subseteq M_i$ for all i. Let n be the order of nilpotency of the quotient M/N, i.e. $\gamma^n(M) \subseteq \sigma^{n*}N$. Repeated application ($n$ times) of Lemma 2.19 yields that $M_n \subseteq N$. Hence we get $N \subseteq M_{n+1} \subseteq M_n \subseteq N$ which implies that $M_{n+1} = M_n$.

(c) $\Rightarrow$ (d) is clear.

(d) $\Rightarrow$ (a) is clear by Proposition 2.17. ☐

This characterization enables us to show the existence of minimal γ-sheaves by reducing to the local case which we proved above.

Theorem 2.21. *Let M be a coherent γ–sheaf. There is a unique subsheaf $\underline{M}$ of M which does not have non-trivial nilpotent quotients.*

Proof. By Proposition 2.20 it is enough to show that the sequence M_i is eventually constant. Let U_i be the subset of X consisting of all $x \in X$ on which $(M_i)_x = (M_{i+1})_x (= (\sigma_\gamma^{-1} M_i)_x)$. By Proposition 2.17 U_i is an open subset of X (in this step I use the key observation Proposition 2.14) an that $(M_i)|_{U_i} = (M_{i+1})|_{U_i}$. By the functorial construction of the M_i's the equalilty $M_i = M_{i+1}$ for one i implies equality for all bigger i. It follows that the sets U_i form an increasing sequence of open subsets of X whose union is X itself by Corollary 2.12 and Proposition 2.20. Since X is noetherian, $X = U_i$ for some i. Hence $M_i = M_{i+1}$ such that the claim follows by Proposition 2.20. ☐

Theorem 2.22. *Let M be a coherent γ–sheaf. Then there is a functorial way to assign to M a* minimal *γ–sheaf $M_{\min}$ in the nil-isomorphism class of M.*

Proof. We may first replace M by the nil-isomorphic quotient $\overline{M}$ which satisfies condition (a) of Definition 2.7. Then replace $\overline{M}$ by its minimal nil-isomorphic submodule $\underline{(\overline{M})}$ which also satisfies condition (b) of Definition 2.7 (and condition (a) because (a) is passed to submodules). Thus the assignment $M \mapsto M_{\min} \overset{\text{def}}{=} \underline{(\overline{M})}$ is a functor since it is a composition of the functors $M \mapsto \overline{M}$ and $M \mapsto \underline{M}$. ☐

Proposition 2.23. *If* $\varphi\colon M \longrightarrow N$ *is a nil-isomorphism, then* $\varphi_{\min}\colon M_{\min} \longrightarrow N_{\min}$ *is an isomorphism.*

Proof. Clearly, $\varphi_{\min}$ is a nil-isomorphism. Since $\ker \varphi_{\min}$ is a nilpotent subsheaf of $M_{\min}$, we have by Definition 2.7 (a) that $\ker \varphi_{\min} = 0$. Since $\operatorname{coker} \varphi_{\min}$ is a nilpotent quotient of $N_{\min}$ it must be zero by Definition 2.7 (b). □

Corollary 2.24. *Let* $\mathcal{M}$ *be a finitely generated unit* $\mathcal{O}_X[\sigma]$*–module. Then* M *has a unique minimal root in the sense of* [Lyu97].

Proof. Let M be any root of $\mathcal{M}$, i.e. M is a coherent γ–sheaf such that γ_M is injective and $\mathsf{Gen}M \cong \mathcal{M}$. Then $M_{\min} = \underline{M}$ is a minimal nil-isomorphic γ–subsheaf of M by Theorem 2.22. By Corollary 2.10 it follows that $M_{\min}$ is the sought after minimal root of $\mathcal{M}$. □

Note that the only assumption needed in this result is that X is F–finite and regular. In particular it does not rely on the finite–length result [Lyu97] Theorem 3.2 which assumes that R is of finite type over a regular local ring (however if does not assume F–finiteness).

Theorem 2.25. *Let* X *be regular and* F*–finite. Then the functor*

$$\mathsf{Gen}\colon \mathbf{Min}_\gamma(X) \longrightarrow \textit{finitely generated unit } \mathcal{O}_X[\sigma]\textit{–modules}$$

is an equivalence of categories.

Proof. The preceding corollary shows that Gen is essentially surjective. The induced map on Hom sets is injective since a map of minimal γ–sheaves f is zero if and only if its image is nilpotent (since minimal γ–sheaves do not have nilpotent sub-modules) which is the condition that $\mathsf{Gen}(f) = 0$. It is surjective since any map between $g : \mathsf{Gen}(M) \longrightarrow \mathsf{Gen}(N)$ is obtained from a map of γ–sheaves $M \longrightarrow \sigma^{e*}N$ for some $e \gg 0$. But this induces a map $M = M_{\min} \longrightarrow (\sigma^{e*}N)_{\min} = N_{\min} = N$. □

3. Applications and Examples

In this section we discuss some further examples and applications of the results on minimal γ–sheaves we obtained so far.

3.1. **γ–crystals.** The purpose of this section is to quickly explain the relationship of minimal γ–sheaves to γ-crystals which were introduced in [BB06]. The category of γ–crystals is obtained by inverting nil-isomorphisms in $\mathbf{Coh}_\gamma(X)$. In [BB06] it is shown that the resulting category is abelian. One has a natural functor

$$\mathbf{Coh}_\gamma(X) \twoheadrightarrow \mathbf{Crys}_\gamma(X)$$

whose fibers we may think of consisting of nil-isomorphism classes of M. Note that the objects of $\mathbf{Crys}_\gamma(X)$ are the same as in $\mathbf{Coh}_\gamma(X)$, however a morphism between γ–crystals $M \longrightarrow N$ is represented by a left-fraction, i.e. a diagram of γ–sheaves $M \Leftarrow M' \rightarrow M$ where the arrow $\Leftarrow$ is a nil-isomorphism.

On the other hand we just constructed the subcategory of minimal γ–sheaves $\mathbf{Min}_\gamma(X) \subseteq \mathbf{Coh}_\gamma(X)$ and showed that there is a functorial splitting $M \mapsto M_{\min}$ of this inclusion. An immediate consequence of Proposition 2.23 is that if M and N are in the same nil-isomorphism class, then $M_{\min} \cong N_{\min}$. The verification of this may be reduced to considering the situation

$$M \Leftarrow M' \Rightarrow N$$

with both maps nil-isomorphisms in which case Proposition 2.23 shows that $M_{\min} \cong M'_{\min} \cong N_{\min}$. One has the following Proposition.

Proposition 3.1. *Let X be regular and F–finite. Then the composition*

$$\mathbf{Min}_\gamma(X) \hookrightarrow \mathbf{Coh}_\gamma(X) \twoheadrightarrow \mathbf{Crys}_\gamma(X)$$

is an equivalence of categories whose inverse is given by sending a γ–crystal represented by the γ–sheaf M to the minimal γ–sheaf $M_{\min}$.

Proof. The existence of $M_{\min}$ shows that $\mathbf{Min}_\gamma(X) \longrightarrow \mathbf{Crys}_\gamma(X)$ is essentially surjective. It remains to show that $\operatorname{Hom}_{\mathbf{Min}_\gamma}(M,N) \cong \operatorname{Hom}_{\mathbf{Crys}_\gamma}(M,N)$. A map $\varphi\colon M \longrightarrow N$ of minimal γ–sheaves is zero in $\mathbf{Crys}_\gamma$ if and only if $\operatorname{image}\varphi$ is nilpotent. But $\operatorname{image}\varphi$ is a subsheaf of the minimal γ–sheaf N, which by Definition 2.7 (a) has no nontrivial nilpotent subsheaves. Hence $\operatorname{image}\varphi = 0$ and therefore $\varphi = 0$. This shows that the map on Hom sets is injective. The surjectivity follows again by functoriality of $M \mapsto M_{\min}$. □

Corollary 3.2. *Let X be regular and F–finite. The category of minimal γ–sheaves $\mathbf{Min}_\gamma(X)$ is an abelian category. If $\varphi\colon M \longrightarrow N$ is a morphism then* $\ker_{\min}\varphi = (\ker\varphi)_{\min} = \underline{\ker\varphi}$ *and* $\operatorname{coker}_{\min}\varphi = (\operatorname{coker}\varphi)_{\min} = \overline{\operatorname{coker}\varphi}$.

Proof. Since $\mathbf{Min}_\gamma(X)$ is equivalent to $\mathbf{Crys}_\gamma(X)$ and since the latter is abelian, so is $\mathbf{Min}_\gamma(X)$. This implies also the statement about ker and coker. □

3.2. The parameter test module.

We give an application to the theory of tight closure. In [Bli04] Proposition 4.5 it was shown that the parameter test module τ_{ω_A} is the unique minimal root of the intersection homology unit module $\mathcal{L} \subseteq H^{n-d}_I(R)$ if $A = R/I$ is the quotient of the regular local ring R (where $\dim R = n$ and $\dim A = d$). Locally, the parameter test module τ_{ω_A} is defined as the Matlis dual of $H^d_m(A)/0^*_{H^d_m(A)}$ where $0^*_{H^d_m(A)}$ is the tight closure of zero in $H^d_m(A)$. The fact that we are now able to construct minimal γ–sheaves globally allows us to give a global candidate for the parameter test module.

Proposition 3.3. *Let $A = R/I$ where R is regular and F–finite. Then there is a submodule $L \subseteq \omega_A = \operatorname{Ext}^{n-d}(R/I,R)$ such that for each $x \in \operatorname{Spec} A$ we have $L_x \cong \tau_{\omega_x}$.*

Proof. Let $\mathcal{L} \subseteq H^{n-d}_I(R)$ be the unique smallest submodule of $H^{n-d}_I(R)$ which agrees with $H^{n-d}_I(R)$ on all smooth points of $\operatorname{Spec} A$. $\mathcal{L}$ exists by [Bli04] Theorem 4.1. Let L be a minimal generator of $\mathcal{L}$, i.e. a coherent minimal γ–sheaf such that $\mathsf{Gen} L = \mathcal{L}$ which exists due to Theorem 2.21. Because of Proposition 2.14 it follows that L_x is also a minimal γ–sheaf and $\mathsf{Gen} L_x \cong \mathcal{L}_x$. But from [Bli04] Proposition 4.5 we know that the unique minimal root of $\mathcal{L}_x$ is $\tau_{\omega_{A_x}}$, the parameter test module of A_x. It follows that $L_x \cong \tau_{\omega_{A_x}}$ by uniqueness. To see that $L \subseteq \operatorname{Ext}^{n-d}(R/I, R)$ we just observe that $\operatorname{Ext}^{n-d}(R/I, R)$ with the map induced by $R/I^{[q]} \longrightarrow R/I$ is a γ–sheaf which generates $H^{n-d}_I(R)$. Hence by minimality of L we have the desired inclusion. □

3.3. **Test ideals and minimal γ–sheaves.** We consider now the simplest example of a γ–sheaf, namely that of a free rank one R–module $M (\cong R)$. That means that via the identification $R \cong \sigma^* R$ the structural map

$$\gamma : M \cong R \xrightarrow{f\cdot} R \cong \sigma^* R \cong \sigma^* M$$

is given by multiplication with an element $f \in R$. It follows that γ^e is given by multiplication by $f^{1+q+\ldots+q^{e-1}}$ under the identification of $\sigma^{e*} R \cong R$

We will show that the minimal γ–subsheaf of the just described γ-sheaf M can be expressed in terms of generalized test ideals. We recall from [BMS] Lemma **??** that the test ideal of a principal ideal (f) of exponent $\alpha = \frac{m}{q^e}$ is given by

$$\tau(f^\alpha) = \text{smallest ideal } J \text{ such that } f^m \in J^{[q^e]}$$

by Lemma **??** of op. cit. $\tau(f^\alpha)$ can also be characterized as σ^{-e} of the $D^{(e)}$–module generated by f^m. We set as a shorthand $J_e = \tau(f^{(1+q+q^2+\ldots+q^{e-1})/q^e})$ and repeat the definition:

$$J_e = \text{smallest ideal } J \text{ of } R \text{ such that } f^{1+q+q^2+\ldots+q^{e-1}} \in J^{[q^e]}$$

and further recall from section 2.5 that

$$M_e = \text{smallest ideal } I \text{ of } R \text{ such that } f \cdot M_{e-1} \subseteq I^{[q]}$$

with $M_0 = M$.

Lemma 3.4. *For all $e \geq 0$ one has $J_e = M_e$.*

Proof. The equality is true for $e = 1$ by definition. We first show the inclusion $J_e \subseteq M_e$ by induction on e.

$$\begin{aligned} M_e^{[p^e]} &\supseteq (f \cdot M_{e-1})^{[q^{e-1}]} = (f^{q^{e-1}} M_{e-1}^{[q^{e-1}]}) \\ &= (f^{q^{e-1}} J_{e-1}^{[q^{e-1}]}) \supseteq f^{q^{e-1}} \cdot f^{1+q+q^2+\ldots+q^{e-2}} \\ &= f^{1+q+q^2+\ldots+q^{e-1}} \end{aligned}$$

since J_e is minimal with respect to this inclusion we have $J_e \subseteq M_e$.

Now we show for all $e \geq 1$ that $f \cdot J_{e-1} \subseteq J_e^{[q]}$. The definition of J_e implies that

$$f^{1+q+\ldots+q^{e-2}} \in (J^{[q^e]} \colon f^{q^{e-1}}) = (J^{[q]} \colon f)^{[q^{e-1}]}$$

which implies that $J_{e-1} \subseteq (J^{[q]} \colon f)$ by minimality of J_{e-1}. Hence $f \cdot J_{e-1} \subseteq J^{[q]}$. Now, we can show the inclusion $M_e \subseteq J_e$ by observing that by induction one has

$$J_e^{[q]} \supseteq f \cdot J_{e-1} \supseteq f \cdot M_{e-1} \,.$$

which implies by minimality of M_e that $M_e \subseteq J_e$. □

This shows that the minimal γ–sheaf $M_{\min}$, which is equal to M_e for $e \gg 0$ by Proposition 2.20, is just the test ideal $\tau(f^{(1+q+q^2+\ldots+q^{e-1})/q^e})$ for $e \gg 0$. As a consequence we have:

Proposition 3.5. *Let M be the γ–sheaf given by $R \xrightarrow{f\cdot} R \cong \sigma^* R$. Then $M_{\min} = \tau(f^{(1+q+q^2+\ldots+q^{e-1})/q^e})$ for $q \gg 0$. In particular, $M_{\min} \supseteq \tau(f^{\frac{1}{q-1}})$ and the F-pure-threshold of f is $\geq \frac{1}{q-1}$ if and only if M is minimal.*

Proof. For $e \gg 0$ the increasing sequence of rational numbers $(1+q+q^2+\ldots+q^{e-1})/q^e$ approaches $\frac{1}{q-1}$. Hence $M_e = \tau(f^{(1+q+q^2+\ldots+q^{e-1})/q^e}) \supseteq \tau(f^{\frac{1}{q-1}})$ for all e. If M is minimal, then all M_e are equal hence the multiplier ideals $\tau(f^\alpha)$ must be equal to R for all $\alpha \in [0, \frac{1}{q-1})$. In particular, the F-pure-threshold of f is $\geq \frac{1}{q-1}$. Conversely, if the F–pure threshold is less than $\frac{1}{q-1}$, then for some e we must have that $\tau(f^{(1+q+q^2+\ldots+q^{e-1})/q^e}) \neq \tau(f^{(1+q+q^2+\ldots+q^e)/q^{e+1}})$ such that $M_e \neq M_{e+1}$ which implies that $M \neq M_1$ such that M is not minimal. □

Remark 3.6. This shows also, after replacing f by f^r, that $\frac{r}{q-1}$ is not an accumulation point of F–thresholds of f for any f in an F–finite regular ring. In [BMS] this was shown for R essentially of finite type over a local ring since our argument there depended on [Lyu97] Theorem 4.2. Even though D–modules appear in the present article, they only do so by habit of the author, as remarked before, they can easily be avoided.

Remark 3.7. Of course, for $r = q-1$ this recovers (and slightly generalizes) the main result in [AMBL05].

Remark 3.8. I expect that this descriptions of minimal roots can be extended to a more general setting using the modifications of generalized test ideals to modules as introduced in the preprint [TT07].

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