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# How is the Hitchin system self-dual? Let us count the ways

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How is the Hitchin system self-dual?
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**Abstract.** The notion of Algebraic Complete Integrability (ACI) of certain mechanical systems, introduced in the early 1980s, has given great impetus to the study of moduli spaces of holomorphic vector bundles over an algebraic curve (or a higher-dimensional variety, with much more ground to be explored). Several notions of "duality" have been the object of much interest in both theories. We highlight the pertinent notions, focusing on the most basic case of a rather 'universal' class of ACIs, (generalized) Hitchin systems. We explore the relationship between dualities of ACIs, occurring both in finite and infinite dimensions. We then highlight variations on the notion of ACI which have also exhibited features of duality. We compare these, our goal being a systematic use of duality to produce new integrable systems from given ones.

## §0. Introduction

The Hitchin system [H] and its generalizations have generated a tremendous amount of activity and constructions, but often the common underlying freatures of two theories remain unrelated.

The purpose of this note is to connect several dualities exhibited by the Hitchin system; in doing so, we present two modifications of the concept of algebraic integrability, because these two were shown, in particular examples, to exhibit the same or a similar kind of duality; we illustrate the concept of Stäckel transformation and of reciprocal transformations (which were recently shown to be essentially the same [BS]) since they connect systems which also exhibit Hitchin-type dualities. We also pose some questions for infinite-dimensional completely integrable systems, given that some specific examples of quantized Hitchin systems exist; again, there doesn't seem to be a worked-out connection between dualities of isospectral type in the Sato Grassmannian and dualities of the Hamiltonian systems, whereas at least in the genus 0 and 1 limits of the Hitchin system, the action-angle dualities would have a very explicit manifestation at the level of the Baker function. Reciprocal transformations have been used for systems of hydrodynamic type, e.g., [AG], which encompass finite and infinite-dimensional commuting flows.

Ultimately we propose, firstly to strengthen the correspondences between the various concepts, and we pose concrete questions below, sketching the method; we hope to complete this work and follow-up with a diagrammatic note that provides a network between the types of dualities and transformations; secondly, we would like to systematize the transformations to produce new integrable systems from known ones.

In this note, we just lay out the proposed links between dualities, both classical and newly constructed (Section 1), between systems including modified versions of algebraically completely integrable (Section 3); we work out the explicit self-duality for the case of moduli of vector bundles in genus 2 (Section 2); and we propose to connect dualities systematically via transformations (Section 4).

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## §1. Aspects of duality

Duality as pertaining to dynamics appears to owe its relevance to Fourier-type transforms via bilinear pairings. It is more difficult, we believe, to understand its 'natural' explanation when it appears in geometry. The goal of this section is to give a list as complete as possible of dualities that occur when integrating equations of motion and to pose the question of relating them.

**1.1 Legendre transform.** We report the first instance from Arnol'd's classical book [Ar, 3.14], where it is sited after a sketch of Calculus of Variations and used to turn the Lagrangian into the Hamiltonian formulation of equations of motion: "The Legendre transformation is a very useful mathematical tool: it transforms functions on a vector space to functions on the dual space. Legendre transformations are related to projective duality and tangential coordinates in algebraic gometry..."

The Legendre transform of a function $f(\underline{x})$ is: in the scalar case, the maximum distance $g(\underline{p})$ between the value of the function at $p = f'(x)$ and the height of the line of slope $p = f'(x)$ at $x$; this can be viewed as a correspondence between a projective line and its dual, if $x$ is regarded as an affine coordinate of $\mathbb{P}^1$ and $p$, the slope of the lines through the origin, as an affine coordinate of $\mathbb{P}^{1*}$; in several variables $(x_1, ..., x_n)$, it is the difference between the inner product $(\underline{p}, \underline{x})$ and the value $f(\underline{x})$, where again $\underline{x}(\underline{p})$ is the value such that $\nabla f(\underline{x}) = \underline{p}$. Note that for this to be uniquely defined, $f$ has to satisfy a convexity condition. It is easy to check that this is an involutive operation. Note also that quadratic forms are self-dual, meaning that $f(\underline{x}(\underline{p})) = g(\underline{p})$ and $g(\underline{p}(\underline{x})) = f(\underline{x})$.

Under this involution, a Lagrangian system of second-order differential equations, defined, in the classical setting, on the tangent bundle to a manifold by a Lagrangian function $L(\underline{q}, \underline{\dot{q}}, t)$ that satisfies:

$$\underline{\dot{p}} = \frac{\partial L}{\partial \underline{q}}, \quad \text{where} \quad \underline{p} = \frac{\partial L}{\partial \underline{\dot{q}}},$$

is converted into a Hamiltonian system:

**Theorem** [A, §15]. *The system of Lagrange's equations is equivalent to Hamilton's equations, namely the system of* $2n$ *first-order equations:*

$$\underline{\dot{p}} = -\frac{\partial H}{\partial \underline{q}}, \qquad \underline{\dot{q}} = \frac{\partial H}{\partial \underline{p}},$$

*where* $H(\underline{p}, \underline{q}, t) = \underline{p}\underline{\dot{q}} - L(\underline{q}, \underline{\dot{q}}, t)$ *is the Legendre transform of the Lagrangian function viewed as a function of* $\underline{\dot{q}}$.

Note that $H$ is defined on the cotangent bundle to the manifold endowed with the natural symplectic structure $d\underline{p} \wedge d\underline{q}$. A recent generalization appears to be a natural duality of "Hessian metrics $g$" [SY], which have an interesting integral property.

While the Lagrangian formulation of the equations stems from a variational energy-minimizing problem, in the 'dual' Hamiltonian version, as generalized below to complex projective manifolds with holomorphic symplectic structure (an area of intensive research together with the concept of duality), perhaps a variational approach is not equally systematized; in particular, projective geometry carries different metric structures than Euclidean geometry, cf. e.g. [Fisch] for a comparison. It is therefore important to make recourse to the variatonal approach as generalized to infinite-dimensional manifolds (specifically loop spaces) by Gardner-Zakharov-Faddeev; the KdV equation could then be derived from the Euler-Lagrange equation and these were generalized to "systems of hydrodynamic type", cf. [Nov] for a concise exposition. It might be interesting to work out a dictionary between the finite-dimensional completely integrable case (esp., the geodesic or Neumann system which will be reviewed below in the context of 'duality') and the PDE case where flows indeed occur on a loop space (esp. the KdV equation whose algebro geometric solutions can be constructed from the geodesic system), to ask whether the variational equations of the classical case are obtained from the hydrodynamic ones; the geometry of the solutions is certainly well understood [DKN], but we are not aware of an explicit Lagrangian in the literature, which would become the much more interesting in the case of the Hitchin system of §2. In the Lagrangian formulation, the pursuit of integration is better achieved by identifying conservation of angular momentum, in the Hamiltonia formulation, conservation of energy: is there a geometric reason why the Hitchin system should have conserved angular momenta?

**1.2 Lie's sphere geometry.** The quadratic complex is key in the integration of the Hitchin system on which we zero in below. An alternative way to Plücker's Grassmannian for parametrizing lines in real projective space, was devised by Sophus Lie. Described over the reals, Lie's line-sphere correspondence does the following [J, Art. 215]. Given a sphere of radius $R$ with equation

$$-2\alpha x - 2\beta x - 2\gamma z + x^2 + y^2 + z^2 + C = 0, \quad \text{where} \quad C = \alpha^2 + \beta^2 + \gamma^2 - R^2$$

in Cartesian coordinates, letting

$$\begin{aligned} C = -\eta, \quad \alpha + \beta\imath = s, \quad R + \gamma = r, \\ \alpha - \beta\imath = \rho, \quad R - \gamma = \sigma, \end{aligned}$$

a correspondence is established between spheres of $\mathbb{R}^3$ and lines whose coordinates $(r, s, \sigma, \rho, \eta)$ satisfying the equation $\eta = r\sigma - s\rho$ are related to the standard Plücker coordinates, namely the six minors of

$$\begin{bmatrix} 1 & 0 & -r & -\rho \\ 0 & 1 & -s & -\sigma \end{bmatrix}.$$

Lie then gave a characterization of linear complexes in terms of differential equations. Certain "special" quadratic complexes were also characterized, where special means that they consist of all the lines tangent to a given surface. The equation is $\sum_{i=0}^{5}(\frac{\partial\phi}{\partial x_i})^2 = 0$, $\phi = \sum_{i=0}^{5}\lambda_i x_i^2$ being the equation of the complex in Klein coordinates (i.e., the Grassmannian has equation: $\sum x_i^2 = 0$). There are 4 co-singular ("confocal" would be the word for quadrics) complexes,

$$\sum_{i=0}^{5}\frac{x_i^2}{\lambda_i + \mu} = 0,$$

that go through any given line $y := [y_0, ..., y_5] \in \mathbb{P}^5$, given by the values $\mu_1, ..., \mu_4$, say, of the parameter. Then, $y$ can be described by the "elliptic coordinates" $[\mu_1, ..., \mu_4]$ in such a way that

$$[y_i] \sim \left[\frac{(\lambda_i+\mu_1)(\lambda_i+\mu_2)(\lambda_i+\mu_3)(\lambda_i+\mu_4)}{f'(-\lambda_i)}\right],\ f(\mu) = \prod_{i=0}^{5}(\lambda_i + \mu)$$

and the PDE for the complex becomes:

$$\sum_{i=1}^{4}\left(\frac{\partial\phi}{\partial\mu_i}\right)^2\frac{f(\mu_i)}{\prod_{j\neq i}(\mu_i - \mu_j)} = 0.$$

Naturally, as soon as you write down a differential equation, you realize that Jacobi already integrated it. Not only do we recognize hyperelliptic genus-2 integrals in his solution:

$$\phi = \sum_{i=1}^{4}\left(\int d\mu_i\frac{\sqrt{(\mu_i - a)(\mu_i - b)}}{\sqrt{f(\mu_i)}}\right) + C,\ \ a, b \in \mathbb{C} \text{ arbitrary},$$

we also see that this system is a transformed Neumann/Geodesic system (cf. §2.3 below), for the parameters of the base (the fixed values of the Hamiltonians) correspond to moving curves $\mu = a, \mu = b$, the family of hyperelliptic curves given by fixing $g + 1 (= 3)$ Weierstrass points (plus a 'normalized' one, such as $\infty$ in the Neumann notation, cf. [M], e.g.) and varying the other $g$.

**1.3** "**Dualities in integrable systems**". We quote *verbatim* from the beginning of [GR], where the authors make our point better than we ever could: "During the last years duality becomes a very fashionable term denoting a lot of different phenomena mainly due to the dramatic development in the string theory. Hence it is necessary to be very precise when speaking in the related issues. That is why we formulate the subject of the paper from the very beginning – the different dualities in the Hitchin type many-body systems as well as their generalizations shall be discussed".

The first appearance of striking examples of duality that connected ACIs, as far as we know, occurred in [FGNR]. In the differentiable category, on (open sets) of a manifold of dimension $2m$, a completely integrable system admits action-angle variables $(I_i, \phi^i)$, $\omega = \sum_{i=1}^{m} dI_i \wedge d\phi^i$ being the symplectic form, given in Darboux coordinates as $\sum_{i=1}^{m} dp_i \wedge dq^i$, $H_j = f_j(\underline{I})$ the Hamiltonians, and

$$\underline{I}(b_2) - \underline{I}(b_1) = \int_\Gamma \omega,$$

where $b_1, b_2$ are two points in the base joined by a path $\gamma$ and $\Gamma \in H_2(\underline{h}^{-1}(\gamma), \underline{h}^{-1}(b_1 \cup b_2); \mathbb{Z})$, $\underline{h}$ is the Hamiltonian map and the lift of $\gamma$ is transported through the Gauss-Manin connection.

**Definition.** [FGNR] Two Hamiltonian systems are dual to each other in the sense of action-coordinate (AC) duality if the action variables $I_i$ of the first system coincide with the coordinates $q^i$ of the second and vice-versa.

**Example.** The two-particle system called Calogero oscillator gives rise in one limit to the Calogero-Moser system, with Hamiltonian: $H(p,q) = \frac{p^2}{2} + \frac{\nu^2}{2q^2}$; we let the parameter $\nu = 1$ for simplicity. The strategy is the following. Step I: find action-angle variables $I = f(H)$, and $\phi$, in terms of $p, q$, so that $dp \wedge dq = dI \wedge d\phi$, with $H = H(\phi)$ a function of $\phi$ only. Step II: find a dual $H^D(I,\phi) = H^D(q)$, a function of $q$ only. For the example,

$$I = \frac{1}{2\pi} \oint pdq = \frac{1}{2\pi} \oint \sqrt{E - \frac{1}{2q^2}} dq, \; d\phi = \frac{dq}{p} \left( \frac{\partial I}{\partial E} \right)_I^{-1}, \text{ so that } \phi = \sqrt{q^2 - \frac{1}{2E}},$$

where $E$ is the value of $H$, the energy, then $H^D(I,\phi) = \frac{q^2}{2} = \frac{\phi^2}{2} + \frac{1}{2I^2}$, interchanging position and momentum.

For the holomorphic-symplectic case the authors give a stronger definition of duality, which they call AA (Action-Angle).

**Definition.** [FGNR] AA duality is obtained by choosing a symplectic basis $(A,B)$ of $H_1(\underline{h}^{-1}(b_1), \mathbb{Z})$, which is also transported via the Gauss-Manin connection along a path $\gamma$ connecting the points $b_1, b_2$ of the base, giving a decomposition of

$$\Gamma \in H_2(\underline{h}^{-1}(\gamma), \underline{h}^{-1}(b_1 \cup b_2), \mathbb{Z}), \; \Gamma = \Gamma_A + \Gamma_B.$$

The dual action variables are:

$$\underline{I}(b_2) - \underline{I}(b_1) = \int_{\Gamma_A} \omega, \quad \underline{I}^D(b_2) - \underline{I}^D(b_1) = \int_{\Gamma_B} \omega.$$

Since the 2-form $\sum dI_i \wedge dI_i^D$ vanishes, (locally) there exists a function $\mathcal{F}$, called prepotential, such that $I_i^D = \frac{\partial \mathcal{F}}{\partial I_i}$.

**Example.** Again in one degree of freedom, the Hamiltonian of the "elliptic Calogero-Moser" system is: $H(p,q) = \frac{p^2}{2} + \nu^2 \wp_\tau(q)$, where $\wp_\tau$ is the Weierstrass function on the elliptic curve $\mathcal{E}_\tau$:

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau) = 4\prod_{i=1}^{3}(x - e_i), \quad x = \wp_\tau(q), \; y = \wp_\tau(q)'.$$

With the notation: $e_0 = \frac{E}{\nu^2} = \frac{H}{\nu^2}$, the action variable is one of the periods of the differential $\frac{pdq}{2\pi}$ on the curve $E = H(p,q)$,

$$I = \frac{1}{2\pi} \oint_A dq \sqrt{2(H - \nu^2 \wp_\tau(q)} = \frac{\imath\nu}{2\sqrt{2}\pi} \oint_A \frac{dx\sqrt{x - e_0}}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}}, \; d\phi = \frac{1}{2\imath T(E)} \frac{dx}{\sqrt{\prod_{i=0}^{3}(x - e_i)}},$$

where

$$T(E) = \frac{1}{4\pi\imath} \oint_A \frac{dx}{\sqrt{\prod_{i=0}^{3}(x - e_i)}}.$$

By introducing the meromorphic function on $E_\tau$,

$$\hat{\text{cn}}_\tau(z) = \sqrt{\frac{x - e_1}{x - e_2}},$$

which up to a rescaling of $z$ is indeed Jacobi's elliptic cosine, and

$$H_D(I,\phi) = \hat{\mathrm{cn}}_\tau(z) = \hat{\mathrm{cn}}_{\tau_E}(\phi)\sqrt{1-\frac{\nu^2(e_1-e_3)}{2E-\nu^2 e_3}},$$

which indeed depends on $z$ only, Action-Angle duality is achieved since

$$\tau_E = \frac{\left(\oint_B \frac{dt}{\sqrt{\prod_{i=1}^3 4(t-t_i)}}\right)}{\left(\oint_A \frac{dt}{\sqrt{\prod_{i=1}^3 4(t-t_i)}}\right)},$$

$$t = \frac{1}{x-e_0} + \frac{1}{3}\sum_{i=1}^{3}\frac{1}{x-e_i},\ t_i = \frac{1}{3}\sum_{j=1}^{3}\frac{e_j-e_i}{(e_0-e_i)(e_0-e_j)}.$$

Not surprisingly, this duality carries more geometry; the general elliptic model, also worked out in [FGNR], with modular functions $g_2, g_3$ viewed as sections of $\mathcal{O}(4n)$ and $\mathcal{O}(6n)$, respectively, over $\mathbb{P}^1$, and Hamiltonian an arbitrary (meromorphic) function over $\mathbb{P}^1$, when dualized has integral manifolds that are Jacobians of dimension 5, thus exemplifying an "embedding of the Liouville tori into (...) abelian varieties of higher rank (...) going back to the original work of S. Novikov and A. Veselov [NV]", who studied the theory of Liouville-integrable Hamiltonian systems on the phase space of the finite-zone potentials. The Poisson brackets obtained from the classical and new integrable dynamical systems are related over these phase spaces. The general Poisson brackets defined by the authors have algebro-geometric significance, and some of these Poisson brackets possess foliations by complex tori which are no longer abelian varieties. The issue will be pursued a bit further in Section 4.

The authors give an example of prepotential for the case $H = \frac{p^2}{2} + \Lambda^2\cos q$, with spectral curve $y^2 = (x-u)(x^2-1)$, $x = \cos q$, $y = \frac{p\sin q}{\sqrt{-2\Lambda}}, u = \frac{H}{\lambda^2}$ which they interpret as $S$-duality (interchanging $SU(2)$ gauge theories near $u=\infty$ and $u=1$, respectively). The dual variables

$$I = \int_{-1}^{1}\sqrt{\frac{x-u}{x^2-1}}dx,\ \ I^D = \int_{1}^{u}\sqrt{\frac{x-u}{x^2-1}}dx$$

obey the Picard-Fuchs equation and $u$ is monodromy-invariant:

$$\left(\frac{d^2}{du^2}+\frac{1}{4(u^2-10)}\right)\begin{pmatrix} I \\ I^D\end{pmatrix} = 0,\ II^D - 2\mathcal{F} = u,\ \mathcal{F}\sim\frac{1}{2}u\log u + ... \sim I^2\log I + \sum_n \frac{c_n}{n}I^{2-4n},\ c_n\in\mathbb{C}.$$

**1.4 Involutions on Sato's Grassmannian.** There are at least two involutions of key importance defined on Sato's Grassmannian. The first, which takes the "Baker function" to its "adjoint", equivalently, a subspace $W\in$Gr to its orthogonal, is the ingredient that yields the link between Gr and the KP hierarchy:

$$\oint_{S^1}\psi_W(g,\zeta)\cdot\psi_{W^\perp}((g')^{-1},\zeta) = 0,$$

$$\psi_W(g,z)|_{z=\zeta} = \tau_W(gq_\zeta)/\tau_W(g)$$

the wave or Baker function is equivalent to:

$$\sum_{j=0}^{\infty} p_j(-2x)p_{j+1}(D_x)e^{\sum_{k=1}^{\infty} y_k D_y^k}\tau(y)\bullet\tau(x) = 0,$$

namely the KP hierarchy in Hirota's bilinear form (we do not provide definitions for this well-known terminology). In Sato's (boson-fermion) correspondence $\partial^{-1} \leftrightarrow z$ (given by $\partial + v_{-1}(x)\partial^{-1} + ...)\psi(x,z) = z\psi$), this corresponds to the classical notion of "adjoint operator", in differential algebra:

$$L = \sum_{j=0}^{N} u_j(x)\partial^j, \quad L^\dagger = \sum_{j=0}^{N} (-\partial)^j u_j(x),$$

the dual Baker function $\psi^\dagger$ to $\psi$ is the Baker function of $\mathcal{L}^\dagger$. More generally, the adjoint operator of differential algebra:

$$(D^* u^*)v = \delta(u^* v) - u^*(Dv), \; u^* \in V^*, \; v \in V,$$

$(V, \delta, D)$ a differential system, namely $D : K \to K$ a derivation on a differential field $(K, \delta)$, $V, V^*$ dual $K$-spaces, was used recently to give algorithmic criteria for the differential system to have a cyclic vector, cf. [CK]. For "algebro-geometric solutions" of the KP hierarchy (the centralizer of $L$ in the ring of differential operators is larger than a polynomial ring $\mathbb{C}[L_0]$), the spectral curves are the same and the eigen-line-bundles of common eigenfunctions of the adjoint rings are the opposite of each other on the Jacobian. However, special points of the Grassmannian, dubbed in the algebro-geometric case "adelic Grassmannian" by G. Wilson, have the property that interchanging the variables $x$ and $z$, they still belong to the Grassmannian: these are the solutions to the "bispectral problem" posed by J.J. Duistermaat and F.A. Grünbaum [DG], namely a classification of differential operators in one variable, with common spectrum, that admit a 'dual' pair of differential operators in their spectral parameter, with spectrum depending on the variable of the original pair.

There seems to be little information on how dualities of ACIs reflect on dualities on the Grassmannian for the PDE problem that corresponds, certainly in a non-trivial way, to the Hamiltonian system (for instance, KdV correspond to the Neumann system, cf. [M]), with the following exception. In [C], Sutherland quantized potentials (in several variables $(x_i)$),

$$-\Delta + \sum_{1 \le i < j \le n} \frac{1}{2} m(m+1) \sinh^{-2}\left(\frac{x_i - x_j}{2}\right),$$

are shown to be bispectral to a commutative ring of difference operators, the rational Ruijsenaars operators, cf. [Ruij] in which Ruijsenaars first detected a duality. We consider the natural question: Is there a bispectral duality corresponding to the quantized Hitchin system and its eigenfunction? In [GT-N-B2], upon finding Hamiltonians for the Hitchin system in genus 2, the authors propose an explicit quantization; the missing link with bispectrality is an explicit wave/Baker function, which Chalykh is able to produce.

## §2. What is this duality?

This section is devoted to the simplest case of the Hitchin system. The phase space is $\mathcal{T}^*\mathcal{SU}_X(2, \mathcal{O}_X)$, the cotangent bundle to the (Seshadri) moduli space of semistable rank-2 bundles over a Riemann surface $X$ of genus 2. This seems to be to date the only case in which explicit Hamiltonians have been written, as quartic polynomials in six variables, cf. [GT-N-B1] where the authors provide a parametrization of the phase space in terms of second-order theta functions and reveal that this case of the Hitchin system is self-dual under interchange of positions and momenta. The aim of this section is to identify this duality in geometric terms. In rephrasing results from [GT-N-B1] we shall be necessarily sketchy, referring to the original paper for much more detail and technical provisos.

**2.1 Wirtinger duality.** Points of the phase space of the system are pairs $(E, \psi)$, where the bundles $E$ can be given as an extension (class)

$$0 \to \ell^{-1} \to E \to \ell \to 0,$$

$\ell$ a line bundle of degree $g - 1 = 1$, and $\psi$ is a holomorphic (1.0) form with values in the bundle of traceless endomorphisms of $E$. By associating to $E$ the subvariety $C_E$ of $\mathrm{Pic}^{g-1}(X)$ consisting of the line bundles $\ell$ such that $H^0(\ell \otimes E) \neq 0$ (equivalently, $E$ is an extension of $\ell$), M.S. Narasimhan and S. Ramanan showed that

$\mathcal{SU}_X(2,\mathcal{O}_X)\overset{\cong}{\to}\mathbb{P}^3=\mathbb{P}H^0(2\Theta_0)$, $E\mapsto\mathbb{C}^*\theta$ where $\theta$ is a second-order theta function that vanishes precisely on the curve $C_E$.

**2.1.1 Two versions.** The classical duality is intrinsic. Indeed, the theta divisor in $\mathrm{Pic}^{g-1}$, $\Theta:=\{\xi\in\mathrm{Pic}^{g-1}:H^0(\xi)\neq 0\}$ is canonically defined, as is the morphism $\mathrm{Pic}^0\to|2\Theta|$, $\eta\mapsto\Theta_\eta+\Theta_{\eta^{-1}}$, and this identifies $|2\Theta|^*$, the linear forms on $|2\Theta|$, with $|2\Theta_0|=H^0(\mathrm{Pic}^0,2\Theta_0)$, because the hyperplane class pulls back to $2\Theta_0$. In [GT-N-B1], this is presented in an analytic way using extrinsic, explicit coordinates that depend on the choice of suitable bases. Upon choosing a (standard) homology basis on $X$, the associated theta function $\vartheta(u)$ gives rise to a second-order theta function (in both $u$ and $u'$) $\vartheta(u'-u)\vartheta(u'+u)$, according to Riemann's addition theorem:

$$\vartheta(u'-u)\vartheta(u'+u)=\sum_{e\in\mathbb{Z}^g}\theta_{2,e}(u')\theta_{2,e}(u),\text{where :}$$

$$\theta_{k,e}(u)=\sum_{n\in BbbZ}^{g}e^{\pi\imath k^t(n+e/k)\tau(n+e/k)+2\pi\imath k(n+e/k)\cdot u}\quad .$$

The linear isomorphism $\iota:H^0(2\Theta_0)^*\to H^0(2\Theta_0)$, $\iota(\phi)(u)=\langle\vartheta(\cdot-u)\vartheta(\cdot+u),\phi\rangle$, is such that by Riemann's addition theorem $\iota$ interchanges the basis $(\theta_{2,e})$ of $H^0(2\Theta_0)$ with the dual basis $(\theta^*_{2,e})$ of $H^0(2\Theta_0)^*$. These dualities coincide, as can be seen by noting that to an element $\phi\in|2\Theta_0|^*$ there should correspond an element $\theta\in|2\Theta_0|=H^0(\mathrm{Pic}^0,2\Theta_0)$ according to the intrinsic duality, and since $\iota(\phi)(u)=\langle\vartheta(\cdot-u)\vartheta(\cdot+u),\phi\rangle$, this is the divisor $\iota(\phi)=\{u:\phi((\Theta_0)_u)+((\Theta_0)_{-u})=0\}$.

In this way, $\iota$ interchages positions $p_1,...,p_4$ and momenta $q_1,...,q_4$ (given by a dual basis) for the standard symplectic form $dp\wedge dq$, which is the canonical one on the cotangent bundle, used by Hitchin. It is by this device that the authors are able to show that, for a Hitchin Hamiltonian $h$, $h(p,q)=h(q,p)$, a property "far from obvious in the original formulation of the Hitchin system" [GT-N-B2, §5.4]. The interchange $(\theta,\phi)\mapsto(\iota(\phi),\iota(\theta))$ takes place on a projective variety, so a natural geometric question is, "what is this duality?" Let's recall how [GT-N-B1] traces the bundle to its projective coordinates $[p_1,...,p_4]\in\mathbb{P}^3$. In the presentation $0\to\ell^{-1}\to E\to\ell\to 0$, $E$ is regarded as the bundle $\ell^{-1}\oplus\ell$ with complex structure ($\overline{\partial}$-operator) $\overline{\partial}_E=\overline{\partial}_{\ell^{-1}\oplus\ell}+B$, $B=\begin{bmatrix}0&b\\0&0\end{bmatrix}$. The duality interchanges the bundle $E$ that corresponds to the curve of zeroes of $\theta$, with the bundle that corresponds to the curve of zeroes of $\iota(\phi)$, $\{u:\langle\vartheta(\cdot+u)\vartheta(\cdot-u),\phi\rangle=0\}$.

More geometrically: for a pair $(\theta,\phi)$ in generic position, namely such that the corresponding Hitchin value is not the square of a differential, equivalently $\eta\neq 0$ in the Higgs field $\psi=\begin{bmatrix}-\mu&\nu\\\eta&\mu\end{bmatrix}$, a generic $\eta$ will not be the product of two elements, one each from $H^0(\ell^2_{u_1})$ and $H^0(K_X)$, where $u_1$ is one of the points associated to $E:\ \theta(u_1)=0$. There is then a 2-dimensional choice of theta functions vanishing at $u_1,u_2,\ u_1\pm u_2,u_1,u_2\notin\Lambda$ , such that the four zeroes $x_1,...,x_4$ of $\vartheta\left(\int_{x_0}^x\omega-u_1\pm u_2-\Delta\right))$ satisfy:

$$u_1+u_2=\int_{x_0}^{x_1}\omega+\int_{x_0}^{x_2}\omega-2\Delta,\quad u_1-u_2=\int_{x_0}^{x_3}\omega+\int_{x_0}^{x_4}\omega.$$

In other words, given $E$ and an associated $\theta$, with a line bundle $\ell_1$ corresponding to a point on it, one chooses a plane through the point of $|2\Theta_0|$ that corresponds to $\theta$ and a $\phi$ in its orthogonal line; the duality now the line meets the Kummer dual in four points $\phi_{\ell_i}$; the divisors of $\ell_1\ell_2$ and $\ell_1\ell_2^{-1}K$ are $x_1+x_2$, $x_3+x_4$, respectively. It is possible to arrange the dual choices in such a way that the associated points $y_i$ are images of the $x_i$ under the hyperelliptic involution.

**Conclusion.** While this duality does not make sense on moduli of bundles, since it also depends on the choice of Higgs field, its geometric meaning is the following: choosing (generically) a point on the curve that corresponds to $E$, and one Fay quadrisecant through that point (the 3-dimensional family of Fay quadrisecants accounts for the choice of $\phi$, in the projectivized cotangent space at $E$, a plane), the defining equations of the 'dual' curve in the pair are obtained in the same way as the original, but after translation by $u_1$ (where the points $u_1,...,u_4$ play symmetrical roles).

Notably, expressing Wirtinger's duality in extrinsic coordinates provides a proof [GT-N-B1, Appendix 3] that the Kummer surface is self-dual (different from the standard geometric argument) and might be useful to investigate similar geometric properties of the higher-genus Kummer variety.

The challenge is then to pursue a duality for the Hitchin system in higher genus. An indication that it should exist, is the AC duality of §1.3: the many-body rational Calogero-Moser systems and trigonometric Ruisenhaars-Schneider systems [RS] (both self-dual) approach each other under two successive deformations of the (dual) parameters, while an AC duality connects the trigonometric Calogero-Moser with the rational Ruisenhaars-Schneider systems [FGNR].

In genus 0 and genus 1, resp., the Gaudin system and the elliptic Calogero-Moser (spin) system have been interpreted as limits of the Hitchin system when the curve $X$ acquires singular points [ER,N]. It is conceivable that the dualities between these systems can also be obtained as a limit of a duality for the Hitchin system in genus $g$; in the $g=2$ case, we are in the process of comparing the [GT-N-B1] duality, in the limit taken as in [ER,Nekr], with the action-angle duality of [FGNR].

An easier project which we propose is to interpret this genus-2 duality of integrable systems in the geometry of the quadratic complex (§1.2): what is the symmetry that an exchange of positions and momenta brings to the Jacobi-Lie differential equation, related as that equation is to the geodesic and Neumann system? Does it interchange these two systems in the sense of [K] in §2.3 below, which too is an exchange of positions and momenta (but also changes the spectral curve, by inversion of the separating variables in the sense of Sklyanin [S]!) In the next subsection, an *ad hoc* Lax-pair model for the system in genus 2 is presented [GT-N-B2], whose spectral curve has genus 3 and Jacobian isomorphic to the Hitchin Prymian where the flows linearize; it does not appear to us that the authors investigated what duality is induced on this Hamiltonian system by their interchange $(\underline{p}, \underline{q}) \leftrightarrow (\underline{q}, \underline{p})$.

**2.2 The Lax pair.** What's more, thanks to these formulas, the authors are able to recognize a symplectomorphism between the phase space of the Hitchin system and that of a Neumann system with $6\times 6$ Lax pair, with first element a $6\times 6$ matrix $L(z)$:

$$L_{mn}(z) = zJ_{mn} + e_n\delta_{mn},$$

where the $15 = \binom{6}{2}$ $J_{ij}$ are defined as follows:

$$J_{12} = \frac{\imath}{2}(q_1p_1 + q_2p_2 - q_3p_3 - q_4p_4), \text{ etc.},$$

$e_i$ are the six branchpoints of the curve $X$, and with spectral curve

$$\prod_{n=1}^{6}(z-e_n)\left(1+\frac{1}{2}\zeta^2 \sum_{n\neq m\in\{1,\dots,6\}} \frac{J_{nm}^2}{(z-e_n)(z-z_m)}\right).$$

Note that this is not the Lax pair with 'constraints' identified by Krichever [Kr1] for the Hitchin system in Tyurin coordinates, for the size of Krichever's matrix is the rank of the bundle (two, in this case).

**2.3 The geodesic and Neumann systems.** Knörrer in [K] identified a correspondence between the Neumann system,

$$\ddot{\xi} = -A\xi + u\xi,\ \xi_1^2 + \dots + \xi_n^2 = 1,\ u = \langle \xi, A\xi\rangle - \langle \dot{\xi}, \dot{\xi}\rangle$$

(assume for simplicity that $A$ is a diagonal matrix with distinct non-zero eigenvalues) and the geodesic motion on the quadric

$$2q(x) = \langle x, Bx\rangle - 1,\ B = \operatorname{diag}\left(\frac{1}{a_1}, \dots, \frac{1}{a_n}\right).$$

He gave a beautiful geometric interpretation (rooted is Riemannian geometry) of the following:

*If $x(t)$ is a geodesic on the quadric, then the zeroes of the polynomial in $z$:*

$$\left(\sum_{i=1}^{n}\frac{\dot{x}_i^2}{a_i - z} - \frac{1}{2}\sum_{i,j=1}^{n}\frac{(x_i\dot{x}_j - x_j\dot{x}_i)^2}{(a_i-z)(a_j-z)}\right)\cdot\prod_{i=1}^{n}(a_i - z)$$

*are independent of t. Calling these zeros* $0, \lambda_1, ..., \lambda_{n-2}$*, then* $0, \frac{1}{\lambda_1}, ..., \frac{1}{\lambda_{n-2}}$ *are the zeroes of the polynomial:*

$$\left(\sum_{i=1}^{n} \frac{\xi_i^2}{\alpha_i - z} - \frac{1}{2}\sum_{i,j=1}^{n} \frac{(\dot{\xi}_i\xi_j - \dot{\xi}_j\xi_i)^2}{(\alpha_i - z)(\alpha_j - z)}\right) \cdot \prod_{i=1}^{n}(\alpha_i - z),\ \alpha_i := \frac{1}{a_i},$$

*and are independent integrals of the Neumann system.*

This being an interchange of positions and momenta, it is again a symplectic transformation. On the other hand, for the 'degenerate' case of geodesics over a manifold of constant curvature, which was shown by J. Moser to be equivalent to Kepler's problem, and by an inversion of the coordinates: $\tilde{x}_i := \frac{x_i}{|\underline{x}|^2}$, the authors of [KBS] show that, this transformation, also canonical, brings the geodesic flows in a parallel fashion to Kepler's flows.

### §3. Aspects of integrability

The goal of this section is to make the mysterious (for us at least) dualities of §2 much more mysterious. To that worthy aim, we present two modifications of the concept of integrability that have been recently introduced.

The elusiveness of the concept of integrability is by now a legend, with volumes recently written on that theme. What may be attainable, is a precise definition at least in the case of a finite-dimensional Hamiltonian system. Yet even in this realm, one has to carefully distinguish between integrability by quadratures, separation of variables, and complete integrability. Since we want to focus on the nuances of complete integrability, let's first review the nomenclature. Liouville integrability and integration by quadratures are taken to be the same in [Ar]; it is a consequence that the integral manifolds are real tori. Separation of variables, which we do not use much here but would assume in Sklyanin's sense [S], "reduces the complicated interacting many-body problems to [a] collection of identical systems with one degree of freedom", and "for the Hitchin-type many-body systems can be recognized within the duality group" [GR]. It is important to note that separability is a local property, while Liouville integrability is usually intended to be global (though not by us: we implicitly and always consider generically completely integrable systems), thus the latter implies the former but not vice versa.

We proceed to highlight the difference between Liouville integrability and algebraic complete integrability (ACI). Then, we present alternative definitions of integrability whose common relation to ACIs are non-linear time transformations.

Consider a Hamiltonian system in $\mathbb{R}^{2n}$

$$\dot{x}_i = \{x_i, H(x,p)\}, \quad \dot{p}_i = \{p_i, H(x,p)\}, \qquad i = 1, \dots, n \tag{3.1}$$

where $\{\cdot,\cdot\}$ is a non-degenerate Poisson bracket, $x = (x_1, \dots, x_n)$, $p = (p_1, \dots, p_n)$ and $H$ is smooth. The system is called integrable if there exist $n$ functionally independent first integrals $H \equiv I_1(x,p), \dots, I_n(x,p)$ in involution $i, j = 1, \dots n$.

If the generic real invariant manifolds $\mathcal{A} = \cap_{l=1}^{n}\{I_l(x,p) = c_l, c_l = \text{const}\}$ are compact, then they are $n$-dimensional tori $\mathbb{T}^n$. More generally, according to Arnold-Liouville's theorem, if the flows are complete (i.e. the solutions to Hamilton equations exist for all times), then there exist $k$ such that $\mathcal{A} \approx \mathbb{T}^K \times \mathbb{R}^{n-k}$.

Integrable Hamiltonian systems with polynomial first integrals whose flows are not complete may be easily obtained and their generic real invariant manifolds are topologically non trivial. We recall two examples: Mumford's example is that of the simplest possible polynomial Hamiltonian that, for geometric reasons, irredeemably blows up at infinity while Flaschka's misses completion by reasons of topological nature.

To give our example we sketch the concept of algebraic complete integrability which will be contextualized more carefully below.

A Hamiltonian system with $n$ degrees of freedom, i.e. defined on a symplectic manifold $M$ of (real) dimension $2n$ is completely integrable if it admits $n$ functions in involution whose differentials are linearly independent (possibly, generically on $M$). (Arnold, [Ar, §49]) When $M$ is a component of the set of real points of an algebraic variety $M_{\mathbb{C}}$ and the symplectic form $\omega$ and hamiltonian function $H$ are rational without poles on

$M$, the concept of algebraic complete integrability can be introduced. For this to be the case, we require that the vector fields corresponding to the hamiltonians in involution still have no poles on a compactification of the fibres on $M_{\mathbb{C}}$.

**Nonexample** (Mumford, [M, §4]):

$$M = \mathbb{R}^2, \ \omega = dx \wedge dy, \ H = x^4 + y^4$$

Here a compactification of the fibre, the affine curve $x^4 + y^4 = c$, is the projective curve $X^4 + Y^4 = cZ^4$, which is smooth (provided $c \neq 0$) and has 4 points at infinity. The vector field $X_H$ defined by $H$, $X_H \rfloor \omega = -dH$,

$$X_H f(x, y) = \frac{\partial H}{\partial y}\frac{\partial f}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial f}{\partial y}$$

is tangent to the fibre in the affine plane, i.e. a multiple of $\chi_*(\frac{\partial}{\partial t})$, if $\chi : t \mapsto (x(t), y(t))$ on the curve, because

$$\chi_*(\frac{\partial}{\partial t})f(x(t), y(t)) = \frac{\partial x}{\partial t}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial f}{\partial y}$$

and differentiating $x^4 + y^4 = c$ gives $\frac{\partial x}{\partial t}x^3 + \frac{\partial y}{\partial t}y^3 = 0$, so $(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}) \propto (y^3, -x^3)$.
At infinity $Z = 0$, $X, Y \neq 0$, the vector field $X_H = 4y^3\frac{\partial}{\partial x} - 4x^3\frac{\partial}{\partial y}$ has a pole, since coordinates at infinity are $(Z/X = 1/x, Y/X = y/x)$ and $\frac{\partial}{\partial x} = -\frac{1}{x^2}\frac{\partial}{\partial(1/x)}$, and since $4y^3/(-x^2) = -4Y/Z$. This is why 4 is the lowest exponent for which this simple nonexample works!

**Note:** In the algebraically completely integrable situation, the fibres are abelian varieties or extensions of such by $\mathbb{C}^{*k}$ for some power $k$.

Following Flasckha [Fl], take $F(p, q) = p^2 - q^3$ and substitute $q = x_1 + iy_1$, $p = x_2 + iy_2$. Then the real and imaginary parts of $F$,

$$I_1 = x_1^2 - y_1^2 - x_2^3 + 3x_2y_2^2, \qquad I_2 = 2y_1x_1 - 3x_2^2y_2 + y_2^3$$

are clearly in involution with respect to the canonical Poisson bracket

$$\{y_i, x_j\} = \delta_{ij}, \qquad \{y_1, y_2\} = \{x_1, x_2\} = 0,$$

the invariant manifold $\mathcal{A}$ may be identified with a complex torus with the points at infinity missing and Arnold-Liouville's theorem is not applicable. This construction may easily be generalized starting with $n$ polynomials $F_1, \ldots, F_n$ in involution in $\mathbb{R}^{2n}$, complexifying and obtaining $2n$ involutive polynomials in $\mathbb{R}^{4n}$. As pointed out by Flaschka [Fl], the core question is then to explain the qualitative difference between presence and absence of first integrals and a natural (open) problem is to find canonical models for the behavior of integrable Hamiltonian systems in two degrees of freedom whose level surfaces are punctured Riemann surfaces.

**3.1 Algebraic complete integrability.** In the following we restrict ourselves to real integrable Hamiltonian systems (3.1) to which the Arnold-Liouville theorem applies. Then following Adler and van Moerbeke [AvM], the system (3.1) is called *algebraically completely integrable* if:

1) the complexified invariant manifolds in $\mathbb{C}^n$, which are noncompact, can be completed into complex Abelian tori $\mathbb{T}^g$, i.e.,

$$\mathcal{A}_C = \mathbb{T}^g \setminus \mathcal{D}_c.$$

Here $\mathbb{T}^g$ is the quotient of $\mathbb{C}^g$ by a lattice $\Lambda_{2g}$ generated by $2g$ period vectors independent over the reals and satisfying Riemann's conditions, $\mathcal{D}_c \subset \mathbb{T}^g$ is a union of codimension one analytic subvarieties, a "divisor". The periods and the divisor depend on the constants of motion $c_1, \ldots, c_{n-g}$. The variables $x$ are meromorphic functions on $\mathbb{T}^g$ with poles along $\mathcal{D}_c \subset \mathbb{T}^g$.

2) In Cartesian coordinates $z_1, \ldots, z_g$ on $\mathbb{C}^g$ the complex trajectories of the system are straight-line projections on $\mathbb{T}^g$, along which (*possibly after a change of time* $t$) the motion is uniform.

**3.2 Deficient Integrability.** Algebraic integrability "with deficiency" was introduced by Pol Vanhaecke [V] in relation to symmetric products of curves and strata of (generalized) Jacobians, and exemplified by Abenda and Fedorov [A,AF1,AF2] in the case of hyperelliptically separable systems. The simplest situation is a real completely integrable Hamiltonian system whose complexified invariant manifolds are codimension 1 subvarieties of an abelian variety.

The generalization of a.c.i. systems proposed by Vanhaecke [V] consists in the association of an integrable system with polynomial invariants on $\mathbb{R}^{2n}$ $(n \geq 1)$ to a pair of polynomials $F(\lambda,\mu)$, $\varphi(\lambda,\mu)$. Indeed any polynomial $\varphi(\lambda,\mu)$ specifies a Poisson bracket on $\mathbb{R}^2$ by $\{y,w\} = \varphi(\lambda,\mu)$, which extends to $(\mathbb{R}^2)^n = \mathbb{R}^2 \times \cdots \mathbb{R}^2$ by taking the product bracket, i.e.

$$\{p_i, x_j\} = \delta_{ij}\varphi(x_j, p_i), \qquad \{x_i, x_j\} = \{p_i, p_j\} = 0.$$

Let $\Delta = \{((x_1,p_1),\dots,(x_n,p_n)) \,|\, x_i = x_j \text{ for some } i \neq j\}$ and consider the map $\mathcal{S} : (\mathbb{R}^2)^n \backslash \Delta \mapsto \mathbb{R}^{2n}$, given by

$$(u(\lambda), v(\lambda)) = \left(\prod_{i=1}^{n}(\lambda - x_i), \sum_{i=1}^{n} p_i \prod_{j\neq i} \frac{\lambda - x_j}{x_i - x_j}\right).$$

$\mathcal{S}$ is invariant for the obvious action of the permutation group $S_d$ on $(\mathbb{R}^2)^n$ and is a $d! : 1$ unramified covering map onto an open subset of $\mathbb{R}^{2n}$. Since the Poisson structure is also invariant under the action of $S_d$, a $\mathcal{C}^\infty$ Poisson bracket $\{\cdot,\cdot\}_n^\varphi$ is defined on the image of $\mathcal{S}$ by requiring that $\mathcal{S}$ is a Poisson map, i.e. that for any $f, g \in \mathcal{C}^\infty(\mathbb{R}^{2n}$, one has $\{f,g\}_n^\varphi \circ \mathcal{S} = \{f \circ \mathcal{S}, g \circ \mathcal{S}\}$. The Poisson bracket $\{\cdot,\cdot\}_n^\varphi$ is given in terms of the coordinates $u_i, v_i$ by

$$\{u(\lambda), u_j\}_n^\varphi = \{v(\lambda), v_j\}_n^\varphi = 0,$$

$$\{u(\lambda), v_j\}_n^\varphi = \{u_j, v(\lambda)\}_n^\varphi = \varphi(\lambda, v(\lambda) \left[\frac{u(\lambda)}{\lambda^{n-j+1}}\right]_+ \bmod u(\lambda), \quad j = 1,\dots,n,$$

where, as usual, $[r(\lambda)]_+$ denotes the polynomial part of the rational function $r(\lambda)$, while $[r(\lambda)]_- = r(\lambda) - [r(\lambda)]_+$. Then all nontrivial Poisson brackets $\{\cdot,\cdot\}_n^\varphi$ are rank $2n$ on a dense subset of $\mathbb{R}^{2n}$ (see [V]).

An arbitrary polynomial $F(\lambda,\mu)$ leads to a natural set of $n$ polynomials on $\mathbb{R}^{2n}$ which have the remarkable property to Poisson commute for all Poisson structures $\{\cdot,\cdot\}_n^\varphi$. These functions are functionally independent (except in the special case $F$ does not depend on $w$), hence they define an integrable system on $\mathbb{R}^{2n}$ for any structure $\{\cdot,\cdot\}_n^\varphi$. Indeed there exists a natural map $\hat{H}_{F,n}; : (\mathbb{R}^2)^n \backslash \Delta \mapsto \mathbb{R}^n$, which assigns to a $n$-tuple $((x_1,p_1),\dots,(x_n,p_n))$ the unique polynomial in $\mathbb{R}[\lambda]$ of degree less than $n$, which takes the value $F(x_i,p_i)$ for $\lambda = x_i$. Since $\hat{H}_{F,n}$ is invariant under the action of $S_d$, $\hat{H}_{F,n} = H_{F,n} \circ \mathcal{S}$, where

$$H_{F,n}(u(\lambda), v(\lambda)) = F(\lambda, v(\lambda)) \bmod u(\lambda) = H_1\lambda^{n-1} + \cdots + H_n.$$

Then $H_1,\dots,H_n$ form a set of $n$ functionally independent polynomials on $\mathbb{R}^{2n}$ which are in involution on the Poisson manifold $(\mathbb{R}^{2n}, \{\cdot,\cdot\}_n^\varphi)$ (see [V]).

If the algebraic $\Gamma \subset \mathbb{C}^2$ defined by $F(\lambda,\mu) = 0$ is smooth, then the fiber

$$\mathcal{A}_{F,n} = \left\{(u(\lambda, v(\lambda)) \in \mathbb{R}^{2n} \,|\, \Big[\frac{F(\lambda, v(\lambda))}{u(\lambda)}\Big]_- = 0\right\} \subset \mathbb{R}^{2n}$$

us also smooth (see [V]). Moreover, its complexification $\mathcal{A}_{F,n}^{\mathbb{C}}$ is an affine part of the $n$-fold symmetric product $\Gamma^{(n)} = \Gamma^n/S_n$. If $n \leq g$ then the algebraic integrable system is deficient.

**Example. Hyperelliptically separable systems** Suppose $\varphi \equiv 1$ and $\Gamma$ is hyperelliptic of genus $g$, i.e. $F(\lambda,\mu) = \mu^2 + \prod_{i=0}^{2g}(\lambda - c_i)$ and let $n \leq g$. Then the complex invariant manifold $\mathcal{A}_{F,n}^{\mathbb{C}}$ is biholomorphic to a smooth affine part of an $n$–dimensional stratum of $W_n \subset \mathrm{Jac}(\Gamma)$. An a.c.i. system is then obtained in the special case $n = g$, since $W_g = \mathrm{Jac}(\Gamma)$, while for $n < g$ the system is algebraically integrable with deficiency (also called hyperelliptically separable by Abenda and Fedorov [AF1]). Notice that in the special case $n = g - 1$ $W_{g-1} \equiv \Theta$, the theta divisor of $\mathrm{Jac}(\Gamma)$.

**3.3 Superintegrability, or Degenerate integrability, and duality.** We underscore the intent of this section by citing Krichever [Kr2]: "It is worth [understanding] if there is [a] general Hamiltonian-type setting, in which these characteristic features of [a symplectic manifold, carrying a family of skew-symmetric 2-forms, each of which is degenerate, while the family is non-degenerate] provide the basis for something that might be the notion of *super-integrable* systems." Notably, Krichever's approach (joint with D.H. Phong), through a universal two-form on a space of meromorphic matrices, encompasses the Hitchin system.

Superintegrability was first considered by Nekhoroshev [Nek]: these are integrable systems in which the number of frequencies of conditionally periodic motions is less than the number $n$ of degrees of freedom. A superintegrable system [Nek] with Hamiltonian $H(p,x)$, $p=(p_1,\dots,p_n), x=(x_1,\dots,x_n)$ is said to be superintegrable if it admits $n+k$, $1\le k\le n-1$ integrals of motion, $n$ of which are pairwise in involution with all of the first integrals. The case $k=n-1$ is also known as maximally superintegrable or degenerate integrable system.

An extensive literature exists on the classification of (classical and quantum) superintegrable systems mainly in the cases where $n=2,3$. In particular, superintegrable systems with a complete set of commuting quadratic integrals of motion are multiseparable, that is the corresponding Hamilton-Jacobi equation allows for separation of variables in more than one system of (orthogonal) coordinates. The classical superintegrable systems with an arbitrary number of degrees of freedom are the harmonic oscillator, the Kepler problem, the Calogero system in a harmonic well and the rational Calogero–Moser system as well as the quantum and spin-generalizations of the latter system.

Reshetikhin in [R] finds a duality between two super-integrable systems. He gives an interpretation of the duality of spin Calogero-Moser and spin Ruijsenaars which is even more promising in view of the Hitchin system, since it is reminiscent of the $n\leftrightarrow r$ duality in Moser's "rank $r$-perturbations", found in [AHH]. It would be striking, in our opinion, if there was a relationship.

Reshetikhin describes the degenerate integrability of rational spin Ruijsenaars by two projections:

$$S\subset T^*G//Ad_G\overset{\tilde\psi}{\to}\tilde\psi(S)\subset(T^*G,p)//Ad_G\overset{\tilde\pi}{\to}G//Ad_G,$$

while the degenerate integrability of the spin Calogero-Moser system is given by:

$$S\subset T^*G//Ad_G\overset{\psi}{\to}\psi(S)\subset(\mathfrak{g}^*\times\mathfrak{g}^*)//Ad^*_G\overset{\pi}{\to}\mathfrak{g}^*//Ad^*_G\simeq\mathfrak{h}^*/W.$$

He shows that the projections $\psi$ and $\psi^*$ are dual in the sense that their fibres meet exactly at one point, thus angle variables for the rational spin Ruijsenaars system are action variables for the corresponding spin Calogero-Moser system and vice versa.

In Moser's system, $r$ is the rank of the perturbation (following Hitchin's work [H], E. Markman defined a Hitchin system with poles, and interpreted the Moser systems as such, with the rank of the bundle, over $\mathbb{P}^1$ in this case, being $r$), and $n$ related to the genus of the spectral curve; the curve in fact can be associated to the determinant of either matrix ($r\times r$ or $n\times n$), as in the following diagram:

$$\begin{array}{ccccccc}
\widetilde{gl(n)}^- & \overset{J^Y_n}{\longleftarrow} & & M & & \overset{J^A_r}{\longrightarrow} & \widetilde{gl(r)}^- \\
 & \overset{J^Y_{n,0}}{\nwarrow} & \swarrow & & \searrow & \overset{J^A_{r,0}}{\nearrow} & \\
\downarrow & M/GL(r)_Y & & & & M/GL(n)_A & \downarrow \\
 & & \searrow & & \swarrow & & \\
\widetilde{gl(n)}_-/GL(n)_A & \longleftarrow & & M/(GL(r)_Y\times GL(n)_A) & & \longrightarrow & \widetilde{gl(r)}^-/GL(r)_Y
\end{array}$$

where we refer to [AHH] for much of the notation, but specifically:
$J^A_r(F,G)=-G^T(A-\lambda)^{-1}F$, $J^Y_n(F,G)=F(Y-z)^{-1}G^T$;
$GL(n)\colon M\to M$ by $g\colon(F,G)\mapsto(gF,(g^T)^{-1}G)$,
$GL(r)\colon M\to M$ by $g\colon(F,G)\to(Fg^{-1},Gg^T)$; $GL(n)_A$ and $GL(r)_Y$ denote the stabilizers under conjugation of $A\in gl(n)$ and $Y\in gl(r)$, resp., and the meaning of the diagram is that we restrict when necessary to

open dense submanifolds (to have, for example, $Gl(n)_A, G(r)_Y$ act freely). The relevant function rings on $M$ which give hamiltonians are:

$$\mathcal{F}^Y = \{\psi \in \mathcal{C}^\infty(M) | \psi(F,G) = \phi(Y + G^T(A-\lambda)^{-1}F), \phi \in I(\widetilde{gl(r)}^*)\}$$
$$\mathcal{F}^A = \{\psi \in \mathcal{C}^\infty(M) | \psi(F,G) = \phi(A + F(Y-z)^{-1}G), \phi \in I(\widetilde{gl(n)}^*)\}$$

and the final statement on the AKS flows (which can be viewed by reduction as flows on the symplectic leaves occurring at the bottom of the diagram, where the arrows are 1:1 Poisson maps) is the following:

**Theorem**. *The two rings $\mathcal{F}^Y$ and $\mathcal{F}^A$ are equal, their elements Poisson commute and their Hamiltonian flows preserve the spectrum of $N(\lambda) = Y + G^T(A-\lambda)^{-1}F$ and $M(z) = A + F(Y-z)^{-1}G$. If $\psi \in \mathcal{F}^Y = \mathcal{F}^A$ is of the form $\psi(F,G) = \phi_1(N) = \phi_2(M)$, the integral curves of the corresponding hamiltonian flow satisfy*

$$\frac{dN}{dt} = [(d\phi_1)_+, N], \quad \frac{dM}{dt} = [(d\phi_2)_+, M].$$

Note that the equality of the rings follows from the usual Weinstein-Aronszajn identity:

$$\det(A-\lambda)\det(Y + G^T(A-\lambda)^{-1}F - z) = \det(Y-z)\det(A + F(Y-z)^{-1}G^T - \lambda),$$

since generators of $\mathcal{F}^Y(\mathcal{F}^A)$ will appear as coefficients in the expansion of the left (right)-hand side in $\lambda$ and $z$; the formula also shows that the two spectral curves are birational.

Is there a dual for the Kepler system, which is known to be superintegrable? More generally, does duality in the sense of §2.1 preserve superintegrability? Is the elliptic spin Calogero-Moser system superintegrable?

## §4. Proposed links

**4.1 Algebro-geometric Poisson structures.** Hyperelliptic separability is a manifestation of the following construction by Veselov and Novikov [VN]. They prescribe a variety $M$ whose points are algebraic curves $\Gamma$ of genus less than or equal to $g$ (e.g. hyperelliptic curves) and for fixed $n$ consider the natural fibering where the fiber other $\Gamma \in M$ is $\Gamma^{(n)}$ (the $n$-th symmetric product).

Points on the fiber bundle $\mathcal{F}$ have the form

$$\left\{ \left(\Gamma, (\lambda_1, \mu_1), \ldots, (\lambda_n, \mu_n)\right), \ (\lambda_i, \mu_i) \in \Gamma, i = 1, \ldots, n, \ \Gamma \in M \right\}.$$

Following [VN], the associated analytic Poisson brackets are determined by i)-iii) below:

i) a subsheaf of rings $A$ in the sheaf of germs of meromorphic functions on $M$, depending only on a point of the base $\Gamma \in M$. Subrings of the form $A_U$ for open domains $U$ play the role of the annihilator of the Poisson bracket which is actually concentrated on subvarieties $\mathcal{F}_A \subset \mathcal{F}$, where $f = \text{const.}$ for all $f \in A$, $\mathcal{F}_A \mapsto M_A \subset M$.

ii) There is a given meromorphic differential 1-form $\Phi(\Gamma)$ depending on $\Gamma$ (or its covering) as a parameter. In local coordinates:

$$\Phi(\Gamma) = \Phi(\Gamma, \lambda) d\lambda,$$

and it is required that the derivatives of $\Phi(\Gamma)$ along all directions of the base tangent to the manifolds $M_A$ be globally defined meromorphic differential one-forms on $\Gamma$.

iii) Allow for the case in which the form $\Phi$ is meromorphic on the covering $\tilde{\Gamma}$ with Abelian monodromy group $\tilde{\Gamma} \mapsto \Gamma$, where the image $\pi_1(\tilde{\Gamma} \mapsto \pi_1(\Gamma) \mapsto H_1(\Gamma, \mathbf{Z})$ is generated by a collection of cycles $a_1, \ldots, a_n$ with pairwise zero index $a_i \cdot a_j = 0$.

**Definition** [VN]: if the closed 2-form

$$\Omega_\Phi = \sum_{i=1}^{n} d\Phi(\Gamma, \lambda_i) \wedge d_{\lambda_i},$$

is non-degenerate in a Zariski-open region of the manifold $\mathcal{F}_A$ where the pair $(A, \Phi)$ satisfies i),ii) and iii) above, then an analytic Poisson bracket with annihilator $A$ is given on an open region of $\mathcal{F}$.

A necessary condition is that the dimension of $\mathcal{F}_A$ is $2n$, and the dimension of $M_A$ is $n$.

By definition, the Poisson bracket is defined by the conditions

$$\{\lambda_i, \lambda_j\} = 0 = \{\Phi(\lambda_i), \Phi(\lambda_j)\}, \quad i, j = 1, \ldots, n,$$

$$\{\Phi(\lambda_i), \lambda_j\} = \delta_{ij}, \quad i, j = 1, \ldots, n,$$

$$\{f, \lambda_j\} = 0 = \{f, \Phi(\lambda_j)\}, \quad j = 1, \ldots, n, f \in A.$$

Clearly, any two function $g, h$ depending only on a point of the base $\Gamma \in M$ are in involution:

$$\{g(\Gamma), h(\Gamma)\} = 0.$$

The interesting case treated in [VN] is when the pair $(A, \Phi)$ is such that all derivatives of $\Phi$ along tangential directions $\tau_1, \ldots, \tau_n$ to the manifold $M_A$ at a general point $\Gamma \in M_A$ form a collection of meromorphic one forms $\nabla_{\tau_i}\Phi$ on $\Gamma$ such that the following holds:

a) the forms $\nabla_{\tau_i}\Phi$ on $\Gamma$ may be represented as

$$\nabla_{\tau_i}\Phi = \omega_i + \tilde{\omega}_i + \sum_{j=1}^{l} \hat{\omega}_{i,j}, \quad i = 1, \ldots, n,$$

where $\omega_i$ are holomorphic on $\Gamma$, $\tilde{\omega}_i$ are meromorphic with zero residues at all poles and $\hat{\omega}_{i,j}$ have a pair of simple poles at $P_j, Q_j$ whose residues differ only in sign;

b) if $n \geq g$ it is required that the forms $\omega_i$ generate the one dimensional cohomology group $H^{1,0}(\Gamma)$ and $\omega_i = 0$, $i > g$.

In the important case $n = g$, the Abel map $\Gamma^{(g)} \mapsto \mathrm{Jac}(\Gamma)$ linearizes the dynamics of all Hamiltonians $H(\Gamma)$ for the Poisson bracket defined by pairs $(A, \Phi)$ possessing the above properties a), b) if and only if $\nabla_{\tau_i}\Phi$, $i = 1, \ldots, g$, form a basis of holomorphic one–forms of the Riemann surface $\Gamma$. Clearly the integrable system is a.c.i. The case of hyperelliptically separable systems corresponds to the case in which $\Gamma$ are hyperelliptic of genus $g$, $n \leq g$.

Note also Veselov's further investigation of non-linear time transformations [V], our next topic.

There are two links between ACI and deficient ACI: 1. In the early examples of Abenda and Fedorov, the integral manifold being an $n$-dimensional stratum of a Jacobian, the deficient systems can be viewed as Dirac reductions of an ACI [AF2]; 2. More recently, Abenda and Grava [AG] apply the theory of reciprocal transformations to link the modulated KdV equations to the modulated Camassa-Holm equations via the average of the reciprocal transformation first introduced by Fuchssteiner [Fu] to link the first negative KdV flow to the Camassa-Holm equation. These transformations, once we restrict to the stationary equation and to the related finite dimensional integrable systems, give rise to nonlinear time transformations which change the Poisson structures while preserving integrability.

**4.2 Nonlinear time transformations.** Let $H$ be the Hamiltonian of the (deficient or completely) integrable system and let the Hamilton equations

$$\frac{d\lambda_i}{dt_0} = \{H, \lambda_i\}, \quad \frac{d\mu_i}{dt_0} = \{H, \mu_i\}, \quad i = 1, \ldots, n.$$

A nonlinear time transformation $dt_0 = \varphi dt_1$ preserves integrability while changing the Poisson structure to $\{\cdot,\cdot\}_1 = \varphi\{\cdot,\cdot\}_0$. In many examples, deficient integrable systems are mapped to a.c.i. systems after such transformation (typically $\phi$ is a symmetric function in the $\lambda_j$s. The typical situation is modelled by the geodesic flow, that is a real completely integrable Hamiltonian system whose complexified invariant manifolds are codimension 1 subvarieties of *generalized* Jacobi varieties (see the example above). After a nonlinear time transformation $dt = \lambda_1 \cdots \lambda_n ds$, the re-parameterized geodesic flow is linearized on on the Jacobi variety $\mathrm{Jac}(\Gamma_c)$.

Of special interest here are the cases in which the finite dimensional integrable system is associated to the stationary solutions to integrable PDEs. In that case the nonlinear time transformation has its counterpart in a reciprocal transformation at the PDEs level. A reciprocal transformation is a closed form which changes the independent variables of the equations and maps conservation laws into conservations laws, but it does not preserve the Poisson structures. Assuming that the evolution equation $u_t = Q(u, u_x, u_{xx}, \ldots)$ admits two conservation laws

$$B(u)_t = A(u)_x, \quad N(u)_t = M(u)_x$$

$(B(u)M(u) - A(u)N(u) \neq 0)$, then we can perform a change of the independent variables by the relations

$$d\hat{x} = B(u)dx + A(u)dt, \qquad d\hat{t} = N(u)dx + M(u)dt.$$

Then the reciprocal equation

$$u_{\hat{t}} = \hat{Q}(u, u_{\hat{x}}, u_{\hat{x}\hat{x}}, \ldots)$$

possesses a bihamiltonian structure and is therefore integrable if and only if the original equation is.

**A case study.** Here we show how the Camassa–Holm equation may be obtained from the first negative Korteweg de Vries flow through a reciprocal transformation (see [Fu]) and the relation between the traveling wave solutions to both systems. In [AG] it is shown that the modulation equations of the Camassa Holm periodic solutions are transformed to the modulation equations of the first negative KdV flow by the averaged reciprocal transformation.

In the following, to distinguish between CH and KdV, we use $(x, t)$ for Camassa-Holm variables and $(y, \tau_-)$ for the KdV variables. The change of dependent variable $\rho^2 = m + \nu$ transforms the Camassa–Holm equation

$$m_t = -2mu_x - um_x - 2\nu u_x, \qquad m = u - u_{xx}. \tag{4.1}$$

into the associated Camassa-Holm equation

$$\rho_t = -\Big(u\rho\Big)_x, \qquad \rho^2 = u - \mathrm{u}_{xx} + \nu,$$

which, via the reciprocal transformation introduced by Fuchssteiner [Fu],

$$dy = \rho dx - u\rho dt, \qquad d\tau_- = dt, \tag{4.2}$$

is finally transformed into

$$u = \rho^2 - \nu - \rho_{y\tau_-} + \frac{\rho_{\tau_-}\rho_y}{\rho}, \qquad \left(\frac{1}{\rho}\right)_{\tau_-} = 2\rho\rho_y - \left(\rho\Big(\log\rho\Big)_{y\tau_-}\right)_y.$$

The above equation is equivalent to the first negative flow of the KdV hierarchy

$$\left(\partial_y^2 + 2U + U_y\partial_y^{-1}\right) U_{\tau_-} = 0, \tag{4.3}$$

under the condition $U_{\tau_-} = -2\rho_y$. Finally, we observe that $\displaystyle\int \rho(x,t)dx$ is a Casimir of the second Hamiltonian operator of the Camassa–Holm equation, $P_2 = m\partial_x + \partial_x m + 2\nu\partial_x$. The Camassa–Holm and the Korteweg-de Vries negative flow equations share many properties. Both of them admit a Lax pair representation, are formally integrable through the inverse scattering method and are elements of Hamiltonian integrable hierarchies. A major difference between the two integrable hierarchies is the absence of a $\tau$–structure in the Camassa–Holm case. As a consequence, the (complex) analytic structure of solutions to (4.1) is more complicated than that of solutions to (4.3).

Let us look for a traveling wave solution to (4.1) of the form

$$u(x,t) = (c - 2\nu) - 2\eta(\zeta), \qquad \zeta = \frac{kx - \omega t + \phi_0}{\epsilon},$$

where $k$ is the wave number, $\omega$ the frequency and $\phi_0$ is a phase to be determined from the initial conditions. Then, the implicit solution is

$$k \int_{\eta_0}^{\eta} \frac{(\xi + \nu) d\xi}{\sqrt{(\xi + \nu) \prod_{i=1}^{3} (\xi - u_i)}} = \zeta, \qquad u_1 + u_2 + u_3 = c - 2\nu. \tag{4.4}$$

Let $-\nu < u_1 < u_2 < u_3$, so that $\eta(\zeta)$ is real periodic in the interval $[u_1, u_2]$. Since $\dfrac{d\zeta}{d\eta}$ has constant sign for $\eta \in [u_1, u_2]$, by standard argument, the (real) inverse function $\eta(\zeta)$ exists and is monotone in $\zeta \in [0, Z]$, where $Z$ is the half period of the travelling wave solution. However, the differential in (4.4) maybe associated to the elliptic curve $\mathcal{E}: \{w^2 = (\xi + \nu) \prod_{i=1}^{3} (\xi - u_i)\}$ and has simple poles at $\infty_{\pm}$. This implies that $\eta(\zeta)$ is *not* meromorphic in $\zeta$.

Nonlinear time transformations play a role also in superintegrability. Below we limit ourseleves to the important example of the Kepler problem and we review some well known facts. We plan to consider the question of discussing the role of such transformations from the algebraic geometric point of view and in its full generality in a subsequent publication.

**The Kepler problem and nonlinear time transformations.** The Kepler problem of a body moving under the influence of the central force field with potential $V(r) = -1/r$ is arguably the best known examples of superintegrable system. In $n$ dimensions the Hamiltonian takes the form

$$H(y,x) = \frac{1}{2}|y|^2 - r^{-1}, \quad |y|^2 = \sum_{i=1}^{n} y_i^2, \;\; r = |x| = (\sum_{i=1}^{n} x_i^2)^{1/2}.$$

Under the nonlinear time rescaling $ds = r^{-1} dt$ (Levi–Civita transformation, see [L]), on the energy surface $H_0 = -1/2$, the rescaled vector field is associated to $K_0 = \dfrac{r}{2}(|y|^2 + 1) = 1$. The energy surface is non compact and does not contain the collision states. Moser's regularization (see [Mo]) carries the geodesic flow on the unit tangent bundle of the pointed $n$–sphere onto the flow in the $n$–dimensional Kepler problem on surface of fixed negative energy. Let $q = (q_0, \dots, q_n)$, $p = (p_0, \dots, p_n)$. Then on

$$T^+ S^n = \{(q,p) \in \mathbb{R}^{2(n+1)}, ||q||^2 = \sum_{i=0}^{n} q_i^2 = 1, \; < p, q >= \sum_{i=0}^{n} q_i p_i = 0, \; p \neq 0\},$$

the Moser diffoemorphism $\mu \; : T^+ S^n \to (\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}^n$ is defined by the formulae

$$x_0 = (||p|| + p_0) q_i - q_0 p_i, \quad y_i = (||p|| + p_0)^{-1} p_i, \quad i = 1, \dots, n$$

and has the property $K_0 \circ \mu = ||p||$. In Moser's regularization, the flow on a surface of fixed negative energy of the Kepler problem is replaced by the geodesic flow on $TS^n$ (see [Mo]) and all functions $\Gamma^{\mu\nu} = q_\mu p_\nu - p_\mu q_\nu$ are first integrals. In particular the pull-back of the $n$-dimensional vector $R = (R_1, \dots, R_n)$, with $R_i = q_0 p_i - p_0 q_i$ takes the form $R_i = \frac{1}{2}(|y|^2 - 1)x_i - y_i \sum_j x_j y_j$, $i =, \dots, n$ and commutes with $K_0$, so that the vector $R + (H_0 + \frac{1}{2})x$ is the generalized Runge-Lenz vector, which gives the $n$ additional integrals of motion besides the $\frac{1}{2}n(n-1)$ components of the angular momentum for the Kepler problem.

**4.3 The Stäckel transform.** A natural questions is then under which conditions, nonlinear time rescalings preserve superintegrability. In [BKM], the authors introduce the "Stäckel transform" to map an integrable system to an integrable system when the Hamiltonian is Stäckel separable and takes the form

$$H(p,q) = \sum_{i,j=1}^{n} g_{ij}(q) p_i p_j + V(q),$$

with $(p,q) \in \mathbb{R}^{2n}$. Then, in the orthogonal separation coordinates they make a change of the independent variable (time) using the potential (which is a Stäckel multiplier by construction) and the transformed system is automatically integrable. In the special case $n=3$ and

$$g_{ij}(q) = \frac{\delta_{ij}}{\lambda(q)}, \quad i,j = 1,\ldots,3,$$

in [KKM] it is proven that $H(p,q)$ is superintegrable with nondegenerate potential on a three dimensional conformally flat space if and only if it is Stäckel equivalent to a superintegrable system on either 3 dimensional flat space or the three dimensional sphere.

The Stäckel transform considered in [BKM] corresponds to the case in which the family of curves is hyperelliptic and the nonlinear time transformation between the Stäckel equivalent systems is a (combination of) elementary symmetric function in the lambda coordinates $\lambda_j$.

The effect of the transformation is to modify the Poisson structure. Consequently, the Stäckel transformation does not preserve algebraic complete integrability in the sense of Adler-van Moerbeke. Vice versa, not every hyperelliptically separable system may be obtained as Stäckel transform from an algebraically integrable system. In particular, the geodesic flow on the ellipsoid (in natural parametrization) is the Stäckel transform of the Neumann system on the sphere.

Next, we show that the Dirac formalism for constrained Hamiltonian systems and reciprocal transformations are useful tools for the algebraic-geometric characterization of deficient and degenerate algebraically integrable systems

**4.4 Dirac constraints** Let $\mathcal{U}$ be a subvariety in the moduli space of curves of genus $g$ and consider an a.c.i. system whose phase space $\mathcal{M}$ can be regarded as a fiber bundle $\mathcal{M} \to \mathcal{U}$ with the base $\mathcal{U}$ parameterizing the corresponding curves and the fibers being (generalized) Jacobians of the curves. Let $\Omega = \sum_{i=1}^{n} d\lambda_i \wedge d\mu_i$ be the canonical symplectic structure, where $n$ is either $g$ or $g+1$. Then on each invariant manifold the pairs of conjugated variables satisfy algebraic relations

$$F(\lambda_i, \mu_i; c) = 0, \quad i = 1,\ldots,n,$$

defining a family of algebraic curves $\Gamma_c$ of genus $g$. $c = (c_1,\ldots,c_n)$ are the independent involutive first integrals of the system and form a basis of coordinates on the base $\mathcal{U}$. Solving $F(\lambda,\mu;c) = 0$ in terms of $\mu$, we obtain the generating function

$$G(\lambda, c) = \sum_i \int_{\lambda_0}^{\lambda_i} \mu(\lambda, c_1, \ldots, c_n)\, d\lambda$$

of another canonical transformation $(\lambda,\mu) \to (c,\phi)$. Since, following [VN], we require that $\dfrac{\partial \mu(\lambda,c)}{\partial c_i}$, $i = 1,\ldots,g$ form a basis of holomorphic differentials on $\Gamma_c$, the canonical transformation is described explicitly by the Abel–Jacobi mapping $\Gamma_c^{(g)} \to \mathrm{Jac}(\Gamma_c)$, if $n = g$ (or by the generalized Abel–Jacobi mapping $\Gamma_c^{(g+1)} \to \mathrm{Jac}^*(\Gamma_c)$, if $n = g+1$) which takes the explicit form

$$\frac{\partial G}{\partial c_i} \equiv \sum_i \int_{\lambda_0}^{\lambda_i} \frac{\partial \mu(\lambda,c)}{\partial c_i}\, d\lambda = \phi_i, \quad i = 1,\ldots,n.$$

$\phi_1,\ldots,\phi_n$ are coordinates on the universal covering of $\mathrm{Jac}(\Gamma_c)$ (resp. of $\mathrm{Jac}^*(\Gamma_c)$) and are also the *complex angle variables*.

Our main observation here (see also [AF2]) is that imposing suitable Dirac constraints to the above a.c.i. systems, one can obtain deficient algebraically integrable systems, whose generic invariant manifolds are strata of (generalized) Jacobi varieties.

Let us constrain the a.c.i. system onto the symplectic subvariety $\mathcal{N} \subset \mathcal{M}$ defined by the $2d$ constraints

$$\lambda_{n-d+1} = C_{n-d+1}, \ \ldots, \ \lambda_n = C_d, \quad C_{n-d+1}, \ldots, C_d = \text{const},$$

$$\mu_{n-d+1} = E_{n-d+1}, \dots, \mu_n = E_n, \quad E_i = \text{const}.$$

Then the following theorem holds

**Theorem** [AF2] *The constraint variety $\mathcal{N}$ intersects the family of (generalized) Jacobians along $(n-d)$-dimensional nonlinear subvarieties (strata), the images of the mapping $\Gamma^{(n-d)} \to Jac(\Gamma_c)$ (respectively $Jac^*(\Gamma_c)$. The latter are complexified invariant manifolds of the system restricted on $\mathcal{N}$.* □

Now the angle variables $\phi_1, \dots, \phi_n$ play the role of redundant coordinates on the strata ( see[AF]).

An interesting question is whether *any* algebraically deficient system can be obtained as a constrained system from an a.c.i. system, even in a generalized sense.

**Example.** The following example shows that even in the case $n = g + 1$ and $d = 1$ the constrained system obtained from an a.c.i. is not a.c.i. in general. The geodesic flow (in its natural parametrization) on an $n$ dimensional ellipsoid may be obtained constraining the free motion in $\mathbb{R}^{n+1}$. Consider a family of confocal quadrics in $\mathbb{R}^{n+1}$ (or in $\mathbb{C}^{n+1}$)

$$Q(s) = \left\{ \frac{X_1^2}{a_1 - s} + \frac{X_2^2}{a_2 - s} + \cdots + \frac{X_{n+1}^2}{a_{n+1} - s} = 1 \right\},$$

$$s \in \mathbb{R}, \quad 0 < a_1 < a_2 < \cdots < a_{n+1}.$$

and the associated ellipsoidal coordinates $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$ such that

$$X_i^2 = \frac{\prod_{j=1}^{n+1}(a_i - \lambda_j)}{\prod_{k \neq i}(a_i - a_k)}, \quad i = 1, \dots, n+1.$$

A free motion of a particle in $\mathbf{R^{n+1}}$ is described in ellipsoidal coordinates by the Hamiltonian

$$H = 2\sum_{i=1}^{n+1} \frac{\prod_{k \neq i}(\lambda_k - \lambda_i)}{\Phi(\lambda_i)}\, \dot{\lambda}_i^2 = \frac{1}{2}\sum_{i=1}^{n+1} \frac{\Phi(\lambda_i)}{\prod_{k \neq i}(\lambda_k - \lambda_i)}\, \mu_i^2,$$

where

$$\Phi(\lambda) = \prod_{i=1}^{n+1}(\lambda - a_i),$$

and $\mu_i$, $i = 1, \dots, n$ are the momenta canonically conjugated to $\lambda_i$. The canonical variables satisfy the algebraic relations

$$\mu_i^2 = \frac{c_0 \prod_{j=1}^{n}(\lambda_i - c_j)}{\Phi(\lambda_i)}, \qquad i = 1 \dots, n+1,$$

defining the genus $n$ hyperelliptic curve $\Gamma_c = \{w^2 = \Phi(\lambda)\prod_{j=1}^{n}(\lambda - c_j) \equiv R(\lambda)\}$. The constants of motion $c_1, \dots, c_n$ have a transparent geometric interpretation: in the configuration space $\mathbb{R}^{n+1}$ the straight line trajectory is tangent to the quadrics $Q(c_1), \dots, Q(c_n)$ of the above confocal family.

The generating function

$$G(\lambda, c) = \sum_{i=1}^{n+1} \int_{\lambda_0}^{\lambda_i} \frac{\sqrt{c_0(\lambda - c_1)\cdots(\lambda - c_n)}}{\sqrt{\Phi(\lambda)}}\, d\lambda$$

produces the following canonical transformation $(\lambda, \mu) \to (\phi, c)$ written in a differential form

$$\sum_{i=1}^{n+1} \frac{d\lambda_i}{2\sqrt{R(\lambda_i)}} = d\phi_1 \quad ,\dots, \quad \sum_{i=1}^{n+1} \frac{\lambda_i^n\, d\lambda_i}{2\sqrt{R(\lambda_i)}} = d\phi_{n+1},$$

where $R(\lambda)$ has been defined above, and the dynamics is linearized on $\text{Jac}^*(\Gamma_c)$ since

$$d\phi_1 = \cdots = d\phi_n = 0, \quad d\phi_{n+1} = dt,$$

(the differentials $d\lambda/w, \dots, \lambda^{n-1}d\lambda/w$ form a basis holomorphic differentials while $\lambda^n d\lambda/w$ is meromorphic of the second kind on the genus $n$ curve $\Gamma_c$). As expected, the free motion of a particle in $\mathbb{R}^{n+1}$ is an a.c.i. system.

Now let us restrict the system onto the symplectic subvariety $\mathcal{N} = \{\lambda_{n+1} = 0,\ \mu_{n+1} = 0\} \subset T\mathbb{R}^{n+1}$. $\mathcal{N}$ coincides with the cotangent bundle of the $n$–dimensional ellipsoid $Q(0)$ on which one of the constants of motion is zero, $c_n = 0$.

The generating function restricted to the geodesic flow

$$G(\lambda, c) = \sum_{i=1}^{n} \int_{\lambda_0}^{\lambda_i} \frac{\sqrt{c_0\lambda(\lambda - c_1)\cdots(\lambda - c_{n-1})}}{\sqrt{\Phi(\lambda)}}\, d\lambda$$

produces the following transformation $(\lambda, \mu) \to (\phi, c)$ written in differential form

$$\sum_{i=1}^{n} \frac{\lambda d\lambda_i}{2\sqrt{R(\lambda_i)}} = d\phi_2 \quad , \dots, \quad \sum_{i=1}^{n+1} \frac{\lambda_i^n\, d\lambda_i}{2\sqrt{R(\lambda_i)}} = d\phi_{n+1}. \tag{4.5}$$

The subvariety $\mathcal{N}$ intersects the family of generalized Jacobians along an $n$-dimensional nonlinear stratum $W_n^*$, which is explicitly described in the angle coordinates $\phi_j$ by

$$W_n^* = \{Z + \partial_{\phi_n} \log \theta(z) = 0\},$$

where $\theta(z)$ is the Jacobi theta function on $\mathrm{Jac}(\Gamma_c)$, $z = (z_1, \dots, z_n)$, $z_j$ are linear combinations of $\phi_1, \dots, \phi_n$ and $Z$ is a linear combination of $\phi_1, \dots, \phi_{n+1}$.

From the mechanical point of view, we have restricted the free motion in the space $\mathbb{R}^{n+1}$ to the geodesic flow on $Q(0) \subset \mathbb{R}^{n+1}$; from the algebraic geometrical point of view, by imposing the constraint, we force the linear motion on the generalized Jacobian to take place on the nonlinear stratum $W_n^*$, where the angle variables $\phi_1, \dots, \phi_{n+1}$ play the role of redundant coordinates. According to the definition of $W_n^*$, $\phi_1$ becomes a transcendental function of $\phi_2, \dots, \phi_{n+1} = t$, which is of course a sign that the system cannot be a.c.i. in the natural length coordinates.

To revisit the geodesic-Neumann correspondence, we observe that the complete Stäckel transformation between the two in separation variables is the composition of two transformations: time-parameter rescaling (*not* symplectic), and a birational transformation.

The geodesic flow on a quadric, after separation of variables (in ellipsoidal coordinates), admits the following quadratures (4.5) with insertion of the time-dependence in $t$:

$$\sum_{i=1}^{n} \frac{\lambda d\lambda_i}{2\sqrt{R(\lambda_i)}} = 0 \quad , \dots, \quad \sum_{i=1}^{n} \frac{\lambda_i^{n-1}\, d\lambda_i}{2\sqrt{R(\lambda_i)}} = 0, \sum_{i=1}^{n} \frac{\lambda_i^n\, d\lambda_i}{2\sqrt{R(\lambda_i)}} = dt. \tag{4.6}$$

which consists of $n-1$ holomorphic differentials and a meromorphic differential on a genus $n$ hyperelliptic curve.

Now let us perform the following nonlinear time rescaling (which is not symplectic):

$$ds = \lambda_1 \dots \lambda_n dt. \tag{4.6}$$

The rescaled quadratures now become:

$$\sum_{i=1}^{n} \frac{d\lambda_i}{2\sqrt{R(\lambda_i)}} = ds, \sum_{i=1}^{n} \frac{\lambda d\lambda_i}{2\sqrt{R(\lambda_i)}} = 0 \quad , \dots, \quad \sum_{i=1}^{n} \frac{\lambda_i^{n-1}}{2\sqrt{R(\lambda_i)}} = 0. \tag{4.7}$$

That is, to give the solution $x_i(t), p_i(t)$ (coordinates and momenta) of the geodesic flow on the quadric in the natural parameter $t$ is equivalent to giving $x_i(s), y_i(s)$ using Jacobi inversion theorem in (4.7) and inverting $s = s(t)$ in (4.5).

Now as Knörrer observed (see $2.3), (4.6) (rescaled geodesic flow) is equivalent to Neumann system in separation variables (up to a birational transformation, known as Moser Trubowitz isomorphism).

Finally, respectively in real $t$ or $s$, the solutions to geodesic flow and Neumann system are both real analytic in their natural parameters, since both transformations are real analytic. However, if we complexify the two systems (in their original parameters) there is a major difference: the Neumann system is a.c.i., while the geodesic flow in its natural parameter is deficient, as observed above.

Of course we conclude with more questions and fewer answers: under which conditions does a nonlinear time transformation preserve (super)integrability? Duality?

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