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Mild Solutions for a Class of Fractional SPDEs and Their Sample Paths

by

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Abstract. In this article we introduce and analyze a notion of mild solution for a class of non-autonomous parabolic stochastic partial differential equations defined on a bounded open subset $D \subset \mathbb{R}^d$ and driven by an infinite-dimensional fractional noise. The noise is derived from an $L^2(D)$-valued fractional Wiener process W^H whose covariance operator satisfies appropriate restrictions; moreover, the Hurst parameter H is subjected to constraints formulated in terms of d and the Hölder exponent of the derivative h' of the noise nonlinearity in the equations. We prove the existence of such solution, establish its relation with the variational solution introduced in [42] and also prove the Hölder continuity of its sample paths when we consider it as an $L^2(D)$–valued stochastic processes. When h is an affine function, we also prove uniqueness. The proofs are based on a relation between the notions of mild and variational solution established in [48] and adapted to our problem, and on a fine analysis of the singularities of Green's function associated with the class of parabolic problems we investigate. An immediate consequence of our results is the indistinguishability of mild and variational solutions in the case of uniqueness.

Keywords: Fractional Brownian motion, stochastic partial differential equation, Green's function, sample path regularity.

AMS Subject Classification
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1 Introduction and Outline

In the very last decades, the interest in fractional Brownian motion, firstly introduced in [28] and referred to as *fBm* in the sequel, has increased enormously, as one important ingredient of fractal models in the sciences. The paper [36] has been one of the keystones that has attracted the attention of part of the probabilistic community to this challenging object. Part of the research on *fBm* has significantly influenced the present state of the art of Gaussian processes (see for instance [4], [5], [6], [8], [40], [44], [47], just to mention a few). An important aspect of the study of *fBm* lies in the domain of stochastic analysis. Since this process is neither a semimartingale nor a Markov process, Itô's theory does not apply. For values of the Hurst parameter H greater that $\frac{1}{2}$ -the regular case- integrals of Young's type and fractional calculus techniques have been considered ([51], [52]). However, for H less than $\frac{1}{2}$ this approach fails. The integral representation of *fBm* as a Volterra integral with respect to the standard Brownian motion has been successfully exploited in setting up a stochastic calculus where classical tools of Gaussian processes along with fractional and Malliavin calculus are combined. Pioneering work in this context is [13], then [2], [7], [12] and also [20]. Since then, there have been many contributions to the subject. Let us refer to [39] for enlightening contents and a pretty complete list of references. Rough path analysis (see [34], [35], [9]) provides a new approach somehow related to Young's approach.

The main reason for developing a stochastic calculus based on *fBm* is mathematical modeling. The theory of ordinary and partial differential equations driven by a fractional noise is nowadays a very active field of research. Some of the motivations come from a number of applications in engineering, biophysics and mathematical finance; to refer only to a few, let us mention [15], [29], [46]. There are also purely mathematical motivations. Problems studied so far go from the existence, the uniqueness, the regularity and the long-time behaviour of solutions to large deviations, support theorems and the analysis of the law of the solutions using Malliavin calculus. Without aiming to be exhaustive, let us refer to [3], [17], [19], [22], [23], [24], [26], [27], [30], [34], [37], [38], [41], [42], [45] and [50]) for a reduced sample of published work.

This paper aims to pursue the investigations of [42], where the authors develop an existence and uniqueness theory of variational solutions for a class of non-autonomous semilinear partial differential equations driven by an infinite-dimensional multiplicative fractional noise through the construction and the convergence of a suitable Faedo-Galerkin scheme.

As is the case for deterministic partial differential equations, a recurrent difficulty is the necessity to decide *ab initio* what solution concept is relevant, since there are several *a priori* non-equivalent possibilities to choose from. Thus, while in [42] two notions of *variational solution* that are subsequently proved to be indistinguishable are introduced, the focus in [23] or [37] is rather on the idea of *mild solution*, that is, vaguely put, a solution which can be expressed as a nonlinear integral equation that involves the linear propagator of the theory without any reference to specific classes of test functions. Consequently, this leaves entirely open the question of knowing whether the variational and mild notions are in some sense equivalent, and indeed we are not aware of any connections between them thus far in this context. For equations of the type considered in this article but driven by standard Wiener processes, this issue was addressed in [48]. In [14] a similar question was analyzed for a class of very general SPDEs driven by a finite-dimensional Brownian motion.

In this article we consider the same class of equations as in [42]. We develop an existence and uniqueness theory of mild solutions along with the indistinguishability of these two kinds of solutions and the Hölder continuity of their sample paths.

Before defining the class of problems we shall investigate, let us make some remarks on notation. Here and below all the functional spaces we introduce are real and we use the standard notations for the usual Banach spaces of differentiable functions, of Hölder continuous functions, of Lebesgue integrable functions and for the related scales of Sobolev spaces defined on regions of Euclidean space used for instance in [1]. For $d \in \mathbb{N}$ let $D \subset \mathbb{R}^d$ be an open and bounded set whose boundary ∂D is of class $\mathcal{C}^{2+\beta}$ for some $\beta \in (0,1)$ (see, for instance, [18] and [31] for a definition of this and related concepts). We will denote by $(.,.)_2$ the standard inner product in $L^2(\mathbb{R}^d)$, by $(.,.)_{\mathbb{R}^d}$ the Euclidean inner product in $\mathbb{R}^d$ and by $|.|$ the associated Euclidean norm.

Let $(\lambda_i)_{i\in\mathbb{N}^+} \subset \mathbb{R}^+_*$ be any sequence of positive numbers such that $\sum_{i=1}^{+\infty} \lambda_i < +\infty$. Let $(e_i)_{i\in\mathbb{N}^+}$ be an orthonormal basis of $L^2(D)$ such that $e_i \in L^\infty(D)$ for each i and $\sup_{i\in\mathbb{N}^+} \|e_i\|_\infty < +\infty$ (the existence of such a basis follows from the standard arguments of [43]). We then define the linear, self-adjoint, positive, non-degenerate trace-class operator C in $L^2(D)$ by $Ce_i = \lambda_i e_i$ for each i. In the sequel we write $\left(\left(B_i^H(t)\right)_{t\in\mathbb{R}^+}\right)_{i\in\mathbb{N}^+}$ for a sequence of one-dimensional, independent, identically distributed fractional Brownian motions with Hurst parameter $H \in (0,1)$, defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and starting at the origin; we introduce the

$L^2(D)$-valued fractional Wiener process $\left(W^H(.,t)\right)_{t\in\mathbb{R}^+}$ by setting

$$W^H(.,t) := \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} e_i(.) B_i^H(t), \tag{1.1}$$

where the series converges a.s. in the strong topology of $L^2(D)$ by virtue of the basic properties of the $B_i^H(t)$'s and the fact that C is trace-class. We also have that $\left(W^H(.,t)\right)_{t\in\mathbb{R}^+}$ is a centered Gaussian process whose covariance is given by

$$\mathbb{E}\left((W^H(.,s),v)_2(W^H(.,t),\hat{v})_2\right) = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right)(Cv,\hat{v})_2$$

for all $s,t \in \mathbb{R}^+$ and all $v,\hat{v} \in L^2(D)$. Let $T \in \mathbb{R}_*^+$ and let us consider the class of real, parabolic, initial-boundary value problems formally given by

$$\begin{aligned}
du(x,t) &= (\operatorname{div}(k(x,t)\nabla u(x,t)) + g(u(x,t)))\,dt + h(u(x,t))W^H(x,dt),\\
&\quad (x,t) \in D \times (0,T],\\
u(x,0) &= \varphi(x), \quad x \in \overline{D},\\
\frac{\partial u(x,t)}{\partial n(k)} &= 0, \quad (x,t) \in \partial D \times (0,T],
\end{aligned} \tag{1.2}$$

where the last relation stands for the conormal derivative of u relative to the matrix-valued field k.

In the sequel we write $n(x)$ for the unit outer normal vector at $x \in \partial D$ and introduce the following set of assumptions:

(C) The square root $C^{\frac{1}{2}}$ of the covariance operator is trace-class, that is, we have $\sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} < +\infty$.

($\mathrm{K}_{\beta,\beta'}$) The entries of k satisfy $k_{i,j}(.) = k_{j,i}(.)$ for all $i,j \in \{1,...,d\}$ and there exists a constant $\beta' \in \left(\frac{1}{2},1\right]$ such that $k_{i,j} \in \mathcal{C}^{\beta,\beta'}(\overline{D}\times[0,T])$ for each i,j. In addition, we have $k_{i,j,x_l} := \frac{\partial k_{i,j}}{\partial x_l} \in \mathcal{C}^{\beta,\frac{\beta}{2}}(\overline{D}\times[0,T])$ for each i,j,l and there exists a constant $\underline{k} \in \mathbb{R}_*^+$ such that the inequality $(k(x,t)q,q)_{\mathbb{R}^d} \geq \underline{k}\,|q|^2$ holds for all $q \in \mathbb{R}^d$ and all $(x,t) \in \overline{D}\times[0,T]$. Finally, we have

$$(x,t) \mapsto \sum_{i=1}^{d} k_{i,j}(x,t)n_i(x) \in \mathcal{C}^{1+\beta,\frac{1+\beta}{2}}(\partial D\times[0,T])$$

for each j and the conormal vector-field $(x,t) \mapsto n(k)(x,t) := k(x,t)n(x)$ is outward pointing, nowhere tangent to ∂D for every t.

(L) The functions $g, h : \mathbb{R} \mapsto \mathbb{R}$ are Lipschitz continuous.

(I) The initial condition satisfies $\varphi \in \mathcal{C}^{2+\beta}(\overline{D})$ and the conormal boundary condition relative to k.

Finally, we shall also need the following hypothesis in order to make it possible to relate the theory we develop below to that of variational solutions put forth in [42]:

($\mathrm{H}_{\gamma,d}$) The derivative $h' : \mathbb{R} \mapsto \mathbb{R}$ of h exists, is Hölder continuous with exponent $\gamma \in (0,1]$ and bounded; moreover, the Hurst parameter satisfies $H \in (1 - \mathsf{n}(\gamma, d), 1)$, where $\mathsf{n}(\gamma, d) := \frac{\gamma}{\gamma+1} \wedge \frac{1}{4d+2}$.

This hypothesis leads to a restricted interval of admissible values for H since $\mathsf{n}(\gamma, d) \in \left(0, \frac{1}{2}\right)$. For instance, if the derivative h' is itself Lipschitz continuous it amounts to assuming $H \in \left(\frac{4d+1}{4d+2}, 1\right)$, that is, larger and larger values of H as d increases; this will be important below with regard to our analysis of the regularity properties of the solutions to (1.2).

It is worth pointing out here that Problem (1.2) is identical to the initial-boundary value problem investigated in [42], up to Hypotheses ($\mathrm{K}_{\beta,\beta'}$) and ($\mathrm{H}_{\gamma,d}$) which imply Hypotheses (K) and (H_γ) of that article, respectively. This immediately entails the existence of what is called there a variational solution of type II for (1.2), henceforth simply coined variational solution.

We organize this article in the following way. In Section 2 we first recall the notion of *variational solution* and introduce a notion of *mild solution* for (1.2) by means of a family of evolution operators in $L^2(D)$ generated by the corresponding deterministic Green's function, whose regularity properties we need are a consequence of Hypothesis ($K_{\beta,\beta'}$). We then proceed by stating our main results concerning the existence and the Hölder regularity of the mild solution along with its uniqueness and its indistinguishability from the variational solution when h is an affine function. The section ends with a discussion about the methods of the proofs. We devote Section 3 to the proof of the results stated in Section 2. In particular, with the same strategy as in [48] we prove the existence of a mild solution indirectly by showing that every variational solution is necessarily mild solution, while we obtain the remaining statements from a sharp control of the singularities on the time-diagonal of Green's function by devising some regularization techniques based on heat kernel estimates.

2 Statement and Discussion of the Results

In the remaining part of this article we write $H^1(D\times(0,T))$ for the isotropic Sobolev space on the cylinder $D\times(0,T)$, which consists of all functions $v\in L^2(D\times(0,T))$ that possess distributional derivatives $v_{x_i}, v_\tau\in L^2(D\times(0,T))$. Define the corresponding norm by

$$\|v\|_{1,2,T}^2 := \int_{D\times(0,T)} dxd\tau\,|v(x,\tau)|^2+\sum_{i=1}^{d}\int_{D\times(0,T)} dxd\tau\,|v_{x_i}(x,\tau)|^2 + \int_{D\times(0,T)} dxd\tau\,|v_\tau(x,\tau)|^2 .$$

The set of all $v\in H^1(D\times(0,T))$ which do not depend on the time variable identifies with $H^1(D)$, the usual Sobolev space on D whose inner product and induced norm we denote by $(.,.)_{1,2}$ and $\|.\|_{1,2}$, respectively. Let us now fix once and for all an $\alpha\in(1-H,\mathsf{n}(\gamma,d))$, which is possible by virtue of the condition on the Hurst parameter in Hypothesis $(\mathrm{H}_{\gamma,d})$. Next we introduce the Banach space $\mathcal{B}^{\alpha,2}(0,T;L^2(D))$ of all Lebesgue-measurable mappings $u:[0,T]\mapsto L^2(D)$ endowed with the norm given by

$$\|u\|_{\alpha,2,T}^2 := \left(\sup_{t\in[0,T]}\|u(t)\|_2\right)^2+\int_0^T dt\left(\int_0^t d\tau\frac{\|u(t)-u(\tau)\|_2}{(t-\tau)^{\alpha+1}}\right)^2<+\infty. \tag{2.1}$$

We first recall the following notion introduced in [42] under a bit more general conditions, in which the function $x\mapsto v(x,t)\in L^2(D)$ is interpreted as the Sobolev trace of $v\in H^1(D\times(0,T))$ on the corresponding hyperplane.

Definition 2.1 *The $L^2(D)$-valued random field $(u_V(.,t))_{t\in[0,T]}$ defined and measurable on $(\Omega,\mathcal{F},\mathbb{P})$ is a variational solution to Problem (1.2) if the following two conditions hold:*

(1) We have $u_V\in L^2(0,T;H^1(D))\cap\mathcal{B}^{\alpha,2}(0,T;L^2(D))$ a.s., which means that the relations

$$\int_0^T dt\,\|u_V(.,t)\|_{1,2}^2=\int_0^T dt\left(\|u_V(.,t)\|_2^2+\|\nabla u_V(.,t)\|_2^2\right)<+\infty$$

and $\|u_V\|_{\alpha,2}<+\infty$ hold a.s.

(2) The integral relation

$$\int_D dx\ v(x,t)u_V(x,t)=\int_D dx\ v(x,0)\varphi(x)+\int_0^t d\tau\int_D dx\ v_\tau(x,\tau)u_V(x,\tau)$$

$$
\begin{aligned}
&- \int_0^t d\tau \int_D dx\, (\nabla v(x,\tau), k(x,\tau)\nabla u_V(x,\tau))_{\mathbb{R}^d} \\
&+ \int_0^t d\tau \int_D dx v(x,\tau) g(u_V(x,\tau)) \\
&+ \int_0^t \int_D dx v(x,\tau) h(u_V(x,\tau)) W^{\mathrm{H}}(x,d\tau)
\end{aligned} \tag{2.2}
$$

holds a.s. for every $v \in H^1(D \times (0,T))$ and every $t \in [0,T]$, where we have defined the stochastic integral as

$$
\begin{aligned}
&\int_0^t \int_D dx\ v(x,\tau) h(u_V(x,\tau)) W^H(x,d\tau) \\
&:= \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t (v(.,\tau), h(u_V(.,\tau))e_i)_2\, B_i^H(d\tau)
\end{aligned} \tag{2.3}
$$

according to (1.1).

From the above hypotheses we easily infer that each term in (2.2) is finite a.s; in particular, since the Hurst parameter satisfies $H \in (1 - \mathsf{n}(\gamma,d), 1) \subset \left(\frac{1}{2}, 1\right)$, we can define each one-dimensional stochastic integral with respect to $B_i^H(t)$ in (2.3) as a pathwise generalized Stieltjes integral as is the case in [37], [41], [42] and [51], to which we refer the reader for the basic definitions and properties. Hypothesis (C) and the fact that h is Lipschitz continuous then imply the absolute convergence of (2.3) a.s.

We now proceed by introducing the parabolic Green's function associated with the principal part of (1.2), that is, the function $G : \overline{D} \times [0,T] \times \overline{D} \times [0,T] \setminus \{s,t \in [0,T] : s \geq t\} \mapsto \mathbb{R}$ which, as a consequence of Hypothesis ($\mathrm{K}_{\beta,\beta'}$), is continuous, twice continuously differentiable in x, once continuously differentiable in t. For every $(y,s) \in D \times (0,T]$, it is also a classical solution to the linear initial-boundary value problem

$$
\begin{aligned}
&\partial_t G(x,t;y,s) = \operatorname{div}(k(x,t)\nabla_x G(x,t;y,s)), \quad (x,t) \in D \times (0,T]\,, \\
&\frac{\partial G(x,t;y,s)}{\partial n(k)} = 0, \quad (x,t) \in \partial D \times (0,T]\,,
\end{aligned} \tag{2.4}
$$

with

$$
\int_D dy G(.,s;y,s)\varphi(y) := \lim_{t \searrow s} \int_D dy G(.,t;y,s)\varphi(y) = \varphi(.),
$$

and satisfies the heat kernel estimates

$$|\partial_x^\mu \partial_t^\nu G(x,t;y,s)| \leq c(t-s)^{-\frac{d+|\mu|+2\nu}{2}} \exp\left[-c\frac{|x-y|^2}{t-s}\right] \tag{2.5}$$

for $\mu = (\mu_1, ..., \mu_d) \in \mathbb{N}^d$, $\nu \in \mathbb{N}$ and $|\mu| + 2\nu \leq 2$, with $|\mu| = \sum_{j=1}^d \mu_j$ (see, for instance, [18] or [31]). In particular, for $|\mu| = \nu = 0$ we have

$$|G(x,t;y,s)| \leq c(t-s)^{-\frac{d}{2}} \exp\left[-c\frac{|x-y|^2}{t-s}\right]. \tag{2.6}$$

We shall refer to (2.6) as the Gaussian property of G. This function allows us to define the following notion of mild solution for (1.2).

Definition 2.2 *The $L^2(D)$-valued random field $(u_M(.,t))_{t\in[0,T]}$ defined and measurable on $(\Omega, \mathcal{F}, \mathbb{P})$ is a mild solution to Problem (1.2) if the following two conditions hold:*

(1) We have $u_M \in L^2(0,T;H^1(D)) \cap \mathcal{B}^{\alpha,2}(0,T;L^2(D))$ a.s.

(2) The relation

$$\begin{aligned} u_M(.,t) = &\int_D dy\ G(.,t;y,0)\varphi(y) + \int_0^t d\tau \int_D dy\ G(.,t;y,\tau) g\left(u_M(y,\tau)\right) \\ &+ \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t \left(\int_D dy\ G(.,t;y,\tau) h\left(u_M(y,\tau)\right) e_i(y)\right) B_i^H(d\tau) \end{aligned} \tag{2.7}$$

holds a.s. for every $t \in [0,T]$ as an equality in $L^2(D)$.

We shall prove in Section 3 that each term in (2.7) indeed defines an $L^2(D)$-valued stochastic process.

The main results of this article are gathered in the next theorem.

Theorem 2.3 *Assume that Hypotheses (C), $(K_{\beta,\beta'})$, (L), (I) and $(H_{\gamma,d})$ hold; then the following statements are valid:*

(a) Problem (1.2) possesses a variational solution u_V and a mild solution u_M such that $u_V(.,t) = u_M(.,t)$ a.s. in $L^2(D)$ for every $t \in [0,T]$.

(b) Every mild solution u_M to Problem (1.2) is Hölder continuous with respect to the time variable in $L^2(D)$; more precisely, there exists a

positive random variable R_α^H *satisfying* $R_\alpha^H < +\infty$ *a.s. such that the estimate*

$$\|u_M(.,t) - u_M(.,s)\|_2 \leq R_\alpha^H \, |t-s|^\theta \left(1 + \|u_M\|_{\alpha,2,T}\right) \tag{2.8}$$

holds a.s. for all $s, t \in [0,T]$ *and every* $\theta \in \left(0, \left(\frac{1}{2} - \alpha\right) \wedge \frac{\beta}{2}\right)$.

(c) If h *is an affine function,* u_V *is the unique variational solution to (1.2) while* u_M *is its unique mild solution.*

Remarks

1. We shall see in Section 3 that the proof of Statement (a) requires the validity of the hypotheses (C),($K_{\beta,\beta'}$), (L), and (I) and the existence of the Hölder continuous derivative h' along with the restriction $H \in \left(\frac{1}{\gamma+1}, 1\right)$, rather than the full strength of Hypothesis ($H_{\gamma,d}$).

2. The existence of a mild solution will be proved here by reference to the existence of a variational solution. This is in contrast with the method of [37], in which the authors prove the existence of mild solutions for a class of *autonomous*, parabolic, fractional stochastic initial-boundary value problems by means of Schauder's fixed point theorem; their method thus requires the construction of a continuous map operating in a compact and convex set of a suitable functional space. If h is an affine function, the arguments of the proof of Statement (c) (see (3.66)) show that a similar approach might be possible for our equations. To the best of our knowledge, there exists as yet no such direct way to prove the existence of mild solutions to (1.2) for a non affine h.

3. As far as the Hölder regularity is concerned, we can obtain another range of values for θ by using the so-called *factorization method*, originally introduced in [11] and extensively used for the analysis of the sample paths of solutions to parabolic stochastic partial differential equations (see, for instance, [48]). In fact, we prove in Section 3 that

$$\|u_M(.,t) - u_M(.,s)\|_2 \leq R \, |t-s|^\theta \left(1 + \|u_M\|_{\alpha,2,T}\right) \tag{2.9}$$

a.s. for some a.s. finite random variable $R \in \mathbb{R}_*^+$, all $s, t \in [0,T]$ and every $\theta \in \left(0, \frac{2}{d+2} \wedge \frac{\beta}{2}\right)$. It is then interesting to compare the ranges of values provided by (2.8) and (2.9), respectively. For instance, if $d = 1$

or $d = 2$ and $\beta \in (0, 1-2\alpha]$ the two ranges are exactly the same, whereas if $\beta \in (1-2\alpha, 1)$ it is (2.9) that gives a larger range than (2.8) does and thereby a better result. For $d \geq 3$ and $\beta \in \left(0, \frac{4}{d+2}\right]$ the ranges are still exactly the same, but if $\beta \in \left(\frac{4}{d+2}, 1\right)$ it is (2.8) that provides the larger interval.

4. If h is an affine function, Theorem 2.3 establishes the complete indistinguishability of mild and variational solutions, although we do not know whether this property still holds for a general h satisfying Hypothesis ($\mathrm{H}_{\gamma,d}$); in fact, the question of uniqueness remains unsettled in this case.

3 Proofs of the Results

In what follows we write c for all the irrelevant deterministic constants that occur in the various estimates. We begin by recalling that the uniformly elliptic partial differential operator with conormal boundary conditions in the principal part of (1.2) admits a self-adjoint, positive realization $A(t) := -\operatorname{div}(k(.,t)\nabla)$ in $L^2(D)$ on the domain

$$\mathcal{D}(A(t)) = \left\{ v \in H^2(D) : (\nabla v(x), k(x,t)n(x))_{\mathbb{R}^d} = 0, \quad (x,t) \in \partial D \times [0,T] \right\} \tag{3.1}$$

(see, for instance, [33]). An important consequence of this property is that the parabolic Green's function G is also, for every $(x,t) \in D \times (0,T]$ with $t > s$, a classical solution to the linear boundary value problem

$$\begin{aligned} &\partial_s G(x,t;y,s) = -\operatorname{div}(k(y,s)\nabla_y G(x,t;y,s)), \quad (y,s) \in D \times (0,T], \\ &\frac{\partial G(x,t;y,s)}{\partial n(k)} = 0, \quad (y,s) \in \partial D \times (0,T], \end{aligned} \tag{3.2}$$

dual to (2.4) (see, for instance, [18] or [21]); this means that along with (2.5) we also have

$$\left| \partial_y^\mu \partial_s^\nu G(x,t;y,s) \right| \leq c(t-s)^{-\frac{d+|\mu|+2\nu}{2}} \exp\left[-c\frac{|x-y|^2}{t-s} \right] \tag{3.3}$$

for $|\mu| + 2\nu \leq 2$. We now use these facts to prove in the next lemma estimates for G, which we shall invoke repeatedly in the sequel to analyze various singular integrals. For the sake of clarity we list those inequalities by their chronological order of appearance in the proofs below.

Lemma 3.1 *Assume that Hypothesis* ($K_{\beta,\beta'}$) *holds. Then, for all* $x, y \in D$ *and for every* $\delta \in \left(\frac{d}{d+2}, 1\right)$ *we have the following inequalities.*

(i) For all $t, \tau, \sigma \in [0, T]$ *with* $t > \tau > \sigma$ *and some* $t^* \in (\sigma, \tau)$,

$$\begin{aligned}&|G(x,t;y,\tau) - G(x,t;y,\sigma)| \\ &\leq c\,(t-\tau)^{-\delta}\,(\tau-\sigma)^{\delta}(t-t^*)^{-\frac{d}{2}} \exp\left[-c\frac{|x-y|^2}{t-t^*}\right].\end{aligned} \tag{3.4}$$

(ii) For all $t, s, \tau \in [0, T]$ *with* $t > s > \tau$ *and some* $\tau^* \in (s, t)$,

$$\begin{aligned}&|G(x,t;y,\tau) - G(x,s;y,\tau)| \\ &\leq c\,(t-s)^{\delta}\,(s-\tau)^{-\delta}(\tau^*-\tau)^{-\frac{d}{2}} \exp\left[-c\frac{|x-y|^2}{\tau^*-\tau}\right]\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&|G(x,t;y,\tau) - G(x,s;y,\tau)|^{\delta} \\ &\leq c\,(t-s)^{\delta}\,(s-\tau)^{-\frac{d+2}{2}\delta+\frac{d}{2}}(\tau^*-\tau)^{-\frac{d}{2}} \exp\left[-c\frac{|x-y|^2}{\tau^*-\tau}\right].\end{aligned} \tag{3.6}$$

(iii) For all $t, s, \tau, \sigma \in [0, T]$ *with* $t > s > \tau > \sigma$,

$$|G(x,t;y,\tau) - G(x,t;y,\sigma)|^{1-\delta} \leq c\,(\tau-\sigma)^{1-\delta}\,(s-\tau)^{-\frac{d+2}{2}(1-\delta)} \tag{3.7}$$

uniformly in t.

Proof. By applying successively (2.6), the mean-value theorem for G and (3.3) with $|\mu| = 0$ and $\nu = 1$ we may write

$$\begin{aligned}&|G(x,t;y,\tau) - G(x,t;y,\sigma)| \\ &\leq (|G(x,t;y,\tau)| + |G(x,t;y,\sigma)|)^{1-\delta}\,|G(x,t;y,\tau) - G(x,t;y,\sigma)|^{\delta} \\ &\leq c\left((t-\tau)^{-\frac{d}{2}} + (t-\sigma)^{-\frac{d}{2}}\right)^{1-\delta}(\tau-\sigma)^{\delta}\,|G_{t^*}(x,t;y,t^*)|^{\delta} \\ &\leq c(t-\tau)^{-\frac{d}{2}(1-\delta)}(t-t^*)^{-\frac{d+2}{2}\delta+\frac{d}{2}}(\tau-\sigma)^{\delta}(t-t^*)^{-\frac{d}{2}} \exp\left[-c\frac{|x-y|^2}{t-t^*}\right] \\ &\leq c\,(t-\tau)^{-\delta}\,(\tau-\sigma)^{\delta}(t-t^*)^{-\frac{d}{2}} \exp\left[-c\frac{|x-y|^2}{t-t^*}\right]\end{aligned}$$

for some $t^* \in (\sigma, \tau)$, since $-\frac{d+2}{2}\delta + \frac{d}{2} < 0$ and $-\frac{d}{2}(1-\delta) - \frac{d+2}{2}\delta + \frac{d}{2} = -\delta$. This proves (3.4). Up to some minor but important changes, the remaining inequalities can all be proved in a similar way. ■

Estimate (3.4) now allows us to prove that our notion of mild solution in Definition 3.2 is indeed well-defined; to this end for arbitrary mappings φ and u defined on D and $D \times [0,T]$, respectively, we introduce the three functions $A(\varphi)$, $B(u)$, $C(u) : D \times [0,T] \mapsto \mathbb{R}$ by

$$A(\varphi)(x,t) := \int_D dy\ G(x,t;y,0)\varphi(y), \tag{3.8}$$

$$B(u)(x,t) := \int_0^t d\tau \int_D dy\ G(x,t;y,\tau) g\left(u(y,\tau)\right), \tag{3.9}$$

$$C(u)(x,t) := \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t \left(\int_D dy\ G(x,t;y,\tau) h\left(u(y,\tau)\right) e_i(y) \right) B_i^H(d\tau), \tag{3.10}$$

and prove the following result.

Lemma 3.2 *Assume that Hypotheses (C), ($K_{\beta,\beta'}$), (L) and (I) hold. Then, for every $u \in \mathcal{B}^{\alpha,2}(0,T;L^2(D))$ we have $A(\varphi)(.,t), B(u)(.,t) \in L^2(D)$, and also $C(u)(.,t) \in L^2(D)$ a.s., for every $t \in [0,T]$.*

Proof. The assertion is evident for $A(\varphi)(.,t)$, by virtue of the fact that φ is bounded and that (2.6) holds. As for $B(u)(.,t)$, we infer from the Gaussian property of G that the measure $d\tau dy\,|G(x,t;y,\tau)|$ is finite on $[0,T]\times D$ uniformly in $(x,t) \in D\times[0,T]$, so that by using successively Schwarz inequality with respect to this measure along with Hypothesis (L) for g we obtain

$$\begin{aligned} |B(u)(x,t)| &\leq \int_0^t d\tau \int_D dy\,|G(x,t;y,\tau) g\left(u(y,\tau)\right)| \\ &\leq c\left(\int_0^t d\tau \int_D dy\,|G(x,t;y,\tau)| \left(1 + |u(y,\tau)|^2\right)\right)^{\frac{1}{2}} \end{aligned}$$

for every $x \in D$. We then get the inequalities

$$\begin{aligned} &\|B(u)(.,t)\|_2^2 = \int_D dx \left| \int_0^t d\tau \int_D dy\ G(x,t;y,\tau) g\left(u(y,\tau)\right)\right|^2 \\ &\leq c\int_0^t d\tau \int_D dy \left(1 + |u(y,\tau)|^2\right) \leq c\left(1 + \int_0^t d\tau\, \|u(.,\tau)\|_2^2\right) < +\infty. \end{aligned}$$

It remains to show that $\|C(u)(.,t)\|_2^2 < +\infty$ a.s. for every $t \in [0,T]$.

Define the functions $f_{i,t}(u) : [0,t) \mapsto L^2(D)$ by

$$f_{i,t}(u)(.,\tau) := \int_D dy\ G(.,t;y,\tau)h\left(u(y,\tau)\right)e_i(y). \tag{3.11}$$

We shall prove that

$$\sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \left\| \int_0^t f_{i,t}(u)(.,\tau)B_i^H(d\tau) \right\|_2 \leq c\ r_\alpha^H \left(1 + \|u\|_{\alpha,2,T}\right), \tag{3.12}$$

a.s., where r_α^H is the a.s. finite and positive random variable defined in (3.20).

Indeed, by using an argument similar to the one above, since h is Lipschitz continuous and $\sup_{i\in\mathbb{N}^+} \|e_i\|_\infty < +\infty$, we first obtain

$$\sup_{i\in\mathbb{N}^+} \|f_{i,t}(u)(.,\tau)\|_2 \leq c\left(1 + \|u(.,\tau)\|_2\right) \tag{3.13}$$

a.s. for every $\tau \in [0,t)$. Furthermore, for every $x \in D$ and all $\sigma, \tau \in [0,t)$ with $\tau > \sigma$ we have

$$\begin{aligned} |f_{i,t}(u)(x,\tau) - f_{i,t}(u)(x,\sigma)| \leq c\Big(&\int_D dy\,|G(x,t;y,\tau)|\,|u(y,\tau)-u(y,\sigma)| \\ &+ \int_D dy\,|G(x,t;y,\tau)-G(x,t;y,\sigma)|\,(1+|u(y,\sigma)|)\Big), \end{aligned}$$

so that we get successively

$$\begin{aligned} &|f_{i,t}(u)(x,\tau) - f_{i,t}(u)(x,\sigma)|^2 \\ &\leq c\int_D dy\,|G(x,t;y,\tau)|\,|u(y,\tau)-u(y,\sigma)|^2 \\ &+ c\int_D dy\,|G(x,t;y,\tau)-G(x,t;y,\sigma)|\left(1+|u(y,\sigma)|^2\right) \\ &\leq c\int_D dy\,|G(x,t;y,\tau)|\,|u(y,\tau)-u(y,\sigma)|^2 \\ &+ c\,(t-\tau)^{-\delta}\,(\tau-\sigma)^{\delta}\int_D dy(t-t^*)^{-\frac{d}{2}}\exp\left[-c\frac{|x-y|^2}{t-t^*}\right]\left(1+|u(y,\sigma)|^2\right) \end{aligned}$$

for some $t^* \in (\sigma,\tau)$ and for every $\delta \in \left(\frac{d}{d+2}, 1\right)$. This is achieved by using Schwarz inequality with respect to the finite measures $dy\,|G(x,t;y,\tau)|$

and $dy\,|G(x,t;y,\tau)-G(x,t;y,\sigma)|$ on D, respectively, along with (3.4). We then integrate the preceding estimate with respect to $x \in D$ and apply the Gaussian property of G to eventually obtain

$$\sup_{i\in\mathbb{N}^+} \|f_{i,t}(u)(.,\tau)-f_{i,t}(u)(.,\sigma)\|_2 \\ \leq c\left(\|u(.,\tau)-u(.,\sigma)\|_2 + (t-\tau)^{-\frac{\delta}{2}}(\tau-\sigma)^{\frac{\delta}{2}}\left(1+\|u(.,\sigma)\|_2\right)\right). \quad (3.14)$$

Therefore, by applying an extended version of Proposition 4.1 of [41] together with Lemma 7.5 in [37], and because of (3.13), (3.14), we infer that there exists a finite positive random variable $\Lambda_\alpha(B_i^H)$, depending only on α, B_i^H and having moments of all orders, such that the sequence of estimates

$$\sum_{i=1}^{+\infty}\lambda_i^{\frac{1}{2}}\left\|\int_0^t f_{i,t}(u)(.,\tau)B_i^H(d\tau)\right\|_2$$

$$\leq \sum_{i=1}^{+\infty}\lambda_i^{\frac{1}{2}}\Lambda_\alpha(B_i^H) \quad (3.15)$$

$$\times\int_0^t d\tau\left(\frac{\|f_{i,t}(u)(.,\tau)\|_2}{\tau^\alpha}+\alpha\int_0^\tau d\sigma\,\frac{\|f_{i,t}(u)(.,\tau)-f_{i,t}(u)(.,\sigma)\|_2}{(\tau-\sigma)^{\alpha+1}}\right)$$

$$\leq c\sum_{i=1}^{+\infty}\lambda_i^{\frac{1}{2}}\Lambda_\alpha(B_i^H) \quad (3.16)$$

$$\times\left(1+\int_0^t d\tau\,\frac{\|u(.,\tau)\|_2}{\tau^\alpha}+\int_0^t d\tau\int_0^\tau d\sigma\,\frac{\|u(.,\tau)-u(.,\sigma)\|_2}{(\tau-\sigma)^{\alpha+1}}\right.$$

$$\left.+\int_0^t d\tau(t-\tau)^{-\frac{\delta}{2}}\int_0^\tau d\sigma(\tau-\sigma)^{\frac{\delta}{2}-\alpha-1}\left(1+\|u(.,\sigma)\|_2\right)\right) \quad (3.17)$$

holds a.s.. Indeed, the choice of α implies that $\tau \mapsto \tau^{-\alpha}$ is integrable at zero. Let us now examine more closely the singular integrals in the above terms. On the one hand, we may write

$$\int_0^t d\tau\,\frac{\|u(.,\tau)\|_2}{\tau^\alpha}+\int_0^t d\tau\int_0^\tau d\sigma\,\frac{\|u(.,\tau)-u(.,\sigma)\|_2}{(\tau-\sigma)^{\alpha+1}} \leq c\,\|u\|_{\alpha,2,T}, \quad (3.18)$$

by using Schwarz inequality relative to the measure $d\tau$ on $(0,t)$ in the last two integrals along with (2.1). On the other hand, since our choice of α also implies $2\alpha < \frac{d}{d+2} < \delta$, we can integrate the singularities of the time increments in the last line of (3.17) and thus get the bound

$$\int_0^t d\tau(t-\tau)^{-\frac{\delta}{2}} \int_0^\tau d\sigma(\tau-\sigma)^{\frac{\delta}{2}-\alpha-1} (1+\|u(.,\sigma)\|_2) \leq c\left(1+\sup_{t\in[0,T]}\|u(.,t)\|_2\right). \tag{3.19}$$

Finally, let

$$r_\alpha^H := \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \Lambda_\alpha(B_i^H). \tag{3.20}$$

Since the B_i^H's are identically distributed, Hypothesis (C) implies that

$$\sum_{i=1}^{\infty} \lambda_i^{\frac{1}{2}} \mathbb{E}(\Lambda_\alpha(B_i^H)) \leq c \sum_{i=1}^{\infty} \lambda_i^{\frac{1}{2}} < +\infty.$$

Therefore, we can substitute (3.18-3.20) into (3.17) to obtain (3.12). ■

In order to relate the notions of variational and mild solution, we now recall that the self-adjoint operator $A(t) = -\operatorname{div}(k(.,t)\nabla)$ defined on (3.1) generates the family of evolution operators $U(t,s)_{0\leq s\leq t\leq T}$ in $L^2(D)$ given by

$$U(t,s)v = \begin{cases} v, & \text{if } s=t, \\ \int_D dy\ G(.,t;y,s)v(y), & \text{if } t>s, \end{cases} \tag{3.21}$$

and that each such $U(t,s)$ is itself self-adjoint (see, for instance, [49]), which means that the symmetry property

$$G(x,t;y,s) = G(y,t;x,s) \tag{3.22}$$

holds for every $(x,t;y,s) \in \overline{D} \times [0,T] \times \overline{D} \times [0,T] \setminus \{s,t \in [0,T] : s \geq t\}$. We now use (3.22) to prove the following result.

Proposition 3.3 *Assume the same hypotheses as in Theorem 2.3. Then, Problem (1.2) possesses a variational solution u_V; moreover, every such variational solution is a mild solution u_M to (1.2). More precisely, for every $t \in [0,T]$, $u_V(.,t) = u_M(.,t)$ a.s. in $L^2(D)$.*

Proof. The existence of a variational solution u_V follows from the Theorem in [42]. In fact, Hypotheses ($K_{\beta,\beta'}$) and ($H_{\gamma,d}$) imply Hypotheses (K) and (H_γ) of [42], respectively.

In order to prove that every variational solution is mild, we can follow the same approach as in Theorem 2 of [48]. For the sake of completeness, we sketch the main ideas.

We shall prove that the $L^2(D)$-valued stochastic process

$$u_V(.,t) - \int_D dy\ G(.,t;y,0)\varphi(y) - \int_0^t d\tau \int_D dy\ G(.,t;y,\tau)g\left(u_V(y,\tau)\right) - \int_0^t \int_D dy\ G(.,t;y,\tau)h\left(u_V(y,\tau)\right)W^H(y,d\tau)$$

is a.s. orthogonal for every $t \in [0,T]$ to the dense subspace $\mathcal{C}_0^2(D)$ consisting of all twice continuously differentiable functions with compact support in D. To this end, for every $v \in \mathcal{C}_0^2(D)$ and all $s,t \in [0,T]$ with $t \geq s$ we define $v^t(.,s) := U(t,s)v$, that is,

$$v^t(x,s) = \begin{cases} v(x), & \text{if } \ s = t, \\ \int_D dyG(y,t;x,s)v(y), & \text{if } \ t > s, \end{cases} \tag{3.23}$$

for every $x \in D$ by taking (3.21) and (3.22) into account. It then follows from (3.2), (3.22) and Gauss' divergence theorem that $v^t \in H^1(D \times (0,T))$, and that for every $t \in [0,T]$, the relation

$$\int_0^t d\tau \int_D dx\ v_\tau^t(x,\tau)u_V(x,\tau) = \int_0^t d\tau \int_D dx \left(\nabla v^t(x,\tau), k(x,\tau)\nabla u_V(x,\tau)\right)_{\mathbb{R}^d} \tag{3.24}$$

holds a.s. Therefore, we may take (3.23) as a test function in (2.2), which, as a consequence of (3.24), leads to the relation

$$(v, u_V(.,t))_2 = (v^t(.,0), \varphi)_2 + \int_0^t d\tau (v^t(.,\tau), g(u_V(.,\tau)))_2 + \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t \left(v^t(.,\tau), h(u_V(.,\tau))e_i\right)_2 B_i^{\mathrm{H}}(d\tau),$$

valid a.s. for every $t \in [0,T]$. After some rearrangements, the substitution of (3.23) into the right-hand side of the preceding expression then leads to the equality

$$(v, u_V(.,t))_2 = \left(v, \int_D dyG(.,t;y,0)\varphi(y)\right)_2 + \left(v, \int_0^t d\tau \int_D dyG(.,t;y,\tau)g\left(u_V(y,\tau)\right)\right)_2 + \left(v, \int_0^t \int_D dyG(.,t;y,\tau)h\left(u_V(y,\tau)\right)W^H(y,d\tau)\right)_2,$$

which holds for every $t \in [0,T]$ a.s. and every $v \in \mathcal{C}_0^2(D)$, thereby leading to the desired orthogonality property. ■

Proof of Statements (a) **and** (b) **of Theorem 2.3**

Let us start with the proof of Statement (b) of Theorem 2.3. For the sake of clarity we investigate each of the functions (3.8)–(3.10) separately.

Proposition 3.4 *Assume that Hypotheses* $(K_{\beta,\beta'})$ *and (I) hold. Then, there exists* $c \in \mathbb{R}_*^+$ *such that the estimate*

$$\|A(\varphi)(.,t) - A(\varphi)(.,s)\|_2 \leq c\,|t-s|^{\theta'} \tag{3.25}$$

holds for all $s,t \in [0,T]$ *and every* $\theta' \in \left(0, \frac{\beta}{2}\right]$.

Proof. Relation (3.8) defines a classical solution to (1.2) when $g = h = 0$, so that the standard regularity theory for linear parabolic equations gives $(x,t) \mapsto A(\varphi)(x,t) \in \mathcal{C}^{\beta,\frac{\beta}{2}}(\overline{D} \times [0,T])$ (see, for instance, [18]), from which (3.25) follows immediately. ■

Regarding (3.9) we have the following result.

Proposition 3.5 *Assume that the same hypotheses as in Theorem 2.3 hold and let* u_M *be any mild solution to (1.2). Then, there exists* $c \in \mathbb{R}_*^+$ *such that the estimate*

$$\|B(u_M)(.,t) - B(u_M)(.,s)\|_2 \leq c\,|t-s|^{\theta''}\left(1 + \sup_{t\in[0,T]} \|u_M(.,t)\|_2\right) \tag{3.26}$$

holds a.s. for all $s,t \in [0,T]$ *and every* $\theta'' \in \left(0, \frac{1}{2}\right)$.

Proof. Without restricting the generality, we may assume that $t > s$. We have

$$\begin{aligned} B(u_M)(.,t) - B(u_M)(.,s) &= \int_s^t d\tau \int_D dy\; G(.,t;y,\tau) g\left(u_M(y,\tau)\right) \\ &+ \int_0^s d\tau \int_D dy \left(G(.,t;y,\tau) - G(.,s;y,\tau)\right) g\left(u_M(y,\tau)\right), \end{aligned} \tag{3.27}$$

and remark that in order to keep track of the increment $t-s$ we can estimate the first term on the right-hand side of (3.27) by using the same kind of arguments as we did in the first part of the proof of Lemma 3.2. For every $x \in D$ this gives

$$\begin{aligned}&\int_s^t d\tau \int_D dy \ \left|G(x,t;y,\tau) g\left(u_M(y,\tau)\right)\right| \\ &\leq c(t-s)^{\frac{1}{2}} \left(\int_s^t d\tau \int_D dy \left|G(x,t;y,\tau)\right| \left(1+\left|u_M(y,\tau)\right|^2\right)\right)^{\frac{1}{2}},\end{aligned}$$

so that we eventually obtain

$$\left\|\int_s^t d\tau \int_D dy\ G(.,t;y,\tau) g\left(u_M(y,\tau)\right)\right\|_2 \leq c(t-s)^{\frac{1}{2}} \left(1+\sup_{t\in[0,T]} \|u_M(.,t)\|_2\right) \tag{3.28}$$

a.s. for all $s,t \in [0,T]$ with $t>s$. In a similar manner, we can keep track of the increment $t-s$ in the second term on the right-hand side of (3.27) by using (3.5). We thus have

$$\begin{aligned}&\left\|\int_0^s d\tau \int_D dy \left(G(.,t;y,\tau)-G(.,s;y,\tau)\right) g\left(u_M(y,\tau)\right)\right\|_2^2 \\ &\leq c\int_0^s d\tau \int_D dy \int_D dx \left|G(x,t;y,\tau)-G(x,s;y,\tau)\right| \left(1+|u_M(y,\tau)|^2\right) \\ &\leq c(t-s)^\delta \int_0^s d\tau (s-\tau)^{-\delta} \left(1+\|u_M(.,\tau)\|_2^2\right) \\ &\leq c(t-s)^\delta \left(1+\sup_{t\in[0,T]} \|u_M(.,t)\|_2^2\right) \qquad (3.29)\end{aligned}$$

for every $\delta \in \left(\frac{d}{d+2}, 1\right)$, a.s. for all $s,t \in [0,T]$ with $t>s$. This last relation holds *a fortiori* for each $\delta \in (0,1)$, so that (3.28) and (3.29) indeed prove (3.26) with $\theta'' = \frac{\delta}{2}$. ■

As for the stochastic term (3.10), we have the following.

Proposition 3.6 *Assume that the same hypotheses as in Theorem 2.3 hold and let u_M be any mild solution to (1.2). Then, there exists $c \in \mathbb{R}_*^+$ such that the estimate*

$$\|C(u_M)(.,t)-C(u_M)(.,s)\|_2 \leq c r_\alpha^H |t-s|^{\theta'''} \left(1+\|u_M\|_{\alpha,2,T}\right) \tag{3.30}$$

holds a.s. for all $s,t \in [0,T]$ and every $\theta''' \in \left(0, \frac{1}{2}-\alpha\right)$.

The proof of Proposition 3.6 is more complicated than that of Proposition 3.5, though based on the same kind of technique. We begin with a preparatory result whose proof is based on inequalities (3.5)–(3.7).

For $0 \leq \tau < s \leq t \leq T$, we set

$$f^*_{i,t,s}(u_M)(.,\tau) := f_{i,t}(u_M)(.,\tau) - f_{i,s}(u_M)(.,\tau), \tag{3.31}$$

where the $f_{i,t}(u_M)$'s are given by (3.11).

Lemma 3.7 *With the same hypotheses as in Theorem 2.3, the estimates*

$$\sup_{i\in\mathbb{N}^+} \left\| f^*_{i,t,s}(u_M)(.,\tau)\right\|_2 \leq c\,(t-s)^{\frac{\delta}{2}}\,(s-\tau)^{-\frac{\delta}{2}}\left(1+\sup_{t\in[0,T]}\|u_M(.,t)\|_2\right) \tag{3.32}$$

and

$$\begin{aligned}&\sup_{i\in\mathbb{N}^+}\left\|f^*_{i,t,s}(u_M)(.,\tau)-f^*_{i,t,s}(u_M)(.,\sigma)\right\|_2\\ &\leq c\,(t-s)^{\frac{\delta}{2}}\,(s-\tau)^{-\frac{\delta}{2}}\,\|u_M(.,\tau)-u_M(.,\sigma)\|_2\\ &+c\,(t-s)^{\frac{\delta}{2}}\,(s-\tau)^{-\frac{1}{2}}\,(\tau-\sigma)^{\frac{1}{2}(1-\delta)}\left(1+\sup_{t\in[0,T]}\|u_M(.,t)\|_2\right)\end{aligned} \tag{3.33}$$

hold a.s. for every $\delta \in \left(\frac{d}{d+2},1\right)$ and for all $\sigma,\tau\in[0,s)$ with $\tau>\sigma$ in (3.33).

Proof. The proof of (3.32) is analogous to that of (3.14) and is thereby omitted. As for (3.33), by using Schwarz inequality relative to the measures $dy\,|G(x,t;y,\tau)-G(x,s;y,\tau)|$ and

$$dy\,|G(x,t;y,\tau)-G(x,s;y,\tau)-G(x,t;y,\sigma)+G(x,s;y,\sigma)|$$

on D along with Hypothesis (L) for h, we get

$$\begin{aligned}&\left\|f^*_{i,t,s}(u_M)(.,\tau)-f^*_{i,t,s}(u_M)(.,\sigma)\right\|_2^2\\ &\leq c\int_D dx\int_D dy\,|G(x,t;y,\tau)-G(x,s;y,\tau)|\,|u_M(y,\tau)-u_M(y,\sigma)|^2\\ &+c\int_D dx\int_D dy\,|G(x,t;y,\tau)-G(x,s;y,\tau)-G(x,t;y,\sigma)+G(x,s;y,\sigma)|\\ &\quad\times\left(1+|u_M(y,\sigma)|^2\right)\\ &\leq c\,(t-s)^{\delta}\,(s-\tau)^{-\delta}\,\|u_M(.,\tau)-u_M(.,\sigma)\|_2^2\end{aligned}$$

$$+\,c\int_D dx\int_D dy\,|G(x,t;y,\tau)-G(x,s;y,\tau)-G(x,t;y,\sigma)+G(x,s;y,\sigma)|$$
$$\times\left(1+|u_M(y,\sigma)|^2\right) \tag{3.34}$$

a.s. for all $s,t,\sigma,\tau\in[0,T]$ with $t\geq s>\tau>\sigma$ and every $\delta\in\left(\frac{d}{d+2},1\right)$, as a consequence of (3.5). It remains to prove that

$$\begin{aligned}
&\int_D dx\int_D dy\,|G(x,t;y,\tau)-G(x,s;y,\tau)-G(x,t;y,\sigma)+G(x,s;y,\sigma)|\\
&\qquad\times\left(1+|u_M(y,\sigma)|^2\right)\\
&\leq c\,(t-s)^{\delta}\,(s-\tau)^{-1}\,(\tau-\sigma)^{1-\delta}\left(1+\sup_{t\in[0,T]}\|u_M(.,t)\|_2^2\right),
\end{aligned} \tag{3.35}$$

for then the substitution of (3.35) into (3.34) leads to (3.33). But (3.35) follows from (3.6)–(3.7), which allow us to get the estimates

$$\begin{aligned}
&|G(x,t;y,\tau)-G(x,s;y,\tau)-G(x,t;y,\sigma)+G(x,s;y,\sigma)|\\
&\leq\left(|G(x,t;y,\tau)-G(x,s;y,\tau)|^{\delta}+|G(x,t;y,\sigma)-G(x,s;y,\sigma)|^{\delta}\right)\\
&\times\left(|G(x,t;y,\tau)-G(x,t;y,\sigma)|^{1-\delta}+|G(x,s;y,\tau)-G(x,s;y,\sigma)|^{1-\delta}\right)\\
&\leq(t-s)^{\delta}\,(s-\tau)^{-\frac{d+2}{2}\delta+\frac{d}{2}}\,(\tau-\sigma)^{1-\delta}\,(s-\tau)^{-\frac{d+2}{2}(1-\delta)}\\
&\times\left((\tau^*-\tau)^{-\frac{d}{2}}\exp\left[-c\frac{|x-y|^2}{\tau^*-\tau}\right]+(\sigma^*-\sigma)^{-\frac{d}{2}}\exp\left[-c\frac{|x-y|^2}{\sigma^*-\sigma}\right]\right)\\
&=c\,(t-s)^{\delta}\,(s-\tau)^{-1}\,(\tau-\sigma)^{1-\delta}\\
&\times\left((\tau^*-\tau)^{-\frac{d}{2}}\exp\left[-c\frac{|x-y|^2}{\tau^*-\tau}\right]+(\sigma^*-\sigma)^{-\frac{d}{2}}\exp\left[-c\frac{|x-y|^2}{\sigma^*-\sigma}\right]\right),
\end{aligned} \tag{3.36}$$

with $\tau^*,\sigma^*\in(s,t)$ and thereby the desired result by the Gaussian property. ■

Proof of Proposition 3.6. For $t>s$ we write

$$C(u_M)(.,t)-C(u_M)(.,s)=\sum_{i=1}^{+\infty}\lambda_i^{\frac{1}{2}}\int_s^t f_{i,t}(u_M)(.,\tau)B_i^H(d\tau)$$

$$
+\sum_{i=1}^{+\infty}\lambda_i^{\frac{1}{2}}\int_0^s f_{i,t,s}^{*}(u_M)(.,\tau)B_i^H(d\tau). \tag{3.37}
$$

In order to estimate the first-term on the right-hand side of (3.37), we can start by using inequalities (3.13) and (3.14). As in (3.17), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{+\infty}\lambda_i^{\frac{1}{2}}\left\|\int_s^t f_{i,t}(u_M)(.,\tau)B_i^H(d\tau)\right\|_2\\
&\leq cr_\alpha^H\left(\int_s^t\frac{d\tau}{(\tau-s)^\alpha}+\int_s^t d\tau\frac{\|u_M(.,\tau)\|_2}{(\tau-s)^\alpha}+\int_s^t d\tau\int_s^\tau d\sigma\frac{\|u_M(.,\tau)-u_M(.,\sigma)\|_2}{(\tau-\sigma)^{\alpha+1}}\right.\\
&\left.+\int_s^t d\tau(t-\tau)^{-\frac{\delta}{2}}\int_s^\tau d\sigma(\tau-\sigma)^{\frac{\delta}{2}-\alpha-1}\left(1+\|u_M(.,\sigma)\|_2\right)\right)
\end{aligned} \tag{3.38}
$$

a.s. for every $s,t\in[0,T]$ with $t>s$ and each $\delta\in\left(\frac{d}{d+2},1\right)$. Furthermore, we now have

$$
\begin{aligned}
&\int_s^t\frac{d\tau}{(\tau-s)^\alpha}+\int_s^t d\tau\frac{\|u_M(.,\tau)\|_2}{(\tau-s)^\alpha}+\int_s^t d\tau\int_s^\tau d\sigma\frac{\|u_M(.,\tau)-u_M(.,\sigma)\|_2}{(\tau-\sigma)^{\alpha+1}}\\
&\leq c\left((t-s)^{1-\alpha}\left(1+\|u_M\|_{\alpha,2}\right)+(t-s)^{\frac{1}{2}}\|u_M\|_{\alpha,2}\right)\\
&\leq c(t-s)^{\frac{1}{2}}\left(1+\|u_M\|_{\alpha,2,T}\right)
\end{aligned} \tag{3.39}
$$

since $\alpha<\frac{1}{2}$. Moreover,

$$
\begin{aligned}
&\int_s^t d\tau(t-\tau)^{-\frac{\delta}{2}}\int_s^\tau d\sigma(\tau-\sigma)^{\frac{\delta}{2}-\alpha-1}\left(1+\|u_M(.,\sigma)\|_2\right)\\
&\leq c\left(1+\sup_{t\in[0,T]}\|u_M(.,t)\|_2\right)\int_s^t d\tau\int_s^\tau d\sigma\ (t-\tau)^{-\frac{\delta}{2}}(\tau-\sigma)^{\frac{\delta}{2}-\alpha-1}\\
&\leq c(t-s)^{2-\alpha}\left(1+\sup_{t\in[0,T]}\|u_M(.,t)\|_2\right),
\end{aligned} \tag{3.40}
$$

by virtue of the convergence of the integral, which can be expressed in terms of Euler's Beta function since $\alpha<\frac{\delta}{2}$. The substitution of (3.39) and (3.40) into (3.38) then leads to the inequality

$$
\sum_{i=1}^{+\infty}\lambda_i^{\frac{1}{2}}\left\|\int_s^t f_{i,t}(u_M)(.,\tau)B_i^H(d\tau)\right\|_2\leq cr_\alpha^H(t-s)^{\frac{1}{2}}\left(1+\|u_M\|_{\alpha,2,T}\right) \tag{3.41}
$$

a.s. for every $s, t \in [0, T]$ with $t > s$.

It remains to estimate the second term on the right-hand side of (3.37). To this end we apply the same adaptation of Proposition 4.1. of [41] as in the proof of Lemma 3.2; this gives the inequality

$$\sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \left\| \int_0^s f^*_{i,t,s}(u_M)(.,\tau) B_i^H(d\tau) \right\|_2 \le \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \Lambda_\alpha(B_i^H)$$
$$\times \int_0^s d\tau \left(\frac{\left\| f^*_{i,t,s}(u_M)(.,\tau) \right\|_2}{\tau^\alpha} + \alpha \int_0^\tau d\sigma \frac{\left\| f^*_{i,t,s}(u_M)(.,\tau) - f^*_{i,t,s}(u_M)(.,\sigma) \right\|_2}{(\tau-\sigma)^{\alpha+1}} \right).$$

By substituting (3.32) and (3.33) we obtain

$$\begin{aligned}
&\sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \left\| \int_0^s f^*_{i,t,s}(u_M)(.,\tau) B_i^H(d\tau) \right\|_2 \\
&\le c r_\alpha^H (t-s)^{\frac{\delta}{2}} \left(\int_0^s d\tau \, (s-\tau)^{-\frac{\delta}{2}} \, \tau^{-\alpha} \left(1 + \sup_{t\in[0,T]} \|u_M(.,t)\|_2 \right) \right. \\
&+ \int_0^s d\tau \, (s-\tau)^{-\frac{\delta}{2}} \int_0^\tau d\sigma \frac{\|u_M(.,\tau) - u_M(.,\sigma)\|_2}{(\tau-\sigma)^{\alpha+1}} \\
&\left. + \int_0^s d\tau \, (s-\tau)^{-\frac{1}{2}} \int_0^\tau d\sigma \, (\tau-\sigma)^{\frac{1}{2}(1-\delta)-\alpha-1} \left(1 + \sup_{t\in[0,T]} \|u_M(.,t)\|_2 \right) \right) \\
&\le c \; r_\alpha^H (t-s)^{\frac{\delta}{2}} \left(1 + \|u_M\|_{\alpha,2,T} \right) \\
&\quad \times \left(1 + \int_0^s d\tau \, (s-\tau)^{-\frac{1}{2}} \int_0^\tau d\sigma \, (\tau-\sigma)^{\frac{1}{2}(1-\delta)-\alpha-1} \right),
\end{aligned} \tag{3.42}$$

where we have got the last estimate using Schwarz inequality with respect to the measure $d\tau$ on $(0, s)$ along with (2.1) in the first two integrals on the right-hand side.

By imposing the additional restriction $\delta < 1 - 2\alpha$, we have

$$\int_0^s d\tau \, (s-\tau)^{-\frac{1}{2}} \int_0^\tau d\sigma \, (\tau-\sigma)^{\frac{1}{2}(1-\delta)-\alpha-1} < +\infty.$$

Thus, we have proved that

$$\sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \left\| \int_0^s f^*_{i,t,s}(u_M)(.,\tau) B_i^H(d\tau) \right\|_2 \le c r_\alpha^H (t-s)^{\frac{\delta}{2}} \left(1 + \|u_M\|_{\alpha,2,T} \right) \tag{3.43}$$

a.s. for all $s,t \in [0,T]$ with $t > s$ and every $\delta \in \left(\frac{d}{d+2}, 1-2\alpha\right)$. The existence of this restricted interval of values of δ is possible by our choice of α. Relations (3.37), (3.41) and (3.43) clearly yield (3.30) with $\theta''' = \frac{\delta}{2} \in \left(0, \frac{1}{2} - \alpha\right)$. ■

It is immediate that Propositions 3.4 to 3.6 imply statement (b) of Theorem 2.3. Notice that $R_\alpha^H = c(1 + r_\alpha^H)$, with r_α^H defined in (3.20). Moreover, this result together with Proposition 3.3 imply statement (a).

Proof of Statement (c) of Theorem 2.3

Under the standing assumptions, we already know from [42] that the variational solution is unique. Since every variational solution is also a mild solution, it suffices to prove that uniqueness holds within the class of mild solutions. To this end, let us write u_M and $\tilde{u}_M$ for any two such solutions corresponding to the same initial condition φ; from (2.7) and (3.8)-(3.10) we have

$$\begin{aligned}&\|u_M(.,t) - \tilde{u}_M(.,t)\|_2 \\ &\leq \|B(u_M)(.,t) - B(\tilde{u}_M)(.,t)\|_2 + \|C(u_M)(.,t) - C(\tilde{u}_M)(.,t)\|_2\end{aligned} \tag{3.44}$$

a.s. for every $t \in [0,T]$. We proceed by estimating both terms on the right-hand side of (3.44). This is easy to achieve for the first one for which we have the following result whose proof is omitted.

Lemma 3.8 *Assume that the same hypotheses as in Theorem 2.3 hold. Then we have*

$$\|B\left(u_M\right)(.,t) - B(\tilde{u}_M)(.,t)\|_2^2 \leq c \int_0^t d\tau \, \|u_M(.,\tau) - \tilde{u}_M(.,\tau)\|_2^2 \tag{3.45}$$

a.s. for every $t \in [0,T]$.

In order to analyze the second term we will need the following preliminary result.

Lemma 3.9 *The hypotheses are the same as in Theorem 2.3 and let the $f_{i,t}(u)$'s be the functions given by (3.11). Then, the estimate*

$$\sup_{(i,t)\in\mathbb{N}^+\times[0,T]} \|f_{i,t}(u_M)(.,\tau) - f_{i,t}(\tilde{u}_M)(.,\tau)\|_2 \leq c\,\|u_M(.,\tau) - \tilde{u}_M(.,\tau)\|_2 \tag{3.46}$$

holds a.s. for every $\tau \in [0,t)$.

Moreover, if h *is an affine function we have*

$$\begin{aligned}
&\sup_{i\in\mathbb{N}^+} \| \, f_{i,t}(u_M)(.,\tau) - f_{i,t}(\tilde{u}_M)(.,\tau) - f_{i,t}(u_M)(.,\sigma) + f_{i,t}(\tilde{u}_M)(.,\sigma)\|_2 \\
&\quad \leq c(t-\tau)^{-\frac{\delta}{2}}(\tau-\sigma)^{\frac{\delta}{2}} \|u_M(.,\sigma) - \tilde{u}_M(.,\sigma)\|_2 \\
&\quad + c\,\|u_M(.,\tau) - \tilde{u}_M(.,\tau) - u_M(.,\sigma) + \tilde{u}_M(.,\sigma)\|_2 \qquad (3.47)
\end{aligned}$$

a.s. for all $t,\tau,\sigma \in [0,T]$ *with* $t > \tau > \sigma$ *and every* $\delta \in \left(\frac{d}{d+2}, 1\right)$.

Proof. Up to minor modifications, we can prove (3.46) as we proved (3.13), while we can still prove (3.47) by applying Schwarz inequality for the relevant measures, the Gaussian property for G along with (3.4) at the appropriate places. ■

The preceding result now leads to the following estimate for the second term on the right-hand side of (3.44).

Lemma 3.10 *Assume that the same hypotheses as in Lemma 3.9 hold and let* h *be an affine function. Then we have*

$$\begin{aligned}
&\|C(u_M)(.,t) - C(\tilde{u}_M)(.,t)\|_2 \\
&\quad \leq c r_\alpha^H \left(\int_0^t d\tau \left(\frac{1}{\tau^\alpha} + \frac{1}{(t-\tau)^\alpha} \right) \|u_M(.,\tau) - \tilde{u}_M(.,\tau)\|_2 \right. \\
&\quad \left. + \int_0^t d\tau \int_0^\tau d\sigma \frac{\|u_M(.,\tau) - \tilde{u}_M(.,\tau) - u_M(.,\sigma) + \tilde{u}_M(.,\sigma)\|_2}{(\tau-\sigma)^{\alpha+1}} \right) \qquad (3.48)
\end{aligned}$$

a.s. for every $t \in [0,T]$.

Proof. From (3.10), (3.11), the same adaptation of Proposition 4.1 of [41] as above, and by using (3.46), (3.47), we have

$$\begin{aligned}
&\|\mathrm{C}(u_M)(.,t) - \mathrm{C}(\tilde{u}_M)(.,t)\|_2 \\
&\leq c r_\alpha^H \left(\int_0^t d\tau \frac{\|u_M(.,\tau) - \tilde{u}_M(.,\tau)\|_2}{\tau^\alpha} \right. \\
&+ \int_0^t d\tau \int_0^\tau d\sigma (t-\tau)^{-\frac{\delta}{2}} (\tau-\sigma)^{\frac{\delta}{2}-\alpha-1} \|u_M(.,\sigma) - \tilde{u}_M(.,\sigma)\|_2 \\
&\left. + \int_0^t d\tau \int_0^\tau d\sigma \frac{\|u_M(.,\tau) - \tilde{u}_M(.,\tau) - u_M(.,\sigma) + \tilde{u}_M(.,\sigma)\|_2}{(\tau-\sigma)^{\alpha+1}} \right) \qquad (3.49)
\end{aligned}$$

a.s. for every $t \in [0, T]$.

Furthermore, by swapping each integration variable for the other in the second term on the right-hand side and by using Fubini's theorem we may write

$$\begin{aligned} &\int_0^t d\tau \int_0^\tau d\sigma (t-\tau)^{-\frac{\delta}{2}} (\tau-\sigma)^{\frac{\delta}{2}-\alpha-1} \|u_M(.,\sigma) - \tilde{u}_M(.,\sigma)\|_2 \\ &= \int_0^t d\tau \, \|u_M(.,\tau) - \tilde{u}_M(.,\tau)\|_2 \int_\tau^t d\sigma (t-\sigma)^{-\frac{\delta}{2}} (\sigma-\tau)^{\frac{\delta}{2}-\alpha-1} \\ &= c \int_0^t d\tau \frac{\|u_M(.,\tau) - \tilde{u}_M(.,\tau)\|_2}{(t-\tau)^\alpha}, \end{aligned}$$

after having evaluated the singular integral explicitly in terms of Euler's Beta function, which is possible since $\alpha < \frac{\delta}{2}$. The substitution of the preceding expression into (3.49) then proves (3.48). ■

In what follows, we write R for all the irrelevant a.s. finite and positive random variables that appear in the different estimates, unless we specify these variables otherwise. The preceding inequalities then lead to the following crucial estimate for $z_M := u_M - \tilde{u}_M$ with respect to the norm in $B^{\alpha,2}(0,t;L^2(D))$.

Lemma 3.11 *Assume that the same hypotheses as in Theorem 2.3 hold and let h be an affine function. Then we have*

$$\begin{aligned} \|z_M\|_{\alpha,2,t}^2 \leq R \Bigg(&\int_0^t d\tau \sup_{\sigma \in [0,\tau]} \|z_M(.,\sigma)\|_2^2 \\ &+ \int_0^t d\tau \left(\int_0^\tau d\sigma \frac{\|z_M(.,\tau) - z_M(.,\sigma)\|_2}{(\tau-\sigma)^{\alpha+1}} \right)^2 \Bigg) \end{aligned} \tag{3.50}$$

a.s. for every $t \in [0, T]$.

Proof. We apply Schwarz inequality relative to the measure $d\tau$ on $(0,t)$ to both integrals on the right-hand side of (3.48). This leads to

$$\begin{aligned} &\|C(u_M)(.,t) - C(\tilde{u}_M)(.,t)\|_2^2 \\ &\leq R \left(\int_0^t d\tau \|z_M(.,\tau)\|_2^2 + \int_0^t d\tau \left(\int_0^\tau d\sigma \frac{\|z_M(.,\tau) - z_M(.,\sigma)\|_2}{(\tau-\sigma)^{\alpha+1}} \right)^2 \right) \end{aligned} \tag{3.51}$$

a.s. for every $t \in [0,T]$. Consequently, from (3.44), (3.45) and (3.51) we obtain

$$\|z_M(.,t)\|_2^2 \leq R\left(\int_0^t d\tau \|z_M(.,\tau)\|_2^2 + \int_0^t d\tau \left(\int_0^\tau d\sigma \frac{\|z_M(.,\tau) - z_M(.,\sigma)\|_2}{(\tau-\sigma)^{\alpha+1}}\right)^2\right)$$

and thereby (3.50) by the very definition of $\|z_M\|_{\alpha,2,t}$. ■

We proceed by analyzing further the second term on the right-hand side of (3.50), so as to eventually obtain an inequality of Gronwall type for $\|z_M\|_{\alpha,2,t}^2$. By reference to (2.7), we may write

$$\begin{aligned}
&z_M(.,\tau) - z_M(.,\sigma) \\
&= \int_\sigma^\tau d\rho \int_D dy G(.,\tau;y,\rho)\left(g(u_M(y,\rho)) - g(\tilde{u}_M(y,\rho))\right) \\
&+ \int_0^\sigma d\rho \int_D dy \left(G(.,\tau;y,\rho) - G(.,\sigma;y,\rho)\right)\left(g(u_M(y,\rho)) - g(\tilde{u}_M(y,\rho))\right) \\
&+ \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_\sigma^\tau \left(f_{i,\tau}(u_M)(.,\rho) - f_{i,\tau}(\tilde{u}_M)(.,\rho)\right) B_i^H(d\rho) \\
&+ \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^\sigma \left(f_{i,\tau,\sigma}^*(u_M)(.,\rho) - f_{i,\tau,\sigma}^*(\tilde{u}_M)(.,\rho)\right) B_i^H(d\rho) \qquad (3.52)
\end{aligned}$$

for all $\sigma, \tau \in [0,t]$ with $\tau > \sigma$ where the $f_{i,\tau}(u_M)$'s and the $f_{i,\tau,\sigma}^*(u_{M,\varphi})$'s are given by (3.11) and (3.31), respectively.

Our next goal is to estimate the $L^2(D)$-norm of each contribution on the right-hand side of (3.52). Regarding the first two terms we have the following result whose proof is quite similar to that of Proposition 3.5 and thereby omitted (see also the proof of Lemma 3.2).

Lemma 3.12 *The hypotheses are the same as in Theorem 2.3; then we have*

$$\begin{aligned}
&\left\|\int_\sigma^\tau d\rho \int_D dy\ G(.,\tau;y,\rho)\left(g(u_M(y,\rho)) - g(\tilde{u}_M(y,\rho))\right)\right\|_2 \\
&\leq c\,(\tau-\sigma)^{\frac{1}{2}} \left(\int_\sigma^\tau d\rho\, \|z_M(.,\rho)\|_2^2\right)^{\frac{1}{2}} \qquad (3.53)
\end{aligned}$$

and

$$\left\|\int_0^\sigma d\rho \int_D dy \left(G(.,\tau;y,\rho) - G(.,\sigma;y,\rho)\right)\left(g(u_M(y,\rho)) - g(\tilde{u}_M(y,\rho))\right)\right\|_2$$

$$\leq c\,(\tau-\sigma)^{\frac{\delta}{2}}\left(\int_0^{\sigma} d\rho\,(\sigma-\rho)^{-\delta}\,\|z_M(.,\rho)\|_2^2\right)^{\frac{1}{2}} \tag{3.54}$$

a.s. for all $\sigma,\tau\in[0,t]$ *with* $\tau>\sigma$ *and every* $\delta\in\left(\frac{d}{d+2},1\right)$.

Next, we turn to the analysis of the third term on the right-hand side of (3.52).

Lemma 3.13 *With the same hypotheses as in Lemma 3.9, we have*

$$\begin{aligned}&\sum_{i=1}^{+\infty}\lambda_i^{\frac{1}{2}}\left\|\int_{\sigma}^{\tau}\left(f_{i,\tau}(u_M)(.,\rho)-f_{i,\tau}(\tilde{u}_M)(.,\rho)\right)B_i^H(d\rho)\right\|_2\\ &\leq R\left(\int_{\sigma}^{\tau} d\rho\left(\frac{1}{(\rho-\sigma)^{\alpha}}+\frac{1}{(\tau-\rho)^{\alpha}}\right)\|z_M(.,\rho)\|_2\right.\\ &\left.+\int_{\sigma}^{\tau} d\rho\int_{\sigma}^{\rho} d\varsigma\frac{\|z_M(.,\rho)-z_M(.,\varsigma)\|_2}{(\rho-\varsigma)^{\alpha+1}}\right)\end{aligned}$$

a.s. for all $\sigma,\tau\in[0,t]$ *with* $\tau>\sigma$.

Proof. In terms of the variables τ,ρ and ς, inequalities (3.46), (3.47) of Lemma 3.9 now read

$$\sup_{(i,\tau)\in\mathbb{N}^+\times[0,T]}\|\,f_{i,\tau}(u_M)(.,\rho)-f_{i,\tau}(\tilde{u}_M)(.,\rho)\|_2\leq c\,\|z_M(.,\rho)\|_2 \tag{3.55}$$

and

$$\begin{aligned}&\sup_{i\in\mathbb{N}^+}\|\,f_{i,\tau}(u_M)(.,\rho)-f_{i,\tau}(\tilde{u}_M)(.,\rho)\;-f_{i,\tau}(u_M)(.,\varsigma)+f_{i,\tau}(\tilde{u}_M)(.,\varsigma)\|_2\\ &\leq c(\tau-\rho)^{-\frac{\delta}{2}}(\rho-\varsigma)^{\frac{\delta}{2}}\,\|z_M(.,\varsigma)\|_2+c\,\|z_M(.,\rho)-z_M(.,\varsigma)\|_2\,,\end{aligned} \tag{3.56}$$

respectively. Hence,

$$\begin{aligned}&\sum_{i=1}^{+\infty}\lambda_i^{\frac{1}{2}}\left\|\int_{\sigma}^{\tau}\left(f_{i,\tau}(u_M)(.,\rho)-f_{i,\tau}(\tilde{u}_M)(.,\rho)\right)B_i^H(d\rho)\right\|_2\\ &\leq\sum_{i=1}^{+\infty}\lambda_i^{\frac{1}{2}}\Lambda_\alpha(B_i^H)\Big(\int_{\sigma}^{\tau} d\rho\frac{\|\,f_{i,\tau}(u_M)(.,\rho)-f_{i,\tau}(\tilde{u}_M)(.,\rho)\|_2}{(\rho-\sigma)^{\alpha}}\\ &+\alpha\int_{\sigma}^{\tau} d\rho\int_{\sigma}^{\rho}\frac{d\varsigma}{(\rho-\varsigma)^{\alpha+1}}\end{aligned}$$

$$\times \left\| f_{i,\tau}(u_M)(.,\rho) - f_{i,\tau}(\tilde{u}_M)(.,\rho) \ - f_{i,\tau}(u_M)(.,\varsigma) + f_{i,\tau}(\tilde{u}_M)(.,\varsigma)\right\|_2 \Big)$$
$$\leq R\left(\int_\sigma^\tau d\rho \frac{\|z_M(.,\rho)\|_2}{(\rho-\sigma)^\alpha} + \int_\sigma^\tau d\rho\,(\tau-\rho)^{-\frac{\delta}{2}}\int_\sigma^\rho d\varsigma\,(\rho-\varsigma)^{\frac{\delta}{2}-\alpha-1}\|z_M(.,\varsigma)\|_2\right.$$
$$\left. + \int_\sigma^\tau d\rho\int_\sigma^\rho d\varsigma \frac{\|z_M(.,\rho)-z_M(.,\varsigma)\|_2}{(\rho-\varsigma)^{\alpha+1}}\right)$$

a.s. for all $\sigma, \tau \in [0,t]$ with $\tau > \sigma$ and every $\delta \in \left(\frac{d}{d+2}, 1\right)$. But the second term on the right-hand side is equal to

$$c\int_\sigma^\tau d\rho\,(\tau-\rho)^{-\alpha}\|z_M(.,\rho)\|_2\,,$$

as can be easily checked by applying Fubini's theorem and by evaluating the resulting inner integral in terms of Euler's Beta function. ∎

As for the analysis of the fourth term on the right-hand side of (3.52) we need the following preparatory result.

Lemma 3.14 *The hypotheses are the same as in Lemma 3.9 and the $f^*_{i,\tau,\sigma}(u)$'s are the functions given by (3.31). Then, the estimates*

$$\sup_{i\in\mathbb{N}^+}\left\| f^*_{i,\tau,\sigma}(u_M)(.,\rho) - f^*_{i,\tau,\sigma}(\tilde{u}_M)(.,\rho)\right\|_2 \leq c(\tau-\sigma)^{\frac{\delta}{2}}(\sigma-\rho)^{-\frac{\delta}{2}}\|z_M(.,\rho)\|_2 \tag{3.57}$$

and

$$\sup_{i\in\mathbb{N}^+}\left\| f^*_{i,\tau,\sigma}(u_M)(.,\rho) - f^*_{i,\tau,\sigma}(\tilde{u}_M)(.,\rho) - f^*_{i,\tau,\sigma}(u_M)(.,\varsigma) + f^*_{i,\tau,\sigma}(\tilde{u}_M)(.,\varsigma)\right\|_2$$
$$\leq c(\tau-\sigma)^{\frac{\delta}{2}}\left((\sigma-\rho)^{-\frac{1}{2}}(\rho-\varsigma)^{\frac{1}{2}(1-\delta)}\|z_M(.,\varsigma)\|_2\right.$$
$$\left. + (\sigma-\rho)^{-\frac{\delta}{2}}\|z_M(.,\rho)-z_M(.,\varsigma)\|_2\right) \tag{3.58}$$

hold a.s. for all $\tau, \sigma, \rho, \varsigma \in [0,T]$ with $\tau > \sigma > \rho > \varsigma$ and every $\delta \in \left(\frac{d}{d+2}, 1\right)$.

Proof. It follows from the same arguments as those outlined in the proof of Lemma 3.9, the key pointwise estimates being this time (3.5) and (3.36). ∎

The last relevant $L^2(D)$-estimate regarding (3.52) is then the following.

Lemma 3.15 *The hypotheses are the same as in Lemma 3.9. Then we have*

$$\sum_{i=1}^{+\infty}\lambda_i^{\frac{1}{2}}\left\|\int_0^\sigma \left(f^*_{i,\tau,\sigma}(u_M)(.,\rho)-f^*_{i,\tau,\sigma}(\tilde{u}_M)(.,\rho)\right)B_i^{\mathrm{H}}(d\rho)\right\|_2$$
$$\leq R(\tau-\sigma)^{\frac{\delta}{2}}\left\{\int_0^\sigma d\rho(\sigma-\rho)^{-\frac{\delta}{2}}\left(\frac{1}{\rho^\alpha}+\frac{1}{(\sigma-\rho)^\alpha}\right)\|z_M(.,\rho)\|_2\right.$$
$$\left.+\int_0^\sigma d\rho(\sigma-\rho)^{-\frac{\delta}{2}}\int_0^\rho d\varsigma\frac{\|z_M(.,\rho)-z_M(.,\varsigma)\|_2}{(\rho-\varsigma)^{\alpha+1}}\right\}$$

a.s. for all $\sigma,\tau\in[0,t]$ with $\tau>\sigma$ and every $\delta\in\left(\frac{d}{d+2},1-2\alpha\right)$.

Proof. Once more by the same adaptation of Proposition 4.1 of [41] as above, together with (3.57), (3.58), we get

$$\sum_{i=1}^{+\infty}\lambda_i^{\frac{1}{2}}\left\|\int_0^\sigma \left(f^*_{i,\tau,\sigma}(u_M)(.,\rho)-f^*_{i,\tau,\sigma}(\tilde{u}_M)(.,\rho)\right)B_i^{\mathrm{H}}(d\rho)\right\|_2$$
$$\leq R(\tau-\sigma)^{\frac{\delta}{2}}\left\{\int_0^\sigma d\rho(\sigma-\rho)^{-\frac{\delta}{2}}\frac{\|z_M(.,\rho)\|_2}{\rho^\alpha}\right.$$
$$+\int_0^\sigma d\rho(\sigma-\rho)^{-\frac{1}{2}}\int_0^\rho d\varsigma(\rho-\varsigma)^{\frac{1}{2}(1-\delta)-\alpha-1}\|z_M(.,\varsigma)\|_2$$
$$\left.+\int_0^\sigma d\rho(\sigma-\rho)^{-\frac{\delta}{2}}\int_0^\rho d\varsigma\frac{\|z_M(.,\rho)-z_M(.,\varsigma)\|_2}{(\rho-\varsigma)^{\alpha+1}}\right\}$$

a.s. for all $\sigma,\tau\in[0,t]$ with $\tau>\sigma$ and every $\delta\in\left(\frac{d}{d+2},1-2\alpha\right)$, so that the result follows from the relation

$$\int_0^\sigma d\rho(\sigma-\rho)^{-\frac{1}{2}}\int_0^\rho d\varsigma(\rho-\varsigma)^{\frac{1}{2}(1-\delta)-\alpha-1}\|z_M(.,\varsigma)\|_2$$
$$=c\int_0^\sigma d\rho(\sigma-\rho)^{-\alpha-\frac{\delta}{2}}\|z_M(.,\rho)\|_2.$$

■

Let us go back to the inequality (3.50). Owing to (3.52) and by using the estimates (3.53), (3.54) together with Lemmas 3.13 and 3.15, we have

$$\|z_M\|^2_{\alpha,2,t}\leq R\int_0^t d\tau\left(\sup_{\rho\in[0,\tau]}\|z_M(.,\rho)\|_2^2+\sum_{k=1}^{6}I_k^2(\tau)\right)\tag{3.59}$$

a.s., for some positive and finite random variable R, where

$$I_1(\tau) = \int_0^\tau \frac{d\sigma}{(\tau-\sigma)^{\frac{1}{2}+\alpha}} \left(\int_\sigma^\tau d\rho \|z_M(.,\rho)\|_2^2\right)^{\frac{1}{2}},$$

$$I_2(\tau) = \int_0^\tau \frac{d\sigma}{(\tau-\sigma)^{-\frac{\delta}{2}+\alpha+1}} \left(\int_0^\sigma d\rho(\sigma-\rho)^{-\delta} \|z_M(.,\rho)\|_2^2\right)^{\frac{1}{2}},$$

$$I_3(\tau) = \int_0^\tau \frac{d\sigma}{(\tau-\sigma)^{\alpha+1}} \int_\sigma^\tau d\rho \left(\frac{1}{(\rho-\sigma)^\alpha} + \frac{1}{(\tau-\rho)^\alpha}\right) \|z_M(.,\rho)\|_2,$$

$$I_4(\tau) = \int_0^\tau \frac{d\sigma}{(\tau-\sigma)^{\alpha+1}} \left(\int_\sigma^\tau d\rho \int_\sigma^\rho d\xi \frac{\|z_M(.,\rho) - z_M(.,\xi)\|_2}{(\rho-\xi)^{\alpha+1}}\right),$$

$$I_5(\tau) = \int_0^\tau \frac{d\sigma}{(\tau-\sigma)^{-\frac{\delta}{2}+\alpha+1}} \int_0^\sigma \frac{d\rho}{(\sigma-\rho)^{\frac{\delta}{2}}} \left(\frac{1}{\rho^\alpha} + \frac{1}{(\sigma-\rho)^\alpha}\right) \|z_M(.,\rho)\|_2,$$

$$I_6(\tau) = \int_0^\tau \frac{d\sigma}{(\tau-\sigma)^{-\frac{\delta}{2}+\alpha+1}} \int_0^\sigma \frac{d\rho}{(\sigma-\rho)^{\frac{\delta}{2}}} \int_0^\rho d\xi \frac{\|z_M(.,\rho) - z_M(.,\xi)\|_2}{(\rho-\xi)^{\alpha+1}}.$$

Set $T_k(t) = \int_0^t d\tau\ I_k^2(\tau)$, $k = 1, \cdots, 6$. The function $\sigma \mapsto (\tau-\sigma)^{-\frac{1}{2}-\alpha}$ is integrable on $(0,\tau)$ for $\alpha \in (0, \frac{1}{2})$. Thus we have

$$T_1(t) \le c \int_0^t d\tau \|z_M(.,\tau)\|_2^2. \tag{3.60}$$

Moreover we have $\delta > 2\alpha$, so that $\sigma \mapsto (\tau-\sigma)^{-\alpha-1+\frac{\delta}{2}}$ is integrable on $(0,\tau)$. By applying Schwarz inequality with respect to the measure given by $(\tau-\sigma)^{-\alpha-1+\frac{\delta}{2}} d\sigma$, and then Fubini's theorem, we obtain

$$\begin{aligned} T_2(t) &\le c \int_0^t d\tau\ \tau^{\frac{\delta}{2}-\alpha} \int_0^\tau d\sigma\ (\tau-\sigma)^{-\alpha-1+\frac{\delta}{2}} \int_0^\sigma d\rho\ (\sigma-\rho)^{-\delta} \|z_M(.,\rho)\|_2^2 \\ &\le c \int_0^t d\rho\ \|z_M(.,\rho)\|_2^2 \int_\rho^t d\tau\ \tau^{\frac{\delta}{2}-\alpha} \int_\rho^\tau d\sigma (\tau-\sigma)^{-\alpha-1+\frac{\delta}{2}} (\sigma-\rho)^{-\delta} \\ &\le c \int_0^t d\rho\ \|z_M(.,\rho)\|_2^2. \end{aligned} \tag{3.61}$$

where in the last inequality we have used the definition of Euler's Beta function.

A trivial integration yields

$$\int_\sigma^\tau d\rho \left(\frac{1}{(\rho-\sigma)^\alpha} + \frac{1}{(\tau-\rho)^\alpha}\right) = \frac{2(\tau-\sigma)^{1-\alpha}}{1-\alpha}.$$

Moreover, the function $\sigma \mapsto (\tau-\sigma)^{-2\alpha}$ is integrable on $(0,\tau)$. Consequently,

$$\begin{aligned}
T_3(t) &\le c\int_0^t d\tau \left(\sup_{\rho\in[0,\tau]} \|z_M(.,\rho)\|_2^2\right)\int_0^\tau d\sigma\ (\tau-\sigma)^{-2\alpha} \\
&\le c\int_0^t d\tau \left(\sup_{\rho\in[0,\tau]} \|z_M(.,\rho)\|_2^2\right). \qquad (3.62)
\end{aligned}$$

For any $\tau\in(0,t)$, set

$$I_\tau = \int_0^\tau d\sigma\ (\tau-\sigma)^{-\alpha-1+\frac{\delta}{2}}\left(\int_0^\sigma d\rho\ (\sigma-\rho)^{-\frac{\delta}{2}}\left(\frac{1}{\rho^\alpha}+\frac{1}{(\sigma-\rho)^\alpha}\right)\right).$$

It is a simple exercise to check that for $\alpha+\frac{\delta}{2}<1$, $\sup_{\tau\in[0,t]} I_\tau<+\infty$. Since

$$T_5(t)\le \int_0^t d\tau\ I_\tau^2 \left(\sup_{\rho\in[0,\tau]} \|z_M(.,\rho)\|_2^2\right),$$

we conclude that

$$T_5(t)\le c\int_0^t d\tau \left(\sup_{\rho\in[0,\tau]} \|z_M(.,\rho)\|_2^2\right). \qquad (3.63)$$

Fix $\eta\in(0,1)$ so that $\sigma\mapsto(\tau-\sigma)^{-\eta}$ is integrable on $(0,\tau)$. Applying Schwarz inequality first with respect to the measure $d\sigma(\tau-\sigma)^{-\eta}$, and then with respect to the Lebesgue measure on the interval (σ,τ) yields

$$\begin{aligned}
T_4(t) &= \int_0^t d\tau\Big(\int_0^\tau \frac{d\sigma}{(\tau-\sigma)^\eta}(\tau-\sigma)^{-\alpha-1+\eta} \\
&\quad\times\Big(\int_\sigma^\tau d\rho\int_\sigma^\rho d\xi\frac{\|z_M(.,\rho)-z_M(.,\xi)\|_2}{(\rho-\xi)^{\alpha+1}}\Big)\Big)^2 \\
&\le c\int_0^t d\tau\ \int_0^\tau \frac{d\sigma}{(\tau-\sigma)^\eta}(\tau-\sigma)^{-2\alpha-2+2\eta} \\
&\quad\times\left(\int_\sigma^\tau d\rho\int_\sigma^\rho d\xi\frac{\|z_M(.,\rho)-z_M(.,\xi)\|_2}{(\rho-\xi)^{\alpha+1}}\right)^2 \\
&\le c\int_0^t d\tau\ \int_0^\tau d\sigma\ (\tau-\sigma)^{\eta-2\alpha-1} \\
&\quad\times\int_\sigma^\tau d\rho\left(\int_\sigma^\rho d\xi\frac{\|z_M(.,\rho)-z_M(.,\xi)\|_2}{(\rho-\xi)^{\alpha+1}}\right)^2.
\end{aligned}$$

By choosing $\eta > 2\alpha$, the function $\sigma \mapsto (\tau-\sigma)^{\eta-2\alpha-1}$ is integrable on $(0,\tau)$. Thus, from the preceding inequalities we obtain

$$\begin{aligned} T_4(t) &\le c\int_0^t d\tau \ \int_0^\tau d\rho\left(\int_0^\rho d\xi \frac{\|z_M(.,\rho)-z_M(.,\xi)\|_2}{(\rho-\xi)^{\alpha+1}}\right)^2 \\ &\le c\int_0^t d\tau \ \|z_M\|^2_{\alpha,2,\tau} \ . \end{aligned} \tag{3.64}$$

By Fubini's theorem and evaluations based upon Euler's Beta function, we have

$$\begin{aligned} T_6(t) &= \int_0^t d\tau\Big(\int_0^\tau d\rho\Big(\int_\rho^\tau d\sigma(\tau-\sigma)^{\frac{\delta}{2}-\alpha-1}(\sigma-\rho^{-\frac{\delta}{2}})\Big) \\ &\quad\times\int_0^\rho d\xi\frac{\|z_M(.,\rho)-z_M(.,\xi)\|_2}{(\rho-\xi)^{\alpha+1}}\Big)^2 \\ &\le c\int_0^t d\tau\left(\int_0^\tau d\rho\left(\int_0^\rho d\xi\frac{\|z_M(.,\rho)-z_M(.,\xi)\|_2}{(\rho-\xi)^{\alpha+1}}\right)^2\right) \\ &\le c\int_0^t d\tau \ \|z_M\|^2_{\alpha,2,\tau}. \end{aligned} \tag{3.65}$$

Finally, inequalities (3.59) to (3.65) imply

$$\|z_M\|^2_{\alpha,2,t} \le R\int_0^t d\tau \ \|z_M\|^2_{\alpha,2,\tau} \tag{3.66}$$

a.s. By Gronwall's lemma, this clearly implies the uniqueness of the mild solution. Now the proof of Theorem 2.3 is complete. ■

We conclude this article by proving (2.9). For this it is sufficient to prove that the inequality

$$\|C(u_M)(.,t)-C(u_M)(.,s)\|_2 \le R|t-s|^{\theta''''}\left(1+\|u_M\|_{\alpha,2,T}\right) \tag{3.67}$$

holds a.s. for all $s,t\in[0,T]$ and every $\theta''''\in\left(0,\frac{2}{d+2}\wedge\frac{1}{2}\right)$, since then the result follows from Propositions 3.4, 3.5 and the fact that $\beta\in(0,1)$. In our case the factorization method we alluded to in Section 2 rests upon the possibility of expressing $C(u_M)(.,t)$ in terms of the auxiliary $L^2(D)$-valued process

$$Y_\varepsilon(u_M)(.,t) := \sum_{i=1}^{+\infty}\lambda_i^{\frac{1}{2}}\int_0^t (t-\tau)^{-\varepsilon} f_{i,t}(u_M)(.,\tau)B_i^H(d\tau)$$

defined for every $\varepsilon \in \left(0, \frac{1}{2}\right)$. In fact, by repeated applications of Fubini's theorem and by using the fundamental property $U(t,\tau)U(\tau,\sigma) = U(t,\sigma)$ for the evolution operators defined in (3.21) we obtain

$$\begin{aligned} C(u_M)(.,t) &= \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t f_{i,t}(u_M)(.,\tau) B_i^H(d\tau) \\ &= \frac{\sin(\varepsilon\pi)}{\pi} \int_0^t d\tau (t-\tau)^{\varepsilon-1} \int_D dy G(.,t;y,\tau) Y_\varepsilon(u_M)(y,\tau) \end{aligned} \tag{3.68}$$

for every $t \in [0,T]$ a.s. We then proceed by estimating the time increments of $C(u_M)$ by means of (3.68) rather than with the expressions of Proposition 3.6. We first notice that the inequality

$$\sup_{t\in[0,T]} \|Y_\varepsilon(u_M)(.,t)\|_2 \le R\left(1 + \|u_M\|_{\alpha,2,T}\right) \tag{3.69}$$

holds a.s. as a consequence of (3.13), (3.14) and estimates similar to those of the proof of Lemma 3.2.

We can now follow the arguments of the proof of (66) in Proposition 6 of [48] to see that, by using (3.68), (3.69), and by choosing $\theta'''' \in \left(0, \frac{2}{d+2} \wedge \frac{1}{2}\right)$ with the additional restriction $\varepsilon \in \left(\theta'''', \frac{2}{d+2} \wedge \frac{1}{2}\right)$, we obtain

$$\begin{aligned} &\|C(u_M)(.,t) - C(u_M)(.,s)\|_2 \\ &\le c\left(\left\| \int_s^t d\tau (t-\tau)^{\varepsilon-1} \int_D dy G(.,t;y,\tau) Y_\varepsilon(u_M)(y,\tau)\right\|_2 \right. \\ &+ \left.\left\| \int_0^s d\tau \int_D dy \left((t-\tau)^{\varepsilon-1} G(.,t;y,\tau) - (s-\tau)^{\varepsilon-1} G(.,s;y,\tau)\right) Y_\varepsilon(u_M)(y,\tau)\right\|_2\right) \\ &\le R\left(|t-s|^\varepsilon + |t-s|^{\theta''''}\right)\left(1 + \|u_M\|_{\alpha,2,T}\right) \le r|t-s|^{\theta''''}\left(1 + \|u_M\|_{\alpha,2,T}\right). \end{aligned}$$

This ends the proof of (2.9). ■

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