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Calibration of Perturbative Black Scholes model with Variance Swaps

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Abstract

We analyze a possible way to calibrate the perturbed Black Scholes model using Variance Swaps. We assume that the market prices of Variance Swaps are coherent, since these are securities very exchanged on the market, we show how calibrate the variance and the bias of volatility parameter using the bid-ask spread on variance swaps and the implied volatility curve.

Key Words: Calibration, Variance Swap, stochastic volatility, error, Dirichlet form, carré du champ operator, bias.

1 Introduction

In this article, we study the calibration of perturbed Black Scholes model, using Variance Swaps securities.

A Variance Swap is a financial security whose payoff is equal to the difference between the realized variance over a span of time and a fixed quantity, known as the variance strike, chosen in order to cancel the derivative premium.

The Perturbed Black Scholes model (PBS), Scotti [11], is a new model based on Black Scholes model (see Black and Scholes [1]), characterized by an uncertain volatility parameter, this uncertainty is treated thanks an error structure, a procedure introduced by Bouleau [2]. We suppose that the uncertainty on volatility parameter is small. The PBS is a stochastic volatility model with closed forms for option pricing, that permits to reproduce a smile on implied volatility and generate automatically a bid-ask spread.

The paper is organized as follows:

In section 2, we resume, shortly, the elements of Perturbed Black Scholes model. In section 3, we analyze the Variance Swap securities. In section 4, we explain how to calibrate the PBS model in accord with Variance Swap options, their bid-ask spreads and at-the-money implied volatility. Finally section 5 resumes and concludes.

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2 Preliminaries

In this paper, we search to calibrate the PBS model, that is based on a recent technique developed by Bouleau [2]. To make the paper self-contained, we resume the classical notations on probability and we make a survey of the key ideas of error theory using Dirichlet forms.

2.1 Notation

In this article we use the following notation:

- $(\Omega,\, \mathcal{F},\, \mathbb{P})$ is the historical probability space, for the sake of brevity denoted with Ω.
- T denotes a fixed positive number.
- $\{\mathcal{F}_t\}_{0\leq t\leq T}$ a filtration of the probability space.
- $\{B_t\}_{0\leq t\leq T}$ is an associated brownian motion, i.e. a brownian motion adapted to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$.
- $\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}}\right)$ is another probability space, used to represent the uncertainty on the volatility parameter, for the sake of brevity denoted with $\widetilde{\Omega}$.
- $\mathbb{E}[\ ]$ and $\mathbb{E}[\ |\mathcal{F}_t]$ denote, respectively, the expectation and the conditional expectation under probability $\mathbb{P}$, while $\widetilde{\mathbb{E}}[\ ]$ denotes the expectation under the probability $\widetilde{\mathbb{P}}$.
- $(P_t)_{t\geq 0}$ denotes a strongly continuous contraction semi-group, $\mathcal{A}$ its generator, with domain $\mathcal{DA}$, and Γ the "carré du champ" operator associated with the Dirichlet form of the semi-group, with domain $\mathbb{D}$.

2.2 Error Theory using Dirichlet Forms

We often work with models characterized by a few parameters, that permit some freedom to reproduce the real data. We must estimate these parameters, the classical way is to fit these on market data thanks to a statistics. However, the result of this fit is not a single value, but a mean value plus an uncertainty, a variance with probability words.

What is the impact of this uncertainty? Generally, we use only the mean value and we forget the variance, since it is small compared with the mean, but we apply no-linear functions, i.e. functions that distort the moments of a random variable. What is the bias yields by the forgetfulness of the probabilistic nature of an estimated parameter?

The answer to this question is not easy, if we consider a no-linear function F and a random variable σ, the expected value of $F(\sigma)$ is different to the function F evaluated at the expected value of σ, but the computation is no very simple, besides the law computation of the random variable $F(\sigma)$ is often unmanageable. This is the reason why the probabilistic nature of uncertainty on a parameter estimation, already known by Gauss, is frequently forgotten.

But, if the main problem is unassailable, we can take advantage from a scale effect, i.e. the variance of an estimated parameter is very small compared with its mean, this fact justify a Taylor expansion, if we study the bias and the variance of $F(\sigma)$ we find:

$$\begin{aligned} \mathbb{E}\left[F(\sigma)-F(\sigma_0)\right] &= \epsilon\left\{F'(\sigma_0)\ A[\sigma]+\tfrac{1}{2}\,F''(\sigma_0)\ \Gamma[\sigma]\right\}+o(\epsilon) \\ \mathbb{E}\left[\left(F(\sigma)-F(\sigma_0)\right)^2\right] &= \epsilon\ \left(F'(\sigma_0)\right)^2\ \Gamma[\sigma]+o(\epsilon) \end{aligned} \tag{2.1}$$

where σ_0 is the estimated value of random variable σ, $\epsilon A[\sigma]$ is its bias, i.e. the difference between σ_0 and $\mathbb{E}[\sigma]$, and $\epsilon\Gamma[\sigma]$ is its variance.

Remark 2.1 *The key difference between a deterministic uncertainty and a probabilistic one is the first relation, the mean of a function evaluated on a random variable is shifted owing to the non-linearity.*

The starting idea of error theory using Dirichlet forms is to assume ϵ very small and to stop the Taylor expansion at the first order, besides to search a theory with two operators, i.e. a bias operator $\mathcal{A}$ and a variance-covariance one Γ, with the chain rule given by equation (2.1); this theory exist and it is the theory of semi-group, the operator $\mathcal{A}$ is the generator of the semi-group and Γ is the "carré du champ" associated with the Dirichlet form of the semi-group.

The main definition of the error theory is the error structure:

Definition 2.1 (Error structure) *An error structure is a term*

$$\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}},\, \mathbb{D},\, \Gamma\right)$$

where

1. $\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}}\right)$ *is a probability space;*

2. $\mathbb{D}$ *is a dense sub-vector space of* $L^2\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}}\right)$*;*

3. Γ *is a positive symmetric bilinear application from* $\mathbb{D} \times \mathbb{D}$ *into* $L^1\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}}\right)$ *satisfying the functional calculus of class* $\mathcal{C}^1 \cap Lip$*, i.e. if* F *and* G *are of class* $\mathcal{C}^1$ *and Lipschitzian,* u *and* $v \in \mathbb{D}$*, we have* $F(u)$ *and* $G(v) \in \mathbb{D}$ *and*

$$\Gamma\left[F(u),\, G(v)\right] = F'(u)G'(v)\Gamma[u,\, v] \quad \widetilde{\mathbb{P}}\ a.s.;$$

4. *the bilinear form* $\mathcal{E}[u,\, v] = \frac{1}{2}\widetilde{\mathbb{E}}\left[\Gamma[u,\, v]\right]$ *is closed;*

5. *The constant function 1 belongs to* $\mathbb{D}$*, i.e. the error structure is Markovian.*

In fact hypotheses 2, 3 and 4 provide that $\mathcal{E}$ is a Dirichlet form, with Γ as carré du champ operator.

When the operator Γ acts two times on the same argument, we use the simplified custom $\Gamma[u] = \Gamma[u,\, u]$. The couple $(\Gamma, \widetilde{\mathbb{P}})$ defines a unique semigroup $(P_t)_{t\geq 0}$ and its generator $\mathcal{A}$ thanks to Hille-Yosida theorem, see Fukushima [8] for a complete proof. The hypothesis 5 assure that the semi-group $(P_t)_{t\geq 0}$ is Markovian.

Therefore we have defined two operators Γ and $\mathcal{A}$ that verify the chain rule (2.1). We finish this section with an useful example, i.e. the Ornstein-Uhlenbeck error structure:

Example 2.1 (Orstein-Uhlenbeck structure)

$$\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}},\, \mathbb{D},\, \Gamma\right) = \left(\mathbb{R},\, \mathcal{B}(\mathbb{R}),\, \mu,\, H^1(\mu), \Gamma[u,\, u] = \{u'\}^2\right)$$

where $\mathcal{B}(\mathbb{R})$ *is the Borel* σ*-field of* $\mathbb{R}$*,* μ *is a gaussian measure and* $H^1(\mu)$ *is the first Sobolev space with respect to the measure* μ*, i.e.* $u \in H^1(\mu)$ *if* $u \in L^2(\mu)$ *and* u' *in the distribution sense belongs to* $L^2(\mu)$*.*

The associate generator has the following domain:

$$\mathcal{DA} = \left\{ u \in L^2(\mu) : \, u'' - x\, f' \text{ in the distribution sense belongs to } L^2(\mu) \right\}$$

and the generator operator is

$$\mathcal{A}[u] = \frac{1}{2}u'' - \frac{1}{2}I \cdot u'$$

where I is the identity map on $\mathbb{R}$.

This example gives the basic idea of an error structure on a parameter, moreover is exists a characterization of all dirichlet forms on $\mathbb{R}$, see Hamza [9].

3 Perturbed Black Scholes model

To make the paper self-contained, we give a survey on the Perturbed Black Scholes model, for a complete analysis see Scotti [11].

The PBS model is based on the classical Black Scholes one without drift, see Black and Scholes [1], the underlying price follows the SDE

$$dS_t = \sigma_0 \, S_t \, dW_t \tag{3.1}$$

where σ_0 is called the volatility and W_t is a Brownian motion.

The BS model present many advantages, in particular the pricing only depends on volatility and we find closed forms for premium and greeks of vanilla options; unluckily the BS model cannot reproduce the market price of call options for all strikes at the same volatility, this effect is called smile.

The basic idea of PBS model is to consider a perturbation of this model by means of an error structure on volatility, in order to reproduce a volatility smile and a bid-ask spread. We make three hypotheses:

1. the real market follows a BS model with fixed and non perturbed volatility σ_0;
2. the trader has to estimate the volatility, so its volatility contains intrinsic inaccuracies, we model this ambiguity by means of an error structure; nonetheless we assume that the stock price S_t is not erroneous. We evaluate the impact of the perturbation, generated by the trader mishandling, on the profit and loss process used by trader to hedge the vanilla option;
3. the trader knows this perturbation and he wants to modify the option prices to take into account the bias induced by the perturbation on volatility.

It is clear that all traders use an "official" BS asset model in order to hedge vanilla options; they use the market price to determine the "fair" values of parameters by inversion of pricing formula. The traders find an observed volatility process ς_t, usually known as implied volatility and hedge their portfolio according to his volatility.

The profit and loss process of a trader has the key role in PBS model, the value of this at the maturity is given by:

$$P\&L = F(\varsigma_0, \, S_0, \, 0) + \int_0^T \frac{\partial F}{\partial x}(\varsigma_t, \, S_t, \, t) dS_t - \Phi(S_T) \tag{3.2}$$

where $F(\varsigma_0,\, S_0,\, 0)$ is the security premium, the integral term represents the hedging strategy, $\Phi(S_T)$ is the Payoff and S_t follows the Black Scholes SDE (3.1). We suppose by simplicity that the trader volatility ς_t is a time independent random variable σ and we define an error structure for the this volatility, therefore this volatility admits the following expansion, with the language of Dirichlet forms:

$$\sigma_0 \longrightarrow \sigma_0 + \epsilon\mathcal{A}[\sigma](\sigma_0) + \sqrt{\epsilon\Gamma[\sigma](\sigma_0)}\widetilde{\mathcal{N}}$$

where $\widetilde{\mathcal{N}}$ is a standard gaussian variable and we assume that this error structure admits a sharp operator.

We estimate the variance and bias error of $\mathbb{E}\left[P\&L\right]$. To perform the calculus, we assume that $\sigma = \sigma_0$ is the right value of the random variable in the sense that $\varsigma_t = \sigma_0$ and $P\&L(\sigma_0) = 0$ almost surely.

Then we can prove, see Scotti [11], that we have the following bias and variance:

$$\begin{aligned} \mathcal{A}[\mathbb{E}[P\&L]] &= \left\{\frac{\partial F}{\partial \sigma}(\sigma_0, x, 0)\mathcal{A}[\sigma](\sigma_0) + \frac{1}{2}\frac{\partial^2 F}{\partial \sigma^2}(\sigma_0, x, 0)\Gamma[\sigma](\sigma_0)\right\} \\ \Gamma\left[\mathbb{E}\left[P\&L\right]\right] &= \left\{\mathbb{E}\left[\frac{\partial F}{\partial \sigma}(\sigma_0, x, 0)\right]\right\}^2 \Gamma[\sigma](\sigma_0) \end{aligned} \tag{3.3}$$

The financial interpretation of this result is that the trader knows the presence of errors in his procedure and wants to neutralize this effect.

We associate:

- the variance of $P\&L$ process to the bid-ask spread of options;
- the bias of $P\&L$ process to a shift of prices of options asked by the trader to the buyer.

Indeed in the classical theory of financial mathematics we assume that all market securities have a single price, if we take into account uncertainty on volatility, we have found that the price of the contingent claim is not unique but we have many possible prices.

Therefore, the trader must modify his prices in order to take into account the two previous effects, namely the variance and the bias, then he fixes a supportable risk probability $\alpha < 0.5$ and accepts to buy the option at the price

$$\text{(Bid Premium)} = \text{(BS Premium)} + \epsilon\, \mathcal{A}\left[\mathbb{E}[P\&L]\right] + \sqrt{\epsilon\, \Gamma\left[\mathbb{E}[P\&L]\right]}\, \mathcal{N}_\alpha$$

where $\mathcal{N}_\alpha$ is the α-quantile of the reduced normal law. Likewise, the trader accepts to sell the option at the price

$$\text{(Ask Premium)} = \text{(BS Premium)} + \epsilon\, A\left[\mathbb{E}[P\&L]\right] + \sqrt{\epsilon\, \Gamma\left[\mathbb{E}[P\&L]\right]}\, \mathcal{N}_{1-\alpha}$$

Since $\mathcal{N}_\alpha + \mathcal{N}_{1-\alpha} = 0$; the mid-premium is

$$\text{(Mid Premium)} = \text{(BS Premium)} + \epsilon\, A\left[\mathbb{E}[P\&L]\right] \tag{3.4}$$

and the bid-ask spread is

$$\text{Bid-Ask spread } = 2\sqrt{\epsilon\, \Gamma\left[\mathbb{E}[P\&L]\right]}\, \mathcal{N}_\alpha \tag{3.5}$$

Now, we concentrate on vanilla options and we study the relative bias. We consider a call option with strike K and maturity T, so the payoff is $(S_T - K)^+$

We can prove, see [11], that the bias of this option is given by:

$$\mathcal{A}[C]|_{\sigma=\sigma_0} = S_0 \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} \left\{ \mathcal{A}\left[\sigma\sqrt{T}\right]|_{\sigma=\sigma_0} + \frac{d_1 d_2}{2\sigma_0\sqrt{T}}\Gamma\left[\sigma\sqrt{T}\right]|_{\sigma=\sigma_0} \right\}. \tag{3.6}$$

where $d_{1,2} = \frac{\ln S_0 - \ln K}{\sigma_0\sqrt{T}} \pm \sigma_0\sqrt{T}$.

This formula remains true when the volatility is a deterministic function of time t under the usual convention

$$\overline{\sigma}_0(T) = \sqrt{\frac{1}{T}\int_0^T \sigma_0^2(t)dt}$$

therefore $\overline{\sigma}_0(T)$ represent an average volatility between 0 and T.

4 Variance Swaps

In this section, we give a survey on Variance Swap securities, for more details see Neuberger [10] and Demeterfi et al. [7].

A Variance Swap is a forward contract on variance of a stock, called the underlying. Its payoff is equal to

$$N\left\{\int_t^T \sigma^2(\dots, s)ds - \mathbb{E}\left[\int_t^T \sigma^2(\dots, s)ds|\mathcal{F}_t\right]\right\} \tag{4.1}$$

where N is a nominal, t is the signature date of the swap, T is the expiration date, while σ is the spot volatility of the underlying estimated at the maturity by the approximation

$$\int_t^T \sigma^2(\dots, s)ds \simeq \sum_{n=1}^{N}\left[\frac{S_{n\frac{T-t}{N}} - S_{(n-1)\frac{T-t}{N}}}{S_{(n-1)\frac{T-t}{N}}}\right]^2 \tag{4.2}$$

where S_t is the underlying price at time t, and N is, generally, the number of days between t and T.

Now we must evaluate the term:

$$\mathbb{E}\left[\int_t^T \sigma^2(\dots, s)ds|\mathcal{F}_t\right]$$

We assume the following hypothesis true:

Hypothesis 4.1 (continuous path) *The underlying evolution is a semi-martingale with continuous path.*

Therefore, the underlying price S_t follows a stochastic differential equation

$$dS_t = S_t\,\mu(\dots, t)\,dt + S_t\,\sigma(\dots, t)\,dW_t \tag{4.3}$$

where W_t is a Brownian motion and $\mu(\dots, t)$ and $\sigma(\dots, t)$ are adapted functions, that can depend on the underlying price S_t and on the realization $\omega \in \Omega$. A consequence of hypothesis 4.1 is that we assume the stock pays no dividends.

By applying Ito's lemma at equation 4.3 we can find the following relation, see Derman [6].

$$\frac{1}{2}\int_t^T \sigma^2(\omega,\, s)\, ds = \int_t^T \frac{dS_s}{S_s} - \ln S_T + \ln S_t \tag{4.4}$$

The first integral can be hedge with a shares position continuously rebalanced to be worth one currency. The second term is a short position on a log-contract. For hedging reasons we want to replicate the log-contract with vanilla options, since these are more liquid. The following identity suggests the decomposition into a combination of out-of-the-money puts and calls and a forward

$$\begin{aligned} \ln S_t - \ln S_T &= -\frac{S_T - S_t}{S_t} && \text{forward contract} \\ &+ \int_0^{S_t} \frac{1}{K^2}\,(K - S_T)^+\, dK && \text{put options} \\ &+ \int_{S_t}^{\infty} \frac{1}{K^2}\,(S_T - K)^+\, dK && \text{call options} \end{aligned} \tag{4.5}$$

Therefore, we remark

Remark 4.1 (absence of model risk) *The price of a Variance Swap between time t and T is known as we know the prices of each vanilla option, e.g. the knowledge of the call price for all strike K is enough, since we have an hedging portfolio make up of forward contract and of a static position on call-put options. In consequence, the prices of Variance Swap have no model risk, i.e. the volatility micro-structure do not change the price of these securities.*

This remark is crucial, first of all, because this fact ensures a necessary condition to be verified when we search to calibrate all financial model, e.g. the perturbed Black Scholes model. Secondly the knowledge of this property has permitted the exchange development over these securities, that has provided an careful pricing, by a real balance between supply and demand, characterized by a tight bid-ask spread.

5 Calibration

In this section we present a procedure for calibrating the PBS model, in accord with Variance Swap securities. We start with an evaluation of the implicit bias of the Variance Swap premium.

The procedure of the calibration in based on three steps:

1. estimation of cumulated volatility $\overline{\sigma}_0(T)\,\sqrt{T}$;
2. estimation of variance of volatility $\Gamma\left[\overline{\sigma}(T)\,\sqrt{T}\right]$;
3. estimation of bias of volatility $\mathcal{A}\left[\overline{\sigma}(T)\,\sqrt{T}\right]$.

The method used to estimate the cumulated volatility and its variance is to replicate the market prices of Variance Swap, instead using the most exchanged vanilla option to fit the bias.

5.1 Variance Swap Constraint

In this subsection we evaluate the expected value of a Variance Swap security and its bid-ask spread according to PBS model.

We have the two following theorems.

Theorem 5.1 (Bias for Variance Swap Premium) *Under the PBS model, the bias of the premium of a Variance Swap security is equal to*

$$\mathcal{A}\left[\int_0^T \sigma^2(t)\,dt\right] = 2\,\overline{\sigma}_0(T)\,\sqrt{T}\,\mathcal{A}\left[\overline{\sigma}(T)\,\sqrt{T}\right] + 2\,\Gamma\left[\overline{\sigma}(T)\,\sqrt{T}\right] \tag{5.1}$$

where each operator is evaluate at $\overline{\sigma}(T) = \overline{\sigma}_0(T)$

Proof: Two proofs are possible, the first is a direct application of bias operator rules, see equation (2.1). The second proof is more financial, we start with the identities (4.4) and (4.5), the bias is a linear operator and is worth zero if its object is the underlying.

$$\mathcal{A}\left[\int_0^T \sigma^2(t)\,dt\right] = \int_0^{S_0} \frac{1}{K^2}\,\mathcal{A}\left[(K - S_T)^+\right]\,dK + \int_{S_0}^{\infty} \frac{1}{K^2}\,\mathcal{A}\left[(S_T - K)^+\right]\,dK$$

The call-put parity shows that the relation for the bias of a call, i.e. the relation (3.6), is true for the put too. So we have to compute

$$\mathcal{A}\left[\int_0^T \sigma^2(t)\,dt\right] = \int_0^{\infty} \frac{S_0}{K^2}\,\frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}}\left\{\mathcal{A}\left[\overline{\sigma}(T)\,\sqrt{T}\right] + \frac{d_1\,d_2}{2\,\overline{\sigma}_0(T)\,\sqrt{T}}\Gamma\left[\overline{\sigma}(T)\,\sqrt{T}\right]\right\}\,dK$$

We make the change of variable $y = \ln K$, we integrate and we find the result 5.1.

$\square$

Theorem 5.2 (Variance for Variance Swap Premium) *Under the PBS model, the variance of the premium of a Variance Swap security is equal to*

$$\Gamma\left[\int_0^T \sigma^2(t)\,dt\right] = 4\,\overline{\sigma}_0^2(T)\,T\,\Gamma\left[\overline{\sigma}(T)\,\sqrt{T}\right] \tag{5.2}$$

where each operator is evaluate at $\overline{\sigma}(T) = \overline{\sigma}_0(T)$

Proof: the proof follows the same idea of the previous theorem, the proof using equation (3.4) requires the employment of sharp operator, see Bouleau [3] or Scotti [11], but the computation follows the same plan of the previous theorem.

$\square$

Remark 5.1 (Variance Swap premium) *Relation (3.4) and theorem 5.1 define the price of Variance Swap, it is given by*

$$\mathbb{E}\left[\int_0^T \sigma^2(t)\,dt\right] = \overline{\sigma}_0^2(T)\,T + 2\,\epsilon\,\overline{\sigma}_0(T)\,\sqrt{T}\,\mathcal{A}\left[\overline{\sigma}(T)\,\sqrt{T}\right] + 2\,\epsilon\,\Gamma\left[\overline{\sigma}(T)\,\sqrt{T}\right] \tag{5.3}$$

Remark 5.2 (Bid-Ask spread of Variance Swap) *Relation (3.5) and theorem 5.2 define the bid-ask spread of Variance Swap, it is given by*

$$\textit{Bid-Ask spread } = 2\sqrt{4\,\epsilon\,\overline{\sigma}_0^2(T)\,T\,\Gamma\left[\overline{\sigma}(T)\,\sqrt{T}\right]}\,\mathcal{N}_\alpha \tag{5.4}$$

A fundamental hypothesis of perturbative approach is that the parameter ϵ is small compared with 1, then a starting estimation of cumulated volatility is the square of mid-premium of Variance Swap.

Relation (5.4) gives us an estimation of $\Gamma\left[\overline{\sigma}(T)\,\sqrt{T}\right]$:

$$\epsilon\,\Gamma\left[\overline{\sigma}(T)\,\sqrt{T}\right] = \frac{(\text{Bid-Ask spread })^2}{16\,\overline{\sigma}_0^2(T)\,T\,(\mathcal{N}_\alpha)^2}$$

where the bid-ask spread can be estimate on the market, the cumulated variance is approximated by the Variance Swap security premium and the only unknown parameter is the quantile α; it is well-known that if we fix $\alpha = 32\%$ (respectively $\alpha = 1\%$) we have $\mathcal{N}_\alpha \simeq 1$ (respectively $\mathcal{N}_\alpha \simeq 3$), therefore we have an estimation of $\epsilon\,\Gamma\left[\overline{\sigma}(T)\,\sqrt{T}\right]$ with a precision of one order of magnitude.

5.2 Bias estimation using ATM volatility

In this subsection we search to estimate the volatility bias of PBS model. In the previous subsection we have find a rough estimation for volatility parameter σ_0 and for its variance $\epsilon\,\Gamma\left[\overline{\sigma}(T)\,\sqrt{T}\right]$. The last parameter in PBS model is the bias of the volatility; we use the At-the-money vanilla options to estimate it.

Remark 5.3 *It is a market evidence that the most exchanged vanilla options have a strike around the forward money.*

We decide to use these securities for the calibration of PBS model. Equations (3.4) and (3.6) give us the expected value of a call in PBS model:

$$\begin{aligned} &\text{PBS Price }\left(S_0,\,K,\,T,\,\overline{\sigma}_0(T),\,\Gamma\left[\overline{\sigma}(T)\,\sqrt{T}\right],\,\mathcal{A}\left[\overline{\sigma}(T)\,\sqrt{T}\right]\right) = \\ &\text{BS Price }\left(S_0,\,K,\,T,\,\overline{\sigma}_0(T)\right) + \epsilon\,S_0\,\frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}}\left\{A\left[\overline{\sigma}(T)\,\sqrt{T}\right] + \frac{d_1 d_2}{2\overline{\sigma}_0\sqrt{T}}\Gamma\left[\overline{\sigma}(T)\,\sqrt{T}\right]\right\} \end{aligned} \tag{5.5}$$

where $d_{1,2} = \dfrac{\ln S_0 - \ln K}{\overline{\sigma}_0\,\sqrt{T}} \pm \overline{\sigma}_0\,\sqrt{T}$.

We choose a traded call with strike K, around the money; we force the PBS Price to be equal to the market price, the BS Price is given by the Black-Scholes formula, The bias $\epsilon\,\mathcal{A}\left[\overline{\sigma}(T)\,\sqrt{T}\right]$ stays the only unknown parameter and, thanks to the linearity, we have a closed form for the bias

$$\epsilon\,\mathcal{A}\left[\overline{\sigma}(T)\,\sqrt{T}\right] = \frac{\sqrt{2\pi}\,e^{\frac{1}{2}d_1^2}}{S_0}\,(\text{PBS Price } - \text{BS Price }) - \frac{d_1\,d_2}{2\overline{\sigma}_0(T)\sqrt{T}}\,\epsilon\,\Gamma\left[\overline{\sigma}(T)\,\sqrt{T}\right] \tag{5.6}$$

5.3 Calibration Procedure

In this subsection we resume the calibration procedure of PBS model using Variance Swap securities and we propose some improvements and remarks.

The basic procedure is the following one:

1. for each maturities T, compute, or find on the market, the value of Variance Swap over the same period, and set the value on cumulated volatility $\overline{\sigma}_0(T)\,\sqrt{T}$ to be equal to the square root of Variance Swap premium;
2. fix a supportable risk probability $\alpha < 0.5$;
3. evaluate the bid-ask spread and use the theorem 5.2 to fix the variance of cumulated volatility $\epsilon\,\Gamma\left[\overline{\sigma}(T)\,\sqrt{T}\right]$;
4. choose a really traded vanilla option with a strike around the money and compute the bias of cumulated volatility $\epsilon\,\mathcal{A}\left[\overline{\sigma}(T)\,\sqrt{T}\right]$, thanks to equation (5.6).

Remark 5.4 *This procedure is clearly a rough estimate and it is inconsistent, since the PBS price for Variance Swap securities depends on the bias, see equation (5.2). An other drawback concerns the choice of the At-The-Money option, the result depends deeply on this selection*

In order to avoid the inconsistence of the previous procedure of calibration, we propose to iterate the previous procedure until the estimated parameters verify the relation (3.4) with a set accuracy. At the end of a loop we hold the previous value for bias and variance of cumulated volatility and we define a new estimation of this one thanks to relation (3.4).

In order to strengthen the calibration, we propose to fix a basket of at-the-money vanilla options and to fit the bias of volatility over this basket rather than to choose a single option.

6 Conclusion

In this paper we propose a calibration procedure for Perturbative Black Scholes model, see Scotti [11] for a main presentation, based on the prices of Variance Swap securities. The Variance Swaps are a no-risk-model assets, so their premium represent implicit constraints to be verify by a calibration.

The article presents an iterative procedure that can be performed quickly thanks to the closed-forms relations of each step.

References

[1] Black, F.; Scholes M. (1973): *The Pricing of Options and Corporate Liabilities*, J. Political Econ. 81 page 637-659.

[2] Bouleau, N. (2003): *Error Calculus for Finance and Physics*, De Gruyter, Berlin.

[3] Bouleau, N. (2003): *Error Calculus and path sensivity in financial models* , Mathematical Finance, 13-1, page 115-134.

[4] Buehler, H. (2006): *Consistent Variance Curve Models* , Finance and Stochastic, 10, pages 178-203.

[5] Carr, P.; Geman, H.; Madan D.B. and Yor M. (2005) , Finance and Stochastic, 9, pages 453-475

[6] Derman, E.; Kamal, M.; Kani, I. and Zou, J. (1996) *Valuing Contracts with Payoffs based on Realized Volatility*, Global Derivatives Quarterly Review, Golman Sachs.

[7] Demeterfi, K.; Derman, E.; Kamal, M. and Zou, J. (1999) *More than you Ever Wanted to Know about Volatility Swaps*, Quantitative Strategies Research Notes, Golman Sachs.

[8] Fukushima, M.; Oshima, Y.; Takeda, M. (1994): *Dirichlet Forms and Markov Process*, De Gruyter, Berlin.

[9] Hamza, M. M. (1975): *Détermination des formes de Dirichlet sur* $\mathbb{R}^n$, PhD thesis, Université d'Orsay, Paris.

[10] Neuberger, A. (1994): *The Log-Contract: A new Instrument to Hedge Volatility*, J. of Portfolio Management, Winter, page 74-80.

[11] Scotti, S. (2007): *Perturbative Approach on Financial Markets*, submitted to Finance and Stochastics.