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An anticipating Itô's formula for Lévy processes

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Abstract

In this paper, we use the Malliavin calculus techniques to obtain an anticipative version of the change of variable formula for Lévy processes. Here the coefficients are in the domain of the anihilation (gradient) operator in the "future sense", which includes the family of all adapted and square-integrable processes. This domain was introduced on the Wiener space by Alòs and Nualart [2]. Therefore, our Itô's formula is not only an extension of the usual adapted formula for Lévy processes, but also an extension of the anticipative version on Wiener space obtained in [2].

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1 Introduction

It is well–known that the Itô's formula, or change of variable formula, is one of the most powerful tools of the stochastic analysis due to its vast range of applications. So, in the few last years, various researchers have studied extensions of the classical Itô's formula for different interpretations of stochastic integral (see, for instance, Alòs and Nualart [2], Di Nunno et al. [4], Moret and Nualart [11], Nualart and Taqqu [15], and Tudor and Viens [21]).

The Malliavin calculus or calculus of variations is also other important tool of the stochastic analysis that allows us to deal with stochastic integrals whose domains include processes that are not necessarily adapted to the underlying filtration. Currently, the interest of this calculus has increased considerably because of its applications in market models (see, for example, Alòs et al. [1], Bally et al. [3], Fournié et al. [5, 6] or Nualart [14]), theoretical applications (see Alòs and Nualart [2], León and Nualart [8], Nualart [13, 14] or Sanz-Solé [17]), among others. This important theory is basically based on the divergence and gradient operators.

The divergence operator has been interpreted as a stochastic integral because it has properties similar to those of the Itô's stochastic integral. For instance, the isometry and local properties, it can be approximated by Riemann sums, the integration by parts formula, etc. (see Nualart [14]). Hence it is important to count on a change of variable formula for the divergence operator in order to improve the applications of the Malliavin calculus to different areas of the human knowledge.

On the Wiener space, the divergence operator was defined by Skorohod [19] and it is an extension of the classical Itô's integral. In order to analyze the properties of the Skorohod integral, the adaptability of the integrands (necesary in the Itô's calculus) is changed by some analytic properties that are used to defined some spaces (called Sobolev spaces), where a fundamental ingredient is the derivative (gradient) operator (see Sections 2.3 and 2.4

below). For instance, Alós and Nualart [2] have considered processes with derivatives "in the future sense". The Skorohod integral can be introduced using different approaches. Namely, the first method is via the Wiener chaos decomposition, and the second one considers the Skorohod integral as the adjoint of the gradient (derivative) operator.

On the Poisson space, the above two methods produce different definitions of stochastic integral (see, for example León and Tudor [9], or Nualart and Vives [16]).

Recently, several approaches to develope a calculus of variations for Lévy processes have been introduced in some articles (see, for instance, Di Nunno et al. [4], Løkka [10] and Solé et al. [20], among others). By all means, it is not necessary to mention on the importance of Lévy processes, and therefore we do not need to say on the importance of a Malliavin calculus for Lévy processes. Now the gradient operator defined utilizing the chaotic decomposition of a square–integrable random variable is not a "derivative operator" (see Section 2.3 below), but it is the sum of a derivative and a difference operators.

The purpose of this paper is to use the Malliavin calculus on the canonical Lévy space given in [20] to prove an anticipating Itô's formula for Lévy processes. The coefficients in this formula have two "derivatives in the future sense". It means, they are in a class of square-integrable processes u such that u_t is in the domain of the gradient operator D at time r for $r > t$, and $D_r u_t$ is also in the domain of D (see Section 2.4). An example of processes satisfying this property is the square-integrable and adapted processes, whose "derivative" is equal to zero.

The paper is organized as follows. In Section 2 we present the framework that we use in this paper, Namely, we introduce some basic facts of the canonical Lévy space and of Malliavin calculus on this space. Finally, the anticipating Itô's formula is studied in Section 3.

2 Preliminaries

In this section we give the framework that will be used in this article. That is, we introduce briefly the canonical Lévy process considered by Solé, Utzet and Vives [20], and the Itô's multiple integrals with respect to it. Then we present some basic facts on the Malliavin calculus for this process. We need to study the anihilation and creation operators corresponding to the Fock space associated with the chaos decomposition on Lévy space, and analyze the Sobolev spaces associated with these operators. Althoug some of these facts are known, we give them for the convenience of the reader.

Throughout we set $\mathbb{R}_0 = \mathbb{R} - \{0\}$. Let ν be a Lévy measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} x^2 d\nu(x) < \infty$ (see Sato [18]), and $T > 0$. The Borel σ–algebra of a set $A \subset \mathbb{R}$ is denoted by $\mathcal{B}(A)$. The jump of a cádlág process Z at time $t \in [0,T]$ is represented by ΔZ_t (i.e., $\Delta Z_t := Z_t - Z_{t-}$).

2.1 Canonical Lévy space

The purpose of this subsection is to present some basic elements of the structure of the canonical Lévy space on the interval $[0,T]$. For a more detailed account of this subject, we refer to Solé, Utzet and Vives [20].

The construction of the canonical Lévy space is divided in three steps, as follows:

Step 1. Here we introduce the canonical space for a compound Poisson process. Toward this end, let Q be a probability measure on $\mathbb{R}$, supported on $S \in \mathcal{B}(\mathbb{R}_0)$, and $\lambda > 0$. Set

$$\Omega_T = \bigcup_{n \geq 0} ([0,T] \times S)^n,$$

with $([0,T] \times S)^0 = \{\alpha\}$, where α is an arbitrary point. The set Ω_T is equipped with the σ–algebra

$$\mathcal{F}_T = \{B \subset \Omega_T : B \cap ([0,T] \times S)^n \in \mathcal{B}(([0,T] \times S)^n), \quad \text{for all} \quad n \geq 1\}.$$

The probability P_T on $(\Omega_T, \mathcal{F}_T)$ is given by

$$P_T(B \cap ([0,T] \times S)^n) = e^{-\lambda T} \frac{\lambda^n (dt \otimes Q)^{\otimes n} (B \cap ([0,T] \times S)^n)}{n!},$$

with $(dt \otimes Q)^0 = \delta_\alpha$.

The space $(\Omega_T, \mathcal{F}_T, P_T)$ is called the canonical space for the compound Poisson process with Lévy measure λQ. A similar definition for the Poisson process was given in Neveu [12], and Nualart and Vives [16]. In $(\Omega_T, \mathcal{F}_T, P_T)$ the process

$$X_t(\omega) = \begin{cases} \sum_{j=1}^{n} x_j 1_{[0,t]}(t_j), & \text{if} \quad \omega = ((t_1, x_1), \cdots, (t_n, x_n)), \\ 0, & \text{if} \quad \omega = \alpha, \end{cases}$$

is a compound Poisson process with intensity λ and jump law given by the probability measure Q.

Step 2. Now we consider the canonical space for a pure jump Lévy process with Lévy measure ν.

Let $S_1 = \{x \in \mathbb{R} : \varepsilon_1 < |x|\}$ and $S_k = \{x \in \mathbb{R} : \varepsilon_k < |x| \leq \varepsilon_{k-1}\}$ for $k > 1$. Here $\{\varepsilon_k : k \geq 1\}$ is a strictly decreasing sequence of positive numbers such that $\varepsilon_1 = 1$, $\lim_{k\to\infty} \varepsilon_k = 0$ and $\nu(S_k) \neq 0$. Note that the fact that ν is a Lévy measure implies that $\nu(S_k) < \infty$ for every $k \geq 1$. Now, the canonical Lévy space with measure ν is defined as

$$(\Omega_J, \mathcal{F}_J, \mathcal{P}_J) = \bigotimes_{k\geq 1} (\Omega^{(k)}, \mathcal{F}^{(k)}, P^{(k)}),$$

where $(\Omega^{(k)}, \mathcal{F}^{(k)}, P^{(k)})$ is the canonical space for the canonical compound Poisson process $\{X_t^{(k)} : t \in [0,T]\}$ with intensity $\lambda_k = \nu(S_k)$ and probability measure $Q_k = \frac{\nu(\cdot \cap S_k)}{\nu(S_k)}$. In this case, for $\omega = (\omega^k)_{k\geq 1} \in \Omega_J$ and $t \in [0,T]$, the limit

$$J_t(\omega) = \lim_{n\to\infty} \sum_{k=2}^{n} (X_t^{(k)}(\omega^k) - t \int_{S_k} x d\nu(x)) + X_t^{(1)}(\omega^1)$$

exists with probability 1 and it is a pure jump Lévy process with Lévy measure ν.

Step 3. The canonical Lévy space on $[0,T]$ with Lévy measure ν is

$$(\Omega, \mathcal{F}, P) = (\Omega_W \otimes \Omega_J, \mathcal{F}_W \otimes \mathcal{F}_J, P_W \otimes P_J),$$

where $(\Omega_W, \mathcal{F}_W, P_W)$ is the canonical Wiener space. Here, for $\omega = (\omega', \omega'') \in \Omega_W \otimes \Omega_J$, the process

$$X_t(\omega) = \gamma t + \sigma \omega'(t) + J_t(\omega'') \tag{2.1}$$

is a Lévy process with triplet (γ, σ^2, ν). For this fact we refer to Sato [18].

2.2 Itô multiple integrals

The construction of multiple integrals with respect to Lévy processes is quite similar to that of multiple integrals with respect to the Brownian motion. The reader can consult Itô [7] for a complete survey on this topic.

Let $X = \{X_t : t \in [0,T]\}$ be a Lévy process with triplet (γ, σ^2, ν). It is well–known that X has the Lévy–Itô representation (see [18])

$$X_t = \gamma t + \sigma W_t + \int_{(0,t]\times\{|x|>1\}} x dJ(s,x) + \lim_{\varepsilon \downarrow 0} \int_{(0,t]\times\{\varepsilon<|x|\leq 1\}} x d\widetilde{J}(s,x).$$

Here the convergence is with probability 1, uniformly on $t \in [0,T]$, $W = \{W_t : t \in [0,T]\}$ is a standard Brownian motion,

$$J(B) = \#\{t : (t, \Delta X_t) \in B\}, \quad B \in \mathcal{B}([0,T]\times \mathbb{R}_0),$$

is a Poisson measure with parameter $dt \otimes d\nu$ and $d\widetilde{J}(t,x) = dJ(t,x) - dt d\nu(x)$.

For $E_1, \ldots, E_n \in \mathcal{B}([0,T]\times\mathbb{R})$ such that $E_i \cap E_j = \emptyset$, $i \neq j$, and

$$\mu(E_i) := \sigma^2 \int_{\{t\in\mathbb{R}_+:(t,0)\in E_i\}} dt + \int_{E_i - (E_i \cap [0,T]\times\{0\})} x^2 dt d\nu(x) < \infty,$$

we define the multiple integral $I_n(1_{E_1\times\cdots\times E_n})$ of order n with respect to M by

$$I_n(1_{E_1\times\cdots\times E_n}) = M(E_1)\cdots M(E_n), \tag{2.2}$$

with

$$M(E_i) = \sigma \int_{\{t\in\mathbb{R}_+:(t,0)\in E_i\}} dW_t + L^2(\Omega) - \lim_{n\to\infty} \int_{\{(t,x)\in E_i: \frac{1}{n}<|x|<n\}} x d\widetilde{J}(t,x).$$

The multiple integral I_n is extended to $L_n^2 := L^2(([0,T]\times\mathbb{R})^n;\ \mathcal{B}(([0,T]\times\mathbb{R})^n); \mu^{\otimes n})$ due to the property

$$\begin{aligned} & E[I_n(1_{E_1\times\cdots\times E_n}) I_m(1_{F_1\times\cdots\times F_m})] \\ = \ & \delta_n(m) n! \int_{([0,T]\times\mathbb{R})^n} \widetilde{1}_{E_1\times\cdots\times E_n} \widetilde{1}_{F_1\times\cdots\times F_m} d\mu^{\otimes n}, \end{aligned} \tag{2.3}$$

where $\widetilde{f}$ is the symmetrization of the function f.

It is well–known that if F is a square–integrable random variable, measurable with respect to the filtration generated by X, then F has the unique representation

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{2.4}$$

where $I_0(f_0) = f_0 = E(F)$ and f_n is a symmetric function in L_n^2.

2.3 The anihilation and creation operators

Henceforth we suppose that the underlying probability space $(\Omega, \mathcal{F}, P)$ is the canonical Lévy space with Lévy measure ν and that X is the Lévy process defined in (2.1).

We say that the square-integrable random variable F given by (2.4) belongs to the domain of the anihilation operator D ($F \in \mathbb{D}^{1,2}$ for short) if and only if

$$\sum_{n=1}^{\infty} nn! ||f_n||^2_{L^2_n} < \infty. \tag{2.5}$$

In this case we define the random field $DF = \{D_z F : z \in [0,T] \times \mathbb{R}\}$ as

$$D_z F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(z, \cdot)).$$

Note that (2.5) yields that the last series converges in $L^2(\Omega \times [0,T] \times \mathbb{R}; P \otimes \mu)$ by (2.3). D is a closed operator from $L^2(\Omega)$ into $L^2(\Omega \times [0,T] \times \mathbb{R}; P \otimes \mu)$, with dense domain. Similarly we can define the iterated derivative $D^n_{z_1,\ldots,z_n} = D_{z_1}, \cdots D_{z_n}$ and its domain $\mathbb{D}^{n,2}$.

The following result is due to Solé et al. [20] and it establishes how we can figure out the random field DF without using the chaos decomposition (2.4). In order to state it, we need the following notation.

Henceforth $W = \{W_t : t \in [0,T]\}$ is the canonical Wiener process and $\mathbb{D}^{1,2}_W(L^2(\Omega_J))$ denotes the family of $L^2(\Omega_J, \mathcal{F}_J, P_J)$–valued random variables that are in the domain of the derivative operator D^W with respect to W. The reader can consult Nualart [14] for the basic definitions and properties of this operator. For $w = (w', ((t_1, x_1), (t_2, x_2), \ldots)) \in \Omega$, $F \in L^2(\Omega)$ and $z = (t,x) \in (0,T] \times \mathbb{R}_0$; we define $w_z = (w', ((t,x), (t_1, x_1), \ldots))$ and

$$(\Psi_{t,x} F)(\omega) = \frac{F(\omega_z) - F(\omega)}{x}.$$

Lemma 2.1 *Let $F \in L^2(\Omega)$ be a random variable such that:*

i) $F \in \mathbb{D}^{1,2}_W(L^2(\Omega_J))$.

ii) $\Psi F \in L^2(\Omega \times [0,T] \times \mathbb{R}_0; P \otimes \mu)$.

Then $F \in \mathbb{D}^{1,2}$ and

$$D_{t,x} F = 1_{\{0\}}(x) \sigma^{-1} D^W_t F + 1_{\mathbb{R}_0}(x) \Psi_{t,x} F.$$

Proof. The proof of this result is an immediate consequence of [20] (Propositions 3.5 and 5.5). ∎

Now we establish an auxiliary tool needed for our results.

Lemma 2.2 *Let $F \in \mathbb{D}^{1,2}$. Then there exists a sequence $\{F_n : n \geq 1\}$ of the form*

$$F_n = \sum_{i=1}^{N} H_{i,n} Z_{i,n} \tag{2.6}$$

such that:

i) $H_{i,n}$ is a smooth functional in $L^2(\Omega_W)$ and $Z_{i,n} \in \mathbb{D}^{2,2} \cap L^\infty(\Omega_J)$ is such that $\Psi Z_{i,n} \in L^2(\Omega \times [0,T] \times \mathbb{R}_0; P \otimes \mu)$.

ii) F_n (resp. DF_n) converges to F (resp. DF) in $L^2(\Omega)$ (resp. $L^2(\Omega \times [0,T] \times \mathbb{R}; P \otimes \mu)$) as $n \to \infty$.

Proof. By the definitions of the space $\mathbb{D}^{1,2}$ and of the multiple integrals, it is enough to show the result for a multiple integral of the form (2.2). That is

$$F = M(E_1) \cdots M(E_n),$$

where $E_1, \cdots, E_n$ are pairwise disjoint borel subsets of $[0,T] \times \mathbb{R}$.

Let $\varphi \in C^\infty(\mathbb{R})$ be a function such that

$$\varphi(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases}$$

Set $\rho_k(x) = x\varphi(\frac{x}{k})$ and

$$F_k = \prod_{i=1}^{n} \left(\rho_k \Big(\int_{\{t:(t,0)\in E_i\}} \sigma dW_t \Big) + \rho_k \left(\lim_{m\to\infty} \int_{\{(t,x)\in E_i: \frac{1}{m} < |x| < m\}} x d\widetilde{J}(t,x) \right) \right).$$

Then $F_k \to F$ in $L^2(\Omega)$ as $k \to \infty$. Now the result follows from the facts that $|\rho_k(x)| \leq |x|$, there is a constant C independent of k such that $|\rho_k'(x)| + |\rho_k''(x)| \leq C$, and

$$\begin{aligned} &\Psi_{t,x}\Big(\rho_k \Big(\lim_{m\to\infty} \int_{\{(t,x)\in E_i: \frac{1}{m} < |x| < m\}} x d\widetilde{J}(t,x) \Big) \Big) \\ &= \frac{1}{x} \Big(\rho_k \Big(x 1_{E_i}(t,x) + \lim_{m\to\infty} \int_{\{(t,x)\in E_i: \frac{1}{m} < |x| < m\}} x d\widetilde{J}(t,x) \Big) \\ &\quad - \rho_k \Big(\lim_{m\to\infty} \int_{\{(t,x)\in E_i: \frac{1}{m} < |x| < m\}} x d\widetilde{J}(t,x) \Big) \Big), \end{aligned}$$

which is a consequence of [20] (Lemma 5.2) or Lemma 2.1. ■

An immediate consequence of the last two lemmas is the following:

Corollary 2.3 *Let F be a random variable in $L^2(\Omega)$. Then $F \in \mathbb{D}^{1,2}$ if and only if $F \in \mathbb{D}^{1,2}_W(L^2(\Omega_J))$ and $\Psi F \in L^2(\Omega \times [0,T] \times \mathbb{R}_0; P \otimes \mu)$.*

Proof. The proof follows from Lemmas 2.1 and 2.2, and from [20] (Proposition 4.8). ■

We will also need the following result.

Lemma 2.4 *Let $F \in \mathbb{D}^{1,2}$ be a bounded random variable. Then $(FG) \in \mathbb{D}^{1,2}$ for every G of the form (2.6).*

Proof. We first observe that $FG \in \mathbb{D}^{1,2}_W(L^2(\Omega_J))$ due to Corollary 2.3.

Finally, we have

$$\Psi_{t,x}(FG) = (\Psi_{t,x}F)G + F\Psi_{t,x}G + (F(\omega_{(t,x)}) - F)\Psi_{t,x}G.$$

Therefore $\Psi(FG) \in L^2(\Omega \times [0,T] \times \mathbb{R}_0; P \otimes \mu)$. Consequently the proof is complete by Lemma 2.1. ■

The creation operator δ is the adjoint of $D : \mathbb{D}^{1,2} \subset L^2(\Omega) \to L^2(\Omega \times [0,T] \times \mathbb{R}; P \otimes \mu)$. It means, u belongs to Dom δ if and only if $u \in L^2(\Omega \times [0,T] \times \mathbb{R}; P \otimes \mu)$ is such that there exists a square–integrable random variable $\delta(u)$ satisfying the duality relation

$$E\Big[\int_{[0,T]\times\mathbb{R}} u_z (D_z F) d\mu(z)\Big] = E(\delta(u)F), \quad \text{for every} \quad F \in \mathbb{D}^{1,2}. \qquad (2.7)$$

It is not difficult to see that this duality relation gives that if u has the chaos decomposition

$$u_z = \sum_{n=0}^{\infty} I_n(u_n(z,\cdot)), \quad z \in [0,T] \times \mathbb{R},$$

where $u_n \in L^2_{n+1}$ is a symmetric function in the last n variables, then $\delta(u)$ has the chaos decomposition

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{u}_n).$$

The creation operator of a process multiplied by a random variable can be calculated via the following two results.

Proposition 2.5 *Let F be a random variable as in Lemma 2.4 and $u \in \mathrm{Dom}\,\delta$ such that*

$$E\Big(\int_{[0,T]\times\mathbb{R}}(u_{t,x}(F+xD_{t,x}F))^2 d\mu(t,x)\Big) < \infty.$$

Then $(t,x) \mapsto u_{t,x}(F+xD_{t,x}F)$ belongs to Dom δ if and only if

$$\Big(F\delta(u) - \int_{[0,T]\times\mathbb{R}} u_{t,x}D_{t,x}F d\mu(t,x)\Big) \in L^2(\Omega).$$

In this case

$$\delta(u_{t,x}F + xu_{t,x}D_{t,x}F) = F\delta(u) - \int_{[0,T]\times\mathbb{R}} u_{t,x}D_{t,x}F d\mu(t,x).$$

Proof. Let G be a random variable as in the right-hand side of (2.6). Then Lemma 2.4 and its proof give

$$\begin{aligned}
&E[GF\delta(u)]\\
&= E\Big[\int_{[0,T]\times\mathbb{R}} u_{t,x}D_{t,x}(FG)d\mu(t,x)\Big]\\
&= E\Big[\sigma^2\int_0^T u_{t,0}D_{t,0}(FG)dt + \int_{[0,T]\times\mathbb{R}_0} u_{t,x}D_{t,x}(FG)d\mu(t,x)\Big]\\
&= E\Big[\sigma^2\int_0^T u_{t,0}(D_{t,0}F)Gdt + \sigma^2\int_0^T u_{t,0}FD_{t,0}Gdt\Big]\\
&\quad +E\Big[\int_{[0,T]\times\mathbb{R}_0} u_{t,x}((D_{t,x}F)G + FD_{t,x}G + x(D_{t,x}F)D_{t,x}G)d\mu(t,x)\Big]\\
&= E\Big[G\int_{[0,T]\times\mathbb{R}} u_{t,x}D_{t,x}Fd\mu(t,x)\Big]\\
&\quad +E\Big[\int_{[0,T]\times\mathbb{R}}(u_{t,x}F + u_{t,x}xD_{t,x}F)D_{t,x}Gd\mu(t,x)\Big].
\end{aligned}$$

Therefore the proof is complete by Lemma 2.2 and by the duality relation (2.7). ∎

The following result is an immediate consequence of the proof of Proposition 2.5.

Corollary 2.6 *Let u and F be as in Proposition 2.5. Moreover assume that $(t,x) \mapsto u_{t,x}xD_{t,x}F$ belongs to Dom δ. Then $Fu \in \mathrm{Dom}\,\delta$ if and only if*

$$F\delta(u) - \delta(u_{t,x}xD_{t,x}F) - \int_{[0,T]\times\mathbb{R}} u_{t,x}D_{t,x}Fd\mu(t,x) \tag{2.8}$$

is a square–integrable random variable. In this case $\delta(Fu)$ is equal to (2.8).

2.4 Sobolev spaces

In this subsection we proceed as in Alòs and Nualart [2] in order to define the spaces that contain the integrands in our Itô's formula.

Let $\mathcal{S}_T$ be the family of processes of the form $u_\cdot = \sum_{j=1}^n F_j h_j(\cdot)$, where F_j is a random variable of the form (2.6) and $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a bounded measurable function. Note that the fact that $\int_{\mathbb{R}} x^2 d\nu(x) < \infty$ implies that $h \in L^2([0,T] \times \mathbb{R}; \mu)$. Denote by $\mathbb{L}^{1,2,f}$ the closure of $\mathcal{S}_T$ with respect to the seminorm

$$||u||_{1,2,f}^2 = E\int_{[0,T]\times\mathbb{R}} u_z^2 d\mu(z) + E\int_{\Delta_1^T} (D_{s,x}u_{t,y})^2 d\mu(s,x)d\mu(t,y),$$

where

$$\Delta_1^T = \Big\{((s,x),(t,y)) \in ([0,T]\times\mathbb{R})^2 : s \geq t\Big\}.$$

A random field $u = \{u(s,y) : (s,y) \in [0,T] \times \mathbb{R}\}$ in $\mathbb{L}^{1,2,f}$ belongs to the space $\mathbb{L}_-^{1,2,f}$ if there is $D^-u \in L^2(\Omega \times [0,T] \times \mathbb{R}; P \otimes \mu)$ such that

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}} \sup_{(s-\frac{1}{n})\vee 0 < r \leq s, y \leq x \leq y+\frac{1}{n}} E(|D_{s,y}u(r,x) - D^-u(s,y)|^2) d\mu(s,y) = 0.$$

The next result will be a useful tool to state the Itô's formula for the operator δ. Remember that we are using the notation $\Delta X_s = X_s - X_{s-}$.

Lemma 2.7 *Let $u = \{u(s,x) : (s,x) \in [0,T] \times \mathbb{R}\}$ be a measurable random field and $\varepsilon_1 > \varepsilon > 0$ such that, for every $(s,y) \in [0,T] \times \{\varepsilon < |x| \leq \varepsilon_1\}$,*

i) There exists a constant $c > 0$ such that $|u(s,y)| < c$.

ii) For any sequences $\{s_n \in [0,s) : n \in \mathbb{N}\}$ and $\{y_n \in \{\varepsilon < |x| \leq \varepsilon_1\} : n \in \mathbb{N}\}$ such that converge to s and y, respectively, we have that the limit

$$u(s-,y) = \lim_{n,m\to\infty} u(s_n, y_m)$$

is well–defined.

iii) $u(\cdot-,\cdot) \in \mathbb{L}_-^{1,2,f}$.

Then

$$\begin{aligned}
&\sum_{0<s\leq t} u(s-,\Delta X_s)\Delta X_s 1_{\{\varepsilon<|\Delta X_s|\leq\varepsilon_1\}} \\
&\quad = \delta((u(s-,y)+yD^-u(s-,y))1_{\{\varepsilon<|y|\leq\varepsilon_1\}}1_{[0,t]}(s)) \\
&\qquad + \int_0^t \int_{\{\varepsilon<|y|\leq\varepsilon_1\}} u(s-,y)yd\nu(y)ds \\
&\qquad + \int_0^t \int_{\{\varepsilon<|y|\leq\varepsilon_1\}} D^-u(s-,y)d\mu(s,y), \quad t\in[0,T].
\end{aligned}$$

Proof. The definition of the space $\mathbb{L}^{1,2,f}$ implies that there exists a sequence $\{u^{(m)}\in\mathcal{S}_T : m\in\mathbb{N}\}$ such that

$$\begin{aligned}
&E\Big[(u(t-,y)-u^{(m)}(t,y))^2 \\
&\qquad + \int_t^T \int_{\mathbb{R}} (D_{s,x}(u(t-,y)-u^{(m)}(t,y)))^2 d\mu(s,x)\Big] \to 0, \qquad (2.9)
\end{aligned}$$

as $m\to\infty$, for μ-a.a. $(t,y)\in[0,T]\times\mathbb{R}$. Hence we can find a sequence $\mathcal{A}_n=\{(s_i^{(n)},y_j^{(n)}) : i,j\in\{1,\ldots,N\}\}$ such that:

- $0\leq s_1^{(n)}<\cdots<s_N^{(n)}\leq T,\ -\varepsilon_1\leq y_1^{(n)}<y_2^{(n)}<\cdots<y_N^{(n)}\leq\varepsilon_1.$
- $0=\lim_{n\to\infty} s_1^{(n)}, T=\lim s_N^{(n)}, -\varepsilon_1=\lim_{n\to\infty} y_1^{(n)}$ and $\varepsilon_1=\lim_{n\to\infty} y_N^{(n)}\cdot$
- $\max_i(s_{i+1}^{(n)}-s_i^{(n)})\to 0$ and $\max_i(y_{i+1}^{(n)}-y_i^{(n)})\to 0$ as $n\to\infty$.
- Property (2.9) holds when we write $(s_i^{(n)},y_{j+1}^{(n)})$ instead of (t,y).

Thus, from the duality relation (2.7), Proposition 2.5, (2.9) and [20] (Theorem 6.1), we obtain

$$\begin{aligned}
&\sum_{i,j=1}^{N-1} u(s_i^{(n)}-,y_{j+1}^{(n)}) \int_{]s_i^{(n)},s_{i+1}^{(n)}]} \int_{y_j^{(n)}}^{y_{j+1}^{(n)}} y1_{\{\varepsilon<|y|\leq\varepsilon_1\}}1_{[0,t]}(s)d\widetilde{J}(s,y) \\
&\quad = \sum_{i,j=1}^{N-1} u(s_i^{(n)}-,y_{j+1}^{(n)})\delta\left(1_{\{\varepsilon<|y|\leq\varepsilon_1\}}1_{]s_i^{(n)},s_{i+1}^{(n)}]}(s)1_{]y_j^{(n)},y_{j+1}^{(n)}]}(y)1_{[0,t]}(s)\right) \\
&\quad = \sum_{i,j=1}^{N-1}\Big\{\delta\Big(1_{[0,t]}(s)1_{\{\varepsilon<|y|\leq\varepsilon_1\}}1_{]s_i^{(n)},s_{i+1}^{(n)}]}(s)1_{]y_j^{(n)},y_{j+1}^{(n)}]}(y)
\end{aligned}$$

$$\times (u(s_i^{(n)}-, y_{j+1}^{(n)}) + yD_{s,y}u(s_i^{(n)}-, y_{j+1}^{(n)}))\Big)$$
$$+ \int_{s_i^{(n)}}^{s_{i+1}^{(n)}} \int_{y_j^{(n)}}^{y_{j+1}^{(n)}} 1_{[0,t]}(s) 1_{\{\varepsilon < |y| \le \varepsilon_1\}} D_{s,y} u(s_i^{(n)}-, y_{j+1}^{(n)}) d\mu(s,y) \Big\}.$$

Indeed, by Proposition 2.5 we have that last equality holds when we change $u(s_i^{(n)}-, y_{j+1}^{(n)})$ by $u^{(m)}(s_i^{(n)}, y_{j+1}^{(n)})$. Consequently, we prove that our claim is true using (2.7) with a random variable as in the right-hand side of (2.6) and letting m goes to ∞. So, we can conclude the proof because of the dominated convergence theorem, the hypotheses of this lemma and the fact that δ is a closed operator. ■

The space $\mathbb{L}_F$ is the closure of $\mathcal{S}_T$ with respect to the norm

$$||u||_F^2 = ||u||_{1,2,f}^2 + E \int_{\Delta_2^T} (D_{r,x} D_{s,y} u_{t,z})^2 d\mu(r,x) d\mu(s,y) d\mu(t,z),$$

with $\Delta_2^T = \{((r,x),(s,y),(t,z)) \in ([0,T] \times \mathbb{R})^3 : r \vee s \ge t\}$.

The following result was stated on the Wiener space by Alòs and Nualart [2].

Lemma 2.8 *Let $u \in \mathbb{L}_F$. Then $u \in \mathrm{Dom}\ \delta$ and*

$$E[\delta(u)^2] \le 2||u||_F^2. \tag{2.10}$$

Proof. We first observe that it is enough to show that (2.10) is true for $u \in \mathcal{S}_T$ because δ is a closed operator. In this case, we have by [20] (Section 6) or by [2],

$$\begin{aligned} E(\delta(u)^2) &= E \int_0^T \int_{\mathbb{R}} u(t,x)^2 d\mu(t,x) \\ &\quad + E \int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} (D_y u(z)) D_z u(y) d\mu(z) d\mu(y). \end{aligned} \tag{2.11}$$

Observe that

$$\begin{aligned} &E\Big(\int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} (D_{s,y} u(t,x)) D_{t,x} u(s,y) d\mu(t,x) d\mu(s,y)\Big) \\ &= 2E\Big(\int_0^T \int_{\mathbb{R}} u(s,y) \delta(1_{[0,s]} D_{s,y} u) d\mu(s,y)\Big) \\ &\le E\Big(\int_0^T \int_{\mathbb{R}} u(s,y)^2 d\mu(s,y)\Big) + E\Big(\int_0^T \int_{\mathbb{R}} [\delta(1_{[0,s]} D_{s,y} u)]^2 d\mu(s,y)\Big) \end{aligned}$$

$$
\begin{aligned}
&\leq E\Big(\int_0^T\int_{\mathbb{R}} u(s,y)^2 d\mu(s,y)\Big)\\
&\quad +E\Big(\int_0^T\int_{\mathbb{R}}\int_0^s\int_{\mathbb{R}}(D_{s,y}u(t,x))^2 d\mu(t,x)d\mu(s,y)\Big)\\
&\quad +E\Big(\int_0^T\int_{\mathbb{R}}\int_{([0,s]\times\mathbb{R})^2} D_{t,x}D_{s,y}u(r,z)d\mu(r,z)d\mu(t,x)d\mu(s,y)\Big).
\end{aligned}
$$

Thus (2.11) yields that (2.10) holds. ■

Inequality (2.10) allows us to consider Lemma 2.7 with $\varepsilon = 0$ or $\varepsilon_1 = \infty$ to obtain the relation between the pathwise integral and the operator δ.

Corollary 2.9 *Let u satisfy the hypotheses of Lemma 2.7 for each $\varepsilon, \varepsilon_1 \in (a,b)$, with $0 \leq a$ and $b \leq \infty$. Moreover assume that the random fields $(s,y) \mapsto u(s-,y), yD^-u(s-,y)$ belong to $\mathbb{L}_F$ and $(s,y) \mapsto u(s-,y)y$ is pathwise integrable with respect to $\widetilde{J}$ on $[0,T]\times\{a<|y|<b\}$. Then*

$$
\begin{aligned}
&\int_{]0,t]}\int_{\{a<|y|<b\}} u(s-,y)yd\widetilde{J}(s,y)\\
&\quad = \delta\Big((u(s-,y)+yD^-u(s-,y))1_{[0,t]}(s)1_{\{a<|y|<b\}}(y)\Big)\\
&\qquad + \int_0^t\int_{\{a<|y|<b\}} D^-u(s-,y)d\mu(s,y), \quad t\in[0,T].
\end{aligned}
$$

Proof. The result is an immediate consequence of Lemmas 2.7 and 2.8. ■

3 The Itô's formula

Here we assume that, for $i \in \{1,\ldots,n\}$,

$$
\begin{aligned}
Y_t^{(i)} &= X_0^{(i)} + \int_0^t u_i(s)dW_s + \int_0^t \sigma_s^{(i)}ds + \int_{]0,t]}\int_{\{|x|>1\}} v_{i1}(s-,x)xdJ(s,x)\\
&\quad + \int_{]0,t]}\int_{\{0<|x|\leq 1\}} v_{i2}(s-,x)xd\tilde{J}(s,x), \quad t\in[0,T].
\end{aligned}
$$

The stochastic integrals with respect to W and J are in the Skorohod and pathwise sense, respectively, and

(H1) $X_0^{(i)} \in \mathbb{D}^{1,2}$.

(H2) $u_i \in \mathbb{L}_F$ is such that $\{\int_0^t u_i(s)dW_s : t\in[0,T]\}$ has continuous paths and there is a constant $M>0$ such that $\int_0^T u_i(s)^2 ds \leq M$ with probability 1.

(H3) $\sigma^{(i)} \in \mathbb{L}^{1,2,f}$ and $\int_0^T (\sigma_s^{(i)})^2 ds \leq M$ with probability 1, for some positive constant M.

(H4) v_{i1} satisfies the assumptions of Corollary 2.9 for $a = 1$ and $b = \infty$. Moreover assume that there is a positive constant M such that $|v_{i1}| < M$ for $(s, y) \in [0, T] \times \{1 < |y| < \infty\}$.

(H5) The hypotheses of Corollary 2.9 hold for v_{i2} with $a = 0$ and $b = 1$, and there is a positive constant M such that $|v_{i2}(s-, y)| \leq M$, for $(s, y) \in [0, T] \times \{0 \leq |y| \leq 1\}$. Moreover assume that $D^- v_{i2} \in \mathbb{L}^{1,2,f}$.

We observe that by Lemma 2.8 and Corollary 2.9, we have that

$$\int_{]0,t]} \int_{\{0<|x|\leq 1\}} v_{i2}(s-, x) x d\widetilde{J}(s, x)$$

belongs to $L^2(\Omega)$, for all $t \in [0, T]$. Also observe that in [2] (Theorem 1) we can find sufficient conditions that guarantee the continuity of the stochastic integral $\{\int_0^t u_i(s) dW_s : t \in [0, T]\}$.

To show our Itô's formula, we first need to assume that our Lévy process defined in (2.1) has no small side jumps. So, for $\varepsilon > 0$, we need to use the notation

$$\begin{aligned} Y_t^{(i),\varepsilon} &= X_0^{(i)} + \int_0^t u_i(s) dW_s + \int_0^t \sigma_s^{(i)} ds + \int_{]0,t]} \int_{\{|x|>1\}} v_{i1}(s-, x) x dJ(s, x) \\ &\quad + \int_{]0,t]} \int_{\{\varepsilon<|x|\leq 1\}} v_{i2}(s-, x) x d\widetilde{J}(s, x), \quad t \in [0, T]. \end{aligned} \tag{3.1}$$

The i–th jump time of the compound Poisson process $\{\int_{]0,t]} \int_{\{\varepsilon<|x|\}} x dJ(s, x) : t \in [0, T]\}$ is denoted by T_i^ε.

Theorem 3.1 *Assume that (H1)–(H5) hold, for $i \in \{1, \ldots, n\}$, and that $F \in C_b^2(\mathbb{R}^n)$. Then, the processes*

$$\Big(\partial_i F(Y_s)(u_i(s) 1_{\{y=0\}} + v_{i2}(s-, y) 1_{\{0<|y|\leq 1\}}) \\ + y 1_{\{0<|y|<1\}} D^-(v_{i2} \partial_i F(Y))(s, y)\Big) 1_{[0,t]}(s)$$

belong to Dom δ *and*

$F(Y_t) - F(X_0)$

$$
\begin{aligned}
= \;& \delta\Big(\Big[\partial_i F(Y_s)(u_i(s)1_{\{y=0\}} + v_{i2}(s-,y)1_{\{0<|y|\le 1\}}) \\
& +y1_{\{0<|y|\le 1\}} D^-(v_{i2}\partial_i F(Y))(s,y)\Big] 1_{[0,t]}(s)\Big) \\
& +\frac{1}{2}\int_0^t \partial_i\partial_j F(Y_s)u_i(s)u_j(s)ds + \int_0^t \partial_i F(Y_s)\sigma_s^{(i)}ds \\
& + \int_0^t \partial_i\partial_j F(Y_s)(D^- Y^{(j)})(s,0)u_i(s)ds \\
& + \int_0^t \int_{\{0<|y|\le 1\}} D^-(\partial_i F(Y)v_{i2})(s,y)d\mu(s,y) \\
& + \sum_{0\le s\le t} \{F(Y_{s-}+\Delta Y_s) - F(Y_{s-}) - \partial_i F(Y_s)v_{i2}(s-,\Delta X_s)\Delta X_s\}1_{\{0<|\Delta X_s|\le 1\}} \\
& + \sum_{0\le s\le t} (F(Y_{s-}+\Delta Y_s) - F(Y_{s-}))1_{\{1<|\Delta X_s|\}}, \quad t\in[0,T].
\end{aligned}
$$

Here we use the convention of summation over repead indixes.

Proof. We first observe that the process $Y^{(i),\varepsilon}$ given by (3.1) evolves as

$$
Y_t^{(i),\varepsilon} = Y_{T_j^\varepsilon}^{(i),\varepsilon} + \int_{T_j^\varepsilon}^t u_i(s)dW_s + \int_{T_j^\varepsilon}^t \sigma_s^{(i)}ds - \int_{]T_j^\varepsilon,t]}\int_{\{\varepsilon<|x|\le 1\}} v_{i2}(s-,x)x\nu(dx)ds
$$

on the stochastic interval $]T_j^\varepsilon, T_{j+1}^\varepsilon[$. Consequently, proceeding as in [2] and using that W and J are independent, and Corollary 2.9, we have that $1_{[0,t]}\partial_i F(Y)u_i$ belongs to Dom δ^W, for $i\in\{1,\dots,n\}$ and

$$
\begin{aligned}
F(Y_t^\varepsilon) \;=\;& F(X_0) + \int_0^t \partial_i F(Y_s^\varepsilon)u_i(s)dW_s + \int_0^t \partial_i F(Y_s^\varepsilon)\sigma_s^{(i)}ds \\
& - \int_0^t \partial_i F(Y_s^\varepsilon)\int_{\{\varepsilon<|x|\le 1\}} v_{i2}(s-,x)xd\nu(x)ds \\
& +\frac{1}{2}\int_0^t \partial_i\partial_j F(Y_s^\varepsilon)u_i(s)u_j(s)ds \\
& + \int_0^t \partial_i\partial_j F(Y_s^\varepsilon)(D^- Y^{(j),\varepsilon})(s,0)u_i(s)ds \\
& + \sum_{0\le s\le t}(F(Y_{s-}^\varepsilon + \Delta Y_s^\varepsilon) - F(Y_{s-}^\varepsilon)), \quad t\in[0,T], \qquad (3.2)
\end{aligned}
$$

with

$$
(D^- Y^{(j),\varepsilon})(s,0) \;=\; D_{s,0}X_0^{(j)} + \int_0^s D_{s,0}u_j(r)dW_r + \int_0^s D_{s,0}\sigma_r^{(j)}dr
$$

$$
\begin{aligned}
&+\delta(D_{s,0}(v_{j2}(r-,y)+yD^-v_{j2}(r-,y))1_{\{\varepsilon<|y|\le 1\}}1_{[0,s]}(r))\\
&+\delta(D_{s,0}(v_{j1}(r-,y)+yD^-v_{j1}(r-,y))1_{\{1<|y|\}}1_{[0,s]}(r))\\
&+\int_0^s\int_{\{\varepsilon<|y|\le 1\}} D_{s,0}(D^-v_{j2}(r-,y))d\mu(r,y)\\
&+\int_0^s\int_{\{1<|y|\}} D_{s,0}(D^-v_{j1}(r-,y))d\mu(r,y)\\
&+\int_0^s\int_{\{1<|y|\}} yD_{s,0}v_{j1}(r-,y)d\nu(y)dr. \qquad (3.3)
\end{aligned}
$$

Now we divide the proof in several steps.

Step 1. Here we see that $Y_t^{(i),\varepsilon}\to Y_t^{(i)}$ in $L^2(\Omega)$ as $\varepsilon\downarrow 0$, for every $t\in[0,T]$.

It follows, from (3.1) and Lemma 2.7,

$$
\begin{aligned}
&Y_t^{(i),\varepsilon}\\
&= X_0^{(i)}+\int_0^t u_i(s)dW_s+\int_0^t \sigma_s^{(i)}ds\\
&\quad+\delta\Big((v_{i1}(s-,y)+yD^-v_{i1}(s-,y))1_{\{1<|y|\}}1_{[0,t]}(s)\Big)\\
&\quad+\delta\Big((v_{i2}(s-,y)+yD^-v_{i2}(s-,y))1_{\{\varepsilon<|y|\le 1\}}1_{[0,t]}(s)\Big)\\
&\quad+\int_0^t\int_{\{\varepsilon<|y|\le 1\}} D^-v_{i2}(s-,y)d\mu(s,y)+\int_0^t\int_{\{1<|y|\}} v_{i1}(s-,y)yd\nu(y)ds\\
&\quad+\int_0^t\int_{\{1<|y|\}} D^-v_{i1}(s-,y)d\mu(s,y). \qquad (3.4)
\end{aligned}
$$

Thus it follows our claim by Corollary 2.9.

Step 2. Now we show that $\partial_i F(Y_\cdot^\varepsilon)v_{i2}(\cdot-,\cdot)$ is in $\mathbb{L}_-^{1,2,f}$.

We first observe that (2.10), (3.4) and [20] (Section 6) yield $Y^{(i),\varepsilon}\in\mathbb{L}_-^{1,2,f}$, $i\in\{1,\dots,n\}$. Therefore, it is clear that

$$
\begin{aligned}
D^-(\partial_i F(Y^\varepsilon)v_{i2}(\cdot-,\cdot))(s,0) &= \partial_i\partial_j F(Y_s^\varepsilon)v_{i2}(s-,0)D^-Y^{(j),\varepsilon}(s,0)\\
&\quad+(\partial_i F(Y_s^\varepsilon)D^-v_{i2}(s-,0). \qquad (3.5)
\end{aligned}
$$

On the other hand, the definition of the operator Ψ leads to write, for $r>t$,

$$
\begin{aligned}
&\Psi_{r,x}(\partial_i F(Y_t^\varepsilon)v_{i2}(t-,y))\\
&\quad= (\Psi_{r,x}\partial_i F(Y_t^\varepsilon))v_{i2}(t-,y)+\partial_i F(Y_t^\varepsilon)\Psi_{r,x}v_{i2}(t-,y)\\
&\qquad+x(\Psi_{r,x}v_{i,2}(t-,y))\Psi_{r,x}\partial_i F(Y_t^\varepsilon)
\end{aligned}
$$

$$
\begin{aligned}
&= v_{i2}(t-,y)\frac{\partial_i F(Y_t^\varepsilon + xD_{r,x}Y_t^\varepsilon) - \partial_i F(Y_t^\varepsilon)}{x} + \partial_i F(Y_t^\varepsilon)D_{r,x}v_{i2}(t-,y) \\
&\quad +(\partial_i F(Y_t^\varepsilon + xD_{r,x}Y_t^\varepsilon) - \partial_i F(Y_t^\varepsilon))D_{r,x}v_{i2}(t-,y),
\end{aligned}
$$

which, together with (3.5) and Corollary 2.3, gives that $\partial_i F(Y^\varepsilon)v_{i2} \in \mathbb{L}_-^{1,2,f}$.

Step 3. From Step 2, Lemma 2.7 and (3.2), we get

$$
\begin{aligned}
&F(Y_t^\varepsilon) \\
&= F(X_0) + \int_0^t \partial_i F(Y_s^\varepsilon)u_i(s)dW_s + \int_0^t \partial_i F(Y_s^\varepsilon)\sigma_s^{(i)}ds \\
&\quad +\delta\left((\partial_i F(Y_s^\varepsilon)v_{i2}(s-,y)\right. \\
&\qquad \left. +y(D^-\partial_i F(Y^\varepsilon)v_{i2})(s,y))1_{\{\varepsilon<|y|\le 1\}}1_{[0,t]}(s)\right) \\
&\quad +\int_0^t\int_{\{\varepsilon<|y|\le 1\}} D^-(\partial_i F(Y_s^\varepsilon)v_{i2})(s,y)d\mu(s,y) \\
&\quad +\frac{1}{2}\int_0^t \partial_i\partial_j F(Y_s^\varepsilon)u_i(s)u_j(s)ds \\
&\quad +\int_0^t \partial_i\partial_j F(Y_s^\varepsilon)(D^-Y^{(j),\varepsilon})(s,0)u_i(s)ds \\
&\quad +\sum_{0\le s\le t}(F(Y_{s-}^\varepsilon + \Delta Y_s^\varepsilon) - F(Y_{s-}^\varepsilon) - \partial_i F(Y_s^\varepsilon) \\
&\qquad \times v_{i2}(s-,\Delta X_s)\Delta X_s)1_{\{\varepsilon<|\Delta X_s|\le 1\}} \\
&\quad +\sum_{0\le s\le t}(F(Y_{s-}^\varepsilon + \Delta Y_s^\varepsilon) - F(Y_{s-}^\varepsilon))1_{\{1<|\Delta X_s|\}}.
\end{aligned} \tag{3.6}
$$

Step 4. Now we analyze the convergence in $L^2(\Omega)$ of the terms in (3.6).

$$
\begin{aligned}
&E\Big(|\sum_{0\le s\le t}(F(Y_{s-}^\varepsilon + \Delta Y_s^\varepsilon) - F(Y_{s-}^\varepsilon))1_{\{1<|\Delta X_s|\}}|^2\Big) \\
&= E|\sum_{0\le s\le t}(F(Y_{s-} + \Delta Y_s) - F(Y_{s-}))1_{\{1<|\Delta X_s|\}}|^2 \\
&\le CE\Big(\sum_{i=1}^n\sum_{0\le s\le t}|v_{i1}(s-,\Delta X_s)\Delta X_s|1_{\{1<|\Delta X_s|\}})^2 \\
&\le n^2CE\Big(\sum_{0\le s\le t}|\Delta X_s|1_{\{1<|\Delta X_s|\}}\Big)^2 \\
&\le CE\Big(\int_{]0,t]}\int_{\{|x|>1\}}|x|d\widetilde{J}(s,x) + \int_{]0,t]}\int_{\{|x|>1\}}|x|d\nu(x)ds\Big)^2
\end{aligned}
$$

$$
\begin{aligned}
&\le C \int_{]0,t]} \int_{\{|x|>1\}} x^2 d\nu(x) ds + \Big(\int_{]0,t]} \int_{\{|x|>1\}} x d\nu(x) ds \Big)^2 \\
&\le C \int_{]0,t]} \int_{\mathbb{R}_0} x^2 d\nu(x) ds < \infty.
\end{aligned}
$$

Also

$$
\begin{aligned}
E&\Big(\sum_{0\le s\le t} (F(Y_{s-} + \Delta Y_s) - F(Y_{s-}) - \partial_i F(Y_s) v_{i2}(s, \Delta X_s)\Delta X_s) 1_{\{0<|\Delta X_s|\le\varepsilon\}} \Big)^2 \\
&\le E\Big(\sum_{i=1}^n \sum_{0\le s\le t} |\Delta Y_s^{(i)}|^2 1_{\{0<|\Delta X_s|\le\varepsilon\}} \Big)^2 \\
&= E\Big(\sum_{i=1}^n \sum_{0\le s\le t} |v_{i2}(s-, \Delta X_s)\Delta X_s|^2 1_{\{0<|\Delta X_s|\le\varepsilon\}} \Big)^2 \\
&\le CE\Big(\sum_{0\le s\le t} |\Delta X_s|^2 1_{\{0<|\Delta X_s|\le\varepsilon\}} \Big)^2 \\
&\le CE\Big(\int_{]0,t]} \int_{\{0<|x|\le\varepsilon\}} x^2 d\widetilde{J}(s,x) \Big)^2 + C\Big(\int_{]0,t]} \int_{\{0<|x|\le\varepsilon\}} x^2 d\nu(x) ds \Big)^2 \\
&\le C \int_{]0,t]} \int_{\{0<|x|\le\varepsilon\}} x^2 d\nu(x) ds \to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}
$$

It is not difficult to deduce, from Step 1,

$$
E\left(\int_0^t |\partial_i F(Y_s) u_i(s) - \partial_i F(Y_s^\varepsilon) u_i(s)|^2 ds \right) \to 0
$$

and, from Step 2,

$$
\begin{aligned}
E&\Big(\int_{]0,t]} \int_{\{0<|y|\le 1\}} \Big| \Big(\partial_i F(Y_s^\varepsilon) v_{i2}(s-, y) + yD^-(\partial_i F(Y^\varepsilon) v_{i2})(s,y) \Big) \\
&\quad \times 1_{\{\varepsilon<|y|\le 1\}} - \partial_i F(Y_s) v_{i2}(s-, y) + yD^-(\partial_i F(Y) v_{i2}) \Big|^2 d\mu(s,y) \Big) \\
&\quad \to 0 \quad \text{as} \quad \varepsilon \downarrow 0.
\end{aligned}
$$

The missing terms can be analyzed similarly.

Step 5. Finally the result follows from the fact that δ is a closed operator and from Steps 1-4. ∎

Theorem 3.2 *Assume that* $\int_{\mathbb{R}_0}|x|d\nu(x)<\infty$*. Then the hypotheses of Theorem 3.1 imply that*

$$\begin{aligned} F(Y_t) &= F(X_0)+\int_0^t \partial_i F(Y_s)u_i(s)dW_s+\int_0^t \partial_i F(Y_s)\sigma_s^{(i)}ds \\ &\quad -\int_0^t \partial_i F(Y_s)\int_{\{0<|x|\le 1\}} v_{i2}(s-,x)xd\nu(x)ds \\ &\quad +\frac{1}{2}\int_0^t \partial_i\partial_j F(Y_s)u_i(s)u_j(s)ds \\ &\quad +\int_0^t \partial_i\partial_j F(Y_s)(D^-Y^{(j)})(s,0)u_i(s)ds \\ &\quad +\sum_{0\le s\le t}(F(Y_{s-}+\Delta Y_s)-F(Y_{s-})),\quad t\in[0,T]. \end{aligned}$$

Proof. The fact that $\int_{\mathbb{R}_0}|x|d\nu(x)<\infty$ yields

$$E\Big(\int_0^t\int_{\{0<|x|\le 1\}}|v_{i2}(s-,x)x|d\nu(x)ds\Big)^2\le C\Big(\int_0^t\int_{\{0<|x|\le 1\}}|x|d\nu(x)ds\Big),$$

which implies

$$\begin{aligned} &E\Big(\int_0^t \partial_i F(Y_s^\varepsilon)\int_{\{\varepsilon<|x|\le 1\}} v_{i2}(s-,x)xd\nu(x)ds \\ &\qquad -\int_0^t \partial_i F(Y_s)\int_{\{0<|x|\le 1\}} v_{i2}(s-,x)xd\nu(x)ds\Big)^2\to 0. \end{aligned}$$

Also we have

$$\begin{aligned} &E\Big(\sum_{0\le s\le t}(F(Y_{s-}+\Delta Y_s)-F(Y_{s-}))1_{\{0<|\Delta X_s|\le\varepsilon\}}\Big)^2 \\ &\le\ CE\Big(\sum_{i=1}^n\sum_{0\le s\le t}|v_{i2}(s-,\Delta X_s)\Delta X_s|1_{\{0<|\Delta X_s|\le\varepsilon\}}\Big)^2 \\ &\le\ CE\Big(\sum_{0\le s\le t}|\Delta X_s|1_{\{0<|\Delta X_s|\le\varepsilon\}}\Big)^2 \\ &\le\ CE\Big(\int_{]0,t]}\int_{\{0<|x|\le\varepsilon\}}|x|d\tilde{J}(s,x)+\int_{]0,t]}\int_{\{0|<|x|\le\varepsilon\}}|x|d\nu(x)ds\Big)^2 \\ &\le\ C\int_{]0,t]}\int_{\{0<|x|\le\varepsilon\}}x^2d\nu(x)ds+C\Big(\int_{]0,t]}\int_{\{0<|x|\le\varepsilon\}}|x|d\nu(x)ds\Big)^2. \end{aligned}$$

Thus the result is a consequence of the proof of Theorem 3.1. ■

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