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# On the distributions of the sup and inf of the classical risk process with exponential claim

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# ON THE DISTRIBUTIONS OF THE SUP AND INF OF THE CLASSICAL RISK PROCESS WITH EXPONENTIAL CLAIM*

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ABSTRACT. The purpose of this article is to use the double Laplace transform of the occupation measure of the classical risk process $X$ with exponential claim to deduce the distributions of the random variables $\sup\{X_s : s \leq t\}$ and $\inf\{X_s : s \leq t\}$, for every $t > 0$. As a consequence, we also get the distributions of the time to ruin in finite time and the first passage of a given level.

## 1. INTRODUCTION

In this paper, we deal with the classical risk process with exponential claim defined on a complete probability space $(\Omega, \mathcal{F}, P)$. More precisely, let

$$X_t = x_0 + ct - \sum_{k=1}^{N_t} R_k, \quad t \geq 0. \tag{1.1}$$

Here $x_0 \geq 0$ is the initial capital, $c > 0$ is the premium income per unit of time, $N = \{N_t, t \geq 0\}$ is an homogeneous Poisson process with rate $\lambda$ and $\{R_k,\ k = 1, 2, \ldots\}$ is a sequence of i.i.d. random variables independent of $N$. Henceforth we suppose that $R_1$ has exponential distribution with mean $1/r$.

Our goal is to calculate explicit expressions for the distributions of the random variables $\sup\{X_s : s \leq t\}$ and $\inf\{X_s : s \leq t\}$, $t > 0$. Toward this end, we apply the complex inversion theorem of the Laplace transform (or Lerch's theorem) to the double Laplace transforms of some occupation measures of $X$ (see Section 2 below). As a consequence, we are also able to give the distribution of the first passage of certain level $x \in \mathbb{R}$ of the process $X$. It means, the distributions of

$$S_x = \inf\{t > 0 : X_t = x\} \quad \text{and} \quad T_x = \inf\{t > 0 : X_t < x\},$$

because the right-continuity of the process $X$ yields

$$\{S_x < t\} = \{\sup_{s \leq t} X_s > x\}, \quad x > x_0$$

and

$$\{T_x \leq t\} = \{\inf_{s \leq t} X_s < x\}, \quad x < x_0.$$

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The study of the distribution of the time to ruin $T_0$ in finite time (i.e., for the case $x = 0$) is extensive in the literature on risk theory due to its applications in business activities. For instance, numerical procedures have been utilized by several authors in the analysis of $T_0$ (see, for example, Dickson and Waters [6] or Seal [12]). This numerical approximations have been improved in several works by deriving expressions for the mentioned distribution (see Asmussen [1, 2], Dickson [4], Dickson et al. [5, 7], Drekic and Willmot [9], and Ignatov and Kaishev [11], among others). In particular, the method used in [5], [7] and [9] is based on the complex inversion theorem, as we does here.

For $x \in \mathbb{R}$, the process $S_x$ have been analyzed by doss Reis [8] and Gerber [10] via a martingale method.

The paper is organized as follows. In Section 2 we relate the Laplace transforms of the functions

$$P\{\sup_{s\leq \cdot} X_s \leq x\} \quad \text{and} \quad P\{\inf_{s\leq \cdot} X_s \leq x\} \tag{1.2}$$

to the double Laplace transform of some occupation measures of $X$. Then, in Section 3 we use the complex inversion theorem to calculate the two probabilities in (1.2).

## 2. Occupation measures

Now we are interested in the Laplace transforms of the occupation measures

$$Y_x(t) = \int_0^t 1_{(x,+\infty)}(X_s)ds \quad \text{and} \quad Y^x(t) = \int_0^t 1_{(-\infty,x)}(X_s)ds,$$

with $x \in \mathbb{R}$ and $t > 0$. So, we assume that the reader is familiar with the elementary properties of the Laplace transform as they are presented, for example, in Spiegel [13].

Throughout, the Laplace transform of a measurable function $h : [0,\infty) \to \mathbb{R}$ is denoted by $\mathcal{L}(h)$. That is,

$$\mathcal{L}(h)(s) = \int_0^\infty e^{-st}h(t)dt,$$

for $s \in \mathbb{R}$ such that this integral is convergent.

The relation between the occupation measures $Y_x$ and $Y^x$, and the probabilities in (1.2) is given by the following result.

**Proposition 1.** *Let $X$ be the classical risk process defined in (1.1) and $x \in \mathbb{R}$. Then for each $t > 0$,*

$$\{Y_x(t) = 0\} = \left\{\sup_{s\leq t} X_s \leq x\right\} \quad \text{and} \quad \{Y^x(t) = 0\} = \left\{\inf_{s\leq t} X_s \geq x\right\}.$$

*Proof.* We first observe that $\{Y_x(t) = 0\} \supset \left\{\sup_{s\leq t} X_s \leq x\right\}$ is trivial. Now we see the reverse inclusion. Let $\omega \in \Omega$ be such that

$$\int_0^t 1_{(x,+\infty)}(X_s(\omega))ds = 0. \tag{2.1}$$

If there is some $s_0 \in (0,t)$ such that $X_{s_0}(\omega) > x$, then, by the right-continuity of $X$, there exists a non-empty open interval $I_{s_0} \subset (0,t)$ such that $s_0 \in I_{s_0}$ and $X_s(\omega) > x$, for all $s \in I_{s_0}$. Consequently,

$$\int_0^t 1_{(x,+\infty)}(X_s(\omega))ds \geq \int_{I_{s_0}} 1_{(x,+\infty)}(X_s(\omega))ds = |I_{s_0}| > 0,$$

where $|I_{s_0}|$ is the length of $I_{s_0}$. But this is a contradiction to (2.1). Therefore $X_s(\omega) \leq x$ for all $s \leq t$, which implies that $\omega$ also belongs to $\left\{\sup_{s\leq t} X_s \leq x\right\}$.

We proceed similarly for the remainder of the proof. $\square$

In order to express the double Laplace transform of $Y^x$ and $Y_x$, we need to introduce the following notation. Let $\tilde{n}$ be a positive real number. The positive and negative roots of the quadratic equation

$$cs^2 + (rc - \lambda - \tilde{n})s - \tilde{n}r = 0$$

are denoted by $v_{\tilde{n}}^+$ and $v_{\tilde{n}}^-$, respectively.

**Proposition 2.** *Let $s$ and $\alpha$ be two positive real numbers. Then*

$$\int_0^\infty e^{-st} E\left[e^{-\alpha Y_x(t)}\right] dt = \begin{cases} \frac{1}{s+\alpha}\left\{1 + \frac{\alpha\left(\frac{1}{c} - \frac{v_s^+}{s}\right)}{v_{s+\alpha}^- - v_s^+ - \frac{\alpha}{c}} e^{(x_0-x)v_{s+\alpha}^-}\right\}, & x_0 \geq x, \\ \frac{1}{s} + \frac{\frac{\alpha sc}{s+\alpha}\left(\frac{1}{c} - \frac{v_s^+}{s}\right)v_{s+\alpha}^- - \alpha\left(v_{s+\alpha}^- - v_s^+ - \frac{\alpha}{c}\right)}{sc\left(v_s^+ + \frac{\alpha}{c}\right)\left(v_{s+\alpha}^- - v_s^+ - \frac{\alpha}{c}\right)e^{(x-x_0)v_s^+}}, & x_0 < x, \end{cases}$$

*and*

$$\int_0^\infty e^{-st} E\left[e^{-\alpha Y^x(t)}\right] dt = \begin{cases} \frac{1}{s+\alpha}\left\{1 + \frac{\alpha\left(\frac{1}{c} - \frac{v_s^-}{s}\right)}{v_{s+\alpha}^+ - v_s^- - \frac{\alpha}{c}} e^{(x_0-x)v_{s+\alpha}^+}\right\}, & x_0 \leq x, \\ \frac{1}{s} + \frac{\frac{\alpha sc}{s+\alpha}\left(\frac{1}{c} - \frac{v_s^-}{s}\right)v_{s+\alpha}^+ - \alpha\left(v_{s+\alpha}^+ - v_s^- - \frac{\alpha}{c}\right)}{sc\left(v_s^- + \frac{\alpha}{c}\right)\left(v_{s+\alpha}^+ - v_s^- - \frac{\alpha}{c}\right)e^{(x-x_0)v_s^-}}, & x_0 \geq x. \end{cases}$$

*Proof.* The proposition is an immediate consequence of Chiu and Yin [3] (Corollary 4.1). $\square$

The following result is a consequence of Proposition 2 and it will be used in Section 3.

**Proposition 3.** *Let $X$ be the classical risk process given by (1.1). Then, for every $s > 0$, we have*

$$\mathcal{L}(P\left(Y_x(\cdot) = 0\right))(s) = \begin{cases} 0, & x_0 \geq x, \\ \frac{1}{s} - \frac{e^{-(x-x_0)v_s^+}}{s}, & x_0 < x, \end{cases} \tag{2.2}$$

*and*

$$\mathcal{L}(P\left(Y^x(\cdot) = 0\right))(s) = \begin{cases} 0, & x_0 \leq x, \\ \frac{1}{s} - \frac{\lambda e^{-(x-x_0)v_s^-}}{cs\left(v_s^+ + r\right)}, & x_0 > x. \end{cases} \tag{2.3}$$

*Proof.* We first deal with equality (2.2).

By the dominated convergence theorem we can write

$$\begin{aligned}
&\lim_{\alpha\to\infty}\int_0^\infty e^{-st}E\left[e^{-\alpha Y_x(t)}\right]dt\\
&=\lim_{\alpha\to\infty}\int_0^\infty e^{-st}\left(\int_{\{Y_x(t)=0\}}+\int_{\{Y_x(t)>0\}}\right)e^{-\alpha Y_x(t)}dPdt\\
&=\int_0^\infty e^{-st}P\left(Y_x(t)=0\right)dt+\int_0^\infty e^{-st}\int_{\{Y_x(t)>0\}}(\lim_{\alpha\to\infty}e^{-\alpha Y_x(t)})dPdt\\
&=\int_0^\infty e^{-st}P\left(Y_x(t)=0\right)dt.
\end{aligned}$$

Therefore, from Proposition 2 we get

$$\begin{aligned}
&\int_0^\infty e^{-st}P\left(Y_x(t)=0\right)dt\\
&\quad=\lim_{\alpha\to\infty}\begin{cases}\frac{1}{s+\alpha}\{1+\frac{\alpha(\frac{1}{c}-\frac{v_s^+}{s})}{v_{s+\alpha}^--v_s^+-\frac{\alpha}{c}}e^{(x_0-x)v_{s+\alpha}^-}\}, & x_0\geq x,\\ \frac{1}{s}+\frac{\frac{\alpha sc}{s+\alpha}(\frac{1}{c}-\frac{v_s^+}{s})v_{s+\alpha}^--\alpha(v_{s+\alpha}^--v_s^+-\frac{\alpha}{c})}{sc(v_s^++\frac{\alpha}{c})(v_{s+\alpha}^--v_s^+-\frac{\alpha}{c})e^{(x-x_0)v_s^+}}, & x_0<x.\end{cases}
\end{aligned}\tag{2.4}$$

Note that the definition of $v_{s+\alpha}^-$ implies that $\lim_{\alpha\to\infty}\left(v_{s+\alpha}^-/\alpha\right)=0$, which leads us to

$$\lim_{\alpha\to\infty}\left(\alpha\left(v_{s+\alpha}^--v_s^+-\frac{\alpha}{c}\right)^{-1}\right)=-c.$$

Hence, the fact that $v_{s+\alpha}^-<0$, for all $\alpha>0$, together with (2.4), yields that equality (2.2) is true for $x_0\geq x$.

On the other hand, for $x_0<x$,

$$\begin{aligned}
&\lim_{\alpha\to\infty}\left(\frac{1}{s}+\frac{\frac{\alpha sc}{s+\alpha}\left(\frac{1}{c}-\frac{v_s^+}{s}\right)v_{s+\alpha}^--\alpha\left(v_{s+\alpha}^--v_s^+-\frac{\alpha}{c}\right)}{sc\left(v_s^++\frac{\alpha}{c}\right)\left(v_{s+\alpha}^--v_s^+-\frac{\alpha}{c}\right)e^{(x-x_0)v_s^+}}\right)\\
&=\lim_{\alpha\to\infty}\left(\frac{1}{s}+\frac{sc\left(\frac{1}{c}-\frac{v_s^+}{s}\right)\frac{v_{s+\alpha}^-}{s+\alpha}-\left(v_{s+\alpha}^--v_s^+-\frac{\alpha}{c}\right)}{sc\left(\frac{v_s^+}{\alpha}+\frac{1}{c}\right)\left(v_{s+\alpha}^--v_s^+-\frac{\alpha}{c}\right)}e^{-(x-x_0)v_s^+}\right)\\
&=\frac{1}{s}-\frac{e^{-(x-x_0)v_s^+}}{s}.
\end{aligned}$$

Thus (2.2) holds.

Finally, in order to see that (2.3) is satisfied, we only need to observe that

$$\lim_{\alpha\to\infty}\left(v_{s+\alpha}^+/\alpha\right)=\frac{1}{c}\quad\text{and}\quad\lim_{\alpha\to\infty}\left(v_{s+\alpha}^+-v_s^--\frac{\alpha}{c}\right)=v_s^++r,$$

and proceed as in the beginning of this proof. □

## 3. The distributions of the sup and inf of $X$ within finite time

The purpose of this section is to apply Lerch's theorem to the Laplace transforms obtained in Proposition 3 to calculate the distributions of the sup and inf of $X$ (see Proposition 1).

In order to state the main result of this paper, we need to introduce the following notation.

Note first that

$$\begin{aligned}(s+\lambda-rc)^2+4crs &= (s+\lambda+rc)^2-4\lambda rc\\ &= (s+\lambda+rc-2\sqrt{\lambda rc})(s+\lambda+rc+2\sqrt{\lambda rc})\\ &= (s-r_1)(s-r_2),\end{aligned}$$

with $r_1=2\sqrt{\lambda rc}-\lambda-rc$ and $r_2=-2\sqrt{\lambda rc}-\lambda-rc$. Hence $r_2<r_1<0$,

$$v_s^+=\frac{s+\lambda-rc+\sqrt{(s-r_1)(s-r_2)}}{2c}, \tag{3.1}$$

and

$$v_s^-=\frac{s+\lambda-rc-\sqrt{(s-r_1)(s-r_2)}}{2c}. \tag{3.2}$$

For sake of simplicity, let us utilize the conventions $a=\frac{x-x_0}{2c}$, $b=\lambda+rc$ and

$$\chi(z)=\frac{\arg(z-r_1)+\arg(z-r_2)}{2},\quad z\in\mathbb{C}.$$

**Theorem 1.** *Let $X$ be the classical risk process (1.1) and $t>0$. Then we have*

$$P\left(\sup_{s\le t}X_s>x\right) = e^{-a(\lambda-rc)}\left(e^{-a\sqrt{r_1r_2}}+\frac{1}{\pi}\int_{r_2}^{r_1}\frac{e^{(t-a)u}\sin(a\,|u-r_1|^{1/2}\,|u-r_2|^{1/2})}{u}du\right), \tag{3.3}$$

*for every $x\in(x_0,x_0+ct)$, and*

$$P\left(\inf_{s\le t}X_s<x\right)=2\lambda e^{-a(\lambda-rc)}\left(\frac{e^{a\sqrt{r_1r_2}}}{b+\sqrt{r_1r_2}} + \frac{1}{\pi}Im\left(\int_{r_2}^{r_1}\frac{e^{(t-a)u-ai|u-r_1|^{1/2}|u-r_2|^{1/2}}}{u\left(u+b-i|u-r_1|^{1/2}|u-r_2|^{1/2}\right)}du\right)\right), \tag{3.4}$$

*for every $x<x_0$.*

**Remarks.**

*i)* Note that

$$P\left(\sup_{s\le t}X_s>x\right)=0 \text{ for } x\ge x_0+ct,\quad\text{and}\quad P\left(\sup_{s\le t}X_s>x\right)=1\text{ for }x<x_0.$$

*ii)* Similarly we have

$$P\left(\inf_{s\leq t} X_s < x\right) = 1 \quad \text{for } x > x_0.$$

In the following two subsections, we separate the proofs of (3.3) and (3.4) for the convenience of the reader because both of them are long and tedious.

3.1. **Proof of equality (3.3).** Let

$$h_M(s) = \frac{e^{-as-a\sqrt{(s-r_1)(s-r_2)}}}{s}, \quad s \in \mathbb{C}\backslash[r_2, r_1],$$

with $\sqrt{(s-r_1)(s-r_2)} = |(s-r_1)(s-r_2)|^{1/2}\exp(i\chi(s))$. Then, by (2.2), (3.1), the inverse theorem of the Laplace transform (see for example [13]) and the fact that $h_M$ is an analytic function on $\mathbb{C}\backslash[r_2, r_1]$, we have, for $\sigma$ large enough,

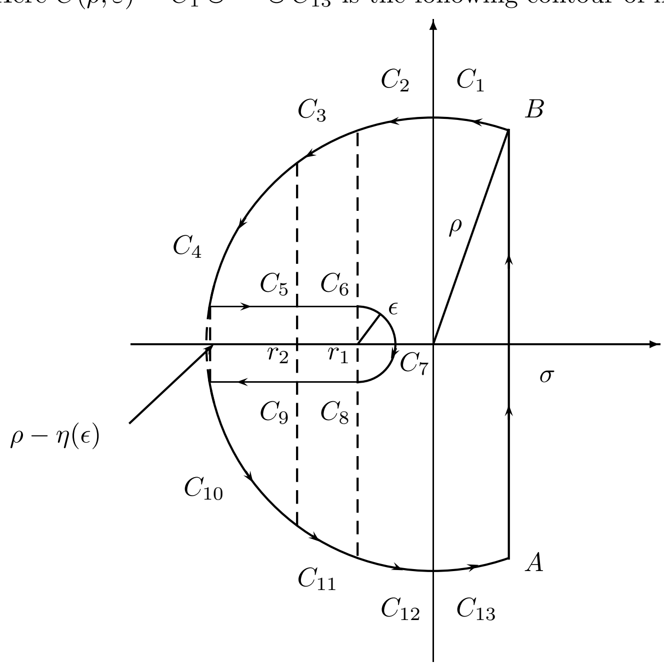
$$\begin{aligned} P\left(Y_x(t)=0\right) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{ts}\left(\frac{1}{s} - \frac{e^{-(x-x_0)v_s^+}}{s}\right)ds \\ &= 1 - \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{ts}\frac{e^{a(rc-\lambda-s-\sqrt{(s-r_1)(s-r_2)})}}{s} \\ &= 1 - \frac{e^{a(rc-\lambda)}}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{ts}h_M(s)ds. \end{aligned}$$

Notice that we can use the inverse theorem because it is not difficult to see that $t \mapsto P(X_t = x) = 0$ is continuous, which follows from the fact that $P(X_t = x) = 0$.

Now observe that 0 is a pole of order one and $r_1$, $r_2$ are branch points of $h_M$. Therefore, by the residue theorem (see [13]),

$$\int_{\sigma-i\infty}^{\sigma+i\infty} e^{ts}h_M(s)ds = 2\pi i e^{-a\sqrt{r_1 r_2}} - \lim_{\rho\to\infty}\lim_{\varepsilon\to 0}\int_{C(\rho,\varepsilon)} e^{ts}h_M(s)ds. \tag{3.5}$$

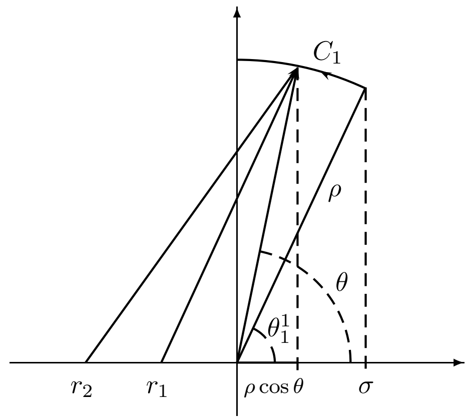
Here $C(\rho, \varepsilon) = C_1 \cup \cdots \cup C_{13}$ is the following contour of integration:



Note that we only need to analyze the integral in the right-hand side of (3.5) on each arc $C_j$, $j \in \{1, \ldots, 13\}$, in order to finish the proof. To do so, now we divide the proof in several steps.

**Step 1.** We begin our study on the arcs $C_1(\rho)$ and $C_{13}(\rho)$.

For $C_1(\rho)$ we take the parametrization $s = \rho e^{i\theta}$, $\theta_1^1 \le \theta \le \pi/2$, where $\rho$ and $\theta_1^1$ are indicated in the following figure:




In this case, it is easy to see that

$$0 \le \arg(\rho e^{i\theta} - r_1) < \frac{\pi}{2} \quad \text{and} \quad 0 \le \arg(\rho e^{i\theta} - r_2) < \frac{\pi}{2},$$

which give $0 \le \chi(\rho e^{i\theta}) < \pi/2$. Since $\cos\theta \ge 0$, when $\theta \in [0, \pi/2]$, we have

$$\mathrm{Re}(\sqrt{(\rho e^{i\theta} - r_1)(\rho e^{i\theta} - r_2)}) = |\rho e^{i\theta} - r_1|^{1/2} |\rho e^{i\theta} - r_2|^{1/2} \cos(\chi(\rho e^{i\theta})) > 0.$$

Using this and the fact that $a > 0$, we can conclude

$$e^{-a\mathrm{Re}(\sqrt{(\rho e^{i\theta} - r_1)(\rho e^{i\theta} - r_2)})} \le 1. \tag{3.6}$$

Now note that $t > a$, due to $x_0 + ct > x$, and that $\rho \cos\theta < \sigma$. Thus,

$$\begin{aligned}
\left| \int_{C_1(\rho)} e^{ts} h_M(s) \right| &= \left| \int_{\theta_1^1}^{\frac{\pi}{2}} \frac{e^{(t-a)\rho e^{i\theta} - a\sqrt{(\rho e^{i\theta} - r_1)(\rho e^{i\theta} - r_2)}}}{\rho e^{i\theta}} \rho i e^{i\theta} d\theta \right| \\
&\le \int_{\theta_1^1}^{\frac{\pi}{2}} e^{(t-a)\rho\cos\theta - a\mathrm{Re}(\sqrt{(\rho e^{i\theta} - r_1)(\rho e^{i\theta} - r_2)})} d\theta \\
&\le \int_{\theta_1^1}^{\frac{\pi}{2}} e^{(t-a)\sigma} d\theta \\
&= e^{(t-a)\sigma} \sin^{-1}\left(\frac{\sigma}{\rho}\right) \le e^{(t-a)\sigma} \left(\frac{\sigma}{\rho}\right) \frac{\pi}{2}.
\end{aligned}$$

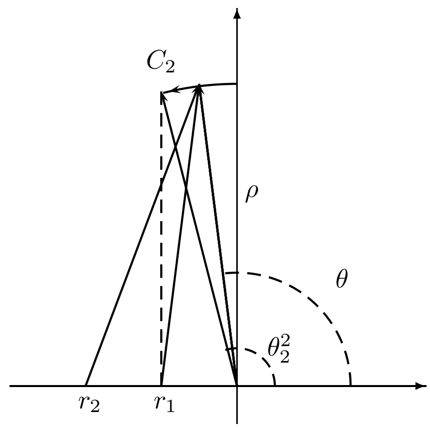
Hence $\lim_{\rho\to\infty} \int_{C_1(\rho)} e^{ts} h_M(s) ds = 0$.

We can proceed in the same way to see that $\lim_{\rho\to\infty}\int_{C_{13}(\rho)} e^{ts}h_M(s)ds$ is also equal to zero.

Note that an important point in this analysis is the fact that $t-a>0$ and $-a<0$. This will be also important in the remaining of this proof.

**Step 2.** Now we consider the integral over $C_2(\rho)$ and $C_{12}(\rho)$.

Over $C_2(\rho)$, we consider the parametrization $s=\rho e^{i\theta}$, $\pi/2\le\theta\le\theta_2^2<\pi$, as it is ilustrated in the next figure:




As in the previous case, the inequality (3.6) is still true. So, taking into account that $\sin\theta\ge 2\theta/\pi$, for $\theta\in[0,\pi/2]$, we conclude

$$\begin{aligned}\left|\int_{C_2(\rho)} e^{ts}h_M(s)ds\right| &\le \int_{\frac{\pi}{2}}^{\theta_2^2} e^{(t-a)\rho\cos\theta - a\mathrm{Re}\left(\sqrt{(\rho e^{i\theta}-r_1)(\rho e^{i\theta}-r_2)}\right)}d\theta\\ &\le \int_{\frac{\pi}{2}}^{\theta_2^2} e^{(t-a)\rho\cos\theta}d\theta\\ &= \int_0^{\theta_2^2-\frac{\pi}{2}} e^{(t-a)\rho\cos\left(\theta+\frac{\pi}{2}\right)}d\theta\\ &= \int_0^{\theta_2^2-\frac{\pi}{2}} e^{-(t-a)\rho\sin\theta}d\theta\\ &\le \int_0^{\theta_2^2-\frac{\pi}{2}} e^{-(t-a)\rho(2\theta/\pi)}d\theta \le \frac{\pi}{2\rho(t-a)}.\end{aligned}$$

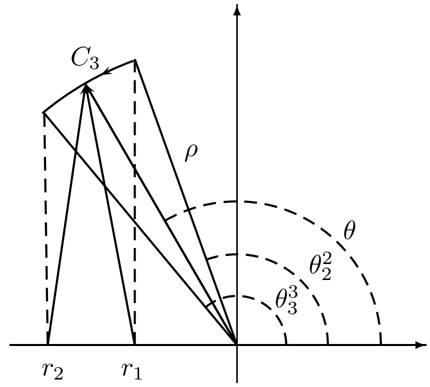
From which $\lim_{\rho\to\infty}\int_{C_2(\rho)} e^{ts}h_M(s)ds=0$.

Similarly, $\lim_{\rho\to\infty}\int_{C_{12}(\rho)} e^{ts}h_M(s)ds=0$.

**Step 3.** Here we will show that

$$\lim_{\rho\to\infty}\int_{C_3(\rho)} e^{ts}h_M(s)ds=0 \quad\text{and}\quad \lim_{\rho\to\infty}\int_{C_{11}(\rho)} e^{ts}h_M(s)ds=0.$$

On $C_3(\rho)$, we still use the parametrization $s = \rho e^{i\theta}$, with $\frac{\pi}{2} < \theta_2^2 \le \theta \le \theta_3^3 < \pi$:




Since $x < x_0 + ct$, then $t > 2a$. Therefore we can take $\eta > 0$ such that

$$t > \left(1 + (1+\eta)^{1/4}\right) a. \tag{3.7}$$

Moreover take $\rho > 0$ such that

$$\eta\rho^2 > |r_2 - r_1|^2 . \tag{3.8}$$

Notice that

$$\frac{\pi}{2} \le \arg(\rho e^{i\theta} - r_1) < \theta \quad \text{and} \quad 0 \le \arg(\rho e^{i\theta} - r_2) \le \frac{\pi}{2}.$$

Hence

$$\frac{\pi}{4} \le \chi(\rho e^{i\theta}) < \frac{\theta + \frac{\pi}{2}}{2} < \theta.$$

Since cos is decreasing on $[\pi/4, \pi]$, we have

$$\mathrm{Re}(\sqrt{(\rho e^{i\theta} - r_1)(\rho e^{i\theta} - r_2)}) \ge \left|\rho e^{i\theta} - r_1\right|^{1/2} \left|\rho e^{i\theta} - r_2\right|^{1/2} \cos\theta. \tag{3.9}$$

On the other hand, the fact that $r_1 \ge \rho\cos\theta$ implies

$$\left|\rho e^{i\theta} - r_1\right| \le \rho. \tag{3.10}$$

And using (3.8), we get

$$\begin{aligned} \left|\rho e^{i\theta} - r_2\right| &\le \sqrt{|r_1 - r_2|^2 + \rho^2 \sin^2(\theta)} \\ &\le \sqrt{|r_1 - r_2|^2 + \rho^2} \\ &\le (\eta+1)^{1/2}\rho. \end{aligned} \tag{3.11}$$

Therefore (3.9), (3.10) and (3.11) yields

$$\begin{aligned} \mathrm{Re}(\sqrt{(\rho e^{i\theta} - r_1)(\rho e^{i\theta} - r_2)}) &\ge \rho^{1/2} (\eta+1)^{1/4} \rho^{1/2} \cos\theta \\ &= (\eta+1)^{1/4} \rho\cos\theta. \end{aligned} \tag{3.12}$$

From this and (3.7) we obtain

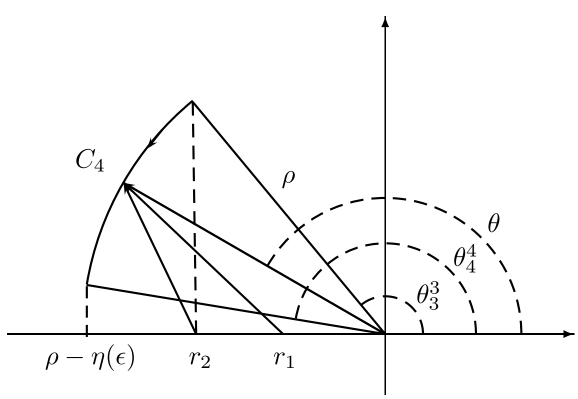
$$\begin{aligned}
\left|\int_{C_3(\rho)} e^{ts}h_M(s)ds\right| &\leq \int_{\theta_2^2}^{\theta_3^3} e^{(t-a)\rho\cos\theta - a\mathrm{Re}(\sqrt{(\rho e^{i\theta}-r_1)(\rho e^{i\theta}-r_2)})}d\theta \\
&\leq \int_{\theta_2^2}^{\theta_3^3} e^{(t-a)\rho\cos\theta - a(\eta+1)^{1/4}\rho\cos\theta}d\theta \\
&= \int_{\theta_2^2}^{\theta_3^3} e^{(t-a(1+(\eta+1)^{1/4}))\rho\cos\theta}d\theta \\
&= \int_{\theta_2^2-\frac{\pi}{2}}^{\theta_3^3-\frac{\pi}{2}} e^{\left(t-a(1+(\eta+1)^{1/4})\right)\rho\cos\left(\theta+\frac{\pi}{2}\right)}d\theta \\
&= \int_{\theta_2^2-\frac{\pi}{2}}^{\theta_3^3-\frac{\pi}{2}} e^{-\left(t-a(1+(\eta+1)^{1/4})\right)\rho\sin\theta}d\theta \\
&\leq \int_{\theta_2^2-\frac{\pi}{2}}^{\theta_3^3-\frac{\pi}{2}} e^{-\left(t-a(1+(\eta+1)^{1/4})\right)\rho\frac{2\theta}{\pi}}d\theta \\
&\leq \frac{\pi}{2\rho(t-a(1+(\eta+1)^{1/4}))}.
\end{aligned}$$

This implies $\lim_{\rho\to\infty}\int_{C_3(\rho)} e^{ts}h_M(s)ds = 0$.

Proceeding as the beginning of this step, we can conclude that we also have $\lim_{\rho\to\infty}\int_{C_{11}(\rho)} e^{ts}h_M(s)ds = 0$.

**Step 4.** Now we deal with the arcs $C_4(\rho,\varepsilon)$ and $C_{10}(\rho,\varepsilon)$.

Here we consider the same parametrization of previous steps. That is, $s = \rho e^{i\theta}$, $\pi/2 \leq \theta_3^3 \leq \theta \leq \theta_4^4 \leq \pi$:




Notice that

$$\frac{\pi}{2} \leq \arg(\rho e^{i\theta} - r_1) \leq \theta \quad \text{and} \quad \frac{\pi}{2} \leq \arg(\rho e^{i\theta} - r_2) \leq \theta.$$

Thus, $\pi/2 \le \chi(\rho e^{i\theta}) \le \theta \le \pi$. Moreover since $r_1, r_2 \ge \rho\cos\theta$ then

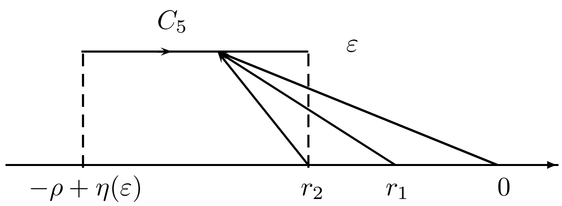
$$\left|\rho e^{i\theta} - r_1\right| \le \rho \quad \text{and} \quad \left|\rho e^{i\theta} - r_2\right| \le \rho,$$

and using the monotony of cos on $[\pi/2, \pi]$, we have

$$\begin{aligned}\mathrm{Re}(\sqrt{(\rho e^{i\theta} - r_1)(\rho e^{i\theta} - r_2)}) &= |\rho e^{i\theta} - r_1|^{1/2}|\rho e^{i\theta} - r_2|^{1/2}\cos(\chi(\rho e^{i\theta})) \\ &\ge |\rho e^{i\theta} - r_1|^{1/2}|\rho e^{i\theta} - r_2|^{1/2}\cos\theta \\ &\ge \rho\cos\theta.\end{aligned}$$

The above estimation is analogous to (3.12). Now the conclusion follows as in Step 3.

**Step 5.** Here we consider $C_5(\rho,\varepsilon)$ and $C_9(\rho,\varepsilon)$.
Over $C_5(\rho,\varepsilon)$ we take the parametrization $s = u + \varepsilon i$, $-\rho + \eta(\varepsilon) \le u \le r_2$ :




Notice that

$$\frac{\pi}{2} \le \arg(u + \varepsilon i - r_1) \le \pi \quad \text{and} \quad \frac{\pi}{2} \le \arg(u + \varepsilon i - r_2) \le \pi,$$

then $\pi/2 \le \chi(u + \varepsilon i) \le \pi$. Since cosine is negative and decreasing on $[\pi/2, \pi]$ we have, for $\varepsilon < |r_1 - r_2|$,

$$\begin{aligned}&-a\mathrm{Re}(\sqrt{(u + \varepsilon i - r_1)(u + \varepsilon i - r_2)}) \\ &\qquad = a|u + \varepsilon i - r_1|^{1/2}|u + \varepsilon i - r_2|^{1/2}(-\cos(\chi(u + \varepsilon i))) \\ &\qquad \le 2^{1/2}a\,|u - r_1|\,.\end{aligned}$$

Hence

$$\begin{aligned}\left|e^{ts}h_M(s)\right| &\le \frac{e^{(t-a)u - a\mathrm{Re}(\sqrt{(u+\varepsilon i - r_1)(u+\varepsilon i - r_2)})}}{|u + \varepsilon i|} \\ &\le \frac{e^{(t-a)r_2 + 2^{1/2}a|\rho - r_1|}}{|r_2|}.\end{aligned} \tag{3.13}$$

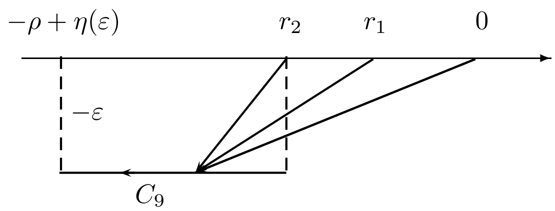
By (3.13) we can apply the dominated convergence theorem and since

$$\arg(u + \varepsilon i - r_1) \to \pi \quad \text{and} \quad \arg(u + \varepsilon i - r_2) \to \pi, \qquad \text{as } \varepsilon \to 0,$$

we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{C_5(\rho,\varepsilon)} e^{ts}h_M(s)ds &= \lim_{\varepsilon\to\infty}\int_{-\rho+\eta(\varepsilon)}^{r_2}\frac{e^{(t-a)(u+\varepsilon i)-a\sqrt{(u+\varepsilon i-r_1)(u+\varepsilon i-r_2)}}}{u+\varepsilon i}du\\ &= \int_{-\rho}^{r_2}\frac{e^{(t-a)u-a|u-r_1|^{1/2}|u-r_2|^{1/2}e^{i\pi}}}{u}du\\ &= \int_{-\rho}^{r_2}\frac{e^{(t-a)u+a|u-r_1|^{1/2}|u-r_2|^{1/2}}}{u}du.\end{aligned}$$

On the other hand, on $C_9(\rho,\varepsilon)$ we use the parametrization $s=-u-\varepsilon i$, $-r_2\leq u\leq \rho-\eta(\varepsilon)$ :




Working as in previous case, and noting that

$$\arg(-u-\varepsilon i-r_1)\to\pi \quad\text{and}\quad \arg(-u-\varepsilon i-r_2)\to\pi, \qquad \text{as } \varepsilon\to 0,$$

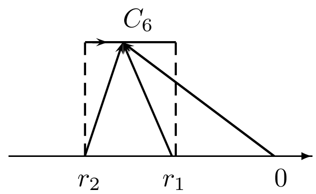
we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{C_9(\rho,\varepsilon)} e^{ts}h_M(s)ds\\ &=\lim_{\varepsilon\to 0}\int_{-r_2}^{\rho-\eta(\varepsilon)}\frac{e^{(t-a)(-u-\varepsilon i)-a\sqrt{(-u-\varepsilon i-r_1)(-u-\varepsilon i-r_2)}}}{-u-\varepsilon i}(-du)\\ &=-\int_{-\rho}^{r_2}\frac{e^{(t-a)u-a|u-r_1|^{1/2}|u-r_2|^{1/2}e^{i\pi}}}{u}du\\ &=-\int_{-\rho}^{r_2}\frac{e^{(t-a)u+a|u-r_1|^{1/2}|u-r_2|^{1/2}}}{u}du.\end{aligned}$$

Therefore,

$$\lim_{\rho\to 0}(\lim_{\varepsilon\to 0}\int_{C_5(\rho,\varepsilon)} e^{ts}h_M(s)ds+\lim_{\varepsilon\to 0}\int_{C_9(\rho,\varepsilon)} e^{ts}h_M(s)ds)=0.$$

**Step 6.** Now we deal with $C_6(\varepsilon)$. To do this, we take the parametrization $s=u+\varepsilon i$, $r_2\leq u\leq r_1$ :




Notice that for $\varepsilon < 1$,

$$
\begin{aligned}
|e^{ts}h_M(s)| &= \left| \frac{e^{(t-a)(u+\varepsilon i)-a\sqrt{(u+\varepsilon i-r_1)(u+\varepsilon i-r_2)}}}{u+\varepsilon i} \right| \\
&= \frac{e^{(t-a)u-a|u+\varepsilon i-r_1|^{1/2}|u+\varepsilon i-r_2|^{1/2}\cos\chi(u+\varepsilon i)}}{|u+\varepsilon i|} \\
&\leq \frac{e^{(t-a)r_1}e^{-a|u+\varepsilon i-r_1|^{1/2}|u+\varepsilon i-r_2|^{1/2}\cos\chi(u+\varepsilon i)}}{|r_1|} \\
&\leq \frac{e^{(t-a)r_1}e^{a|u+\varepsilon i-r_1|^{1/2}|u+\varepsilon i-r_2|^{1/2}}}{|r_1|} \\
&\leq \frac{e^{-(t-a)r_1+a(1+|r_1-r_2|^2)^{1/2}}}{|r_1|}. \qquad (3.14)
\end{aligned}
$$

Since

$$\arg(u+\varepsilon i-r_1)\to\pi \quad \text{and} \quad \arg(u+\varepsilon i-r_2)\to 0, \quad \text{as } \varepsilon\to 0,$$

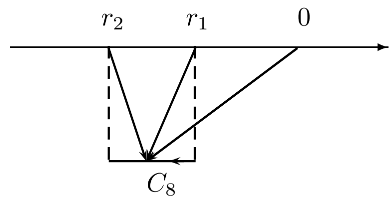
then

$$
\begin{aligned}
\sqrt{(u+\varepsilon i-r_1)(u+\varepsilon i-r_2)} &= |u+\varepsilon i-r_1|^{1/2}\,|u+\varepsilon i-r_2|^{1/2}\exp\left(i\chi(u+\varepsilon i)\right) \\
&\to |u-r_1|^{1/2}\,|u-r_2|^{1/2}\,i, \text{ as } \varepsilon\to 0.
\end{aligned}
$$

Due to (3.14) we are able to apply the dominated convergence theorem:

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\int_{C_6(\varepsilon)} e^{ts}h_M(s)ds &= \lim_{\varepsilon\to 0}\int_{r_2}^{r_1}\frac{e^{(t-a)(u+\varepsilon i)-a\sqrt{(u+\varepsilon i-r_1)(u+\varepsilon i-r_2)}}}{u+\varepsilon i}du \\
&= \int_{r_2}^{r_1}\frac{e^{(t-a)u-a|u-r_1|^{1/2}|u-r_2|^{1/2}i}}{u}du.
\end{aligned}
$$

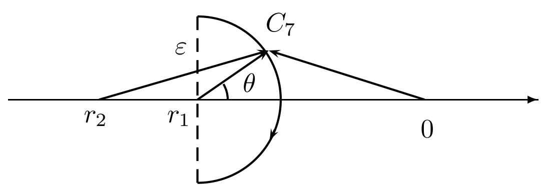
**Step 7.** Now we suppose that $C_8(\varepsilon)$ is defined by $s=-u-\varepsilon i$, $-r_1\leq u\leq -r_2$ :




we can imitate Step 6 to we get

$$\lim_{\varepsilon\to 0}\int_{C_8(\varepsilon)} e^{ts}h_M(s)ds = -\int_{r_2}^{r_1} \frac{e^{(t-a)u+a|u-r_1|^{1/2}|u-r_2|^{1/2}i}}{u}du.$$

**Step 8.** Finally we deal with $C_7(\varepsilon)$. Here, we put $s = r_1 + \varepsilon e^{-i\theta}$, $-\pi/2 \leq \theta \leq \pi/2$ :




Thus

$$\chi(\varepsilon e^{-i\theta} + r_1) \in \left[\frac{3}{2}\pi, 2\pi\right] \cup \left[0, \frac{1}{2}\pi\right].$$

Consequently, using the fact that $\cos\theta \geq 0$, for $\theta \in [\frac{3}{2}\pi, 2\pi] \cup [0, \frac{1}{2}\pi]$, we obtain

$$\mathrm{Re}(\sqrt{\varepsilon e^{-i\theta}(\varepsilon e^{-i\theta} + r_1 - r_2)}) = \varepsilon^{1/2}|\varepsilon e^{-i\theta}+r_1-r_2|^{1/2}\cos\chi(\varepsilon e^{-i\theta}+r_1) \geq 0. \tag{3.15}$$

Also, for $0 < \varepsilon < -r_1/2$, we have

$$\begin{aligned} \left|r_1 + \varepsilon e^{-i\theta}\right| &= \sqrt{r_1^2 + 2r_1\varepsilon\cos(-\theta) + \varepsilon^2} \\ &\geq |r_1 + \varepsilon| = -r_1 - \varepsilon > -\frac{r_1}{2}. \end{aligned} \tag{3.16}$$

The estimations (3.15), (3.16) and $t > a$ yield

$$\begin{aligned}\left|\int_{C_7(\varepsilon)} e^{ts} h_M(s)ds\right| &\leq \varepsilon e^{(t-a)r_1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\frac{e^{(t-a)\varepsilon e^{-i\theta}-a\sqrt{\varepsilon e^{-i\theta}(\varepsilon e^{-i\theta}+r_1-r_2)}}}{r_1+\varepsilon e^{-i\theta}}\right| d\theta \\ &\leq \frac{2\varepsilon e^{(t-a)r_1}}{|r_1|} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|e^{(t-a)\varepsilon e^{-i\theta}-a\sqrt{\varepsilon e^{-i\theta}(\varepsilon e^{-i\theta}+r_1-r_2)}}\right| d\theta \\ &= \frac{2\varepsilon e^{(t-a)r_1}}{|r_1|} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(t-a)\varepsilon\cos(-\theta)-a\mathrm{Re}(\sqrt{\varepsilon e^{-i\theta}(\varepsilon e^{-i\theta}+r_1-r_2)})} d\theta \\ &\leq \frac{2\varepsilon e^{(t-a)(r_1+\varepsilon)}}{|r_1|}\pi.\end{aligned}$$

Therefore $\lim_{\varepsilon\to 0}\int_{C_7(\varepsilon)} e^{ts}h_M(s)ds = 0$.

**Step 9.** To finish the proof, we only need to take into account Steps 1-8, together with (3.5), and Propositions 1 and 3. □

3.2. **Proof of (3.4).** Let us define

$$K(s) = s + b + \sqrt{(s-r_1)(s-r_2)}, \quad s \in \mathbb{C}\backslash[r_2, r_1].$$

Since

$$2b + r_1 + r_2 = 0 \neq -4\lambda rc = r_1 r_2 - b^2,$$

then

$$(s+b)^2 \neq (s-r_1)(s-r_2), \ \forall s \in \mathbb{C}.$$

This implies that $1/K$ is analytic over $\mathbb{C}\backslash[r_2, r_1]$. Moreover, it is not difficult to see that, for $\rho$ large enough, we have

$$|K(s)| \geq \begin{cases} 2\sqrt{\lambda rc}, & s \in C_2 \cup C_4 \cup C_5 \cup C_7 \cup C_9 \cup C_{10} \cup C_{12}, \\ b, & s \in C_1 \cup C_{13}, \\ (\rho+r_1)^{1/2}(\rho+r_2)^{1/2}\sin\frac{\pi}{4}, & s \in C_3 \cup C_{11}, \\ \varepsilon + |\mathrm{Re}(s)-r_1|^{1/2}\,|\mathrm{Re}(s)-r_2|^{1/2}\sin\frac{\pi}{4}, & s \in C_6 \cup C_8. \end{cases} \tag{3.17}$$

As in the proof of (3.3) we have by (3.2)

$$\begin{aligned} P\left(Y^x(t)=0\right) &= 1 - \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{ts}\frac{\lambda e^{-(x-x_0)v_s^-}}{cs\left(v_s^+ + r\right)}ds \\ &= 1 - \frac{\lambda e^{-a(\lambda-rc)}}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{ts}h_m(s)ds, \end{aligned}$$

where

$$h_m(s) = \frac{e^{-as+a\sqrt{(s-r_1)(s-r_2)}}}{sK(s)}.$$

Finally, the result follows from Proposition 1, (3.17) and the proof of (3.3). Observe that $a < 0$ and $t - 2a > 0$, together with (3.17), allow us to copy, line by line, the proof of (3.3) to show that (3.4) is also true. □

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