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REPORT No. 9, 2007/2008, fall

ISSN 1103-467X

ISRN IML-R- -9-07/08- -SE+fall

ORNSTEIN-UHLENBECK PROCESSES ON LIE GROUPS

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ABSTRACT. We consider Ornstein-Uhlenbeck processes (OU-processes) related to hypoelliptic diffusion on finite-dimensional Lie groups: let \mathcal{L} be a hypoelliptic, left-invariant “sum of the squares”-operator on a Lie group G with associated Markov process X , then we construct OU-type processes by adding horizontal gradient drifts of functions U . In the natural case $U(x) = -\log p(1, x)$, where $p(1, x)$ is the density of the law of the Markov process X starting at the identity e at time $t = 1$ with respect to the right-invariant Haar measure on G , we show the Poincaré inequality by applying the Driver-Melcher inequality for “sum of the squares” operators on Lie groups.

The Markov process associated to $-\log p(1, x)$ is called the OU-process related to the given hypoelliptic diffusion on G . We prove the global strong existence of this OU-process on G . The Poincaré inequality for a large class of potentials U is then shown by perturbation methods and used to obtain a hypoelliptic equivalent of the standard result on cooling schedules for simulated annealing. The relation between local results on \mathcal{L} and global results for the constructed OU-process is widely used in this study.

1. PREPARATIONS FROM FUNCTIONAL ANALYSIS

We consider a finite-dimensional, connected Lie group G with Lie algebra \mathfrak{g} , its right-invariant Haar measure μ and a family of left-invariant vector fields $V_1, \dots, V_d \in \mathfrak{g}$. We assume that Hörmander’s condition holds, i.e. the sub-algebra generated by V_1, \dots, V_d coincides with \mathfrak{g} .

We consider furthermore a stochastic basis $(\Omega, \mathcal{F}, \mathbf{P})$ with a d -dimensional Brownian motion B and the Lie group valued process

$$dX_t^x = \sum_{i=1}^d V_i(X_t^x) \circ dB_t^i, \quad X_0^x = x \in G. \quad (1.1)$$

The generator of this process is denoted by \mathcal{L} and we have

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^d V_i^2,$$

where we interpret the vector fields as first order differential operators on $C^\infty(G, \mathbb{R})$. Furthermore, we define a semigroup P_t acting on bounded measurable functions $f : G \rightarrow \mathbb{R}$ by

$$P_t f(x) = \mathbf{E}(f(X_t^x)).$$

Key words and phrases. Lie group, hypo-elliptic diffusion, spectral gap, Poincaré inequality, simulated annealing.

The first and third author gratefully acknowledge the support from the FWF-grant Y 328 (START prize from the Austrian Science Fund). The second author gratefully acknowledges the support from the EPSRC fellowship EP/D071593/1. All authors are grateful for the warm hospitality at the Mittag-Leffler Institute in Stockholm and at the Hausdorff Institute in Bonn.

This semigroup can be extended to a strongly continuous semigroup on $L^2(G, \mu)$, which we will denote by the same letter P_t . The carré du champ operator Γ is defined for functions f , where it makes sense, by

$$\Gamma(f, g) = \mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f. \quad (1.2)$$

In our particular case, we obtain immediately

$$\Gamma(f, f) = \sum_{i=1}^d (V_i f)^2.$$

Notice that the carré du champ operator does not change if we add a drift to the generator \mathcal{L} .

Due to the right invariance of the Haar measure μ and the left-invariance of the vector fields V_i , the operator \mathcal{L} is symmetric (reversible) with respect to μ and therefore μ is an infinitesimal invariant measure in the sense that $\int \mathcal{L}f(x) \mu(dx) = 0$ for all smooth compactly supported functions f . Furthermore, due to the symmetry of \mathcal{L} and the invariance of μ , we have from (1.2) the relation

$$2 \int f \mathcal{L}g \mu = - \int \Gamma(f, g) \mu \quad (1.3)$$

for all $f \in \mathcal{C}_0^\infty(G)$ and $g \in \mathcal{C}^\infty(G)$.

Let now $U : G \rightarrow \mathbb{R}$ be an arbitrary smooth function such that

$$\int_G \exp(-U(x)) \mu(dx) < \infty,$$

and consider the modified generator

$$\mathcal{L}^U := \mathcal{L} - \frac{1}{2} \Gamma(U, \cdot).$$

Notice that $\mu^U = \exp(-U) \mu$ is an invariant (finite) measure for \mathcal{L}^U , since, by (1.3),

$$\begin{aligned} \int \mathcal{L}^U f \mu^U &= \int (\mathcal{L}f) \exp(-U) \mu - \frac{1}{2} \int \Gamma(U, f) \exp(-U) \mu \\ &= -\frac{1}{2} \int \Gamma(f, \exp(-U)) \mu - \frac{1}{2} \int \Gamma(f, U) \exp(-U) \mu = 0. \end{aligned}$$

Here, the last step uses the fact that $\Gamma(f, \cdot)$ is a derivation.

We have the following observation on the existence of a spectral gap at 0:

Proposition 1.1. *The operator \mathcal{L}^U has a spectral gap at 0 of size $a > 0$ in $L^2(\mu^U)$ if and only if*

$$\int_G \Gamma(f, f)(x) \mu^U(dx) \geq 2a \int_G f(x)^2 \mu^U(dx),$$

for all compactly supported smooth functions f on G satisfying

$$\int_G f(x) \mu^U(dx) = 0.$$

Proof. \mathcal{L}^U has a spectral gap of size $a > 0$ at 0 if

$$\int_G f(x) \mathcal{L}^U f(x) \mu^U(dx) \leq -a \int_G f(x)^2 \mu^U(dx)$$

for test functions with $\int_G f(x)\mu^U(dx) = 0$. Now, by construction, the integral on the left hand side can be partially integrated, hence

$$-\frac{1}{2} \int_G \Gamma(f, f)(x)\mu^U(dx) \leq -a \int_G f(x)^2 \mu^U(dx)$$

for any test function f satisfying the integral constraint. This proves the desired inequality. \square

Remark 1.2. *Assume that there is a spectral gap. Then the largest $a > 0$ in the previous inequality is the modulus of the smallest non-zero spectral value of \mathcal{L}^U .*

Remark 1.3. *If we want to write an inequality for all test functions f , it reads like*

$$\int_G \Gamma(f, f)(x)\mu^U(dx) \geq 2a \left(\int_G f(x)^2 \mu^U(dx) \int_G \mu^U(dx) - \left(\int_G f(x)\mu^U(dx) \right)^2 \right) \quad (1.4)$$

for all test functions $f \in \mathcal{C}_0^\infty(G)$.

2. STRONG EXISTENCE OF OU-PROCESSES WITH VALUES IN LIE GROUPS

Let G be a finite-dimensional, connected Lie group. We consider now the special case where we take as our ‘potential’ $W_t(x) = -\log p(t, x)$, $t > 0$, where $p(t, x)$ is the density of the law of X_t^e with respect to μ . By Hörmander’s Theorem [Hör67, Hör07], the function $(t, x) \mapsto p(t, x)$ is a positive and smooth function, hence the potential W_t is as in the previous section. We write for short \mathcal{L}^t instead of \mathcal{L}^{W_t} and we call the associated Markov process the **OU-process on G** . We show that we have in fact global strong solutions for the corresponding Stratonovich SDE with values in G . The next proposition is slightly more general.

Proposition 2.1. *Consider a smooth potential $U : G \rightarrow \mathbb{R}$ such that*

$$\int \exp(-U(x)) \mu(dx) < \infty.$$

Consider the following Stratonovich SDE with values in G :

$$dY_t^y = V_0(Y_t^y)dt + \sum_{i=1}^d V_i(Y_t^y) \circ dB_t^i, \quad Y_0^y = y \in G, \quad (2.1)$$

where $V_0 f = -\frac{1}{2}\Gamma(U, f)$ for smooth test functions f . Then there is a global strong solution to (2.1) for all initial values $y \in G$.

Proof. Since the coefficients defining (2.1) are smooth by assumption, there exists a unique strong solution up to the explosion time

$$\zeta_y = \inf\{t : \lim_{\tau \rightarrow t} Y_\tau^y = \infty\}.$$

We then define a semigroup \mathcal{P}_t on $L^2(G, \mu^U)$ by

$$(\mathcal{P}_t f)(y) = \mathbf{E}(f(Y_t^y) \mathbf{1}_{\zeta_y > t}). \quad (2.2)$$

It can be shown in the exact same way as in [Che73, Li92] that \mathcal{P}_t is a strongly continuous contraction semigroup and that its generator \mathcal{A} coincides with \mathcal{L}^U on the set $\mathcal{C}_0^\infty(G)$ of compactly supported smooth functions.

On the other hand, setting $\mathcal{D}(\mathcal{L}^U) = \mathcal{C}_0^\infty(G)$, one can show as in [Che73, Li92] that \mathcal{L}^U is essentially self-adjoint, so that one must have $\mathcal{A} = \overline{\mathcal{L}^U} = (\mathcal{L}^U)^*$. In particular, since the constant function $\mathbf{1}$ belongs to $L^2(G, \mu^U)$ by the integrability

of $\exp(-U)$ and since $\int (\mathcal{L}^U \psi)(x) \mu^U(dx) = 0$ for any test function $\psi \in \mathcal{C}_0^\infty(G)$, $\mathbf{1}$ belongs to the domain of $(\mathcal{L}^U)^*$ and therefore also to the domain of \mathcal{A} . This then implies that $\mathcal{P}_t \mathbf{1} = \mathbf{1}$ by the same argument as in [Li92]. In particular, coming back to the definition (2.2) of \mathcal{P}_t , we see that $\mathbf{P}(\zeta_y = \infty) = 1$ for every y , which is precisely the non-explosion result that we were looking for. \square

Remark 2.2. *While this argument shows that, given a fixed initial condition y , there exists a unique global strong solution Y_t^y to (2.1), it does not prevent more subtle kinds of explosions, see for example [Elw78].*

By Proposition 2.1 and since $p(t, x)$ is smooth and integrable, it follows immediately that the OU-process exists globally in a strong sense.

Corollary 2.3. *For any given $\tau > 0$, the process*

$$dY_t^y = V_0(Y_t^y)dt + \sum_{i=1}^d V_i(Y_t^y) \circ dB_t^i, \quad Y_0^y = y \in G,$$

with $V_0 f = -\frac{1}{2}\Gamma(W_\tau, f)$ has a global strong solution.

Remark 2.4. *More traditional Lyapunov-function based techniques seem to be highly non-trivial to apply to this situation, due to the lack of information on the behaviour of $U(y)$ at large y . In view of [BA88, Léa87b, Léa87a], it is tempting to conjecture that one has the asymptotic*

$$\lim_{\tau \rightarrow 0} \tau^2 \partial_\tau \log p(\tau, y) = d^2(e, y), \quad (2.3)$$

and that the limit holds uniformly over compact sets K that do not contain the origin e . (Note that it follows from [BA88] that this is true provided that K does not intersect the cut-locus.) If it were the case that (2.3) holds, possible space-time scaling properties of $p(\tau, x)$ could imply that, for every $\tau > 0$, there exists a compact set K such that $\mathcal{L}p(\tau, x) = \partial_\tau p(\tau, x) > 0$ for $x \notin K$. On the other hand, one has

$$\mathcal{L}^\tau W_\tau = -\frac{1}{2}\Gamma(W_\tau, W_\tau) + \mathcal{L}W_\tau = -\frac{\mathcal{L}p(\tau, \cdot)}{p(\tau, \cdot)},$$

so that this would imply that W_τ is a Lyapunov function for the corresponding OU-process.

3. SPECTRAL GAPS FOR OU-PROCESSES

Next we consider the question if \mathcal{L}^t admits a spectral gap on $L^2(p_t \mu)$ for $t > 0$, which turns out to be a consequence of the Driver-Melcher inequality (see [Mel04]).

Theorem 3.1. *The following assertions are equivalent:*

- *The operator \mathcal{L}^t has a spectral gap of size $a_t > 0$ on $L^2(p_t \mu)$ for $t > 0$, and a positive H^1 -function $a : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$.*
- *The local estimate*

$$P_t(\Gamma(f, f))(g) \geq 2a_t((P_t f^2)(g) - ((P_t f)(g))^2)$$

holds true for all test functions $f : G \rightarrow \mathbb{R}$, for all $t > 0$ and a positive H^1 -function $a : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ at one (and therefore all) point $g \in G$.

Furthermore, if we know that

$$\Gamma(P_t f, P_t f)(e) \leq \varphi(t) P_t(\Gamma(f, f))(e)$$

holds true for all test functions $f \in \mathcal{C}_0^\infty(G)$, all $t \geq 0$, and a strictly positive locally integrable function $\varphi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$, then we can choose a_t by

$$a_t \int_0^t \varphi(t-s) ds = \frac{1}{2},$$

for $t > 0$ and the two equivalent assertions hold true.

Proof. Since μ^{W_t} is equal to the law of X_t^e , one has $\int f \mu^{W_t} = P_t f(e)$ for every $f \in \mathcal{C}_0^\infty(G)$. The equivalence of the first two statements then follows from (1.4) and the fact that the translation invariance of (1.1) implies that if the bound holds at some g , it must hold for all $g \in G$. We fix a test function $f : G \rightarrow \mathbb{R}$ as well as $t > 0$ and consider

$$H(s) = P_s((P_{t-s} f)^2)$$

for $0 \leq s \leq t$. The derivative of this function equals

$$H'(s) = P_s(\Gamma(P_{t-s} f, P_{t-s} f))$$

and therefore – assuming the third statement – we obtain

$$H'(s) \leq \varphi(t-s) P_t(\Gamma(f, f)).$$

Whence we can conclude

$$H(t) - H(0) \leq \int_0^t \varphi(t-s) ds P_t(\Gamma(f, f)),$$

which is the second of the two equivalent assertions for an appropriately chosen a . \square

Remark 3.2. We can replace the Lie group G by a general manifold M , on which we are given a hypo-elliptic, reversible diffusion X with “sum of the squares” generator \mathcal{L} . Then the completely analogous statement holds on M , in particular local Poincaré inequalities on M for \mathcal{L} lead to a spectral gap for the OU-type process \mathcal{L}^t with $t > 0$.

Corollary 3.3. Let G be a free, nilpotent Lie group with d generators e_1, \dots, e_d and denote by X the canonical diffusion process on G , i.e.

$$dX_t = \sum_{i=1}^d X_t e_i \circ dB_t^i.$$

Then the operator \mathcal{L}^t has a spectral gap of size $a_t = \frac{1}{2Kt}$ on $L^2(p_t \mu)$ for some constant $K > 0$.

Proof. Due to the results of the very interesting Ph.D.-thesis [Mel04] (see also [DM07]), there is a constant K such that the bound

$$\Gamma(P_t f, P_t f)(e) \leq K P_t(\Gamma(f, f))(e)$$

holds true for all test functions $f \in \mathcal{C}_0^\infty(G)$ and for all times $t \geq 0$. Furthermore, due to the scaling properties of P_t , there exists a best constant K such that this bound is sharp for all $t \geq 0$. This shows that $a_t K t = \frac{1}{2}$, due to the assertions of Theorem 3.1. \square

3.1. Generalisation to homogeneous spaces. Consider now M a homogeneous space with respect to the Lie group G , i.e. we have a (right) transitive action $\hat{\pi}: G \times M \rightarrow M$ of G on M . We assume that there exists a measure μ^M on M which is invariant with respect to this action. We also assume that we are given a family V_1, \dots, V_d of left-invariant vector fields on G that generate its entire Lie algebra \mathfrak{g} as before. These vector fields induce vector fields V_i^M on M by means of the action $\hat{\pi}$.

By choosing an ‘origin’ $o \in M$, we obtain a surjection $\pi: G \rightarrow M$ by $\pi(g) = \hat{\pi}(g, o)$. Due to the invariance of μ^M with respect to the action $\hat{\pi}$, the vector fields V_i^M are anti-symmetric operators on $L^2(\mu^M)$ and the generator

$$\mathcal{L}^M = \frac{1}{2} \sum_{i=1}^d (V_i^M)^2$$

is consequently symmetric on $L^2(\mu^M)$. In particular we have

$$(V_i^M f) \circ \pi = V_i(f \circ \pi)$$

for $i = 1, \dots, d$. The local Driver-Melcher inequality on G translates to the same inequality on M by means of

$$P_i^M(f) \circ \pi = P_i(f \circ \pi)$$

for test functions $f: M \rightarrow \mathbb{R}$, hence we obtain the corresponding Driver-Melcher inequality on M with the same constants, too.

4. A SIMPLE RESULT ON SIMULATED ANNEALING

For applications to simulated annealing, our main tool will be the following simple Theorem:

Theorem 4.1. *Let $U: G \rightarrow \mathbb{R}$ be a potential U with*

$$|U + \log p(\varepsilon, \cdot)| \leq D_\varepsilon$$

for some $\varepsilon > 0$ and some constant $D_\varepsilon > 0$. Assume furthermore that a Poincaré inequality holds for \mathcal{L}^ε , i.e.

$$P_\varepsilon(f^2)(e) \leq K_\varepsilon P_\varepsilon(\Gamma(f, f))(e) \quad (4.1)$$

for test functions $f \in \mathcal{C}_0^\infty(G)$ with $P_\varepsilon f(e) = 0$ and some constant $K > 0$. Then one has $\exp(-U) \in L^1(\mu(dx))$ and the Poincaré inequality

$$\int f^2(x) \exp(-U(x)) \mu(dx) \leq C_\varepsilon \int \Gamma(f, f)(x) \exp(-U(x)) \mu(dx) \quad (4.2)$$

holds for all test functions $f \in \mathcal{C}_0^\infty(G)$ with $\int f(x) \exp(-U(x)) \mu(dx) = 0$ and some constant $C_\varepsilon = K_\varepsilon \exp(2D_\varepsilon) > 0$. In particular, this leads to a spectral gap of size at least $\frac{1}{C_\varepsilon}$ for \mathcal{L}^U .

Proof. It follows immediately from the inequality

$$p(\varepsilon, x) = \exp(-U(x)) \exp(U(x)) p(\varepsilon, x) \geq \exp(-D) \exp(-U(x))$$

that $\exp(-U) \in L^1(\mu)$. Furthermore,

$$\exp(-U(x)) = \frac{1}{p(\varepsilon, x) \exp(U(x))} p(\varepsilon, x) \geq \exp(-D) p(\varepsilon, x)$$

for all $x \in G$ by assumption. Hence we deduce (4.2) with $C_\varepsilon = K_\varepsilon \exp(2D_\varepsilon)$ from (4.1). \square

Remark 4.2. See [BLW07] for results on unbounded perturbations, where one can hope for similar conclusions.

Throughout the remainder of this section we assume that M is a nilmanifold, that is a homogeneous space with respect to a nilpotent Lie group G . We consider the same structures as in Section 3.1 on M , but we omit the index M in the vector fields and measures in order to improve readability. We shall furthermore assume that M satisfies the following global estimate:

Assumption 4.3. *There is a constant \tilde{D} such that*

$$|d(x_0, x)^2 + t \log p(t, x_0, x)| \leq \tilde{D} \quad (4.3)$$

for all $0 < t < 1$ and all $x \in M$ and some (and therefore all by translation invariance) $x_0 \in M$. Here, d denotes the Carnot-Carathéodory distance on M (see [Léa87b, Léa87a] and [SC84] for the proper definitions and further references).

Remark 4.4. *If M is a compact nilmanifold, we can apply Léandre's beautiful result [Léa87b, Léa87a] that*

$$\lim_{t \rightarrow 0} t \log p(t, x_0, x) = -d(x_0, x)^2$$

holds true uniformly on M , which implies Assumption 4.3. An abelian, non-compact example where this estimate holds is given by \mathbb{R}^d . On the simplest non-compact and non-abelian example, the Heisenberg group G_d^2 , the previous estimate is wrong, see a recent preprint of H.-Q. Li [Li].

We prove a quantitative simulated annealing result under the previous assumption 4.3 on the nilmanifold M and follow closely the lines of [HS88]. The idea is to introduce a parameter ε in the operators,

$$\mathcal{L}^{U, \varepsilon} = \mathcal{L} - \frac{1}{2} \Gamma\left(\frac{U}{\varepsilon^2}, \cdot\right),$$

such that the associated invariant measure $\exp(-\frac{U}{\varepsilon^2})\mu$ concentrates around the minima of U . Notice that the previous operator satisfies

$$\varepsilon^2 \mathcal{L}^{U, \varepsilon} = \varepsilon^2 \mathcal{L} - \frac{1}{2} \Gamma(U, \cdot),$$

and a spectral gap for $\varepsilon^2 \mathcal{L}^{U, \varepsilon}$ is a spectral gap for the diffusion process

$$dY_t^y = V_0(Y_t^y)dt + \sum_{i=1}^d \varepsilon V_i(Y_t^y) \circ dB_t^i, \quad Y_0^y = y \in G,$$

with $V_0 f = -\frac{1}{2} \Gamma(U, f)$. Consequently we know – given strong existence – that the law of Y_t^y converges to $\exp(-\frac{U}{\varepsilon^2})\mu$ and concentrates a posteriori around the minima of U .

In the following theorem we try to quantify this behaviour. We denote by $\mu^{U, \varepsilon}$ the probability measure invariant for $\mathcal{L}^{U, \varepsilon}$ and we use the notation

$$\text{var}_\varepsilon(f) = \langle (f - \langle f \rangle_\varepsilon)^2 \rangle_\varepsilon$$

with $\langle f \rangle_\varepsilon = \int_M f(g) \mu^{U, \varepsilon}(dg)$.

Theorem 4.5. *Let $U : M \rightarrow \mathbb{R}$ be a smooth function such that there exist a constant D and a point $x_0 \in M$ such that*

$$|U(x) - d(x_0, x)^2| \leq D,$$

for all $x \in M$. Then there exist constants $R, c > 0$ such that for $\varepsilon(t) = \frac{c}{\sqrt{\log(R+t)}}$ and

$$\text{var}_{\varepsilon(t)}(f) \leq K(R+t) \langle \frac{1}{2} \Gamma(f, f) \rangle_{\varepsilon(t)},$$

for all test functions $f \in C_0^\infty(M)$ and $t \geq 0$.

Proof. We can start to collect results. Combining Corollary 3.3 and Assumption 4.3 with Theorem 4.1, we obtain that spectral gap for the operator $\mathcal{L}^{U, \varepsilon}$ has size at least

$$\frac{1}{K\varepsilon^2} \exp\left(-\frac{2(D+\tilde{D})}{\varepsilon^2}\right)$$

for $0 < \varepsilon < 1$, so that $\varepsilon^2 \mathcal{L}^{U, \varepsilon}$ has a spectral gap of size at least

$$\frac{1}{K} \exp\left(-\frac{2(D+\tilde{D})}{\varepsilon^2}\right).$$

We choose $c^2 = 2(D+\tilde{D})$ and R sufficiently big so that $\varepsilon(t) < 1$ for $t \geq 0$, and we conclude that

$$K \exp\left(\frac{2(D+\tilde{D})}{\varepsilon(t)^2}\right) \leq K(R+t)$$

for all $t \geq 0$, which yields the desired result. \square

We denote by Z the process with cooling schedule $t \mapsto \varepsilon(t)$ as in the previous theorem,

$$dZ_t^z = V_0(Z_t^z)dt + \sum_{i=1}^d \varepsilon(t) V_i(Z_t^z) \circ dB_t^i.$$

Then the previous conclusion leads to the following proposition.

Proposition 4.6. *Let f_t denote the Radon-Nikodym derivative of the law of Z_t^z with respect to $\mu_t = \mu^{U, \varepsilon(t)}$ and let*

$$u(t) := \|f_t - 1\|_{L^2(\mu_t)}^2$$

denote the distance in $L^2(\mu_t)$, then

$$u'(t) \leq -\frac{1}{K(R+t)}u(t) + \frac{1}{c^2(R+t)}u(t) + \frac{1}{c^2(R+t)}\sqrt{u(t)}$$

for the constants R, c and K from Theorem 4.5.

Remark 4.7. *If we choose $c^2 > K$, which is always possible, we conclude that $\sup_{t \geq 0} u(t)$ is bounded from above by a constant depending on f_0, c, R and K .*

Proof. The proof follows closely the lines of [HS88]. By assumption we know that $u(t) = \|f_t\|_{L^2(\mu_t)}^2 - 1$ and hence with the notation $\beta(t) = \frac{1}{\varepsilon(t)^2}$,

$$\begin{aligned} u'(t) &= -2\langle \Gamma(f_t, f_t) \rangle_{\varepsilon(t)} - \beta'(t) \int (U - \langle U \rangle_{\varepsilon(t)}) f_t^2 \mu_t \\ &= -2\langle \Gamma(f_t, f_t) \rangle_{\varepsilon(t)} - \beta'(t) \int (U - \langle U \rangle_{\varepsilon(t)}) (f_t - 1)^2 \mu_t \end{aligned}$$

$$\begin{aligned}
& -2\beta'(t) \int (U - \langle U \rangle_{\varepsilon(t)})(f_t - 1)\mu_t \\
& \leq -\frac{1}{K(R+t)}u(t) + \frac{1}{c^2(R+t)}u(t) + \frac{1}{c^2(R+t)}\sqrt{u(t)}.
\end{aligned}$$

Here, we used the Cauchy-Schwarz inequality and the conclusions of the previous Theorem 4.5. \square

Theorem 4.8. *Assume that we are in the previous settings with $c^2 > K$, so that $\sup_{t \geq 0} \|f_t\|_{L^2(\mu_t)} \leq M < \infty$. Define $U_0 = \inf_{x \in M} U(x)$ and, for every $\delta > 0$, denote by A_δ the set $A_\delta = \{x \in M \mid U(x) \geq U_0 + \delta\}$. Then we can conclude that*

$$\mathbf{P}(Z_t^z \in A_\delta) \leq M\sqrt{\mu_t(A_\delta)}$$

for every $t > 0$ and every $\delta \geq 0$.

Proof. It follows from the Cauchy-Schwarz inequality that $\mathbf{P}(Z_t^z \in A_\delta) = \int_{A_\delta} f_t \mu_t \leq M\sqrt{\mu_t(A_\delta)}$, as required. \square

Remark 4.9. *Since $\lim_{\varepsilon \rightarrow 0} \mu^{U, \varepsilon}(A_\delta) = 0$ for every $\delta > 0$, we obtain that for all continuous bounded test functions f , we have*

$$\mathbf{E}(f(Z_t^z)) \rightarrow f(x_0),$$

provided that there is only one element $x_0 \in M$ such that $U(x_0) = U_0$.

Remark 4.10. *We can improve the previous result from L^2 -estimates to L^q -estimates for $\|f_t - 1\|_{L^q(\mu_t)}$, for $q > 2$: this follows [HKS89, Theorem 2.2] and the fact that the proof of [HKS89, Theorem 2.7] applies due to a valid Sobolev inequality, i.e. for every $q > 2$ there is a constant $C_0 > 0$ such that*

$$\|f\|_{L^q(\mu)}^2 \leq C_0 \left(\frac{1}{2} \int_M \Gamma(f, f) \mu + \|f\|_{L^2(\mu)}^2 \right)$$

holds for all test functions $f : M \rightarrow \mathbb{R}$. Such a Sobolev inequality can be found for instance in [Hör67, Section 3] or in [SC84].

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