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Regularity of a degenerated convolution semi-group without to use the Poisson process

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Abstract

We translate in semi-group theory the proof of regularity of a degenerated semi-group we got a long time ago by using the Malliavin Calculus of Bismut type for jump processes

1 Introduction

Malliavin [21] recovered Hoermander's theorem by using the Brownian motion. Malliavin for that used an heavy apparatus of functional analysis: Bismut avoided to use this heavy apparatus in order to recover Hoermander's theorem by probabilistic methods [3].

Bismut's way of the Malliavin Calculus for diffusions was translated in [13], [14], [15], [16], [17] and [18] in semi-group theory.

Bismut [3] considered Poisson processes and stochastic differential equations driven by them in order to state some regularity theorems for Markov semi-groups generated by integro-differential operators. This allowed to Léandre to generalize for jump processes Hoermander's theorem ([9], [10], [11], [12]). For developments of the Malliavin Calculus for jump processes, we refer to [1], [2], [5], [7, [8], [22] and [23].

Léandre has translated Bismut's way of the Malliavin Calculus for jump processes [3] in semi-group theory ([19], [20]). This allows to [20] to recover one of the results of Léandre [9], [10], [11] [12] by avoiding the Poisson process.

Our goal is to recover another result of Léandre by using only the semi-group theory.

We will state our result, with simplified hypothesis in order to simplify the exposition.

Let $g_j(z)$ be m smooth functions from $\mathbb{R}^*$ into $\mathbb{R}^+$ such that

$$\int_{\mathbb{R}}(z^2 \wedge 1)g_j(z)dz < \infty \tag{1}$$

and on a neighborhood of 0

$$g_j(z) = \frac{C}{|z|^{1+\alpha_j}} \tag{2}$$

for some $\alpha_j \in]0,1[$.

We consider m smooth curves $\gamma_j(z)$ into $\mathbb{R}^d$ with bounded derivatives at each order such that

$$\gamma_j(0) = 0 \tag{3}$$

We do the following hypothesis:

Hypothesis H: There exists a k such that $\cup_{j,l\leq k} \frac{d^l}{dz^l}\gamma_j(0)$ spanns $\mathbb{R}^d$.

We consider the Markov generator acting on $C_b^\infty(\mathbb{R}^d)$

$$Lf(x) = \sum \int_{\mathbb{R}} (f(x+\gamma_j(z)) - f(x) \\ - 1_{|\gamma_j(z)|<1} < \gamma_j(z), grad_x f(x) >) g_j(z)dz \tag{4}$$

L spanns a convolution semi-group P_t. Our main result is the following

Theorem 1 *(Léandre [9])Under hypothesis H, the semi group P_t has a smooth heat-kernel $y \to p_t(x,y)$ for $t > 0$*

$$P_t f(x) = \int_{\mathbb{R}^m} p_t(x,y)f(y)dy \tag{5}$$

The proof of this theorem is the translation in semi-group theory of the proof of the same theorem we got in [9] by using the Malliavin Calculus of Bismut type for Poisson processes.

We consider a smooth function $\nu(z)$ with compact support with values in $\mathbb{R}^*$ equals to z^4 in a neighborhood of 0. We consider the space $\mathbb{R}^d \times \mathbb{M}_d$ where $\mathbb{M}_d$ is the space of symmetric matrices on $\mathbb{R}^d$. $(x,V) \in \mathbb{R}^d \times \mathbb{M}_d$. We consider the Malliavin generator acting on test functions $\hat{f}$ on this space

$$\hat{L}\hat{f}(x,V) = \sum \int_{\mathbb{R}} (\hat{f}(x+\gamma_j(z), V + \nu(z) < ., \gamma_j'(z) >^2) - \hat{f}(x,V) \\ - 1_{|\gamma_j(z)|<1} < \gamma_j(z), grad_x \hat{f}(x,V) >) g_j(z)dz \tag{6}$$

V is called the Malliavin matrix. The next theorem will allow us to prove the main theorem of this work.

Theorem 2 *(Bismut [3])If the Malliavin condition is checked*

$$\hat{P}_t[V^{-p}](x,0) < \infty \tag{7}$$

for all even positive integer p and for all $t > 0$, then the convolution semi-group P_t has a smooth heat-kernel.

We thank the warm hospitality of Institut Mittag Leffler, where this work was done at the occasion of an activity about stochastic partial differential equations.

2 Proof of Bismut's theorem without to use the Poisson process

Since the proof is very similar to the proof of Theorem 1 of [20], we will give only the scheme. But since we consider a convolution semi-group, the algebra is much more simple as in [20]. We will begin as in [20], part I, by elementary considerations.

Let $\hat{L}$ be the generator on $\mathbb{R}^d \times \mathbb{R}$ ($(x,u) \in \mathbb{R}^d \times \mathbb{R}$)

$$\hat{L}\hat{f}(x,u) = \sum_j \int_{\mathbb{R}} (\hat{f}(x+\gamma_j(z), u+h(z)) - \hat{f}(x,u)$$
$$- 1_{|\gamma_j(z)|<1} < \gamma_j(z), grad_x \hat{f}(x,u) >) g_j(z)dz \quad (8)$$

where $h(z) = -\nu'(z) - \nu(z)g_j'(z)/g_j(z)$ is supposed smooth.

We get as in [20] 13

$$\hat{P}_t[fu](x,0) = \int_0^t P_{t-s}[\sum_j \int_{\mathbb{R}} P_s[f](x+\gamma_j(z))h(z)g_j(z)dz] \quad (9)$$

But the right side hand of (9) is equal to

$$\int_0^t P_{t-s}[\int_{\mathbb{R}} < grad_x P_s[f](x+\gamma_j(z), \gamma_j'(z) > \nu(z)g_j(z)dz \quad (10)$$

Therefore we get:

Proposition 3 *(Bismut [3])We have the elementary integration by parts formula*

$$\hat{P}_t[fu](x,0) = \int_0^t P_{t-s}[\sum \int_{\mathbb{R}} < grad_x P_s[f](x+\gamma_j(z), \gamma_j'(z) > \nu(z)g_j(z)dz] \quad (11)$$

We put $E^l = \mathbb{R}^d \times \mathbb{M}_d \times \mathbb{R}^{d_2} \times .. \times \mathbb{R}^{d_l}$ $x^l = (x_1, v, x_2, .., x_l)$ is the generic element of this big space . We consider a bounded map $g(z)$ from $\mathbb{R}^m$ into E^l. Its values in $\mathbb{R}^d$ are $\gamma(z) = \sum \gamma_j(z)$ and the others components have compact support and are bounded by $C|z|^2$. $d\mu(z) = \sum g_j(z_j)dz_j$.

We consider the generator

$$L^l f^l(x^l) = \int_{\mathbb{R}^m} (f^l(x^l + g(z))$$
$$- f^l(x^l) - 1_{|\gamma(z)|<1} < \gamma(z), grad_{x_1} f^l(x^l) >) d\mu(z) \quad (12)$$

It generates a Markov semi-group P_t^l. The main remark is that we consider a first order operator with constant coefficient D_{x^l}, D_{x^l} commute with L^l: Therefore

$$D_{x^l} P_t^l[f^l](x^l) = P_t^l[D_{x^l} f^l](x^l) \quad (13)$$

which gives a simplified version of Proposition 7 of [?]. We suppose only in (13) that f^l has all derivatives bouded.

The second remark whose proof follows the same lines than the proof of Theorem 6 of [20] is the following theorem:

Theorem 4 *For all f^l with polynomial growth*

$$P_t^l[|f^l|](x^l) < \infty \tag{14}$$

The only change with respect of the proof of Theorem 6 of [20] is that $\gamma_j(z)$ are bounded.

We deduce that if $f^l = h(x_1)h_1(x^l)$ where h^1 is a polynomial in the others component than x_1 and h is smooth with bounded derivatives in x_1, (13) is still true.

We put $g_1(z) = \gamma'(z)\nu(z)$ for the first component and for the others component $g_1(z) = g'(z)\nu(z)$. $g^1(z)$ satisfies still the hypothesis of Theorem?: We consider the generator on $E^l \times E^l$

$$\begin{aligned}\tilde{L}^l f^l(x^l, V^l) = \\ &\int_{\mathbb{R}^m} (f^l(x^l + g(z),, V^l + g_1(z)) \\ &- f^l(x^l, V^l) - 1_{|\gamma(z)|<1} < \gamma(z), grad_{x_1} f^l(x^l, V) >) d\mu(z)\end{aligned} \tag{15}$$

This generator is of type studied before.

We consider the generator on $E^l \times \mathbb{R}$

$$\begin{aligned}\hat{L}^l \hat{f}^l(x^l, u) = &\int_{\mathbb{R}^m} (\hat{f}^l(x^l + g(z), u + h(z)) \\ &- \hat{f}^l(x^l, u) - 1_{|\gamma_j((z)|<1} < \gamma_j(z), grad_{x^1} \hat{f}^l(x^l, u) >) g_j(z) dz\end{aligned} \tag{16}$$

where $h(z) = -\nu'(z) - \nu(z)g_j'(z)/g_j(z)$ It generates a semi group $\hat{P}_t^l$.

We get the extension of Proposition 3.

Theorem 5 *We have the integration by parts formula*

$$\hat{P}_1^t[f^l u](x, 0) = \tilde{P}_t^l[< D_{x^l} f^l, V^l >](x^l, 0^l) \tag{17}$$

if f^l has a polynomial growth in each variable.

Proof: We remark first of all that $\tilde{L}^l$ commute with first order differential operator with constant coefficient. Therefore these differential operator commute with $\tilde{P}_t^l$: So $\tilde{P}_t^l$ defines a semi-group on affine functions of V^l. We consider the generator

$$\begin{aligned}\overline{L} f^l(x^l, V^l) = &\int_{\mathbb{R}^m} (f^l(x^l + g(z), V^) - \\ &f^l(x^l, V^l) - 1_{|\gamma(z)|<1} < \gamma(z), grad_{x_1} f^l(x^l, V^l) >) d\mu(z)\end{aligned} \tag{18}$$

and the generator acting on affine functions in V^l

$$\overline{L}^{l,1} f^l(x^l, V^l) = \int_{\mathbb{R}^m} (f^l(x^l + g(z), g^1(z)) - f^l(x^l + g(z), 0^l)) d\mu(z) \quad (19)$$

We have clearly on affine functions in V^l:

$$\tilde{L}^l = \overline{L}^l + \overline{L}^{l,1} \quad (20)$$

Therefore by Volterra expansion we get:

$$\tilde{P}_t^l[< D_{x^l} f^l, V^l >](x^l, 0^l) = \overline{P}_t^l[< D_{x^l} f^l, V^l >](x^l, 0^l) + \sum_n \int_{0<s_1<\ldots<s_n<t} \overline{P}_{s_1}^l \overline{L}^{l,1} \ldots . \overline{L}^{l,1} \overline{P}_{t-s_n}^l [< D_{x^l} f^l, V^l >](x^l, 0^l) ds_1 \ldots ds_n \quad (21)$$

But $\overline{P}_t^l[< D_{x^l} f^l, V^l >](x^l, V^l)$ is an affine function in V^l such that if we apply $\overline{L}^{l,1}$ to this quantity, it does not depend on V^l. Therefore the Volterra expansion reads:

$$\tilde{P}_t^l[< D_{x^l} f^l, V^l|](x^l, 0^l) = \overline{P}_t^l[< D_{x^l} f^l, V^l](x^l, 0^l) + \int_0^t P_s^l \overline{L}^{l,1} P_{t-s}^l [< D_{x^l} f^l, V^l](x^l, 0^l) ds \quad (22)$$

This last equality is nothing else that (11) in Proposition 3 but with this more complicated semi-group.
$\diamondsuit$

Proof of Bismut's theorem: We consider the Markov generator

$$L^1 f(x_1, V) = \int_{\mathbb{R}^m} f(x_1 + \gamma(z), V + \sum \gamma'(z_j) \nu(z_j) < \gamma'(z_j), . >) - f(x_1, V) - 1_{|\gamma(z)|<1} < \gamma(z), grad_{x_1} f(x, V) >) d\mu(z) \quad (23)$$

We consider the function of Bismut type

$$(x_1, V) \to < D_{x_1} f(x_1), V^{-1} e_k > \quad (24)$$

where e_k is the standard basis of $\mathbb{R}^d$ and we apply iteratively the previous integration by parts to this test function in order to get for all multi-index (α)

$$P_t[D^{(\alpha)} f](x) = P_t^l[f\theta](x, 0) \quad (25)$$

where P_t^l is a semi-group of the type considered before and θ a polynomial in the extra-variables and in the inverse of the Malliavin matrix.
$\diamondsuit$

3 Proof of Léandre's theorem without to use the Poisson process

Let us recall the uniform Tauberian theorem of [10] which are an improvement of the Tauberian theorems of [3], [6].

Let F be a set of smooth maps from $\mathbb{R}$ into $\mathbb{R}$ equal to 0 in 0 such for all k

$$\sup_{f\in F, z\in\mathbb{R}} |f^{(k)}(z)| < \infty \tag{26}$$

and such that there exists K such that

$$\inf_{f\in F}(\sup_{k\leq K} |f^{(k)}(0)|) > C > 0 \tag{27}$$

Let us define

$$\tau_f(\beta) = \int_{\mathbb{R}} (\exp[-\beta|f(z)|] - 1)g(z)dz \tag{28}$$

where $\int_{\mathbb{R}}(z^2 \wedge 1)g(z)dz < \infty$ and where $g(z) \geq 0$ Let us suppose that for some $\alpha \in]0,2[$

$$\underline{\lim}_{z\to 0}|z|^{1+\alpha}g(z) > 0 \tag{29}$$

Then

Theorem 6 *If (26), (27), and (28) are checked, there exists $\alpha_1 > 0$ such that*

$$\overline{\lim}_{\beta\to\infty}(\sup_{f\in F}\frac{\tau_f(\beta)}{\beta^{\alpha_1}}) < 0 \tag{30}$$

Leu us recall that for all $p > 0$

$$\hat{P}_t[V^p](x,0) < \infty \tag{31}$$

such that is enough to show for all $p > 0$

$$\sup_{|\xi|=1} \hat{P}_t[< V, \xi >^{-p}] < \infty \tag{32}$$

We consider the semi-group P_t^ξ associated to the generator L^ξ on $\mathbb{R}^d\times\mathbb{R}$, $(x,u) \in \mathbb{R}^d \times \mathbb{R}$:

$$\begin{aligned} L^\xi \hat{f}(x,u) = \sum \int_{\mathbb{R}} (\hat{f}(x+\gamma_j(z), u+\nu(z) < \xi, \gamma_j'(z) >^2) - \hat{f}(x,u) \\ - 1_{|\gamma_j(z)|<1} < \gamma_j(z), grad_x \hat{f}(x,u) >)g_j(z)dz \end{aligned} \tag{33}$$

We have as it was remarked by Bismut ([3])

$$\hat{P}_t[< V, \xi >^{-p}](x,0) = P_t^\xi[u^{-p}](x,0) = \Gamma(p)^{-1}\int_0^\infty \beta^{p-1}P_t^\xi[\exp[-\beta u]](x,0)d\beta \tag{34}$$

We consider the generator L^β acting on functions on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ ($(x, u, v) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$)

$$\begin{aligned} L^\beta \hat{f}(x,u,v) = \sum \int_{\mathbb{R}} (\hat{f}(x+\gamma_j(z), u + <\xi, \gamma'_j(z)>^2 \nu(z), v) - \hat{f}(x,u,v) \\ - 1_{|\gamma_j(z)|<1} < \gamma_j(z), grad_x \hat{f}(x,u,v) >] g_j(z) dz \\ + \sum \int_{\mathbb{R}} < \exp[-2\beta\nu(z) < \xi, \gamma'_j(z)>^2] - 1, grad_v \hat{f}(x,u,v) > g_j(z) dz \end{aligned} \quad (35)$$

Let us consider the function $\exp[-2\beta u - v]$. If we apply L^β to it, we find zero: Therefore

$$P_1^\beta[\exp[-2\beta u - v]](x, 0, 0) = 1 \quad (36)$$

Since we have Markov semigroups, we get the following propostion proved in [?] by using martingales theory:

Proposition 7 *(Bismut [3])For all positive p*

$$\hat{P}_t[< V, \xi >^{-p}](x, 0) \leq \Gamma(p)^{-1} \int_P^\infty \beta^{p-1} (P_1^\beta[\exp[v]](x,0,0))^{1/2} d\beta \quad (37)$$

But

$$P_t^\beta[\exp[v]](x,0,0) = \exp[\sum \int_0^t \int_{\mathbb{R}} (\exp[-2\beta\nu(z) < \gamma'_j(z), \xi >^2] - 1) g_j(z) dz \quad (38)$$

and we deduce our results by the previous uniform Tauberian theorem.

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