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ABSOLUTELY CONTINUOUS LAWS OF JUMP-DIFFUSIONS IN FINITE AND INFINITE DIMENSIONS WITH APPLICATIONS TO MATHEMATICAL FINANCE

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Abstract. In mathematical Finance calculating the Greeks by Malliavin weights has proved to be a numerically satisfactory procedure for finite-dimensional Itô-diffusions. The existence of Malliavin weights relies on absolute continuity of laws of the projected diffusion process and a sufficiently regular density. In this article we first prove results on absolute continuity for laws of projected jump-diffusion processes in finite and infinite dimensions, and a general result on the existence of Malliavin weights in finite dimension. In both cases we assume Hörmander conditions and hypotheses on the invertibility of the so-called linkage operators. The message is that for the construction of numerical procedures for the calculation of the Greeks in fairly general jump-diffusion cases one can proceed as in a pure diffusion case. We also show how the given results apply to infinite dimensional questions in mathematical Finance. There we start with a Vasiček model, and add – by pertaining no arbitrage – a jump diffusion component. We prove that we can obtain in this case an interest rate model, where the law of any projection is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^M$.

1. Introduction

We shall consider in this article the question whether the law of $\mathbf{l}(X_t^x)$, for a finite dimensional projection $\mathbf{l} : H \to \mathbb{R}^M$, is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^M$, where X_t^x is the solution of the stochastic (partial) differential equation

$$dX_t^x = (AX_{t^-}^x + \alpha(X_{t^-}^x))dt + \sum_{i=1}^{d} V_i(X_{t^-}^x)dB_t^i + \sum_{j=1}^{m} \delta_j(X_{t^-}^x)(dL_t^j - \lambda_j dt), \tag{1.1}$$

$$X_0^x = x \in H, \tag{1.2}$$

and H is a possibly infinite dimensional separable Hilbert space. We refer to the previous equation loosely speaking as a jump-diffusion on the Hilbert space H pointing out that the involved Lévy processes are of finite type. For sake of simplicity we shall always work with the càglàd integrand $t \mapsto X_{t^-}^x$, even though for the dt

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and dB_t integrals this is superfluous. In the infinite dimensional setting we are not aware of results on absolute continuity of the projected process in the jump diffusion case. Related work has been done in [5] for the construction of first variation processes. In the diffusion case we refer to the work [3] and the references therein, and in particular to the recently published inspiring results of Jonathan Mattingly, see for instance [1].

Very satisfying results in the finite dimensional setting with Lévy processes of infinite type have been obtained by Thomas Cass in [10] through a generalization of the Norris Lemma to Lévy processes. This builds upon our results presented in this work for the finite activity case (see Section 7). Substantial work with respect to absolute continuity and smoothness of the density has been already published in the eighthies, where the most prominent ones are [7] and [6]. Therein several questions of extension of hypo-ellipticity results (and Malliavin Calculus) to jump-processes are discussed and completely solved, however, the problem of a hypo-elliptic diffusion together with a finite-activity jump-structure remained open. It has to be pointed out here that – in contrast to [6] – we do not need any extension of Malliavin Calculus to jump processes for our results (see also the discussion in Remark 5). This gap was somehow filled by the announced results of [22], but we found this work very difficult to understand. Recently – motivated by questions from financial mathematics, see Section 8 for an outline of the problem – there has been increasing interest in those results, see the the works [2], [12] and [13] and the references therein.

From the point of view of existence and uniqueness for jump-diffusions in infinite dimensions our main reference is [14], and the references therein. Since we consider jump-diffusions as concetenated diffusions on Hilbert space we mention [11] as the main reference for existence and uniqueness results, but also [4] and [9] for many interesting constructions and ideas.

There are two applications added to this work. The first one is the HJM-equation (as presented in [14]), where we show that the innocent Vasiček model (see for instance the seminal work of [8]) with a certain jump structure triggered by a one-dimensional Poisson process yields – under no-arbitrage assumtions – a model, where not only no finite dimensional realizations do exist, but where every projection into a finite dimensional subspace admits a density (compare with the notion of generic interest rate evolutions from [3]. The second application is concerned with concrete formulas for the calculation of Malliavin weights. There we the message is that one can think Poisson-trajectory-wise, when applying the results.

When we analyse jump-diffusions with values in Hilbert spaces then loosely speaking the following facts hold true:

- between two consecutive jumps of the jump-diffusion we are given an ordinary diffusion.
- at a jump we add to the left limit the jump size (which usually depends on the left limit, too). In [21], Chapter V.10, Hypothesis (H3), this operation is formalized by the so called *linkage operators*. We shall apply this notion here, too.

Hence the following picture arises:

- In order to obtain absolute continuity of the projected diffusion process, we need the Hörmander condition to be in force. Otherwise we cannot expect

– conditioned on the event of positive probability that no jump occurs – that there is absolute continuity.

- In order to preserve absolute continuity we need the linkage operators to be invertible in a proper sense.

Remark 1. Both conditions are 'sine qua non', since it is easy to imagine counterexamples.

In Section 2 we fix the general setting of this article (see [21], Chapter V.3 for all necessary details on existence and uniqueness of solutions). We shall deal with Lévy processes of finite type as drivers of the stochastic differential equations, even though we believe that one should be able to prove similar results in the case of many small jumps, too. We also state the main assumptions of this work in Section 2 for later use.

In Section 3 we prove a decomposition theorem, which tells that solving a jump-diffusion SDE is the same as solving associated diffusion SDEs and concatenating the solutions by linkage operators. In Section 4 we show that we can also prove results on first variation processes in the spirit of the decomposition theorem. We prove that under our analytic requirements there is in fact a sufficiently regular first variation process.

In Section 5 we show by means of Malliavin calculus for a d-dimensional Brownian motion that the law of a projected jump-diffusion is absolutely continuous with respect to Lebesgue measure. In Section 6 we show an example from mathematical Finance, where we see very directly the phenomenon of absolute continuity arising from the introduction of jumps and the no-arbitrage condition. In Section 7 we restrict to the finite dimensional setting to show that the density of the absolutely continuous law is in fact smooth, which amounts to saying that the inverse of the covariance matrix has p-th moments for all $p \geq 1$. In Section 8 we apply the invertibility of the covariance matrix to the calculation of Greeks in Bismut's spirit. Section 9, an Appendix, proves an additional estimate for a proof of D. Nualart's book [20]. Since our observation of the additional estimate, which is implicitly already visibly, is best explained by redoing the proof, we decided to do so.

2. Setting and Assumptions

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space where the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. Let $(B_t)_{t\geq 0}$ be a d-dimensional Brownian motion and $(L_t^j)_{t\geq 0}$, $j = 1, \ldots, m$ be m independent compound Poisson processes given by

$$L_t^j := \sum_{k=1}^{N_t^j} Z_k^j,$$

where N_t^j denotes a Poisson process with jump intensity $\widetilde{\lambda_j} > 0$ and $Z^j = (Z_k^j)_{k\geq 1}$ is an i.i.d. sequence of random variables with distribution μ_j for $j = 1, \ldots, m$, such that each μ_j admits all moments. Furthermore, we want to introduce the compensated compound Poisson process

$$L_t^j - E(L_t^j) = L_t^j - \lambda_j t,$$

where $\lambda_j = E(Z_1^j)\widetilde{\lambda_j}$ is the average jump size times the jump rate.

We assume that all sources of randomness are mutually independent and that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration with respect to $(B_t, L_t^1, \ldots, L_t^m)_{t\geq 0}$.

Let H be a separable Hilbert space. We fix furthermore a strongly continuous semi-group S on H with generator A. Let $\alpha, V_1, \dots, V_d$, the diffusion vector fields, and $\delta_1, \dots, \delta_m$, the jump vector fields, be C^∞-bounded on H, that is, the vector fields are infinitely often differentiable with bounded partial derivatives of all proper orders $n \geq 1$. We consider the mild càdlàg solution $(X_t^x)_{t\geq 0}$ of a stochastic differential equation

$$dX_t^x = (AX_{t-}^x + \alpha(X_{t-}^x))dt + \sum_{i=1}^{d} V_i(X_{t-}^x)dB_t^i + \sum_{j=1}^{m} \delta_j(X_{t-}^x)(dL_t^j - \lambda_j dt), \tag{2.1}$$

$$X_0^x = x \in H. \tag{2.2}$$

See [14] for all necessary details on existence and uniqueness of the previous equation.

The previous conditions are slightly more than standard for existence and uniqueness of mild solutions, i.e. in [14] the authors need Lipschitz conditions on the vector fields, whereas we assume them to be C^∞-bounded. In order to speak about absolute continuity of projections to $\mathbb{R}^M$ we shall need more assumptions, in particular for conclusions drawn from the geometry of the given vector fields $\alpha, V_1, \dots, V_d, \delta_1, \dots, \delta_m$ several quite strong analytic requirements are necessary. We group the assumptions in three groups and indicate in each section, which assumptions we shall need.

Let $\mathbf{l} : H \to \mathbb{R}^M$ be a projection, then we want to know whether the law of $\mathbf{l}(X_t^x)$ is absolutely continuous with respect to Lebesgue measure and if – in case – the density is smooth. Following the short discussion in the introduction we need the Hörmander conditions to be in force and we need to suppose invertibility on linkage operators.

We apply the following notations for Hilbert spaces $\mathrm{dom}(A^k)$,

$$\begin{aligned}
\mathrm{dom}(A^k) &:= \left\{ h \in H \,|\, h \in \mathrm{dom}(A^{k-1}) \text{ and } A^{k-1}h \in \mathrm{dom}(A) \right\}, \\
||h||^2_{\mathrm{dom}(A^k)} &:= \sum_{i=0}^{k} ||A^i h||^2, \\
\mathrm{dom}(A^\infty) &= \bigcap_{k\geq 0} \mathrm{dom}(A^k),
\end{aligned}$$

which we need in order to specify the analytic conditions.

Assumption 1. We assume that the generator A of S generates in fact a *strongly continuous group*. We assume furthermore that $\alpha, V_1, \dots, V_d$, the diffusion vector fields, and $\delta_1, \dots, \delta_m$, the jump vector fields, are C^∞-bounded on the Hilbert spaces $\mathrm{dom}(A^k)$ for $k \geq 0$, that is, the vector fields are infinitely often differentiable with bounded partial derivatives of all proper orders $n \geq 1$ on the Hilbert space $\mathrm{dom}(A^k)$ for $k \geq 0$.

Assumption 2. We take Assumption 1 for granted, i.e. we can consider all vector fields on the space $\mathrm{dom}(A^k)$ for $k = 0, \dots, \infty$. For a proper statement of the Hörmander condition we need the drift to be the 'real' one. Therefore, we define

$$V_{\mathrm{real}}(x) := V_0(x) - \sum_{j=1}^{m} \lambda_j \delta_j(x).$$

Here we apply the notion

$$V_0(x) = Ax + \alpha(x) - \frac{1}{2}\sum_{i=1}^{d} \mathrm{T}\, V_i(x) \cdot V_i(x)$$

for $x \in \mathrm{dom}(A)$ and call V_0 the Stratonovich drift of the diffusion. Recall the (tangent) directional derivative operator T defined through

$$\mathrm{T}\, V(x) \cdot v = \frac{d}{d\epsilon}\Big|_{\epsilon=0} V(x + \epsilon v).$$

Lie brackets can only be calculated on the Fréchet space $\mathrm{dom}(A^\infty)$ and there we formulate the Hörmander condition. We assume that the distribution $\mathcal{D}(x)$ generated by the vector fields

$$\begin{aligned} &V_1(x), \ldots, V_d(x), \quad [V_i(x), V_j(x)]\ (i, j = \mathrm{real}, 1, \ldots, d), \\ &[V_i(x), [V_j(x), V_k(x)]]\ (i, j, k = \mathrm{real}, 1, \ldots, d), \quad \ldots\ldots \end{aligned} \tag{2.3}$$

is dense in H for one $x \in \mathrm{dom}(A^\infty)$.

Assumption 3. We assume that the inverse of $x \mapsto x + z\delta_j(x)$ exists and is C^∞-bounded on each $\mathrm{dom}(A^k)$ for $z \in \mathrm{supp}(\mu_j)$, $j = 1, \ldots, m$ and $k \geq 0$ (recall that μ_j was the distribution of the random variable Z_j).

Remark 2. As far as Assumption 1 is concerned we do believe that the assertions of this paper also work for semigroups, but we do not have a proof for this assertion so far.

Remark 3. The Hörmander condition could not be formulated without Assumption 1.

Example 1. In order to show examples of vector fields, which are C^∞-bounded on $\mathrm{dom}(A^k)$ consider the following structure. Let H be a separable Hilbert space and A the generator of a strongly continuous semigroup. We know that $\mathrm{dom}(A^\infty)$ is a Fréchet space and an injective limit of the Hilbert spaces $\mathrm{dom}(A^k)$ for $k \geq 0$. Following the analysis as developed in [15] (see also [18] and [17] were the analytic concepts have been originally developed), we can consider the following vector fields $V : U \subset H \to \mathrm{dom}(A^\infty)$. If V is smooth in the sense explained in [15] and has the property that its derivatives of proper order $n \geq 1$ are bounded on $U \subset H$, then V is obviously a C^∞-bounded vector field and additionally $V|_{\mathrm{dom}(A^\infty)}$ is a Banach-map-vector field in the sense of [15]. Such vector fields constitute a class, where Assumptions 1–3 can be readily checked.

3. Decomposition Theorem for Jump-Diffusions on Hilbert Spaces

In order to properly understand how to apply the Malliavin calculus, we state the following rather obvious structure theorem on jump-diffusions, which simply takes into account that stochastic integration with respect to the Poisson process follows the rules of Lebesgue-Stieltjes integration (see for instance [21] for a general exposition). The observation seems to be new in the infinite dimensional setting. Here we only need that the vector fields are C^∞-bounded on H in order to guarantee existence and uniqueness of the respective equations.

Theorem 1. *Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space and let $(B_t)_{t\geq 0}$ be a d-dimensional Brownian motion and $(L_t^j)_{t\geq 0}$ be m independent compound Poisson processes with compensator $(\lambda_j t)_{t\geq 0}$ for $j = 1, \ldots, m$, such that the filtration is the natural filtration with respect to $(B_t, L_t^1, \ldots, L_t^m)_{t\geq 0}$. Let S be a strongly continuous semigroup with generator A on H. Let $\alpha, V_1, \ldots, V_d$, the diffusion vector fields, and $\delta_1, \ldots, \delta_m$, the jump vector-fields, be C^∞-bounded on H and consider the càdlàg solution $(X_t^x)_{0\leq t\leq T}$ of a stochastic differential equation*

$$dX_t^x = (AX_{t^-}^x + \alpha(X_{t^-}^x))dt + \sum_{i=1}^{d} V_i(X_{t^-}^x)dB_t^i + \sum_{j=1}^{m} \delta_j(X_{t^-}^x)(dL_t^j - \lambda_j dt), \tag{3.1}$$

$$X_0^x = x. \tag{3.2}$$

Let η denote a piecewise constant, càdlàg trajectory $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ of the compound Poisson process L with finitely many jumps on compact intervals and starting at 0. We consider

$$dY_{s,t}^{x,\eta} = \Big(AY_{s,t^-}^{x,\eta} + \alpha(Y_{s,t^-}^{x,\eta}) - \sum_{j=1}^{m} \lambda_j \delta_j(Y_{s,t^-}^{x,\eta})\Big)dt + \sum_{i=1}^{d} V_i(Y_{s,t^-}^{x,\eta})dB_t^i + \sum_{j=1}^{m} \delta_j(Y_{s,t^-}^{x,\eta})d\eta^j(t), \tag{3.3}$$

$$Y_{s,s}^{x,\eta} = x. \tag{3.4}$$

Then $(Y_{s,t}^{x,\eta})_{s\geq 0}$ can be given explicitly in terms of the jump times τ_n of η for $n \geq 0$ and the diffusion process between two consecutive jumps:

$$Y_t^{x,\eta} := Y_{0,t}^{x,\eta} \quad \text{for } 0 \leq t < \tau_1$$

$$Y_t^{x,\eta} := Y_{\tau_1,t}^{y,\eta}\Big|_{y=Y_{0,\tau_1^-}^{x,\eta} + \sum_{j=1}^m \delta_j(Y_{0,\tau_1^-}^{x,\eta})\Delta\eta^j(\tau_1)} \quad \text{for } \tau_1 \leq t < \tau_2$$

$$\vdots$$

$$Y_t^{x,\eta} := Y_{\tau_{n-1},t}^{y,\eta}\Big|_{y=Y_{0,\tau_{n-1}^-}^{x,\eta} + \sum_{j=1}^m \delta_j(Y_{0,\tau_{n-1}^-}^{x,\eta})\Delta\eta^j(\tau_{n-1})} \quad \text{for } \tau_{n-1} \leq t < \tau_n$$

Here we write $\Delta\eta(t) := \eta(t) - \eta(t^-)$ for $t \geq 0$. We define the process $(Y_t^{x,L})_{t\geq 0}$ by inserting the compound Poisson process L for η in $(Y_t^{x,\eta})_{t\geq 0}$. The resulting process $(Y_t^{x,L})_{t\geq 0}$ is then indistinguishable from $(X_t^x)_{t\geq 0}$.

Proof. For the proof we refer to [21], Chapter V.10, Theorem 57, in particular with respect to the conditioning on the jump part. The proof remains unchanged in the infinite dimensional setting, see [14] for the existence and uniqueness proof on separable Hilbert spaces. □

Remark 4. For future use we shall always assume that the first jumping time of η is strictly positive $\tau_1 > 0$ and that each time corresponds to the jump of exactly one coordinate process L^j, which is true for almost all trajectories of the compound Poisson process L. Notice that the dependence of $(Y_t^{x,\eta})_{t\geq 0}$ on the jump times of η is continuous, but certainly not smooth.

Remark 5. Notice that one can also interpret the result in the following way: consider the solution $(X_t^x)_{t\geq 0}$ of equation (3.1) as an element of $L^2(\Omega_1 \times \Omega_2; H)$, where Ω_1 carries the Brownian motion part (with natural filtration), Ω_2 carries the

Poisson part (with natural filtration) and $\Omega_1 \times \Omega_2$ is equipped with the respective product σ-algebras. Then we know by Fubini's Theorem that

$$L^2(\Omega_1 \times \Omega_2; H) = L^2(\Omega_2; L^2(\Omega_1; H)).$$

The previous theorem only clarifies the jump-diffusion structure of the dependence on Ω_2. In other words, between jumps we have ordinary diffusions, and at a jump we link by linkage operators.

4. First Variation Processes

In order to calculate Malliavin derivatives, which is crucial for arguments on absolute continuity, we need precise statements on first variation processes of jump diffusions. For later purposes, but also in order to see results on the inverse of the first variation process easily, we write our equations in the Stratonovich notation. This is not innocent in infinite dimensions, since mild solutions are in general **not** semi-martingales and therefore the Stratonovich notation fails to be applicable in general. However, by Assumption 1 we are able to determine whether we are given a semi-martingale or not by analysing the initial value of the process.

In this section we need Assumption 1 and 3 to be in force.

Given the stochastic differential equation (3.1), we can switch to Stratonovich notation for $x \in \operatorname{dom}(A)$, and obtain

$$dX_t^x = V_0(X_{t^-}^x)dt + \sum_{i=1}^{d} V_i(X_{t^-}^x) \circ dB_t^i + \sum_{j=1}^{m} \delta_j(X_{t^-}^x)(dL_t^j - \lambda_j dt)$$

with the Stratonovich drift given by

$$V_0(x) := Ax + \alpha(x) - \frac{1}{2}\sum_{i=1}^{d} \mathrm{T}\, V_i(x) \cdot V_i(x)$$

for $x \in \operatorname{dom}(A)$. Recall the tangent (derivative) operator T

$$\mathrm{T}\, V(x) \cdot v = \frac{d}{d\epsilon}\Big|_{\epsilon=0} V(x + \epsilon v).$$

We do also consider the stochastic differential equation (3.3) with respect to one trajectory η and switch to Stratonovich notation, too. The following theorem states the result on the first variation process along one trajectory η, which yields in the sequel the same result by inserting the compound Poisson process L for η. Notice that the trajectory η is such that the first jumping time is strictly positive and that at each jumping time τ_n for $n \geq 1$ only one coordinate jumps.

Theorem 2. *Assume Assumption 1 and 3. We fix $k \geq 0$. The* first variation process $(J_{s\to t}(x,\eta))_{t\geq s}$ *associated with* $(Y_t^{x,\eta})_{t\geq 0}$ *on* $\operatorname{dom}(A^k)$ *is well defined and* *satisfies the stochastic differential equation*

$$
\begin{aligned}
(4.1)\qquad dJ_{s\to t}(x,\eta)\cdot h = \Big(& AJ_{s\to t-}(x,\eta)\cdot h + \mathrm{T}\,\alpha(Y^{x,\eta}_{s,t^-})\cdot J_{s\to t^-}(x,\eta)\cdot h \\
& - \sum_{j=1}^{m} \lambda_j\, \mathrm{T}\,\delta_j(Y^{x,\eta}_{s,t^-})\cdot J_{s\to t^-}(x,\eta)\cdot h\Big) dt \\
& + \sum_{i=1}^{d} \Big(\mathrm{T}\,V_i(Y^{x,\eta}_{s,t^-})\cdot J_{s\to t^-}(x,\eta)\cdot h\Big) dB^i_t \\
& + \sum_{j=1}^{m} \Big(\mathrm{T}\,\delta_j(Y^{x,\eta}_{s,t^-})\cdot J_{s\to t^-}(x,\eta)\cdot h\Big) d\eta^j(t), \\
(4.2)\qquad J_{s\to s}(x,\eta)\cdot h = h, &
\end{aligned}
$$

for $h, x \in \mathrm{dom}(A^k)$ and $t \geq s$. The Itô equation has a unique global mild solution for $h, x \in \mathrm{dom}(A^k)$ and $J_{s\to t}(x,\eta)$ defines a continuous linear operator on $\mathrm{dom}(A^k)$, which is invertible if $x \in \mathrm{dom}(A^{k+1})$.
The Stratonovich equation on $\mathrm{dom}(A^k)$ in turn is only well-defined for $h, x \in \mathrm{dom}(A^{k+1})$. We apply the (formal) notation here

$$
\mathrm{T}\,V_0(x)v = Av + \mathrm{T}\,\alpha(x)v - \frac{1}{2}\,\mathrm{T}(x \mapsto \sum_{i=1}^{d} \mathrm{T}\,V_i(x)\cdot V_i(x))v
$$

for $x \in \mathrm{dom}(A)$ and $v \in \mathrm{dom}(A)$.

$$
\begin{aligned}
dJ_{s\to t}(x,\eta)\cdot h = \Big(& \mathrm{T}\,V_0(Y^{x,\eta}_{s,t^-})\cdot J_{s\to t^-}(x,\eta)\cdot h - \\
& - \sum_{j=1}^{m} \lambda_j\, \mathrm{T}\,\delta_j(Y^{x,\eta}_{s,t^-})\cdot J_{s\to t^-}(x,\eta)\cdot h\Big) dt \\
(4.3)\qquad & + \sum_{i=1}^{d} \Big(\mathrm{T}\,V_i(Y^{x,\eta}_{s,t^-})\cdot J_{s\to t^-}(x,\eta)\cdot h\Big) \circ dB^i_t \\
& + \sum_{j=1}^{m} \Big(\mathrm{T}\,\delta_j(Y^{x,\eta}_{s,t^-})\cdot J_{s\to t^-}(x,\eta)\cdot h\Big) d\eta^j(t), \\
J_{s\to s}(x,\eta)\cdot h = h, &
\end{aligned}
$$

for $h, x \in \mathrm{dom}(A^{k+1})$ and $t \geq s$. The adjoint of the inverse

$$
Z^{x,h}_t := \left(J_{s\to t}(x,\eta)^{-1}\right)^* \cdot h,
$$

if it exists, should satisfy the following Stratonovich equation at the point x in direction h,

$$
\begin{aligned}
dZ_t^{x,h} = &-\Big(\mathrm{T}\,V_0(Y_{s,t^-}^{x,\eta})^* \cdot Z_{t^-}^{x,h} - \sum_{j=1}^m \lambda_j\,\mathrm{T}\,\delta_j(Y_{s,t^-}^{x,\eta})^* \cdot Z_{t^-}^{x,h}\Big)dt- \\
&- \sum_{i=1}^d \Big(\mathrm{T}\,V_i(Y_{s,t^-}^{x,\eta})^* \cdot Z_{t^-}^{x,h}\Big) \circ dB_t^i - \\
&- \sum_{j=1}^m \Big(\mathrm{T}\,\delta_j(Y_{s,t^-}^{x,\eta})^* \cdot Z_{t^-}^{x,h}\Big) d\eta^j(t) \\
&+ \sum_{j=1}^m \Big(\Big(\mathrm{T}\,\delta_j(Y_{s,t^-}^{x,\eta})^2\Big)^* \cdot \Big(\Big(id_H + \Delta\eta^j(t)\,\mathrm{T}\,\delta_j(Y_{s,t^-}^{x,\eta})\Big)^{-1}\Big)^* \cdot \\
&\qquad \cdot Z_{t^-}^{x,h}(\Delta\eta^j(t))^2\Big),
\end{aligned} \tag{4.4}
$$

for $h, x \in \mathrm{dom}(A^{k+1})$ and $t \geq s \geq 0$ (Here we applied the notions of [21]*).*

Remark 6. The completely analogous theorem holds when we replace η by a compound Poisson process L. We do not state this theorem again, but we point out that we even have moment estimates for the respective processes, which is the only additional relevant information. To be precise, the first variation process $J_{s\to t}(x) \cdot h$, which equals $J_{s\to t}(x, L)$ by construction, has bounded second moments by [14].

Proof. Under our Assumption 1 the regularity in the initial values is clear by well-known results from [11] and the chain rule on Hilbert spaces (recall that the linkage operators are smooth). We are allowed to pass to the Stratonovich decomposition since we integrate semi-martingales by Itô's formula on Hilbert spaces for $x, h \in \mathrm{dom}(A^{k+1})$ due to the arguments of [3]: the core assertion is here that we can replace H by each $\mathrm{dom}(A^k)$ for some $k \geq 0$, which means in turn if we start in $\mathrm{dom}(A^{k+1})$ and obtain a mild solution there, it is indeed a strong solution considered on $\mathrm{dom}(A^k)$, for $k \geq 0$. It remains to show the invertibility results on the respective first variation processes.

Left invertibility of the first variation $J_{s\to t}(y, .)$ follows by Itô's formula since we have càdlàg trajectories with finitely many jumps. Calculating the semi-martingale decomposition of $(Z_t^{y,\cdot})^* \cdot J_{0\to t}(x, \eta)$ given by equations (4.3) and (4.4) yields the result

$$(Z_t^{x,\cdot})^* J_{0\to t}(x, \eta) = id_{\mathrm{dom}(A^k)}.$$

Thus, the solution of equation (4.4) is the left inverse of $J_{s\to t}$.

We prove that the left inverse is also the right inverse by the same reasoning as in the proof of Proposition 2 in [3]. Therefore, we choose an orthonormal basis $(g_i)_{i\geq 1}$ of $\mathrm{dom}(A^k)$ which lies in $\mathrm{dom}(A^{k+1})$. Then we can compute the semi-martingale decomposition of

$$
\begin{aligned}
&\sum_{i=1}^N \langle (Z_t^{x,h_1})^*, g_i\rangle_{\mathrm{dom}(A^k)} \langle g_i, J_{s\to t}(x,\eta)^* \cdot h_2\rangle_{\mathrm{dom}(A^k)} \quad = \\
&\quad \sum_{i=1}^N \langle h_1, Z_t^{x,g_i}\rangle_{\mathrm{dom}(A^k)} \langle J_{s\to t}(x,\eta) \cdot g_i, h_2\rangle_{\mathrm{dom}(A^k)},
\end{aligned}
$$

for $h_1, h_2 \in \operatorname{dom}(A^{k+1})$ and $N \geq 1$. Applying the Stratonovich decomposition and by adjoining we can free the g_i's and pass to the limit, which yields vanishing finite variation and martingale part. Hence

$$\langle J_{s\to t}(x,\eta)(Z_t^{x,h_1})^*, h_2\rangle_{\operatorname{dom}(A^k)} =$$

$$\lim_{N\to\infty} \sum_{i=1}^{N} \langle (Z_t^{x,h_1})^*, g_i\rangle_{\operatorname{dom}(A^k)} \langle g_i, J_{s\to t}(x,\eta)^* \cdot h_2\rangle_{\operatorname{dom}(A^k)}$$
$$= \langle h_1, h_2, \rangle_{\operatorname{dom}(A^k)},$$

which is what a right inverse should satisfy. □

5. Absolutely continuous laws in finite and infinite dimensions

In this section we assume Assumptions 1, 2 and 3. We want to determine by means of the Malliavin calculus whether the law of $\mathbf{l}(Y_t^{x,\eta})$ is absolutely continuous with respect to Lebesgue measure for $t > 0$.

For details on Malliavin calculus see [19] and [20], where in particular the derivative operator and the Skorohod integral for Malliavin Calculus with respect to a d-dimensional Brownian motion are defined.

Our first task is the calculation of the Malliavin derivative for a fixed càdlàg path η. In a second step, we consider the composed problem, where we replace η by a compound Poisson process L as outlined before. Therefore, we first fix a piecewise constant, càdlàg trajectory $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ of the process $(L_t^1, \ldots, L_t^m)_{t\geq 0}$ with finitely many jumps on compact intervals starting at 0.

Theorem 3. *We take Assumptions 1, 2 and 3 for granted, where $x \in \operatorname{dom}(A^\infty)$ denotes the point where the Hörmander condition (2.3) holds true. Let $(Y_t^x)_{t\geq 0}$ denote the unique càdlàg solution of equation (3.3). Then for projections $\mathbf{l} : H \to \mathbb{R}^M$ the law of $\mathbf{l}(Y_t^x)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^M$ for $t > 0$.*

Proof. Fix $t > 0$. We are able to write the Malliavin derivative of Y_t^x for each Poissonian trajectory η,

$$D_s^i(\mathbf{l} \circ Y_t^{x,\eta}) = \mathbf{l} \circ J_{0\to t}(x) J_{0\to s}(x)^{-1} V_i(Y_{s^-}^{x,\eta}) 1_{[0,t]}(s).$$

We can calculate the *Malliavin covariance matrix* γ as

$$\langle \gamma(\mathbf{l} \circ Y_t^{x,\eta})\xi, \xi\rangle := \sum_{i=1}^{d} \int_0^t \langle \mathbf{l} \circ J_{0\to t}(x) J_{0\to s}(x)^{-1} V_i(Y_{s^-}^{x,\eta}), \xi\rangle^2 ds.$$

Consequently, the covariance matrix $\gamma(\mathbf{l} \circ Y_t^{x,\eta})$ can be calculated in the usual way via the reduced covariance matrix

$$\langle C_t\xi, \xi\rangle := \sum_{i=1}^{d} \int_0^t \left\langle J_{0\to s}(x)^{-1} V_i(Y_{s^-}^{x,\eta}), \xi\right\rangle^2 ds.$$

with the relation $\gamma(\mathbf{l} \circ Y_t^{x,\eta}) = (\mathbf{l} \circ J_{0\to t}(x)) C_t (\mathbf{l} \circ J_{0\to t}(x))^*)$, where $*$ denotes as usual the adjoint operator with respect to the Hilbert space structures on H and $\mathbb{R}^M$. We assume n jumps of η on $[0,t]$ and we denote by $0 = \tau_0 < \tau_1 < \ldots < \tau_n \leq t$ the sequence of jump times of η. For convenience, we denote the last point in time t by τ_{n+1}, even if $\tau_{n+1} = \tau_n$, which can in principle happen. Hence we can decompose,

$$\langle C_t\xi, \xi\rangle := \sum_{k=0}^{n}\sum_{i=1}^{d}\int_{\tau_k}^{\tau_{k+1}} \left\langle J_{0\to s}(x)^{-1}V_i(Y^{x,\eta}_{s^-}), \xi\right\rangle^2 ds = \sum_{k=0}^{n}\langle C^k_t\xi, \xi\rangle.$$

Each of the summands determines a symmetric matrix C^k_t and can be interpreted as a reduced covariance matrix coming from a diffusion between τ_k and τ_{k+1} with initial value $Y^x_{\tau_k^-}$ for $k = 0, \dots, n$. We do not know whether the Hörmander condition is true everywhere. Therefore, we do not know whether C^k_t is a positive definite operator for all $k \geq 0$. From [3], Theorem 1, we do know, however, that each C^0_t is a positive definite operator and there exist null sets N_0 such that on N_0^c the matrix C^0_t is invertible. Hence the law of $(\mathbf{l}\circ Y^{x,\eta}_t)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^M$, since $J_{0\to t}(x)$ is invertible and therefore $\gamma(\mathbf{l}\circ Y^{x,\eta}_t)$ has empty kernel (Theorem 2.1.2 in [20], p. 86). □

Remark 7. The same conclusions hold for $Y^{x,\eta}_{t^-}$: notice that $Y^{x,\eta}_t = Y^{x,\eta}_{t^-}$, if there is no jump at t. Otherwise $Y^{x,\eta}_t = Y^{x,\eta}_{t^-} + \sum_{j=1}^m \delta_j(Y^{x,\eta}_{0,t^-})\Delta\eta^j(t)$, but invertible diffeomorphisms transform absolutely continuous laws in absolutely continuous ones.

Now we wish to extend this theorem to the jump-diffusion process $(X^x_t)_{t\geq 0}$, which is easy since – conditioned on one trajectory η we do have an absolutely continuous law and this property is not perturbed by integration due to Fubini's Theorem.

Theorem 4. *We take Assumptions 1, 2 and 3 for granted, where $x \in \mathrm{dom}(A^\infty)$ denotes the point where the Hörmander condition (2.3) holds true. Let $(X^x_t)_{t\geq 0}$ denote the unique càdlàg solution of equation (3.1). Then for projections $\mathbf{l} = (l_1, \dots, l_k) : H \to \mathbb{R}^M$ the law of $\mathbf{l}(X^x_t)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^M$ for $t > 0$. Notice that $\mathbf{l}(X^x_t)$ and $\mathbf{l}(X^x_{t^-})$ have the same distribution.*

Proof. The proof applies the following simple corollary of Fubini's theorem on $\mathbb{R}^M$ with Lebesgue measure λ and a probability space $(\Omega, \mathcal{F}, P)$: let ν be a probability measure on $\mathbb{R}^M \times \Omega$ such that there is random density $p : \mathbb{R}^M \times \Omega \to \mathbb{R}_{\geq 0}$ with

$$\int_{\mathbb{R}^M\times\Omega} f(x,\eta)p(x,\eta)(\lambda\otimes P)(dx, d\eta) = \int_{\mathbb{R}^M\times\Omega} f(x,\eta)\nu(dx, d\eta),$$

then the marginal of ν on $\mathbb{R}^M$ is absolutely continuous with density

$$p(x) = \int_\Omega p(x,\eta)P(d\eta)$$

for almost all $x \in \mathbb{R}^M$. In our case we know that the law of X^x_t is absolutely continuous for almost all trajectories η of the compound Poisson processes L (precisely those with $\tau_1 > 0$, only one coordinate jumps at each jumping time, and finitely many jumps on compact intervals), the probability measure ν corresponds to the distribution of (X^x_t, L), where we choose Ω the space of càdlàg trajectories on $\mathbb{R}_{\geq 0}$ with values in $\mathbb{R}^m$. Finally we have that the law of X^x_t is $p(x)\lambda(dx)$. □

6. Applications of the infinite dimensional result to Interest Rate Theory

We consider the following HJM-model with jumps

$$\begin{cases} dr_t &= \left(\frac{d}{dx}r_{t^-} + \alpha_{HJM}(r_{t^-}) + \beta(r_{t^-})\right)dt + \sigma(r_{t^-})dB_t + \delta(r_{t^-})dN_t \\ r_0 &= r^* \in H \end{cases}$$

on some Hilbert space H of forward rate curves as constructed in [3] and [14]. See also the seminal article [8], which influenced these lines of research very much. $(B_t)_{t\geq 0}$ is a standard Brownian motion and $(N_t)_{t\geq 0}$ is a standard Poisson process with intensity $\widetilde{\lambda} > 0$ and jump measure $\mu = \delta_1$ (hence $\widetilde{\lambda} = \lambda$). Define

$$\begin{aligned} \Psi_1(z) &\equiv \ln \mathbb{E}\left[e^{zW_1}\right] = \ln\left(e^{\frac{z^2}{2}}\right) = \frac{z^2}{2} \quad \text{and} \\ \Psi_2(z) &\equiv \ln \mathbb{E}\left[e^{zN_1}\right] = \ln\left(\exp\left(\lambda\left(e^z - 1\right)\right)\right) = \lambda\left(e^z - 1\right). \end{aligned}$$

Then we know from [14], equation (2.4), that

$$\begin{aligned} \alpha_{HJM}(r)(x) &= -\sigma(r)(x)\Psi_1'\left(-\int_0^x \sigma(r)(y)dy\right) \\ &= \sigma(r)(x)\int_0^x \sigma(r)(y)dy \end{aligned}$$

and

$$\begin{aligned} \beta(r)(x) &= -\delta(r)(x)\Psi_2'\left(-\int_0^x \delta(r)(y)dy\right) \\ &= -\lambda\delta(r)(x)\exp\left(-\int_0^x \delta(r)(y)dy\right). \end{aligned}$$

For an explicit example we choose

$$\sigma(r)(x) = \sigma(x) > 0 \quad \text{and} \quad \delta(r)(x) = -\frac{d}{dx}\ln(Br)(x),$$

where the vector field B will be determined later. Then we have

$$\alpha_{HJM}(r)(x) = \sigma(x)\int_0^x \sigma(y)dy$$

and

$$\beta(r)(x) = \lambda\frac{d}{dx}\ln(Br)(x)\frac{(Br)(x)}{(Br)(0)} = \lambda\frac{\frac{d}{dx}(Br)(x)}{(Br)(0)}.$$

We choose B such that $(Br)(x)$ is positive on an open subset $U \subset H$, for $x \in \mathbb{R}$ and $B(r)(0) = 1$ for all $r \in U$. Hence δ is well defined. Thus, for such $r \in U$ we have

$$\beta(r)(x) = \lambda\frac{d}{dx}(Br)(x)$$

and

$$\delta(r)(x) = -\frac{d}{dx}\ln(Br)(x).$$

A particular choice in the spirit of Remark 1 is given through

$$B(r)(x) = \psi(x, l(r)),$$

where the maps $y \mapsto \frac{d}{dx}\psi(., y)$ and $y \mapsto \frac{1}{\psi(., y)}$ from $\mathbb{R}$ to $\mathrm{dom}(A^\infty) \subset H$ are supposed to be C^∞-bounded with $\psi(0, y) = 1$ for all $y \in \mathbb{R}$. The map l denotes here a non-vanishing linear functional $l : H \to \mathbb{R}$. Hence δ and β are well-defined

C^∞-bounded vector fields on the whole Hilbert space and we have global existence of mild solutions.

We know that the Vasiček-model (the model without jumps) admits a finite dimensional realizations as for

$$V_{\mathrm{real}}(r)(x) = \frac{d}{dx} r(x) + \alpha_{HJM}(r)(x)$$

we have

$$\dim(\{V_{\mathrm{real}}, \sigma\}_{LA}(r)) \leq 2$$

with an appropriate σ at any point $r \in \mathrm{dom}((\frac{d}{dx})^\infty)$. If we switch on the jump structure and choose ψ generic, this two dimensional structure (a regular finite dimensional realization in the sense of [15]) is destroyed, since then the drift changes due to no-arbitrage. We obtain a dense Lie algebra if we choose the vector field B generically.

7. Smooth densities for the law X_T^x on $\mathbb{R}^M$

In the sequel we consider the case $\dim H = M$ and $\mathbf{l} = \mathrm{id}$ and we choose a coordinate representation $H = \mathbb{R}^M$. We then want to show that the p-th power of the inverse of the Malliavin covariance matrix of X_t^x for $t > 0$ can be integrated even with respect to Poisson trajectory η. We therefore need an extension of the Hörmander condition, which is called the uniform Hörmander condition:

Following [20], we define

$$\Sigma'_0 := \{V_1, \dots, V_d\}$$

$$\Sigma'_n := \left\{ [V_k, V], k = 1, \dots, d, V \in \Sigma'_{n-1}; [V_{\mathrm{real}}, V] + \frac{1}{2} \sum_{i=1}^{d} [V_i, [V_i, V]], V \in \Sigma'_{n-1} \right\}$$

for $n \geq 1$. We assume that there exists j_0 and $c > 0$ such that

$$\inf_{\xi \in S^{M-1}} \sum_{j=0}^{j_0} \sum_{V \in \Sigma'_j} \langle V(x), \xi \rangle^2 \geq c$$

uniformly in $x \in \mathbb{R}^M$.

Theorem 5. *Assume that* $\dim H < \infty$*. We take Assumptions 2 and 3 for granted, but assume that the Hörmander condition (2.3) holds true uniformly on* $\mathbb{R}^M$*. Let* $(X_t^x)_{t \geq 0}$ *denote the unique càdlàg solution of equation (3.1) and fix* $t > 0$*. Then the random variable* X_t^x *admits a smooth density with respect to Lebesgue measure on* $\mathbb{R}^M$*. Furthermore, the covariance matrix of* X_t^x *is invertible with p-integrable inverse for all* $p \geq 1$*.*

Proof. We write the Malliavin derivative of X_t^x,

$$D_s^i X_t^x = J_{0 \to t}(x) J_{0 \to s}(x)^{-1} V_i(X_{s-}^x) 1_{[0,t]}(s).$$

and calculate the reduced covariance matrix

$$\langle C_t \xi, \xi \rangle = \sum_{i=1}^{d} \int_0^t \left\langle J_{0 \to s}(x)^{-1} V_i(X_{s-}^x), \xi \right\rangle^2 ds.$$

We now apply the result from Theorem 1 and condition on the trajectories of the compound Poisson process (2)

$$
\begin{aligned}
(7.1)\qquad & \sup_{\xi\in S^{M-1}} P(\langle C_t\xi,\xi\rangle<\epsilon)\\
&= \sup_{\xi\in S^{M-1}} \sum_{n_1,\dots,n_m\geq 0}\Big[\prod_{k=1}^m P(N_t^k=n_k)\Big] P(\langle C_t\xi,\xi\rangle<\epsilon | N_t^j=n_j \text{ for } j=1,\dots,m).
\end{aligned}
$$

As in the proof of Theorem 3, we can decompose $\langle C_t\xi,\xi\rangle$ into

$$\langle C_t\xi,\xi\rangle=\sum_{k=0}^{\infty}\sum_{i=1}^{d}\int_{\tau_k\wedge t}^{\tau_{k+1}\wedge t}\left\langle J_{0\to s}(x)^{-1}V_i(X_{s-}^x),\xi\right\rangle^2 ds,$$

where $\tau_0=0<\tau_1<\ldots<\tau_n\leq\ldots$ denotes the sequence of jump times of $(N_t)_{0\leq t\leq T}$. Hence, we obtain for $n=n_1+\cdots+n_m$

$$
\begin{aligned}
&\sup_{\xi\in S^{M-1}} P(\langle C_t\xi,\xi\rangle<\epsilon | N_t^j=n_j,\ j=1,\dots,m)\leq\\
&\sup_{\xi\in S^{M-1}} P\left(\sum_{i=1}^{d}\int_{\tau_k\wedge t}^{\tau_{k+1}\wedge t}\left\langle J_{0\to s}(x)^{-1}V_i(X_{s-}^x),\xi\right\rangle^2 ds<\epsilon\Big| N_t^j=n_j,\ j=1,\dots,m\right),
\end{aligned}
$$

for all $0\leq k\leq n$. Observing that $\max_{0\leq k\leq n}(\tau_{k+1}-\tau_k)^{K(p)}\geq(\frac{t}{n})^{K(p)}$ (after all we only have n jumps, so the maximal distance between two consecutive jumps is bigger than $\frac{t}{n}$), we finally obtain

$$P(\langle C_t\xi,\xi\rangle<\epsilon|N_t^j=n_j \text{ for } j=1,\dots,m)\leq\epsilon^p$$

for $0\leq\epsilon\leq(\frac{t}{n})^{K(p)}\epsilon_0(p)$ due to the calculations outlined in the Appendix. Note that we can apply the calculations from the Appendix, since $J_{0\to s}(x)^{-1}$ is well-defined and bounded due to boundedness of $(id+zd\delta_j)^{-1}$ for $z\in\operatorname{supp}(\mu_j)$ and $j=1,\dots,m$. Hence integration with respect to the measures μ_j is possible and yields finite bounds. Recall also that μ_j has moments of all orders, hence X_t^x is L^p and so is $J_{0\to s}(x)^{-1}$ (see [21] for all details on SDEs).

Let $\Lambda=\inf_{\xi\in S^{M-1}}\langle C_t\xi,\xi\rangle$ be the smallest eigenvalue of the reduced covariance matrix C_t. Following the steps of [20], Lemma 2.3.1, we know that

$$P(\Lambda<\epsilon\mid N_t^j=n_j \text{ for } j=1,\dots,m)\leq const\cdot\epsilon^p$$

for any $p\geq 2$ and $0\leq\epsilon\leq\left(\frac{t}{n}\right)^{K(p+2M)}\epsilon_0(p+2M)=:\epsilon_{max}$, where the constant depends on the p-norm of C_t. In the sequel we shall denote any constant of this type by D. We denote by ρ the law of Λ conditioned on $N_t^j=n_j$ for $j=1,\dots,m$.

Consequently, for $j = 1, \ldots, m$, we have by Fubini's Theorem

$$\begin{aligned} E\left(\frac{1}{\Lambda^{p-1}} \;\Big|\; N_t^j = n_j\right) &= E\left(\frac{1}{\Lambda^{p-1}} \cdot \mathbf{1}_{\{\Lambda > \epsilon_{max}\}} \;\Big|\; N_t^j = n_j\right) \\ &\quad + E\left(\frac{1}{\Lambda^{p-1}} \cdot \mathbf{1}_{\{\Lambda \le \epsilon_{max}\}} \;\Big|\; N_t^j = n_j\right) \\ &\le \frac{1}{\epsilon_{max}^{p-1}} + \int_0^{\epsilon_{max}} \frac{1}{z^{p-1}} \rho(dz) \\ &= \frac{1}{\epsilon_{max}^{p-1}} + \int_0^{\epsilon_{max}} (p-1) \int_z^{\infty} \frac{1}{t^p} dt \rho(dz) \\ &= \frac{1}{\epsilon_{max}^{p-1}} + (p-1) \int_0^{\epsilon_{max}} \frac{1}{z^p} \int_0^z \rho(dt) dz + \\ &\quad + (p-1) \int_{\epsilon_{max}}^{\infty} \frac{1}{z^p} \int_0^{\epsilon_{max}} \rho(dt) dz \\ &\le \frac{D}{\epsilon_{max}^{p-1}} + D \underbrace{\int_0^{\epsilon_{max}} \frac{1}{z^p}\, z^p dz}_{=\epsilon_{max}} \\ &\le D(\frac{t}{n})^{K(p+2M)} \epsilon_0(p+2M) + \\ &\quad + \frac{D}{[(\frac{t}{n})^{K(p+2M)} \epsilon_0(p+2M)]^{p-1}}. \end{aligned}$$

Hence through the decomposition (7.1),

$$\begin{aligned} E\left(\frac{1}{\Lambda^{p-1}}\right) &\le \sum_{n_1, \ldots, n_m > 0} \prod_{k=1}^{m} P(N_t^k = n_k) \cdot \\ &\cdot D \cdot \left[\left(\frac{t}{n}\right)^{K(p+2M)} \epsilon_0(p+2M) + \frac{1}{[(\frac{t}{n})^{K(p+2M)} \epsilon_0(p+2M)]^{p-1}} \right] < \infty \end{aligned}$$

the result follows by $n = n_1 + \cdots + n_m$ and by the following fact for any real number K,

$$\sum_{n_1, \ldots, n_m > 0} \frac{\lambda_1^{n_1} \cdots \lambda_m^{n_m}}{n_1! \cdots n_m!} e^{-\lambda_1 n_1 - \cdots - \lambda_m n_m} (n_1 + \cdots + n_m)^K < \infty. \qquad \square$$

8. Calculating the Greeks in finite dimension

In the sequel we consider the case $\dim H = M$ and $\mathbf{l} = \mathrm{id}$ as in the previous section. Once we are given an invertible Malliavin covariance matrix with p-integrable inverse such as in Theorem 5, we can easily calculate derivatives with respect to initial values and obtain explicit formulas for so-called Malliavin weights (see [16] for successful applications of this method in mathematical finance). We sum up quickly the main idea: in mathematical Finance the gradient of the function $x \mapsto E(f(X_t^x))$ has the ratios of sensitivities, which control the hedging portfolios away from jumps. Hence for any calculation of the price $E(f(X_t^x))$ of a certain derivative at maturity $t > 0$ it is crucial to also know $\nabla E(f(X_t^x))$ to set up hedging portfolios (for the diffusion component).

Very often pricing results in a weak-approximation-scheme for the process X. So in general the only way to obtain the gradient for measurable, non-differentiable claims f is to perform a numerical differentiation, which can get cumbersome in many cases. If we have a Malliavin weight, we can calculate along the same scheme during the same simulation the Malliavin weight, and therefore obtain finally the $n+1$ values $E(f(X_t^x)))$ and $\nabla E(f(X_t^x))$ after only one simulation. Hence the precise knowledge on existence and structure of the Malliavin weight might be very useful. Numerical implementations of this procedure for jump-diffusion will be done in a different work. In the following section we provide explicit formulas for Malliavin weights in the case of jump diffusions.

We denote in this sequel the Skorohod integral (resp. the divergence operator) by δ and its domain by $\operatorname{dom}(\delta)$.

Definition 1. Assume that $H = \mathbb{R}^M$, fix $t > 0$ and a direction $v \in \mathbb{R}^M$. We define a set of Skorohod-integrable processes

$$\mathbb{A}_{t,x,v} = \left\{ a \in \operatorname{dom}(\delta) \text{ such that } \sum_{i=1}^{d} \int_0^t J_{0\to s}(x)^{-1} V_i(X_{s-}^x) a_s^i ds = v \right\}$$

and call it the set of path-perturbations with target-value v.

Remark 8. In the previous definition such as in the whole section assertions on Skorohod-integrability are meant Poissonian-trajectory-wise.

Proposition 1. *Assume that $H = \mathbb{R}^M$. We take Assumption 3 for granted. Fix $t > 0$ and a direction $v \in \mathbb{R}^M$. Assume furthermore uniform ellipticity, i.e. $M = d$ and there is $c > 0$ such that*

$$\inf_{\xi \in S^{M-1}} \sum_{k=1}^{M} \langle V_k(x), \xi \rangle^2 \geq c.$$

Then $\mathbb{A}_{t,x,v} \neq \emptyset$ and there exists an integrable, real valued random variable π (which depends linearly on v) such that for all bounded random variables f we obtain

$$\frac{d}{d\epsilon}|_{\epsilon=0} E(f(X_t^{x+\epsilon v})) = E(f(X_t^x)\pi).$$

Such a random variable π is called a Malliavin weight.

Proof. Here the proof is particularly simple, since we can take a matrix $\sigma(x) := (V_1(x), \dots, V_M(x))$, which is uniformly invertible with bounded inverse. We define

$$a_s := \frac{1}{t} \sigma(X_{s-}^x)^{-1} \cdot J_{0\to s}(x) \cdot v$$

for $0 \leq s \leq t$ and obtain that $a \in \mathbb{A}_{t,x,v}$. Furthermore – as in [16] and [12] – we obtain

$$\pi = \sum_{i=1}^{M} \int_0^t a_s^i dB_s^i,$$

since the Skorohod integrable process a is in fact adapted, left-continuous and hence Itô-integrable. □

Theorem 6. *Assume that $H = \mathbb{R}^M$. We take Assumptions 2 and 3 for granted, but assume that the Hörmander condition (2.3) holds true uniformly on $\mathbb{R}^M$. Fix $t > 0$ and a direction $v \in \mathbb{R}^M$. Then $\mathbb{A}_{t,x,v} \neq \emptyset$ and there exists an integrable, real* *valued random variable π (which depends linearly on v) such that for all bounded random variables f we obtain*

$$\frac{d}{d\epsilon}|_{\epsilon=0} E(f(X_t^{x+\epsilon v})) = E(f(X_t^x)\pi).$$

We can choose π to be the Skorohod integral of any element $a \in \mathbb{A}_{t,x,v} \neq \emptyset$ and call it a Malliavin weight.

Proof. We take f bounded with bounded first derivative, then we obtain

$$\frac{d}{d\epsilon}|_{\epsilon=0} E(f(X_t^{x+\epsilon v})) = E(df(X_t^x) J_{0\to t}(x) \cdot v).$$

If there is $a \in \mathbb{A}_{t,x,v}$, we obtain

$$\begin{aligned} E(df(X_t^x) J_{0\to t}(x) \cdot v) &= E(df(X_t^x) \sum_{i=1}^d \int_0^t J_{0\to t}(x) J_{0\to s}(x)^{-1} V_i(X_{s^-}^x) a_s^i ds) \\ &= E(\sum_{i=1}^d \int_0^t df(X_t^x) J_{0\to t}(x) J_{0\to s}(x)^{-1} V_i(X_{s^-}^x) a_s^i ds) \\ &= E(\sum_{i=1}^d \int_0^t D_s^i f(X_t^x) a_s^i ds) \\ &= E(f(X_t^x)\delta(a)). \end{aligned}$$

Here we cannot assert that the strategy is Itô-integrable, since it will be anticipative in general. In order to see that $\mathbb{A}_{t,x,v} \neq \emptyset$ we construct an element, namely

$$a_s^i := \left\langle J_{0\to s}(x)^{-1} V_i(X_{s^-}^x), (C_t)^{-1} v \right\rangle,$$

where C_t denotes the reduced covariance matrix from Theorem 5. Indeed

$$\begin{aligned} &\sum_{i=1}^d \left\langle \int_0^t J_{0\to s}(x)^{-1} V_i(X_{s^-}^x) a_s^i ds, \xi \right\rangle \\ &= \sum_{i=1}^d \int_0^t \left\langle J_{0\to s}(x)^{-1} V_i(X_{s^-}^x), \xi \right\rangle \left\langle J_{0\to s}(x)^{-1} V_i(X_{s^-}^x), (C_t)^{-1} v \right\rangle ds \\ &= \left\langle \xi, C_t (C_t)^{-1} v \right\rangle = \langle \xi, v \rangle \end{aligned}$$

for all $\xi \in \mathbb{R}^M$, since C_t is a symmetric random operator defined via

$$\langle \xi, C_t \xi \rangle = \sum_{i=1}^d \int_0^t \left\langle J_{0\to s}(x)^{-1} V_i(X_{s^-}^x), \xi \right\rangle^2 ds$$

for $\xi \in \mathbb{R}^M$. □

For any other derivative with respect to parameters ϵ, we consider a modified set, namely

$$\mathbb{B}_{t,x,v} = \left\{ b \in \operatorname{dom}(\delta) | \sum_{i=1}^d \int_0^t J_{0\to s}(x)^{-1} V_i(X_{s^-}^x) b_s^i ds = J_{0\to t}(x)^{-1} \frac{d}{d\epsilon}|_{\epsilon=0} X_t^{x,\epsilon} \right\}.$$

Here we are given a parameter-dependent process $X_t^{x,\epsilon}$, where all derivatives with respect to ϵ can be calculated nicely. Also in this case we can construct – if the reduced covariance matrix is invertible and regular enough – an element, namely

$$b_s^i := \Big\langle J_{0\to s}(x)^{-1} V_i(X_{s-}^x), (C_t)^{-1} J_{0\to t}(x)^{-1} \frac{d}{d\epsilon}|_{\epsilon=0} X_t^{x,\epsilon} \Big\rangle .$$

This is a consequence of the following reasoning,

$$\sum_{i=1}^d \Big\langle \int_0^t J_{0\to s}(x)^{-1} V_i(X_{s-}^x) b_s^i ds, \xi \Big\rangle = \sum_{i=1}^d \int_0^t \Big\langle J_{0\to s}(x)^{-1} V_i(X_{s-}^x), \xi \Big\rangle \cdot$$
$$\cdot \Big\langle J_{0\to s}(x)^{-1} V_i(X_{s-}^x), (C_t)^{-1} J_{0\to t}(x)^{-1} \frac{d}{d\epsilon}|_{\epsilon=0} X_t^{x,\epsilon} \Big\rangle ds$$
$$= \Big\langle \xi, C_t (C_t)^{-1} J_{0\to t}(x)^{-1} \frac{d}{d\epsilon}|_{\epsilon=0} X_t^{x,\epsilon} \Big\rangle = \Big\langle \xi, J_{0\to t}(x)^{-1} \frac{d}{d\epsilon}|_{\epsilon=0} X_t^{x,\epsilon} \Big\rangle,$$

due to symmetry of C_t.

9. Appendix

Theorem 7. *Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space and let $(B_t)_{t\geq 0}$ be a d-dimensional Brownian motion adapted to the filtration (which is not necessarily generated by the Brownian motion). Let $V, V_1, \ldots, V_d$, the diffusion vector fields be C^∞-bounded on $\mathbb{R}^M$ and consider the continuous solution $(X_t^x)_{0\leq t\leq T}$ of a stochastic differential equation (in Stratonovich notation). V_0 denotes the Stratonovich corrected drift term,*

$$dX_t^x = V_0(X_t^x)dt + \sum_{i=1}^d V_i(X_t^x) \circ dB_t^i, \tag{9.1}$$

$$X_0^x = x. \tag{9.2}$$

Assume that the uniform Hörmander condition holds true (see the proof for the precise statement). Then for any $p \geq 1$ there exist numbers $\epsilon_0(p) > 0$ and an integer $K(p) \geq 1$ such that for each $0 < t < T$

$$\sup_{\xi \in S^{M-1}} P(\langle C_t \xi, \xi \rangle < \epsilon) \leq \epsilon^p$$

holds true for $0 \leq \epsilon \leq t^{K(p)} \epsilon_0(p)$. The result holds uniformly in x.

Remark 9. The time-dependence of the estimate $0 \leq \epsilon \leq t^{K(p)} \epsilon_0(p)$ is new in comparison to [20] and results from re-doing the proof. It is heavily applied in Section 7 and the main technical ingredient of the given proof.

Proof. The proof of the theorem is a careful re-reading of the Norris Lemma and the classical proof of the Hörmander theorem in probability theory (see [19] or [20]). We shall sketch this path in the sequel (see [20], pp.120–123):

(1) Consider the random quadratic form

$$\langle C_t \xi, \xi \rangle = \sum_{i=1}^d \int_0^t \big\langle J_{0\to s}(x)^{-1} V_i(X_s^x), \xi \big\rangle^2 ds.$$

Following [20], we define

$$
\begin{aligned}
\Sigma_0' &:= \{V_1, \dots, V_d\} \\
\Sigma_n' &:= \left\{ [V_k, V], k = 1, \dots, d, V \in \Sigma_{n-1}'; [V_0, V] + \frac{1}{2} \sum_{i=1}^{d} [V_i, [V_i, V]], V \in \Sigma_{n-1}' \right\}
\end{aligned}
$$

for $n \geq 1$. We assume that there exists j_0 and $c > 0$ such that

$$
\inf_{\xi \in S^{M-1}} \sum_{j=0}^{j_0} \sum_{V \in \Sigma_j'} \langle V(x), \xi \rangle^2 \geq c
$$

uniformly in $x \in \mathbb{R}^M$.

(2) We define $m(j) := 2^{-4j}$ for $0 \leq j \leq j_0$ and the sets

$$
E_j := \Big\{ \sum_{V \in \Sigma_j'} \int_0^t \left\langle J_{0 \to s}(x)^{-1} V(X_s^x), \xi \right\rangle^2 ds \leq \epsilon^{m(j)} \Big\}.
$$

We consider the decomposition

$$
\begin{aligned}
E_0 &= \{\langle C_t \xi, \xi \rangle \leq \epsilon\} \subset (E_0 \cap E_1^c) \cup (E_1 \cap E_2^c) \cup \dots \cup (E_{j_0-1} \cap E_{j_0}^c) \cup F, \\
F &= E_0 \cap \dots \cap E_{j_0}.
\end{aligned}
$$

and proceed with

$$
P(F) \leq C \epsilon^{\frac{q\beta}{2}},
$$

for $\epsilon \leq \epsilon_1$ and any $q \geq 2$ with a constant C depending on q and the norms of the derivatives of the vector fields $V_0, \dots, V_d$. Furthermore $0 < \beta < m(j_0)$. The number ϵ_1 is determined by the following two (!) equations

$$
\begin{aligned}
(j_0 + 1) \epsilon_1^{m(j_0)} &< \frac{c \epsilon_1^{\beta}}{4}, \\
\epsilon_1^{\beta} &< t.
\end{aligned}
$$

Hence ϵ_1 depends on j_0, c, t and the choice of β, via

$$
\epsilon_1 < \min \left(t^{\frac{1}{\beta}}, \left(\frac{c}{4(j_0 + 1)} \right)^{\frac{1}{m(j_0) - \beta}} \right).
$$

This little observation additional to the proof in [20] is key for our proof.

(3) We obtain furthermore that

$$P(E_j \cap E_{j+1}^c) = P\Bigg(\sum_{V \in \Sigma'_j} \int_0^t \langle J_{0\to s}(x)^{-1} V(X_s^x), \xi \rangle^2 ds \le \epsilon^{m(j)},$$

$$\sum_{V \in \Sigma'_{j+1}} \int_0^t \langle J_{0\to s}(x)^{-1} V(X_s^x), \xi \rangle^2 ds > \epsilon^{m(j+1)} \Bigg)$$

$$\le \sum_{V \in \Sigma'_j} P\Bigg(\int_0^t \left\langle J_{0\to s}(x)^{-1} V(X_s^x), \xi \right\rangle^2 ds \le \epsilon^{m(j)},$$

$$\sum_{k=1}^d \int_0^t \left\langle J_{0\to s}(x)^{-1} [V_k, V](X_s^x), \xi \right\rangle^2 ds +$$

$$+ \int_0^t \left\langle J_{0\to s}(x)^{-1} \Big([V_0, V] + \frac{1}{2} \sum_{i=1}^d [V_i, [V_i, V]] \Big)(X_s^x), \xi \right\rangle^2 ds > \frac{\epsilon^{m(j+1)}}{n(j)} \Bigg),$$

where $n(j) = \#\Sigma'_j$. Since we can find the bounded variation and the quadratic variation part of the martingale $(\langle J_{0\to s}(x)^{-1} V(X_s^x), \xi \rangle)_{0\le s\le t}$ in the above expression, we are able to apply Norris Lemma (see [20], Lemma 2.3.2). We observe that $8m(j+1) < m(j)$, hence we can apply it with $q = \frac{m(j)}{m(j+1)}$.

(4) We obtain for $p \ge 2$ – still by the Norris Lemma – the estimate

$$P(E_j \cap E_{j+1}^c) \le d_1 \left(\frac{\epsilon^{m(j+1)}}{n(j)} \right)^{rp} + d_2 \exp\left(-\Big(\frac{\epsilon^{m(j+1)}}{n(j)} \Big)^{-\nu} \right)$$

for $\epsilon \le \epsilon_2$. Furthermore $r, \nu > 0$ with $18r + 9\nu < q - 8$, the numbers d_1, d_2 depend on the vector fields $V_0, \dots, V_d$, and on p, T. The number ϵ_2 can be chosen as $\epsilon_2 = \epsilon_3 t^{k_1}$, where ϵ_3 does not depend on t anymore.

(5) Putting all together we take the minimum of ϵ_1 and ϵ_2 to obtain the desired dependence on t. □

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