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Stochastic Partial Differential Equations with a Noised Starting Condition

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Abstract In this paper we analyze how uncertainty on the starting condition of a stochastic partial differential equation is transferred to its solution. We model the noise in the starting condition with a recent technique developed by Bouleau (2003). We prove that the variance and the bias of the solution can be easily estimated thanks to two linear stochastic partial differential equations depending on the solution of the SPDE itself.

1 Introduction

Stochastic partial differential equations appeared in the mid-1960s to model random phenomena analyzed in biology, e.g evolution of populations, and in physics, e.g. wave propagation in random media. Nowadays, SPDEs have taken a crucial role in climatology, after the seminal paper of Hasselmann [13]: this methodology is used to create the Stochastic Climate Models, a new and prolific branch in climatology.

In all previous studies the stochastic nature is imposed only at the evolution equation whereas the starting condition is assumed to be deterministic. However the starting condition of the problem is generally not perfectly known. As a matter of fact, if we consider the case of climate models, it is not so easy to define the situation of the earth today, even if we limit our study to some macro parameters.

The aim of this paper is not to introduce a new particular stochastic model in physics, biology or climatology, but to propose a method to take into account the presence of a perturbation in the starting condition.

Using a recent technique of Bouleau [3], we suppose that all perturbations are very small: this fact allows us to expand the perturbation in a series and to stop it at the two first corrections, i.e. the bias and the variance. The error theory using the language of Dirichlet forms defines a correct mathematical framework to analyze how uncertainty passes through a stochastic partial differential equation.

The structure of this article is the following. The second section presents a survey of stochastic partial differential equations theory. Section 3 aims at describing the error theory using the language of Dirichlet forms. Section 4 shows how the two mathematical tools, i.e. SPDE and Dirichlet forms, can interact in order to describe the diffusion of uncertainty through an SPDE. Finally section 5 resumes and concludes.

2 Stochastic Partial Differential Equations

We consider a general parabolic stochastic partial differential equation for all (t, x) belongs to $\mathbb{R}^+ \times \mathbb{R}^d$,

I am grateful to Nicolas Bouleau for his helpful comments, this work was conceived during a visit to Institut Mittag-Leffler (Djursholm, Sweden), I would like to thank Daniel Conus and Robert Dalang for introducing me in the subject of SPDEs, for a copy of lecture notes at EPFL and for suggestions. My special gratitude goes to Carlo Coen whose help was determinant to clarify this paper. All remaining errors are my fault.

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$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \alpha(t, x, u(t, x))\, \dot{C}(t, x) + \beta(t, x, u(t, x)) \\ \\ u(0, x) = f(x) \end{cases} \tag{2.1}$$

where, Δ denotes the Laplacian operator in $\mathbb{R}^d$, $\dot{C}(t, x)$ is a Gaussian noise, see Dalang et al. [8], with spatial correlation $k(\cdot, \cdot)$, i.e.

$$\mathbb{E}[\dot{C}(t, x)\, \dot{C}(s, y)] = \delta_0(t - s)\, k(x, y)$$

and $\alpha, \beta : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are functions that satisfy the classical properties, see Da Prato and Zabczyk [9], and Lipschitz in the last variable.

This problem admits a unique mild solution, see Da Prato [10] and Zabczyk [17], which is the following

$$\begin{aligned} u(t, x) = & \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \left[\alpha(s, y, u(s, y))\, \dot{C}(s, y) + \beta(s, y, u(s, u)) \right] ds\, dy \\ & + \int_{\mathbb{R}^d} G(t, x - y)\, f(y)\, dy \end{aligned} \tag{2.2}$$

where $G(t, x)$ is the Green function of the associated PDE, in this case

$$G(t, x) = (2\, \pi\, t)^{-\frac{d}{2}}\, e^{-\frac{|x|^2}{2t}} \tag{2.3}$$

It is clear that a notion of stochastic integral is needed for the term involving the noise and we decide to follow the approach of Da Prato and Zabczyk, see [9].

2.1 Gaussian Noise

We fix a measurable space $(E, \mathcal{E}, \mu)$ where μ is a σ-finite measure.

Definition 2.1 (White Noise)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathbb{A}$ be the set of the subsets A of E such that $\mu(A)$ is finite. Then the white noise associated with μ is a function W defined on $\mathbb{A}$ with values in Ω with the following properties:

- $\forall A \in \mathbb{A}$, $W(A)$ has a gaussian law $\mathcal{N}(0, \mu(A))$;
- $\forall A, B$ such that $A \cap B = \varnothing$, the two random variable $W(A)$ and $W(B)$ are independent and $W(A \cup B) = W(A) + W(B)$.

In our particular case we can fix $(E, \mathcal{E}, \mu) = \big(\mathbb{R}^+ \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^d), dt\, dx\big)$ and we can define the stochastic integral

Definition 2.2 (Stochastic Integral)
We fix a basis A_i of disjoint subsets of E such that $\mu(A_i) < \infty$ and let $f(t, x)$ a simple function in $\mathbb{R}^+ \times \mathbb{R}^d$, i.e.

$$f(t, x) = \sum \theta_i\, \mathbb{I}_{A_i}(t, x)$$

Then we define:

$$\int_{\mathbb{R}^+ \times \mathbb{R}^d} f(t, x)\, W(dt, dx) \equiv \sum_i \theta_i\, W(A_i)$$

Thanks to the fact that this integral is an isometry between the simple functions into the $L^2(\Omega)$-space, we can extend the integral at the closedness of the simple functions space, i.e. the $L^2(\mathbb{R}^+ \times \mathbb{R}^d)$ space.

The definition of White Noise can be generalized at a Gaussian Noise if we suppose that it exist a spatial correlation between two disjoint set in space, however we keep the white noise hypothesis in time, therefore the time correlation is a delta function, see Dalang et al. [8].

Generally the spatial Gaussian noise is denoted by the covariance function $f(x, y)\, \delta(t - s)$, a Gaussian noise is said homogenous when $f(x, y) = \overline{f}(x - y)$ and the spectral measure is given by the Fourier transform of $\overline{f}$.

2.2 Particular cases

Before the analysis of the general case, given by the SPDE (2.1), we have some interesting particular cases. The first one is homogenous diffusion where $\alpha \equiv 1$ and $\beta \equiv 0$, in this case the mild solution (2.2) is the strong solution

$$\widehat{u}(t,\,x) = \int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,C(ds,\,dy) + \int_{\mathbb{R}^d} G(t,x-y)\,f(y)\,dy \tag{2.4}$$

if and only if the stochastic integral is well-defined, i.e. Fourier transform of the Green function $G(t,\,x)$ belongs to $L^2(\overline{f})$ where $\overline{f}$ is the spectral measure of the noise, see Dalang in [8] page 47.

A second interesting case is when α is a constant but we release the hypothesis on β, in this case the mild solution (2.2) becomes

$$\begin{aligned}\widetilde{u}(t,\,x) = &\int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,C(ds,\,dy)\\ &+\int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\beta(s,\,y,\,\widetilde{u}(s,\,y))\,ds\,dy + \int_{\mathbb{R}^d} G(t,x-y)\,f(y)\,dy\end{aligned} \tag{2.5}$$

also in this case the stochastic integral is well-defined under the same hypothesis of the homogeneous diffusion, whereas the solution can be estimated thanks to a Picard iteration scheme:

$$\begin{aligned}&\widetilde{u}_0(t,\,x) = \int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,\dot{C}(ds,\,dy) + \int_{\mathbb{R}^d} G(t,x-y)\,f(y)\,dy\\ &\qquad\text{and}\\ &\widetilde{u}_n(t,\,x) = \int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,C(ds,\,dy)\\ &\qquad+\int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\beta(s,\,y,\,\widetilde{u}_{n-1}(s,\,u))\,ds\,dy + \int_{\mathbb{R}^d} G(t,x-y)\,f(y)\,dy\end{aligned} \tag{2.6}$$

It hold that $\widetilde{u}_n(t,\,x)$ converges in L^p to $\widetilde{u}(t,\,x)$ if $p \geq 0$, see Nualart in [8].

2.3 General case

In the general case the solution of the SPDE (2.1) is given by the limit of the following Picard iteration

$$\begin{aligned}&u_0(t,\,x) = \int_{\mathbb{R}^d} G(t,x-y)\,f(y)\,dy\\ &\qquad\text{and}\\ &u_n(t,\,x) = \int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,\sigma(s,\,y,\,u_{n-1}(s,\,y))C(ds,\,dy)\\ &\qquad+\int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\beta(s,\,y,\,u_{n-1}(s,\,u))\,ds\,dy + \int_{\mathbb{R}^d} G(t,x-y)\,f(y)\,dy\end{aligned} \tag{2.7}$$

and the solution converges to $u(t,\,x)$ in L^p for any $p \geq 2$, see Nualart in [8].

We have a last interesting case, analyzed by Dalang, the linear case, we assume that $\alpha(t,\,x,\,u) = \overline{\alpha}(t,\,x)u$ and $\beta(t,\,x,\,u) = \overline{\beta}(t,\,x)u$, in this case we have a better approximation of $u(t,\,x)$

Theorem 2.3 (Dalang)
Let $\overline{u}(t,\,x)$ be the solution of the SPDE

$$\begin{cases}\dfrac{\partial\,\overline{u}}{\partial\,t} - \varDelta\,\overline{u} = \overline{\alpha}(t,\,x)\,\overline{u}(t,\,x)\;\dot{C}(t,\,x) + \overline{\beta}(t,\,x)\,\overline{u}(t,\,x)\\[2ex] \quad\overline{u}(0,\,x) = f(x)\end{cases} \tag{2.8}$$

where $\dot{C}(t, x)$ is a Gaussian noise with spectral measure μ, suppose that for all t, the Fourier transform of the Green function of the heat diffusion belongs to $L^2(\mu \times ds)$ where ds is the Lebesque measure over the interval $[0, t]$ and suppose that the product of the Fourier transform of $G(t, x)$ and the Fourier transform of $\beta(t, x)$ belongs to L^1. Then the solution $\overline{u}(t, x)$ is given by

$$\overline{u}(t, x) = \sum_{m=0}^{\infty} I_m(t, x) \tag{2.9}$$

where

$$\begin{aligned} I_0(t, x) &= \int_{\mathbb{R}^d} G(t, x-y)\, f(y)\, dy \\ I_{m+1}(t, x) &= \int_{[0,\, t]} \int_{\mathbb{R}^d} G(t-s,\, x-y)\, I_m(s,\, y)\, [C(ds,\, dy) + \beta(s,\, y)\, ds\, dy] \end{aligned} \tag{2.10}$$

and the convergence is uniform in $L^2(\Omega, \mathcal{A}, \mathbb{P})$.

3 Error Theory

We consider a function F, depending on a parameter x and we consider a perturbation of the parameter by means of a gaussian distribution. We analyze the impact of this uncertainty on $F(x)$; we suppose that $F \in C^2$ with respect to the parameter x.

This approach may be performed with elementary limit calculations in a finite dimensional framework:

3.1 One-dimensional Case

Assume that $x \in \mathbb{R}$. We model the perturbation on the parameter x by means of the following transformation:

$$x \to X = x + \sqrt{\epsilon\gamma}g + \epsilon a$$

where g is a random variable, following the centered reduced normal law and independent of all the other random variables present in the model, a is a constant and ϵ is a very small parameter. In other words we replace the parameter with a random variable with mean $x + \epsilon a$ and variance $\epsilon\gamma$. We can interpret the term γ as the re-normalized variance of the variable X, subject to the perturbation; a is the re-normalized bias of the parameter induced by the perturbation.

We evaluate the bias and the variance of the perturbation of F, i.e. $F(x + \sqrt{\epsilon\gamma}g + \epsilon a) - F(x)$.

$$\begin{aligned} \mathbb{E}\left[F(X) - F(x)\right] &= \mathbb{E}\left[F'(x)\left(\sqrt{\epsilon\gamma}g + \epsilon a\right) + \frac{1}{2}F''(x)\left(\sqrt{\epsilon\gamma}g + \epsilon a\right)^2 + o(\epsilon)\right] \\ &= \epsilon\left\{a\,\mathbb{E}\left[F'(x)\right] + \frac{1}{2}\gamma\,\mathbb{E}\left[F''(x)\right]\right\} + o(\epsilon) \\ \mathbb{E}\left[\left(F(X) - F(x)\right)^2\right] &= \mathbb{E}\left[\left(F'(x)\right)^2\left(\sqrt{\epsilon\gamma}g + \epsilon a\right)^2\right] + o(\epsilon) \\ &= \epsilon\gamma\mathbb{E}\left[\left(F'(x)\right)^2\right] + o(\epsilon) \end{aligned} \tag{3.1}$$

Finally we can remark:

Remark 1 *The bias and variance have the two following chain rules:*

1. *the bias of the function F, induced by the parameter uncertainty, depends both on the bias and the variance of the parameter. Indeed, this bias presents two terms. The first term, given by the bias of the perturbed parameter, is proportional to the first derivative. The second term is related to the convexity of the function F and proportional to the variance of the parameter; it has a purely probabilistic origin (see Bouleau [?] and [4]).*

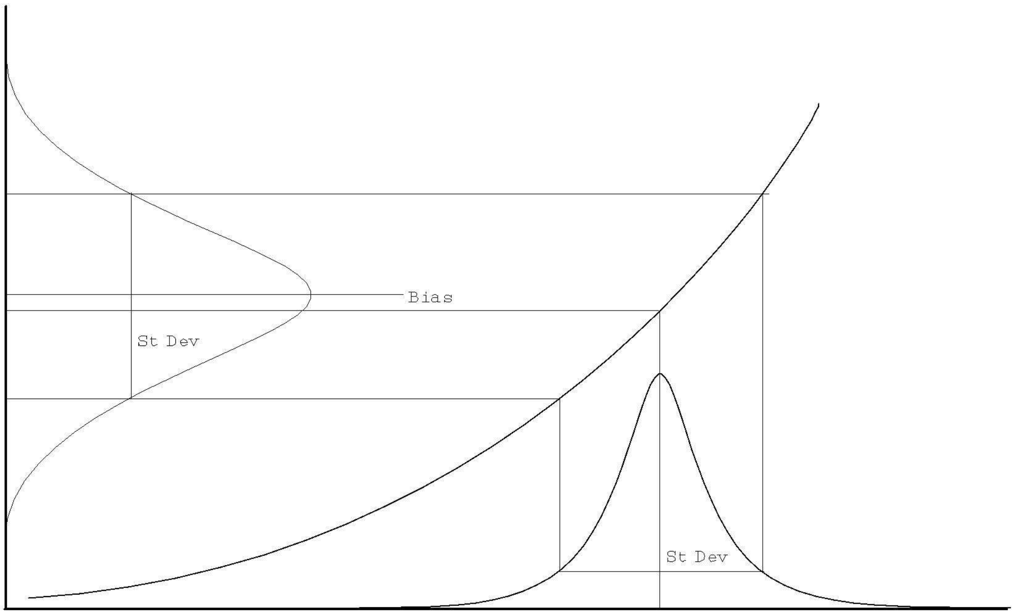

Fig. 3.1 Impact of uncertainty on a parameter through a non-linear function.

2. *the variance of the function F is proportional to the variance of the perturbed parameter and to the first derivative of the function.*

The figure 3.1 resumes the two impacts.

The previous result can be easily extended at a d-dimensional parameter, but it is not so easy to extend this approach to infinite dimensional parameters. This is compulsory since we are generally concerned with perturbation of a stochastic process or noises.

Another main problem of the classical approach is to define correctly the limit when ϵ goes to zero in a probabilistic framework. Which type of convergence involves? Almost surely, in law or in probability? The answer is not clear, see Azencott [2] for a complete analysis.

One way to avoid the problem of convergence is based on Dirichlet Forms which possess several convenient features for computing the propagation of perturbations, the closedness of the form assures the wellness of the limit and this theory can be extended to the infinite dimensional case.

3.2 Dirichlet Forms Approach

Then we present the Dirichlet Forms approach, we start by recalling the essential ingredients of this method, see Bouleau [3] for an exhaustive analysis.

We introduce the idea of error structure:

Definition 1 (Error Structure) *An error structure is a term*

$$\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \mathbb{D}, \Gamma\right)$$

where

- $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ *is a probability space;*
- $\mathbb{D}$ *is a dense sub-vector space of* $L^2\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$*;*
- Γ *is a positive symmetric bilinear application from* $\mathbb{D} \times \mathbb{D}$ *into* $L^1\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ *satisfying the functional calculus of class* $\mathcal{C}^1 \cap Lip$*, i.e. if F and G are of class* $\mathcal{C}^1$ *and Lipschitzian, u and $v \in \mathbb{D}$, we have $F(u)$ and $G(v) \in \mathbb{D}$ and*

$$\Gamma\left[F(u),\, G(v)\right] = F'(u)G'(v)\Gamma[u,\, v] \;\; \widetilde{\mathbb{P}}\; a.s.;$$

- *the bilinear form* $\mathcal{E}[u,\, v] = \frac{1}{2}\widetilde{\mathbb{E}}\left[\Gamma[u,\, v]\right]$ *is closed;*
- *The constant function 1 belongs to* $\mathbb{D}$*, i.e. the error structure is Markovian.*

In mathematical literature the form $\mathcal{E}$ is called a "local Dirichlet form" that possesses a “carré du champ” operator Γ.

We can associate to the error structure $\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}},\, \mathbb{D},\, \Gamma\right)$ a unique strongly continuous contraction semigroup $(P_t)_{t\geq 0}$ thanks to the Hille-Yosida theorem (see Albeverio [1] pages: 9-11 and 20-26). This semigroup has a unique generator $(A,\, \mathcal{D}A)$; it is a self-adjoint operator that satisfies, for $F \in \mathcal{C}^2$, $u \in \mathcal{D}A$ and $\Gamma[u] \in L^2(\widetilde{\mathbb{P}})$:

$$A\left[F(u)\right] = F'(u)A[u] + \frac{1}{2}F''(u)\Gamma[u] \;\; \widetilde{\mathbb{P}}\; a.s.$$

The functional calculus extends the ideas of classical Gauss error theory, the idea is to consider the perturbation as an error. In analogy with the classical approach of error theory we associate the carré du champ operator Γ to the normalized[1] variance of the error. Similarly, the generator describes the error biases after normalization (for more details we refers to the book of Bouleau [3] chapters III and V or to [4]).

The operator Γ is bilinear, therefore the computations become awkward to perform, to solve this problem we can introduce a new operator called sharp that is, in some sense, a square root of Γ, i.e. a linear version of the standard deviation of the error.

Definition 2 (Sharp operator) *Let* $\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}},\, \mathbb{D},\, \Gamma\right)$ *an error structure and* $\left(\widehat{\Omega},\, \widehat{\mathcal{F}},\, \widehat{\mathbb{P}}\right)$ *a copy of the probability space* $\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}}\right)$*. Under the Mokobodzki hypothesis, i.e. the space* $\mathbb{D}$ *is separable, there exists an operator sharp* $(\,)^{\#}$ *with these three properties:*

- $\forall\, u \in \mathbb{D},\; u^{\#} \in L^2(\widetilde{\mathbb{P}} \times \widehat{\mathbb{P}})$;
- $\forall\, u \in \mathbb{D},\; \Gamma[u] = \widehat{\mathbb{E}}\left[\left(u^{\#}\right)^2\right]$;
- $\forall\, u \in \mathbb{D}^n$ *and* $F \in \mathcal{C}^1 \cap Lip,\; (F(u_1,\, \ldots,\, u_n))^{\#} = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i} \circ u\right) u_i^{\#}.$

Therefore we have defined two operators Γ and $\mathcal{A}$ that verify the chain rule (3.1). We finish this section with an useful example, i.e. the Ornstein-Uhlenbeck error structure:

Example 3.1 (Orstein-Uhlenbeck structure)

$$\left(\widetilde{\Omega},\, \widetilde{\mathcal{F}},\, \widetilde{\mathbb{P}},\, \mathbb{D},\, \Gamma\right) = \left(\mathbb{R},\, \mathcal{B}(\mathbb{R}),\, \mu,\, H^1(\mu),\, \Gamma[u,\, u] = \{u'\}^2\right)$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ-field of $\mathbb{R}$, μ is a gaussian measure and $H^1(\mu)$ is the first Sobolev space with respect to the measure μ, i.e. $u \in H^1(\mu)$ if $u \in L^2(\mu)$ and u' in the distribution sense belongs to $L^2(\mu)$.

The associate generator has the following domain:

$$\mathcal{D}\mathcal{A} = \left\{u \in L^2(\mu) :\, u'' - x\, f' \text{ in the distribution sense belongs to } L^2(\mu)\right\}$$

and the generator operator is

$$\mathcal{A}[u] = \frac{1}{2}u'' - \frac{1}{2}I \cdot u'$$

where I is the identity map on $\mathbb{R}$.

This example gives the basic idea of an error structure on a parameter, moreover is exists a characterization of all dirichlet forms on $\mathbb{R}$, see Hamza [12].

[1] the variance divided by the small parameter ϵ.

4 Uncertainties in SPDEs due to the boundary function

In this section we analyze how an uncertainty in the boundary function is transmitted to the solution of the stochastic partial differential equation.

We consider that the boundary function $f(x)$ is characterized by an uncertainty on the value, we define an error structure for the space of function L^p from $\mathbb{R}^d$ into $\mathbb{R}$, we follow the approach of Bouleau, see [3] pages 83-85.

We consider a basis $\phi_n(x)$ of the function space and we represent the function $f(x)$ thanks to the vector of coefficients

$$f(x) = \sum_n a_n \, \phi_n(x)$$

with setting the coefficients a_n to be random with an error structure on each sub-space, then we have an error structure on the function space, thanks to an infinite product of structures, see Bouleau [3] pages 59-65.

Remark 2 (Correlation) *We do not assume any independence between two error structures related with two different sub-spaces, since we can perform the computation without this hypothesis. However under the hypothesis of $\mathbb{D}$-independence see Bouleau [6], the numerical evaluations of the variance-covariance and the bias are more simple.*

We assume that this error structure admits a sharp operator, so we have the following representation

$$f^{\#}(x) = \sum_n a_n^{\#} \, \phi_n(x) \tag{4.1}$$

and the related variance-covariance and bias

$$\begin{aligned} \Gamma[f(x),\, f(y)] &= \sum_{n,\, m} \phi_n(x)\phi_m(y)\Gamma[a_n,\, a_m] \\ A[f(x)] &= \sum_n \phi_n(x) A[a_n] \end{aligned} \tag{4.2}$$

To simplify the study we assume that only a finite number of variables a_n are erroneous, this hypothesis permits to make the proof easier. On the other hand, this assumption is very restrictive and a large part of the results remains true. A possible way to force a finite number of erroneous variables, without external hypothesis, is presented in Bouleau [3] page 84.

In the next two subsections we study the variance-covariance and the bias of the solution.

4.1 Variance and Covariance of the Solution

In this subsection, we study the operator Γ on the solution $u(t,\, x)$ of the SPDE (2.1), we use the sharp operator and we start with the SPDE verified by the sharp.

Theorem 4.1 (SPDE for Sharp Operator) *Under the hypothesis that the functions $\alpha(t,\, x,\, u)$ and $\beta(t,\, x,\, u)$ belong to C^1 w.r.t. the variable u and the functions with their derivatives are bounded and Lipschitzian. The sharp of the solution of the SPDE (2.1) verifies the following SPDE*

$$\begin{cases} \dfrac{\partial u^{\#}}{\partial t} - \Delta\, u^{\#} = \dfrac{\partial \alpha}{\partial u}(t,\, x,\, u(t,\, x))\; u^{\#}(t,\, x)\; \dot{C}(t,\, x) + \dfrac{\partial \beta}{\partial u}(t,\, x,\, u(t,\, x))\; u^{\#} \\ \qquad u^{\#}(0,\, x) = f^{\#}(x) \end{cases} \tag{4.3}$$

and the solution exists and it is unique.

Proof We give the proof following the sequence of particular cases of section 2, that is we compute the sharp of the solution $\widehat{u}(t,\, x)$, $\widetilde{u}(t,\, x)$ and $u(t,\, x)$.

In the homogeneous diffusion case, we can apply the sharp operator directly on the solution (2.4), so we find

$$\widehat{u}^{\#}(t,\, x) = \int_{\mathbb{R}^d} G(t, x-y)\, f^{\#}(y)\, dy$$

thanks to the properties of sharp operator, see definition 2. In this case the stochastic integral is not perturbed, therefore the sharp operator has no impact on the stochastic integral and we can use the error theory using Dirichlet form without any further proof.

Clearly the previous result is the solution of the SPDE (4.3) when $\alpha = 1$ and $\beta = 0$.

When a non-homogeneous drift exists, i.e. the SPDE (2.5), we can apply the bias operator on the Picard series (2.6), so we find

$$\begin{aligned} &\widetilde{u}_0^{\#}(t,\,x) = \int_{\mathbb{R}^d} G(t, x-y)\, f^{\#}(y)\, dy \\ &\qquad \text{and} \\ &\widetilde{u}_n^{\#}(t,\,x) = \int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,\frac{\partial\beta}{\partial u}(s,\,y,\,\widetilde{u}_{n-1}(s,\,y))\,\widetilde{u}_{n-1}^{\#}(s,\,y)\,ds\,dy + \int_{\mathbb{R}^d} G(t, x-y)\, f^{\#}(y)\, dy \end{aligned} \tag{4.4}$$

we have to prove that this series converges to a fix point. We remark that $\widetilde{u}_n(s,\,y)$ converges to $\widetilde{u}(s,\,y)$ in L^p-norm, see Nualart in [8], so we can change $\widetilde{u}_n$ with $\widetilde{u}$ thanks to the Lipschitzian coefficients and the difference can be controlled.

We use the contraction property, thanks to the bounded parameters, and this limit is the sharp of the solution $\widetilde{u}^{\#}(t,\,x)$ that verifies the equation

$$\widetilde{u}^{\#}(t,\,x) = \int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,\frac{\partial\beta}{\partial u}(s,\,y,\,\widetilde{u}(s,\,u))\,\widetilde{u}^{\#}(s,\,u)\,ds\,dy + \int_{\mathbb{R}^d} G(t, x-y)\, f(y)^{\#}\, dy$$

that is the integral form of the SPDE (4.3). As well in this case the stochastic integral does not produce any effect, the error theory using Dirichlet forms assure the well-posedness of the SPDE (4.3) and its solution.

Finally we analyze the general case given by the SPDE (2.1). We use the same strategy seen when $\beta \neq 0$, we apply the sharp operator to the Picard scheme (2.7)

$$\begin{aligned} &u_0^{\#}(t,\,x) = \int_{\mathbb{R}^d} G(t, x-y)\, f^{\#}(y)\, dy \\ &\qquad \text{and} \\ &u_n^{\#}(t,\,x) = \int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,\frac{\partial\sigma}{\partial u}(s,\,y,\,u_{n-1}(s,\,y))\,u_{n-1}^{\#}(s,\,y)\,C(ds,\,dy) \\ &\qquad + \int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,\frac{\partial\beta}{\partial u}(s,\,y,\,u_{n-1}(s,\,u))\,u_{n-1}^{\#}(s,\,y)\,ds\,dy + \int_{\mathbb{R}^d} G(t, x-y)\, f^{\#}(y)\, dy \end{aligned} \tag{4.5}$$

and we use a fix point argument to assure the existence of the solution and to verify the SPDE (4.11).

The difference with respect to the previous case is the presence of a stochastic integral depending on the sharp of the solution, we must verify that at for any n the integral

$$\int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,\frac{\partial\sigma}{\partial u}(s,\,y,\,u_{n-1}(s,\,y))\,u_{n-1}^{\#}(s,\,y)\,C(ds,\,dy)$$

is well-posed. We remark that, thanks to the hypothesis, $\sigma_u(t,\,x,\,u)$ is bounded, and $G(t,\,x)$ belongs to C^{∞}, therefore the Gronwall lemma assure the convergence of the Picard series, see Da Prato [11].

□

The theorem 4.1 presents an interesting similarity with a result of Da Prato, see [10] page 64, about the derivative of the solution of a SPDE with respect to the initial datum.

The theorem 4.1 has a direct consequence, we have an easy representation of the sharp, we state this property in the following lemma.

Lemma 4.2 (Series for the sharp of the solution) *Under the hypotheses of theorems 2.3 and 4.1 we have the following series for the sharp of the solution.*

$$u^{\#}(t,\, x) = \sum_{m=0}^{\infty} I_m^{(u^{\#})}(t,\, x) \tag{4.6}$$

where

$$\begin{aligned} I_0^{(u^{\#})}(t,\, x) &= \int_{\mathbb{R}^d} G(t, x-y)\, f^{\#}(y)\, dy \\ I_{m+1}^{(u^{\#})}(t,\, x) &= \int_{[0,\, t]} \int_{\mathbb{R}^d} G(t-s,\, x-y)\, I_m^{(u^{\#})}(s,\, y)\, \frac{\partial \sigma}{\partial u}(s,\, y,\, u(s,\, y))\, C(ds,\, dy) \\ &\quad + \int_{[0,\, t]} \int_{\mathbb{R}^d} G(t-s,\, x-y)\, I_m^{(u^{\#})}(s,\, y)\, \frac{\partial \beta}{\partial u}(s,\, y,\, u(s,\, y))\, ds\, dy \end{aligned} \tag{4.7}$$

Proof We remark that the SPDE (4.3) verified by the sharp of the solution is linear, the statement of the lemma is now a direct consequence of the theorem 2.3.

□

We can rewrite the sharp thanks to the decomposition. (4.1)

Result 1 (sharp of the SPDE) *The sharp of the solution of the SPDE 2.1 is given by*

$$\begin{aligned} J_0^{(n)}(t,\, x) &= \int_{\mathbb{R}^d} G(t, x-y)\, \phi_n(y)\, dy \\ J_{m+1}^{(n)}(t,\, x) &= \int_{[0,\, t]} \int_{\mathbb{R}^d} G(t-s,\, x-y)\, J_m^{(n)}(s,\, y)\, \frac{\partial \sigma}{\partial u}(s,\, y,\, u(s,\, y))\, C(ds,\, dy) \\ &\quad + \int_{[0,\, t]} \int_{\mathbb{R}^d} G(t-s,\, x-y)\, J_m^{(n)}(s,\, y)\, \frac{\partial \beta}{\partial u}(s,\, y,\, u(s,\, y))\, ds\, dy \\ u^{\#}(t,\, x) &= \sum_{n,\, m} a_n^{\#}\, J_m^{(n)}(t,\, x) \end{aligned} \tag{4.8}$$

Proof This result is a direct consequence of the lemma 4.2, the linearity of the sharp operator and the linearity of the SPDE for the sharp 4.3.

□

Now we can make a remark.

Remark 4.3 (Choice of the basis) The relation (4.8) shows the importance of the choice of the basis $\phi_n(x)$. As a matter of fact, it stands to reason that the computation of $J_m^{(n)}(t,\, x)$ for many n becomes numerically expensive. Therefore the essential thing is that the expansion over the basis $\phi_n(x)$ would can be cut off. In this sense, a good choice can be a wavelets basis, see Scotti [15] for a complete analysis.

Thanks to this result, we have the following characterization for the variance of the solution.

Result 2 (Gamma of the solution) *The variance-covariance of the solution is given by*

$$\Gamma[u(t,\, x),\, u(s,\, y)] = \sum_{j,\, k,\, n,\, m} \Gamma[a_n,\, a_k]\, J_m^{(n)}(t,\, x)\, J_j^{(k)}(s,\, y) \tag{4.9}$$

Proof This lemma is a direct consequence of the result 1 and the properties of sharp operator, see definition 2.

□

we conclude our analysis with two remarks.

Remark 4.4 (Linearity) The equation (4.9) shows that the variance covariance operator admits an easy decomposition into two terms. The first one $\Gamma[a_n, a_k]$ is the covariance between the two erroneous coefficients of the decomposition of the starting function $f(x)$ into the basis $\phi_n(x)$. The second term $J_m^{(n)}(t, x)\, J_j^{(k)}(s, y)$ catches the evolution of the solution through the SPDE but it is unrelated with the error on the function $f(x)$.

This decomposition depends crucially on the linearity of the SPDE verified by the sharp (4.3), besides this characteristic of the sharp is intrinsic, in PDE analysis the sharp is a generalization of the tangent linear problem, see Choi [7] or Tallagrand et al. [16] for a description of tangent linear problem and Scotti [14] for the study of the uncertainty diffusion through a PDE.

Remark 4.5 (Independence) If we assume that the error structures on each sub-space, on which the functions space has been split, are independent, then the relation for the variance-covariance (4.9) becomes more simple.

$$\Gamma[u(t, x), u(s, y)] = \sum_{j,\, n,\, m} \Gamma[a_n]\, J_m^{(n)}(t, x)\, J_j^{(n)}(s, y) \tag{4.10}$$

Therefore, a good choice for the basis ϕ_n would be a basis that exploits the information about the uncertainty on the boundary function.

4.2 Bias of the solution

Given the function of variance-covariance we can study the bias.

Theorem 4.6 (SPDE for Bias Operator) *Under the hypothesis that the functions $\alpha(t, x, u)$ and $\beta(t, x, u)$ belong to C^2 w.r.t. the variable u and the functions with their derivatives are bounded and Lipschitz. The bias of the solution of the SPDE (2.1) verifies the following SPDE*

$$\begin{cases} \dfrac{\partial A[u]}{\partial t} - \Delta A[u] = \dfrac{\partial \alpha}{\partial u}(t, x, u(t, x))\, A[u](t, x)\, \dot{C}(t, x) + \dfrac{\partial \beta}{\partial u}(t, x, u(t, x))\, A[u](t, x) \\ \qquad + \dfrac{1}{2}\dfrac{\partial^2 \alpha}{\partial u^2}(t, x, u(t, x))\, \Gamma[u](t, x)\, \dot{C}(t, x) + \dfrac{1}{2}\dfrac{\partial^2 \beta}{\partial u^2}(t, x, u(t, x))\, \Gamma[u](t, x) \\ A[u](0, x) = A[f](x) \end{cases} \tag{4.11}$$

where $\Gamma[u](t, x)$ is given by equation (4.9). Moreover the solution of this SPDE exists and it is unique.

Proof We use the same argument applied in theorem 4.1. We apply the bias operator A on the Picard iteration (2.7), thanks to the chain rule (3.1) we find

$$A[u_0](t, x) = \int_{\mathbb{R}^d} G(t, x - y)\, A[f](y)\, dy$$

and

$$\begin{aligned} A[u_n](t, x) &= \int_{[0,\, t]\times\mathbb{R}^d} G(t - s, x - y)\, \frac{\partial \sigma}{\partial u}(s, y, u_{n-1}(s, y))\, A[u_{n-1}]\, C(ds, dy) \\ &+ \frac{1}{2}\int_{[0,\, t]\times\mathbb{R}^d} G(t - s, x - y)\, \frac{\partial^2 \sigma}{\partial u^2}(s, y, u_{n-1}(s, y))\, \Gamma[u_{n-1}](s, y)\, C(ds, dy) \\ &+ \int_{[0,\, t]\times\mathbb{R}^d} G(t - s, x - y) \frac{\partial \beta}{\partial u}(s, y, u_{n-1}(s, y))\, A[u_{n-1}](s, y)\, ds\, dy \\ &+ \frac{1}{2}\int_{[0,\, t]\times\mathbb{R}^d} G(t - s, x - y) \frac{\partial^2 \beta}{\partial u^2}(s, y, u_{n-1}(s, y))\, \Gamma[u_{n-1}](s, y)\, ds\, dy \\ &+ \int_{\mathbb{R}^d} G(t, x - y)\, A[f](y)\, dy \end{aligned} \tag{4.12}$$

where $\Gamma[u_{n-1}](s,\,y)$ is given by the Picard iteration (4.5) and the properties of the sharp operator, see definition 2. But $u_n(t,\,x)$ converges in L^p-norm to $u(t,\,x)$, therefore, we can control the error given by the exchange between $u_n(t,\,x)$ and $u(t,\,x)$ in Picard iteration (4.12), thanks to the Lipschitzian coefficients. Following the same idea we can exchange $\Gamma[u_{n-1}](s,\,y)$ with $\Gamma[u](s,\,y)$, given by the result 2.

Now we have a Picard iteration depending only on the term $A[u_{n-1}](s,\,y)$. The iteration is linear in $A[u_{n-1}](s,\,y)$ and all coefficients are bounded. The Gronwall lemma assure the convergence of the Picard iteration to a unique solution $A[u](t,\,x)$ and this solution verifies the stochastic partial differential equation (4.11).

□

We make an interesting remark.

Remark 4.7 (Biased solution) The equation (4.11) show that the solution of a stochastic partial differential equation can be biased even if the starting condition is unbiased. It is sufficient to suppose that the SPDE (2.1) is non-linear, in the sense that $\sigma(t,\,x,\,u)$ or $\beta(t,\,x,\,u)$ have a non-zero second derivative with respect to u. In this case the SPDE (4.11) verified by the bias present an exogenous term proportional to variance of the solution. Therefore the study of the bias is very important and can modify the behavior of the model.

We can remark that the Picard iteration (4.12) has an interesting structure in particular

$$A[u_0](t,\,x) = \sum_n A[a_n]\; J_0^{(n)}(t,\,x) \tag{4.13}$$

where $J_0^{(n)}(t,\,x)$ is given by equation (4.8). this property can be generalized at each term of the Picard iteration (4.12). We introduce the following notation

Definition 4.8 We define a series of functions $\Psi_m^{(n)}(t,\,x)$ thanks to the following relations. If $A[a_n] \neq 0$ then

$$\begin{aligned}
\Psi_m^{(n)}(t,\,x) = &\int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,\frac{\partial\sigma}{\partial u}(s,\,y,\,u(s,\,y))\;J_{m-1}^{(n)}(s,\,y)\;C(ds,\,dy)\\
&+\sum_k \Theta_k^{(n)}\int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,\frac{\partial^2\sigma}{\partial u^2}(s,\,y,\,u(s,\,y))\;J_{m-1}^{(n)}(s,\,y)\,J_{m-1}^{(k)}(s,\,y)\;C(ds,\,dy)\\
&+\int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,\frac{\partial\beta}{\partial u}(s,\,y,\,u(s,\,y))\;J_{m-1}^{(n)}(s,\,y)\;ds\,dy\\
&+\sum_k \Theta_k^{(n)}\int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,\frac{\partial^2\beta}{\partial u^2}(s,\,y,\,u(s,\,y))\,J_{m-1}^{(n)}(s,\,y)\,J_{m-1}^{(k)}(s,\,y)\;ds\,dy\\
&+\int_{\mathbb{R}^d} G(t,x-y)\,\phi_n(y)\,dy
\end{aligned} \tag{4.14}$$

where

$$\Theta_k^{(n)} = \frac{\Gamma[a_n,\,a_k]}{2\,A[a_n]} \tag{4.15}$$

otherwise, if $A[a_n]=0$, then

$$\begin{aligned}
\Psi_m^{(n)}(t,\,x) = &\sum_k \theta_k^{(n)}\int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,\frac{\partial^2\sigma}{\partial u^2}(s,\,y,\,u(s,\,y))\;J_{m-1}^{(n)}(s,\,y)\,J_{m-1}^{(k)}(s,\,y)\;C(ds,\,dy)\\
&+\sum_k \theta_k^{(n)}\int_{[0,\,t]\times\mathbb{R}^d} G(t-s,\,x-y)\,\frac{\partial^2\beta}{\partial u^2}(s,\,y,\,u(s,\,y))\,J_{m-1}^{(n)}(s,\,y)\,J_{m-1}^{(k)}(s,\,y)\;ds\,dy
\end{aligned} \tag{4.16}$$

where

$$\theta_k^{(n)} = \frac{\Gamma[a_n,\,a_k]}{2} \tag{4.17}$$

Thanks to this definition we have a characterization for the bias solution of the SPDE (4.11).

Theorem 4.9 *The series of functions*

$$\Xi_m(t,\,x) = \sum_n \left\{ A[a_n]\,\Psi_m^{(n)}(t,\,x)\,\mathbb{I}_{A[a_n]\neq 0} + \Psi_m^{(n)}(t,\,x)\,\mathbb{I}_{A[a_n]=0} \right\} \tag{4.18}$$

converges to $A[u]$ *when* m *goes to infinity in* L^2*-norm.*

Proof We remark that the sum over n involves only a finite number of elements, thanks to the hypothesis of a finite number of erroneous coefficients a_n.

The crucial argument of this proof is to remark that $\Xi_m(t,\,x)$ is a rewriting of the Picard iteration of the SPDE (4.11). In particular, we recall that this SPDE is linear in the bias $A[u_n]$. The starting condition admits a decomposition, see equation (4.2), also the operator Γ admits the same decomposition, see equation (4.9). Therefore each term of the SPDE (4.11) can be separated into a sum. Clearly the Picard iteration preserves this property, and the decomposition gives birth to the functions $\Psi_m^{(n)}(t,\,x)$.

The proof ending using the fact that the Picard iteration converges to the solution of the SPDE 4.11 in $L^2 - norm$.

□

We conclude this analysis with a particular case:

Remark 4.10 (independence) If we assume that the error structures on each sub-space, on which the functions space has been split, are independent, then the decomposition, given by the functions $\Psi_m^{(n)}(t,\,x)$, splits the bias of the solution into independent sub-spaces, in the sense that the two coefficients $\Theta_k^{(n)}$ and $\theta_k^{(n)}$, defined in equations (4.15) and (4.17) respectively, become proportional to a Dirac delta $\delta_{n,\,k}$. In this case the computation becomes more easy.

5 Conclusion

In this article we have studied how an uncertainty on the starting condition pass on the solution of a stochastic partial differential equation. We have considered a stochastic partial differential equation of heat diffusion type, with a colored noise, in order to simplify the proof of the well-posedness of the solutions of the SPDEs.

In order to describe the uncertainty on the starting condition and to compute the uncertainty on the solution, we have used the error theory using Dirichlet forms, technique introduced by Bouleau. We have assumed that the uncertainty is very small with respect to the value of the functions, so we have used a hieratic strategy, we have computed the solution of the SPDE without uncertainty, then the variance-covariance of this solution and finally the bias induced by the non-linearity.

We have find that the variance can be easy estimated thanks to the sharp, a linear version of the standard deviation of the uncertainty, that verifies a linear stochastic partial differential equation. We have proved that the variance admits a decomposition in a series.

The study of the bias has allowed to show the existence of a stochastic partial differential equation verified by the bias of the solution. Furthermore, we have proved that the bias of the solution can be decomposed into a sum of terms that lives into the subspaces used to break up the starting condition.

We have also showed that the bias exists even if the starting condition are unbiased, it is enough that the stochastic partial equation would be non-linear. This result is very interesting, as a matter of fact, the bias induced only by the variance has a purely probabilistic origin and this fact modify the behavior of the model below the stochastic partial differential equation.

In physics and climatology the noises play a central role, however also the starting condition has to be estimated and this estimate is afflicted by an uncertainty, it is possible to evaluate the its variance thanks to the Fisher information matrix, see Bouleau and Chorro [5] for an analysis of the relation between information matrix and error theory. The variance and the bias induced by this uncertainty have an important role in the forecast of a model.

This paper is a first study in the combination of the error theory using Dirichlet forms and stochastic partial differential equations, many others ways has to be examined, in particular, we work on the uncertainty in the functional structure of the SPDE, i.e. it is possible to consider an uncertainty on the functions σ and β. Another interesting analysis will be the study of the transmission of an uncertainty on the diffusion coefficient, in this case the error will be propagated through the Green function. Finally a more theoretical study will be the definition of an error structure on the colored noise itself.

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