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**INSTITUT
MITTAG-LEFFLER**

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

**Integration by parts formula for locally
smooth laws and applications to equations
with jumps II**

V. Bally

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An integration by parts formula for locally smooth laws and applications to equations with jumps II.

Vlad Bally

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Abstract

We consider the stochastic equation $dX_t = g(t, X_t)dt + c(t, a, X_{t-})dN(t, a)$ where N is a poisson point measure on an abstract measurable space. We settel a differential calculus based on the jump times in order to give sufficient conditions in order that the law of X_t is absolutely continuous and has a smooth density.

Key words: Intégration by parts formula, Malliavin Calculus, Stochastic Equations, Poisson Point Measures.

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1 Introduction

We consider the one dimensional equation

$$X_t = x + \int_0^t c(u, a, X_{u-})dN(u, a) + \int_0^t g(u, X_u)du \quad (1)$$

where N is a Posson point measure of intensity measure μ on some abstract measurable space E . We assume that c and g are infinitely differentiable with respect to t and x , have bounded derivatives of any order and have linear growth with respect to x . Moroever we assume that the derivatives of c are bounded by a function \bar{c} such that $\int \bar{c}d\mu < \infty$. Under these hypothesis the equation has a unique solution - and the stochastic integral $\int cdN$ is a Stildjers integral.

Our aim is to give sufficient conditions in order that the law of X_t is absolutely continuous with respect to the Lebesgue measure and has a smooth density. If $E = R^m$ and $\mu(da) = h(a)da$ with a smooth density h then one may developp a Malliaivn calculus based on the amplitudes of the jumps in order to solve this problem. This has been done first in [Bis] and then in [B.G.J]. But if μ is a singular measure this approach fails and one has to use the noise given by the jump times of the Poisson point measure in order to settle an analogues of the Malliaivn calculus. This is a much more delicate problem and several approches have been proposed. A first step is to prove that the law of X_t is absolutely continuous with respect to the Lebesgue measure, without bothering about the regularity. A first result in this sense was obtained by Carlen and Pardoux in [C.P] and was followed by number of other papers (see [D],[E.T],[K1],[N.S]). The second step is to obtain the regularity of the density. Recently two reults of this type has been obtained by Ishikawa and

Kunita in [I.K] and by Kulik in [K.2]. In both cases one deals with an equation of the form

$$dX_t = g(t, X_t)dt + f(u, X_{u-})dU_t \quad (2)$$

where U is a Lévy process. The above equation is multi-dimensional (let us mention that the method presented in our paper may be used in the multi-dimensional case as well, but then some technical problems related to the control of the Malliaivn covariance matrix have to be solved - and for simplicity we preferred to live out this qaind of difficulties in this paper). Ishikawa and Kunita in [I.K] used the finite difference approach given by J. Picard in [P] in order to obtain sufficient conditions for the regularity of the density of the solution of an equation of type (1) (in a somehow more particular form, closed to linear equations). The result in that paper produces a large class of examples in which we get a smooth desnity even for an intensity measure which is singular with respect to the Lebesgue measure. The second approach is due to Kulik [K]. He settled a Malliaivn type calculus based on perturbations of the time structure in order to give sufficient conditions for the smoothness of the density. In his paper the coefficient f is equal to one so the non degeneracy comes form the drift term g only. As befor, he obtains the regularity of the density even if the intensity measure μ is singular. He also proves that under some appropriate conditions, the density is not smooth for a small t so that one has to wait befor the regularization effect of the noise produces a regular density.

The result proved in our paper is the following. We consider the function

$$\alpha(u, a, x) = g(x) - g(x + c(u, a, x)) + (g\partial_x c + \partial_u c)(u, a, x).$$

Except the regularity and boundedness conditions on g and c we consider the following non degeneracy assupmtion. There exists a measurable function α such that $|\alpha(u, a, x)| \geq \underline{\alpha}(a) > 0$ for every $(u, a, x) \in R_+ \times E \times R$ and, given $q \in N$, there exists a sequence of subsets $E_n \uparrow E$ such that $\mu(E_n) < \infty$ and

$$\lim_{n \rightarrow \infty} \frac{1}{\mu(E_n)} \ln \left(\int_{E_n} \frac{1}{\alpha^q(a)} d\mu(a) \right) = \theta_q$$

for some $\theta_q < \infty$. Then, for $t > (2q + 4)\Theta_{8(2q+4)^2}$ the law of X_t is absolutely continuous and has a density of class C_b^q . In particular, if $\theta_q = 0$ then the density is of class C_b^q for every $t > 0$.

In the paper of Kulik one takes $c(u, a, x) = u$ so $\alpha(u, a, x) = g(x) - g(x + c(u, a, x))$. Then the non degeneracy condition concerns just the drift coefficinet g . And in the paper of Ishikawa and Kunita the baisic example (which corresponds to the geometric Lévy process) is $c(u, a, x) = xa(e^a - 1)$ and g constant. So $\alpha(u, a, x) = a(e^a - 1) \sim a^2$ as $a \rightarrow 0$. The drift coefficient does not contribute to the non degeneracy condition (which is analogues to the uniform ellipticity condition).

The main ingredient in our approch is the following well known property of the jump times $T_k^n, k \in N$ of a Poisson process $J_t^n, t > 0$ (the reason of being of $n \in N$ will apaar in a moment). Conditionaly to $J_t^n = m$, the law of (T_1^n, \dots, T_m^n) is uniform on $(0, T)$. In particular, for each $p = 1, \dots, m$, the law of T_p^n conditionaly to $\mathcal{G}_p := \sigma(J_t^n = m, T_q^n, q \neq p)$ is uniform on (T_{p-1}^n, T_{p+1}^n) . It follows that for every $\phi \in C^1$

$$E_{\mathcal{G}_p}(\phi'(T_p^n)) = \frac{\phi(T_{p+1}^n) - \phi(T_{p+1}^n-)}{T_{p+1}^n - T_{p-1}^n} - \frac{1}{T_{p+1}^n - T_{p-1}^n} \int_{T_{p-1}^n}^{T_{p+1}^n} \phi(u) du. \quad (3)$$

And our problem is to put to work this elementary integration by parts formula. This is a non trivial matter because we will work with "small jumps" so that $T_{p+1}^n - T_{p-1}^n$ becomes an infinitesimal quantity. In order to do it we set up a finite dimensional Malliavin calculus analogues to the one given in [B] (a variant of this calculus has already been used in [B.BM] for numerical applications). Then we approximate X_t with some X_t^N which solves the equation (1) in which the dN is replaced with $dN_N = 1_{E_N} dN$. Since $\mu(E_N) < \infty$, the number of jumps before t is finite, and so X_t^N is a function of a finite number of $T_k^N, k \leq J_t^N$. For this finite dimensional functional we use a finite dimensional differential calculus. This calculus is not based on all the random variables $T_k^N, k \leq J_t^N$ but only on part of them, namely on $T_k^n, k \leq J_t^n$ with $n \leq N$ to be chosen in an appropriate way. This is a variant of the calculus developed in [B] but the significant difference is that in the integration by parts formula (3) the border terms $\phi(T_{p+1}^n+)$ and $\phi(T_{p+1}^n-)$ appear. In [B] we use some weights π_p which have the property $\pi_p(T_{p+1}^n+) = \pi_p(T_{p+1}^n-) = 0$ in order to cancel the border terms. But in the framework here this does not seem to be possible (see the remark in the proof of Theorem 7). So our basic duality formula (see Proposition 1) contains border terms. And this leads to some specific difficulties. As a consequence, the restriction to a finite variation integral with respect to dN (and not a compensated stochastic integral) is crucial because we need some very strong upper bounds in order to handle the border terms.

Once we are able to use the differential calculus for X_t^N we obtain an integration by parts formula of the type $E(\phi'(X_t^N)) = E(\phi(X_t^N)H_{N,n})$ for some integrable random variable $H_{N,n}$. And this gives $|E(\phi'(X_t^N))| \leq \|\phi\|_\infty E(|H_{N,n}|)$. If $\sup_{N,n} E(|H_{N,n}|) \leq C < \infty$, then $|E(\phi'(X_t^N))| \leq C \|\phi\|_\infty$ and a classical argument based on the Fourier transform guarantees that the law of X_t is absolutely continuous and has a smooth density. But in our case $\sup_{N,n} E(|H_{N,n}|) = \infty$. Nevertheless we are able to use an argument based on the Fourier transform inspired from [F]. Such an argument has been used in [B] also.

The paper is organized as follows. In Section 2 we establish the finite dimensional differential calculus, we obtain the basic integration by parts formula and its iterations and we give some upper bounds for the quantities which are involved in this formula. In section 3 we apply this calculus to the solution of the equation (1). And in the appendix we give some elementary facts about the derivatives of the solution of the equation (1) with respect to the jump times.

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2 Malliavin calculus for simple functionals

We consider a probability space (Ω, \mathcal{F}, P) , a sub σ -algebra $\mathcal{G} \subseteq \mathcal{F}$ and a sequence of random variables $V_i, i = 1, \dots, m$ where $m \in \mathbb{N}$ is fixed. We denote $V = (V_1, \dots, V_m)$ and $\widehat{V}_i = (V_1, \dots, V_{i-1}, V_{i+1}, \dots, V_m)$. We also put $\mathcal{G}_i = \mathcal{G} \vee \sigma(V_j, j \neq i)$. Our calculus is based on the variables $V_i, i = 1, \dots, m$ and \mathcal{G} represents the σ algebra which contains the noise which is not involved in the calculus. We assume that $V_i \in \cap_p L^p$.

We also consider some functions $\pi_i : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}_+$ which are $\mathcal{G} \times B(\mathbb{R}^m)$ measurable and which will be the weights that we use in our calculus.

The functionals which we consider will not be differentiable on the whole space

but only on some intervals that we define now. For each i we consider two functions $a_i, b_i : \Omega \times R^{m-1} \rightarrow R_+ \cup \{-\infty, \infty\}$ which are $\mathcal{G} \times B(R^{m-1})$ measurable and such that

$$\pi_i(\omega, v) \mathbf{1}_{\{b_i(\omega, \widehat{v}_i) \leq a_i(\omega, \widehat{v}_i)\}} = 0, \quad \pi_i(\omega, v) \mathbf{1}_{\{v_i \leq a_i(\omega, \widehat{v}_i)\} \cup \{b_i(\omega, \widehat{v}_i) \leq v_i\}} = 0.$$

So π_i is zero outside (a_i, b_i) and $b_i > a_i$ if $\pi_i \neq 0$. Notice that in [B] we assume that $\pi_i(\omega, \widehat{v}_i, b_i-) = \pi_i(\omega, \widehat{v}_i, a_i+) = 0$. And this is necessary in order to cancel the border terms in the integration by parts formula. But here we do not ask this and so we keep the border terms.

We denote by $C_{a,b}^\infty$ the space of $\mathcal{G} \times B(R^m)$ measurable functions $f : \Omega \times R^m \rightarrow R$ such that, for every $i = 1, \dots, m$, every $\widehat{v}_i \in R^{m-1}$ and every $\omega \in \Omega$, the function $v_i \rightarrow f(\omega, \widehat{v}_i, v_i)$ is of class C^∞ on $(a_i(\omega, \widehat{v}_i), b_i(\omega, \widehat{v}_i))$. Moreover we assume that this function and its derivatives have finite right hand side (respectively left hand side) limits in a_i (respectively in b_i) and we denote

$$R_i^+ f(\omega, \widehat{v}_i) := \lim_{v_i \uparrow b_i(\omega, \widehat{v}_i)} f(\omega, \widehat{v}_i, v_i), \quad R_i^- f(\omega, \widehat{v}_i) := \lim_{v_i \downarrow a_i(\omega, \widehat{v}_i)} f(\omega, \widehat{v}_i, v_i).$$

We also assume that f and all its derivatives have polynomial growth: there exists some constants C and p such that $|f(\omega, v)| \leq C(1 + |v|^p)$.

Our basic hypothesis is that the conditional law of V_i with respect to \mathcal{G}_i is absolutely continuous with respect to the Lebesgue measure and the density belongs to $C_{a,b}^\infty$. So we assume:

H1. For each $i = 1, \dots, m$ there exists $p_i \in C_{a,b}^\infty$ such that for every $\mathcal{G} \times B(R^m)$ measurable function f one has

$$E_{\mathcal{G}_i}(f(\omega, V)) = \int_R f(\omega, \widehat{V}_i, v) p_i(\omega, \widehat{V}_i, v) dv$$

where $E_{\mathcal{G}_i}$ designis the conditional expectation. We denote

$$\bar{p}_i = \bar{p}_i(\omega, V) = \mathbf{1}_{(a_i(\omega, \widehat{V}_i), b_i(\omega, \widehat{V}_i))} (V_i) \partial_{v_i} \ln p_i(\omega, V).$$

We introduce now the objects which come on in the Malliavin calculus.

Simple functionals. A random variable F is called a simple functional if $F = f(\omega, V)$ with $f \in C_{a,b}^\infty$. We denote by \mathcal{S} the space of the simple functionals. It is an algebra.

Simple processes. A simple process of length m is sequence of random variables $U = (U_i)_{i \leq n}$ such that $U_i(\omega) = u_i(\omega, V)$, $u_i \in C_{a,b}^\infty$. We denote by \mathcal{P} the space of the simple processes. And we consider the scalar product

$$\langle U, W \rangle_\pi = \sum_{i=1}^m \pi_i U_i W_i.$$

We define now the differential operators which appear in Malliavin's calculus.

□ **The Malliavin derivative** $D : \mathcal{S} \rightarrow \mathcal{P}$: if $F = f(\omega, V_1, \dots, V_m)$ then

$$\begin{aligned} D_i F & : = \mathbf{1}_{(a_i(\omega, V), b_i(\omega, V))} (V_i) \partial_{v_i} f(\omega, V(\omega)), \\ DF & = (D_i F)_{i \leq m} \in \mathcal{P}. \end{aligned}$$

For $F = (F^1, \dots, F^d) \in \mathcal{S}^d$ the Malliavin covariance matrix is defined by

$$\sigma^{i,j}(F) = \langle DF^i, DF^j \rangle_\pi = \sum_{k=1}^m \pi_k D_k F^i D_k F^j.$$

More generally we consider d simple processes $\xi_j = (\xi_j^i)_{i=1,m} \in \mathcal{P}, j = 1, \dots, d$ and we define

$$\sigma_{\xi}^{i,j}(F) = \langle DF^i, \xi_j \rangle_{\pi} = \sum_{k=1}^m \pi_k D_k F^i \times \xi_j^k.$$

If we take $\xi_j^k = D_k F^j$ we obtain the standard Malliavin covariance matrix. We may interpret ξ_1, \dots, ξ_d as directions in which the derivative is taken.

□ **The Skorohod integral** Let $U = (U_i)_{i \leq m} \in \mathcal{P}$ with $U_i = u_i(\omega, V)$. We define $\delta : \mathcal{P} \rightarrow \mathcal{S}$ by

$$\begin{aligned} \delta_i(U) &: = -(D_i(\pi_i u_i) + \pi_i u_i \bar{p}_i)(\omega, V), \\ \delta(U) &= \sum_{i=1}^m \delta_i(U) = - \sum_{i=1}^m (D_i(\pi_i u_i) + \pi_i u_i \bar{p}_i)(\omega, V). \end{aligned}$$

We recall that $\bar{p}_i = 1_{(a_i, b_i)}(V_i) \partial_{v_i} \ln p_i(\omega, V)$.

□ **The border terms.** For $F = f(\omega, V) \in \mathcal{S}$ and $U = (u_i(\omega, V))_{i=1, \dots, m} \in \mathcal{P}$ we define

$$\begin{aligned} [F, U]_{\pi} &= \sum_{i=1}^m (R_i^+(FU_i \pi_i p_i) - R_i^-(FU_i \pi_i p_i)) \\ &= \sum_{i=1}^m ((f \times u_i)(\omega, V_1, \dots, V_{i-1}, b_i-, V_{i+1}, \dots, V_m)(\pi_i p_i)(\omega, b_i-) \\ &\quad - (f \times u_i)(\omega, V_1, \dots, V_{i-1}, a_i+, V_{i+1}, \dots, V_m)(\pi_i p_i)(\omega, a_i+)). \end{aligned}$$

In our framework the duality between δ and D is given by the following lemma.

Proposition 1 *We assume that **H.1** holds true. Then for every $F \in \mathcal{S}$ and $U \in \mathcal{P}$*

$$E_{\mathcal{G}}(\langle DF, U \rangle_{\pi}) = E_{\mathcal{G}}(F \delta(U)) + E_{\mathcal{G}}([F, U]_{\pi}). \quad (4)$$

Proof. Using the hypothesis **H1** and the fact that $v_i \rightarrow \pi_i(\omega, v)$ is null outside (a_i, b_i) we obtain

$$\begin{aligned} E_{\mathcal{G}}(D_i F \times (U_i \pi_i)) &= E_{\mathcal{G}}(E_{\mathcal{G}_i}(D_i F \times (U_i \pi_i))) \\ &= E_{\mathcal{G}}\left(\int_{a_i}^{b_i} (\partial_{v_i} f \times \pi_i u_i)(\omega, \widehat{V}_i, v) p_i(\omega, \widehat{V}_i, v) dv\right). \end{aligned}$$

Using the integration by parts formula the above integral is equal to

$$\begin{aligned} &R_i^+(f \pi_i u_i p_i)(\omega, \widehat{V}_i) - R_i^-(f \pi_i u_i p_i)(\omega, \widehat{V}_i) - \int_{a_i}^{b_i} f \times (\partial_{v_i}(\pi_i u_i) p_i + \pi_i u_i \partial_{v_i} p_i)(\omega, \widehat{V}_i, v) dv_i \\ &= R_i^+(f \pi_i u_i p_i)(\omega, \widehat{V}_i) - R_i^-(f \pi_i u_i p_i)(\omega, \widehat{V}_i) - \int_{a_i}^{b_i} f \times (\partial_{v_i}(\pi_i u_i) p_i + \pi_i u_i \bar{p}_i) p_i(\omega, \widehat{V}_i, v) dv_i \\ &= R_i^+(f \pi_i u_i p_i)(\omega, \widehat{V}_i) - R_i^-(f \pi_i u_i p_i)(\omega, \widehat{V}_i) - E_{\mathcal{G}_i}(f \times (\partial_{v_i}(\pi_i u_i) p_i + \pi_i u_i \bar{p}_i)(\omega, V)). \end{aligned}$$

We come back and we obtain

$$\begin{aligned} \sum_{i=1}^m E_{\mathcal{G}}(D_i F \times (U_i \pi_i)) &= \sum_{i=1}^m E_{\mathcal{G}}(R_i^+(f \pi_i u_i p_i)(\omega, \widehat{V}_i) - R_i^-(f \pi_i u_i p_i)(\omega, \widehat{V}_i)) \\ &\quad - E_{\mathcal{G}}(f \times \sum_{i=1}^m (\partial_{v_i}(\pi_i u_i) p_i + \pi_i u_i \bar{p}_i)(\omega, V)). \end{aligned}$$

□

We have the following straightforward computation rules.

Lemma 2 *Let $\phi : R^d \rightarrow R$ be a C^∞ function with polynomial growth and let $F = (F^1, \dots, F^d) \in \mathcal{S}$. Then $\phi(F) \in \mathcal{S}$ and*

$$D\phi(F) = \sum_{k=1}^d \partial_k \phi(F) D F^k. \quad (5)$$

If $F \in \mathcal{S}$ and $U \in \mathcal{P}$ then

$$\delta(FU) = F\delta(U) - \langle DF, U \rangle_\pi. \quad (6)$$

The proof of the first equality is trivial and the proof of the second one is obtained using the duality formula and the chain rule.

The main result in this section is the following.

Theorem 3 *We assume that **H.1** holds true. Let $F = (F^1, \dots, F^d) \in \mathcal{S}^d$ and $G \in \mathcal{S}$. We suppose that there exist d simple processes $\xi_j, j = 1, \dots, d$ such that the matrix $\sigma_\xi(F)(\omega)$ is invertible and we denote $\gamma_\xi(F) = \sigma_\xi^{-1}(F)$. We assume that*

$$E(|\det \gamma_\xi(F)|^p) < \infty \quad \forall p \in \mathbb{N}. \quad (7)$$

Let $\phi : R^d \rightarrow R$ be a C^∞ function with polynomial growth. Then, for every $i = 1, \dots, d$, one has

$$E_{\mathcal{G}}(\partial_i \phi(F) G) = E_{\mathcal{G}}(\phi(F) H_\xi^i(F, G)) + \sum_{j=1}^d E_{\mathcal{G}}(\left[\phi(F), G \gamma_\xi^{ji}(F) \xi_j \right]_\pi) \quad (8)$$

with

$$H_\xi^i(F, G) = \sum_{j=1}^d \delta(G \gamma_\xi^{ji}(F) \xi_j) = \sum_{j=1}^d (G \gamma_\xi^{ji}(F) \delta(\xi_j) - \langle \xi_j, D(\gamma_\xi^{ji}(F) G) \rangle_\pi). \quad (9)$$

Proof. Using the chain rule

$$\begin{aligned} \langle D\phi(F), \xi_j \rangle_\pi &= \sum_{r=1}^m \pi_r D_r \phi(F) \times \xi_j^r \\ &= \sum_{i=1}^d \partial_i \phi(F) \sum_{r=1}^m \pi_r D_r F^i \xi_j^r = \sum_{i=1}^d \partial_i \phi(F) \sigma_\xi^{ij}(F) \end{aligned}$$

so that $\partial_i \phi(F) = \sum_{j=1}^d \langle D\phi(F), \xi_j \rangle_\pi \gamma_\xi^{ji}(F)$. Since $F \in \mathcal{S}$ it follows that $\phi(F) \in \mathcal{S}$ and since $\xi \in \mathcal{P}$ it follows that $\sigma_\xi^{ij}(F) \in \mathcal{S}$. Moreover, since $\det \gamma_\xi \in \cap_{p \in N} L^p$ it follows that $\gamma_\xi^{ij}(F) \in \mathcal{S}$. So $G\gamma_\xi^{ji}(F)\xi_j \in \mathcal{P}$ and we may use the duality formula:

$$\begin{aligned} E_G(\partial_i \phi(F)G) &= \sum_{j=1}^d E_G(\langle D\phi(F), G\gamma_\xi^{ji}(F)\xi_j \rangle_\pi) \\ &= E_G(\phi(F)\delta(G\gamma_\xi^{ji}(F)\xi_j)) + \sum_{j=1}^d E_G([\phi(F), G\gamma_\xi^{ji}(F)\xi_j]_\pi) \end{aligned}$$

and the proof is completed. \square

2.1 Iterated integration by parts formula

We will iterate the integration by parts formula from the previous section. Our idea is to use different directions for each integration by parts. We fix $L \in N$ and we have in mind that we want to iterate the integration by parts formula L times. We consider a sequence of directions $\xi_l = (\xi_{l,1}, \dots, \xi_{l,d})$, $l = 1, \dots, L$ with $\xi_{l,i} = (\xi_{l,i}^p)_{p=1,m} \in \mathcal{P}$ which will be fixed throughtout this section. We define $D_{\xi_{l,i}} : \mathcal{S} \rightarrow \mathcal{S}$ by

$$D_{\xi_{l,i}} F = \sum_{p=1}^m \pi_p \xi_{l,i}^p D_p F = \langle \xi_{l,i}, DF \rangle_\pi.$$

The fact that $D_{\xi_{l,i}} F \in \mathcal{S}$ allows us to take a new derivative on this functional. So the reason of being of the directional derivatives is to avoid to define higher order derivatives.

For $1 \leq l \leq L$ we denote by Q_l the set of the multi-indexes $A = (A_1, \dots, A_r)$, $1 \leq r \leq l$ with $A_k = (l_k, i_k)$, with $1 \leq l_1 < \dots < l_r \leq l$ and $i_k \in \{1, \dots, d\}$, $k = 1, \dots, r$. We denote $|A| = r$. And for $A \in Q_l$ we define

$$D_\xi^A F := D_{\xi_{A_r}} \dots D_{\xi_{A_1}} F. \quad (10)$$

Notice that since $\pi_k, \xi_{l,i}^p \in \mathcal{S}$ the above definition makes sense. It is easy to check that

$$D_\xi^A (F \times G) = \sum_{B \cup B' = A} D_\xi^B F \times D_\xi^{B'} G \quad (11)$$

with B and B' disjoint. Moreover, for any function $\phi \in C^l(R^d, R)$, $F = (F^1, \dots, F^d) \in \mathcal{S}^d$ and $A \in Q_l$ we have

$$D_\xi^A \phi(F) = \sum_{r=1}^{|A|} \sum_{|\alpha|=r} \sum_{B_1 \cup \dots \cup B_r = A} \partial^\alpha \phi(F) \prod_{j=1}^r D_\xi^{B_j} F^{\alpha_j} \quad (12)$$

wher $\sum_{B_1 \cup \dots \cup B_r = A}$ is the sum over all the partitions of length r of A .

We introduce the following norms. For $F = f(\omega, V)$ we define

$$|F|_{\max} = \max_{i=1,m} \sup \left\{ \left| f(\omega, \widehat{V}_i, v_i) \right| : a_i(\omega, \widehat{V}_i) < v_i < b_i(\omega, \widehat{V}_i) \right\}$$

and

$$|F|_l := |F|_{\max} + \sum_{A \in Q_l} |D_\xi^A F|_{\max} \quad (13)$$

As a consequence of (11) and (12) we have

$$|F \times G|_l \leq |F|_l \times |G|_l \quad \text{and} \quad (14)$$

$$|\phi(F)|_l \leq \max_{0 \leq |\alpha| \leq l} |\partial^\alpha \phi(F)|_{\max} \times |F|_l^l. \quad (15)$$

For a functional $F = (F_1, \dots, F_d) \in \mathcal{S}^d$ we denote $\sigma_l(F) = \sigma_{\xi_l}(F)$ and

$$C_l(F) = |F|_l + \sum_{1 \leq k \leq l} \sum_{i,j=1}^d \left| \sigma_k^{ij}(F) \right|_k. \quad (16)$$

Lemma 4 *Suppose that $\sigma_l(F)$ is invertible and let $\gamma_l(F) := \sigma_l^{-1}(F)$. Then*

$$\left| \gamma_l^{ij}(F) \right|_l \leq l!(1 + C_l(F))^{d(l+1)} (1 + |\det \gamma_l(F)|_{\max})^l. \quad (17)$$

Proof. We write $\gamma_l^{i,j}(F) = (\det \sigma_l(F))^{-1} \times \hat{\sigma}_l^{ij}(F)$ where $\hat{\sigma}_l^{ij}(F)$ is the algebraic complement. Using (14) we obtain $\left| \gamma_l^{ij}(F) \right|_l \leq |(\det \sigma_l(F))^{-1}|_l \times C_l(F)^{d-1}$. Using (15) with $\phi(x) = x^{-1}$ we obtain

$$\begin{aligned} \left| (\det \sigma_l(F))^{-1} \right|_l &\leq |\det \gamma_l(F)|_{\max} + \max_{1 \leq r \leq l} \left| \phi^{(r)}(\det \sigma_l(F)) \right|_{\max} \times (1 + |\det \sigma_l(F)|_l)^l \\ &\leq l!(1 + |\det \gamma_l(F)|_{\max})^l \times (1 + C_l(F))^{dl} \end{aligned}$$

and (17) is proved. \square

We define now

$$d_l(\xi) = \sum_{1 \leq k \leq l} \sum_{i=1}^d \left| \delta(\xi_{k,i}) \right|_{k-1}. \quad (18)$$

In the previous section we have defined the random variables $H_\xi^i(F, G)$, $i = 1, \dots, d$. In order to iterate the integration by parts formula we define recursively $H_1^i(F, G) = H_{\xi_1}^i(F, G)$ and

$$H_{l+1}^{(\alpha_1, \dots, \alpha_{l+1})}(F, G) = H_l^{(\alpha_1, \dots, \alpha_l)}(F, H_{\xi_{l+1}}^{\alpha_{l+1}}(F, G)). \quad (19)$$

Lemma 5 *There exists some constants C (depending on l and d) such that*

$$\begin{aligned} i) \quad \left| H_{\xi_l}^i(F, G) \right|_{l-1} &\leq C |G|_l (1 + d_{l-1}(\xi)) (1 + C_l(F))^{d(l+1)} (1 + |\det \gamma_l(F)|_{\max})^l \\ ii) \quad |H_l^\alpha(F, G)| &\leq C |G|_l (1 + d_l(\xi))^l (1 + C_l(F))^{dl^2} \left(1 + \sum_{l'=1}^l |\det \gamma_{l'}(F)|_{\max} \right)^{l^2}. \end{aligned} \quad (20)$$

Remark 1 *In the previous inequalities we put $1 + d_l(\xi)$ in order to precise that we use $\max(1, d_l(\xi))$. So, if $d_l(\xi)$ is very small, $\left| H_{\xi_l}^i(F, G) \right|_{l-1}$ is not obliged to be small. The same*

for $C_l(F)$ and for $|\det \gamma_l(F)|$. At the contrary, we use $|G|_l$ and so if this quantity is small then $\left| H_{\xi_l}^i(F, G) \right|_{l-1}$ is also small. This will be important in the sequel.

Proof. Recall that

$$H_{\xi_l}^i(F, G) = G \times \sum_{j=1}^d (\gamma_l^{ij}(F) \delta(\xi_{l,j}) - D_{\xi_{l,j}} \gamma_l^{ij}(F)) + \sum_{j=1}^d \gamma_l^{i,j}(F) D_{\xi_{l,j}} G.$$

Then using (14) and (17) we obtain *i*). And *ii*) follows by recurrence. \square

We will need the following supplementary hypothesis. We consider some non-negative random variables $C_\xi, C_F, C_G, \eta_p, p = 1, \dots, m$ and \bar{F} which are \mathcal{G} measurable. This means that they do not depend on $V_i, i = 1, \dots, m$, and in this sense they may be considered as consatnts. We also assume that

$$E((C_\xi + C_F + C_G)^p) < \infty \quad \forall p \in N. \quad (21)$$

We assume that for every $p = 1, \dots, m, l = 1, \dots, L, i = 1, \dots, d$

$$A_L(\xi) \quad i) \quad d_L(\xi) \leq C_\xi, \quad (22)$$

$$ii) \quad \left| \xi_{l,i}^p \pi_p \right|_{l-1} \leq \eta_p \quad \text{and} \quad \sum_{p=1}^m \eta_p \leq C_\xi.$$

Moroever we assume that there exists some disjoint sets $I_l \subset \{1, \dots, m\}, l = 1, \dots, L$ such that

$$iii) \quad \xi_{l,i}^p = 0 \quad \text{if} \quad p \notin I_l. \quad (23)$$

This means that the derivatives in the directions $\xi_{l,i}, i = 1, \dots, d$ involve the variables $V_p, p \in I_l$ only. Consequently the integration by parts formula with ξ_l is based on this set of variables. Since I_l are disjoint sets we use different sets of variables for each integration by parts. Notice that in this case $|F|_l$ involves only derivatives of order one with respect to each V_p (because I_{l_1}, \dots, I_{l_r} are disjoint). But $\left| \sigma_k^{ij}(F) \right|_k$ involves derivatives of second order with respect to the variables $V_p, p \in I_k$ (because we have $\sigma_k^{ij}(F) = D_{\xi_{k,j}} F^i$ and then we apply one more derivative in the directions $\xi_{k,h}, h = 1, \dots, d$).

On F we assume that

$$A_L(F) \quad i) \quad C_L(F) \leq C_F, \quad (24)$$

$$ii) \quad |\det \gamma_k(F)|_{\max} \leq C_F, \quad k = 1, \dots, L.$$

$$iii) \quad |F| \leq \bar{F},$$

and on G we assume that

$$A_L(G) \quad |G|_L \leq C_G. \quad (25)$$

Proposition 6 *We assume that H.1 holds true. Let $F = (F^1, \dots, F^d) \in \mathcal{S}^d$ and $G \in \mathcal{S}$. We suppose that for every $l = 1, \dots, L, \sigma_l(F)$ is invertible and we assume that the hypothesis $A_L(\xi), A_L(F)$ and $A_L(G)$ hold true. We consider a function $\phi \in C^\infty(\mathbb{R}^d, \mathbb{R})$ such that $|\phi(x)| \leq \psi(|x|)$ where ψ is a non decreasing function. Then there exists some universal constants C and q such that for every multi-index $\alpha = (\alpha_1, \dots, \alpha_l) \in \{1, \dots, d\}^l, l \leq L$ one*

has

$$|E_{\mathcal{G}_{(l+1)}}(\partial^\alpha \phi(F)G)| \leq C\psi(\bar{F})C_G C_\xi^l C_F^{(d+1)l^2}. \quad (26)$$

where $\mathcal{G}_{(l)} := \mathcal{G} \vee \sigma(V_p, p \in \cup_{l'=l}^L I_{l'})$.

Proof. We proceed by recurrence. Suppose first that $l = 1$ so that $\alpha = \{i\}, i \in \{1, \dots, d\}$. We use the integration by parts formula (8) with the direction ξ_1 . Since $\xi_1^p = 0$ for $p \notin I_1$ we employ derivatives with respect to the variables $V_p, p \in I_1$ only. So $\mathcal{G}_{(2)}$ may be considered as the σ algebra of the noise which is not involved in the calculus and we may use the integration by parts formula (8) using the conditional expectation $E_{\mathcal{G}_{(2)}}$ instead of E_G . We obtain

$$\begin{aligned} E_{\mathcal{G}_{(2)}}(\partial_i \phi(F)G) &= E_{\mathcal{G}_{(2)}}(\phi(F)H_{\xi_1}^i(F, G)) + \sum_{j=1}^d \sum_{p \in I_1} E_{\mathcal{G}_{(2)}}(R_p^+(\phi(F)G\gamma_{\xi_1}^{ji}(F)\xi_{1,j}^p \pi_p p_p)) \\ &\quad - \sum_{j=1}^d \sum_{p \in I_1} E_{\mathcal{G}_{(2)}}(R_p^-(\phi(F)G\gamma_{\xi_1}^{ji}(F)\xi_{1,j}^p \pi_p p_p)). \end{aligned}$$

Using (20), i) and our hypothesis we obtain

$$\begin{aligned} |H_{\xi_1}^i(F, G)| &\leq C|G|_1(1 + d_1(\xi))(1 + C_1(F))^d(1 + |\det \gamma_1(F)|_{\max}) \\ &\leq C \times C_G C_\xi C_F^{d+1} \end{aligned}$$

for some universal constant C . We also have $|\phi(F)| \leq \psi(|F|) \leq \psi(\bar{F})$ so we obtain

$$|E_{\mathcal{G}_{(2)}}(\phi(F)H_{\xi_1}^i(F, G))| \leq C\psi(\bar{F})C_G C_\xi C_F^{d+1}.$$

We fix now $p \in I_1$ and we estimate

$$\left| R_p^+(\phi(F)G\gamma_{\xi_1}^{ji}(F)\xi_{1,j}^p \pi_p p_p) \right| = |\phi(R_p^+ F)| \times |R_p^+ G| \times \left| R_p^+ \gamma_{\xi_1}^{ji}(F) \right| \times |R_p^+(\xi_{1,j}^p \pi_p p_p)|.$$

We have $|\phi(R_p^+ F)| \leq \psi(|R_p^+ F|) \leq \psi(|R_p^+ \bar{F}|) = \psi(\bar{F})$. The last equality is due to the fact that \bar{F} does not depend on V_p . Since $G \leq C_G$ we obtain $|R_p^+ G| \leq |R_p^+ C_G| = C_G$ and in the same way $\left| R_p^+ \gamma_{\xi_1}^{ji}(F) \right| \leq C \times C_F^d$ and $|R_p^+(\xi_{1,j}^p \pi_p p_p)| \leq \eta_p$. So

$$\sum_{p \in I_1} \left| R_p^+(\phi(F)G\gamma_{\xi_1}^{ji}(F)\xi_{1,j}^p \pi_p p_p) \right| \leq C\psi(\bar{F})C_G C_F^d \sum_{p=1}^m \eta_p \leq C\psi(\bar{F})C_G C_F^d C_\xi.$$

The same inequality holds for R_p^- and we obtain (26).

We suppose now that the inequality holds true for $\alpha = (\alpha_1, \dots, \alpha_l)$ and we prove it for $\alpha' = (\alpha_1, \dots, \alpha_l, \alpha_{l+1})$. We employ the integration by parts formula (8) with the direction

ξ_{l+1} and we obtain

$$\begin{aligned} E_{\mathcal{G}_{(l+2)}}(\partial^{\alpha_{l+1}} \partial^\alpha \phi(F)G) &= E_{\mathcal{G}_{(l+2)}}(\partial^\alpha \phi(F)H_{\xi_{l+1}}^{\alpha_{l+1}}(F, G)) \\ &+ \sum_{j=1}^d \sum_{p \in I_{l+1}} E_{\mathcal{G}_{(l+2)}}((R_p^+(\partial^\alpha \phi(F)G\gamma_{\xi_{l+1}}^{j, \alpha_{l+1}}(F)\xi_{l+1, j}^p \pi_p p_p)) \\ &- \sum_{j=1}^d \sum_{p \in I_{l+1}} E_{\mathcal{G}_{(l+2)}}(R_p^-(\partial^\alpha \phi(F)G\gamma_{\xi_{l+1}}^{j, \alpha_{l+1}}(F)\xi_{l+1, j}^p \pi_p p_p))). \end{aligned}$$

We evaluate the first term. We check that $G' := H_{\xi_{l+1}}^{\alpha_{l+1}}(F, G)$ verifies the hypothesis $A_L(G')$ with $C_{G'} = C \times C_G C_\xi C_F^{(d+1)l}$. And this follows from (20) and $A_L(G), A_L(F), A_L(\xi)$. So we may use the recurrence hypothesis and we obtain

$$\left| E_{\mathcal{G}_{(l+1)}}(\partial^\alpha \phi(F)H_{\xi_{l+1}}^{\alpha_{l+1}}(F, G)) \right| \leq C\psi(\bar{F})C_G C_\xi^{l+1} C_F^{(d+1)(l^2+l)} \leq C\psi(\bar{F})C_G C_\xi^{l+1} C_F^{(d+1)(l+1)^2}.$$

Since $\mathcal{G}_{(l+2)} \subset \mathcal{G}_{(l+1)}$ we may replace $E_{\mathcal{G}_{(l+1)}}$ by $E_{\mathcal{G}_{(l+2)}}$. So this term verifies (26).

We fix now $p \in I_{l+1}$ and we denote

$$h_p = h_p(V_1, \dots, V_m) = \partial^\alpha \phi(F)G\gamma_{\xi_{l+1}}^{j, \alpha_{l+1}}(F)\xi_{l+1, j}^p \pi_p p_p.$$

We look to

$$E_{\mathcal{G}_{(l+2)}}(R_p^+(\partial^\alpha \phi(F)G\gamma_{\xi_{l+1}}^{j, \alpha_{l+1}}(F)\xi_{l+1, j}^p \pi_p p_p)) = E_{\mathcal{G}_{(l+2)}}(E_{\mathcal{G}_{(l+1)}}(R_p^+(h_p))).$$

Since $\partial^\alpha \phi$ is bounded, we have $|h_p| \leq C \times C_G \eta_p (1 + C_\xi + C_F)^q$, which does not depend on V_p and which is integrable. So we may use Lebesgue's theorem and we obtain

$$E_{\mathcal{G}_{(l+1)}}(R_p^+(h_p)) = E_{\mathcal{G}_{(l+1)}}(\lim_{v_p \uparrow b_p} h_p(\widehat{V}_p, v_p)) = \lim_{v_p \uparrow b_p} E_{\mathcal{G}_{(l+1)}}(h_p(\widehat{V}_p, v_p)).$$

Now we have to evaluate $E_{\mathcal{G}_{(l+1)}}(h_p(\widehat{V}_p, v_p)) = E_{\mathcal{G}_{(l+1)}}(\partial^\alpha \phi(F_{v_p})G'_{v_p})$ with $F_{v_p} := f(\omega, (\widehat{V}_p, v_p))$ and $G'_{v_p} := (G\gamma_{\xi_{l+1}}^{j, \alpha_{l+1}}(F)\xi_{l+1, j}^p \pi_p p_p^{-1} |_{V_p=v_p})$. For every $v_p \in (a_p, b_p)$ we have

$$\left| G'_{v_p} \right|_l \leq \left| G_{v_p} \right|_l \times \left| (\gamma_{\xi_{l+1}}^{j, \alpha_{l+1}}(F))_{v_p} \right|_l \times \left| (\xi_{l+1, j}^p \pi_p p_p)_{v_p} \right|_l \leq C_G (1 + C_F)^{d(l+1)} \eta_p =: C_{G'_{v_p}}.$$

Notice that here comes on the fact that we work with the norms $|\cdot|_{\max}$. And this is the reason of being of these norms. Using the recurrence hypothesis we obtain

$$\begin{aligned} \left| E_{\mathcal{G}_{(l+1)}}(h_p(V_1, \dots, V_m)) \right| &= \left| E_{\mathcal{G}_{(l+1)}}(\partial^\alpha \phi(F_{v_p})G'_{v_p}) \right| \leq C\psi(\bar{F})C_{G'_{v_p}} C_\xi^l C_{F_{v_p}}^{dl^2} \\ &\leq C\psi(\bar{F})C_G C_\xi^l C_F^{d(l+1)^2} \eta_p. \end{aligned}$$

Passing to the limit with $v_p \uparrow b_p$ we get $\left| E_{\mathcal{G}_{(l+1)}}(R_p^+(h_p)) \right| \leq C\psi(\bar{F})C_G C_\xi^l C_F^{d(l+1)^2} \eta_p$ and finally

$$\sum_{p=1}^m \left| E_{\mathcal{G}_{(l+1)}}(R_p^+(h_p)) \right| \leq C\psi(\bar{F})C_G C_\xi^l C_F^{d(l+1)^2} \sum_{p=1}^m \eta_p \leq C\psi(\bar{F})C_G C_\xi^{l+1} C_F^{d(l+1)^2}.$$

We replace now $E_{\mathcal{G}_{(l+1)}}$ by $E_{\mathcal{G}_{(l+2)}}$ and the proof is completed. \square

Remark 2 In the last inequality comes on $\sum_{p=1}^m \eta_p$. This is why we need to have $|G|_l$ and not $1 + |G|_l$ in (20). If not we would have $\sum_{p=1}^m (1 + \eta_p) \geq m$ and this is not convinient for us because in concrete applications $m \uparrow \infty$ (notice that m does not appear in the right hand side of (26)).

3 Stochastic eqations

We consider a Poission point process p with measurable state space $(E, B(E))$. We reffer to [I.W] for notation. We denote by N the counting measure associated to p so $N_t(A) := N((0, t] \times A)$ represents the number of points $p_s \in A, s < t$. The intensity measure is $dt \times \mu(da)$ where μ is a sigma-finite measure on $(E, B(E))$. We consider a sequence $E_n \uparrow E$ such that $\mu(E_n) < \infty$ and $\mu(E_{n+1}) \leq \mu(E_n) + c, \forall n \in N$ for some constant $c > 0$. This sequence is choosen in the begining and the non-degeneracy assumption bellow depends on it.

We consider the one dimensional stochastic equation

$$X_t = x + \int_0^t \int_E c(u, a, X_{u-}) dN(du, da) + \int_0^t g(u, X_u) du. \quad (27)$$

Our aim is to give sufficient conditions on the coefficients c and g in order that the law of X_t is absolutely continuous with respect to the Lebesgue measure and has a smooth density.

H 3.1 We assume that the functions $(s, x) \rightarrow c(s, a, x), a \in E$, and $(s, x) \rightarrow g(s, x)$ are infinitely differentiable and there exists some bounded functions $c_L : E \rightarrow R, L \in N$ and some constants $g_L, L \in N$ such that

$$|c(u, a, x)| \leq c_1(a)(1 + |x|), \quad \sum_{k=1}^L |\partial_x^k c(u, a, x)| \leq c_L(a),$$

$$|g(u, x)| \leq g_1(1 + |x|), \quad \sum_{k=1}^L |\partial_x^k g(u, x)| \leq g_L.$$

Under the above hypothesis the above equation admits a unique solution and $\int_0^t \int_E c(s, a, S_{u-}) dN(ds, da)$ is a Stildjer integral.

This is the regularity assumption. We introduce now the non degeneracy condition. First of all we give a sufficient condition under which the tangent flow is invertible.

H. 3.2 We assume that there exists a measurable function $\widehat{c} : E \rightarrow R_+$ such that $\int_E \widehat{c}(a) d\mu(a) < \infty$ and

$$\|\partial_x c \times (I + \partial_x c)^{-1}(u, a, x)\| \leq \widehat{c}(a), \quad \forall (u, a, x) \in R_+ \times E \times R^d. \quad (28)$$

For notational convinience we take $\widehat{c} = c_L$ (it suffice to take the max for example).

We give now the non-degeneracy condition iself. We consider the function

$$\alpha(s, a, x) = g(s, x) - g(s, x + c(s, a, x)) + (g\partial_x c + \partial_t c)(s, a, x). \quad (29)$$

H. 3.3 We assume that there exists a measurable functions $\underline{\alpha} : E \rightarrow R_+$ such that

$|\alpha(s, a, x)| \geq \underline{\alpha}(a)$ and for every $q \in N$ one has

$$\liminf_n \frac{1}{\mu(E_n)} \ln \left(\int_{E_n} \frac{1}{\underline{\alpha}(a)^q} \mu(da) \right) = \theta_q < \infty. \quad (30)$$

For notational convenience we assume that $\underline{\alpha} \leq 1$.

Theorem 7 *We assume that **H.3.1**, **H.3.2** and **H.3.3** hold true. Let $q \in N$. For $t > (2q + 4)\theta_{8(2q+4)^2}$ the law of X_t is absolutely continuous with respect to the Lebesgue measure and the density is of class C_b^q . In particular, if $\theta_q = 0$ for every $q \in N$ then the law of X_t is absolutely continuous with respect to the Lebesgue measure and the density is of class C_b^∞ for every $t > 0$.*

In order to prove this result we will use the integration by parts formula from the previous section. We consider a discrete version of the above equation:

$$X_t^N = x + \int_0^t \int_{E_N} c(u, a, X_{u-}^N) dN(du, da) + \int_0^t g(u, X_u^N) du.$$

We fix $n < N$. We denote by $T_k^n, k \in N$ (resepctively by $T_k^{N,n}$) the jump times of the Poisson process $J_t^n = N((0, t], E_n)$ (respectively $J_t^{N,n} = N((0, t], E_N \setminus E_n)$) and we put $\Delta_k^n = p_{T_k^n}$ and $\Delta_k^{N,n} = p_{T_k^{N,n}}$. Then the above equation reads

$$X_t^N = x + \sum_{k=1}^{J_t^n} c(T_k^n, \Delta_k^n, X_{T_k^n-}^n) + \sum_{k=1}^{J_t^{N,n}} c(T_k^{N,n}, \Delta_k^{N,n}, X_{T_k^{N,n}-}^N) + \int_0^t g(u, X_u^N) du.$$

We fix $L \in N$ and we have in mind that we will use L integration by parts. Then we denote $t_k = tk/L$ and we fix some integrs $0 = m_0 < m_1 < \dots < m_L$. We will work on the set $\{J_{t_1}^n = m_1, \dots, J_{t_L}^n = m_L\}$ and the random variables on which the calculus is based are T_1^n, \dots, T_m^n with $m = m_L$. The σ -algebra which contains the noise which is not involved in the calculus is $\mathcal{G} = \sigma(\{J_{t_1}^n = m_1, \dots, J_{t_L}^n = m_L\}, \Delta_k^n, T_k^{N,n}, \Delta_k^{N,n}, k \in N)$ and we will take conditional expectations with respect to $\mathcal{G}_p = \sigma(T_i^n, i \neq p, i \leq m) \vee \mathcal{G}$. The law of $(T_{m_k}^n, \dots, T_{m_{k+1}-1}^n)$ conditionally to $J_{t_k}^n = m_k$ and $J_{t_{k+1}}^n = m_{k+1}$ is uniform on (t_k, t_{k+1}) so that the law of T_p^n conditionally to \mathcal{G}_p is $p_p(\omega, v) dv$ with

$$p_p(\omega, v) = \frac{1}{T_{p+1} - T_{p-1}} 1_{(T_{p-1}, T_{p+1})}(v).$$

Here and in the sequel we denote

$$\begin{aligned} T_p &= T_p^n \quad \text{for } m_k \leq p < m_{k+1}, \\ T_{m_k-1} &= t_k, \quad \text{and } T_{m_{k+1}} = t_{k+1}. \end{aligned}$$

We denote

$$h_{n,k} = \frac{t_{k+1} - t_k}{2(m_{k+1} - m_k)}$$

and we will work with the weights

$$\pi_p(\omega, T_{m_k}, \dots, T_{p+1}) = \prod_{i=m_k}^{p-1} 1_{(0, h_{n,k}]}(T_{i+1} - T_{i-1}) \times 1_{(h_{n,k}, \infty)}(T_{p+1} - T_{p-1}).$$

The reason of being of $h_{n,k}$ is the following. There are $m_{k+1} - m_k$ times in the interval (t_k, t_{k+1}) so that at list for one p one has $T_{p+1} - T_{p-1} > h_{n,k}$. So we conclude that exactly one of π_p is non nul (and in this case it is equal to one). We denote p_k the index for which $\pi_{p_k} = 1$.

Moreover we put $a_p = T_{p-1}$ and $b_p = T_{p-2} + h_{n,k}$. Let us notice that $\pi_p 1_{\{b_p < a_p\}} = 0$. Indeed, if $b_p < a_p$ then $h_{n,k} < T_{p-1} - T_{p-2} < T_p - T_{p-2}$ so that $\pi_p = 0$. Notice that if $T_{p-1} = a_p < T_p < b_p = T_{p-2} + h_{n,k}$ then

$$\pi_p(\omega, T_{m_k}, \dots, T_{p+1}) = \prod_{i=m_k}^{p-2} 1_{(0, h_{n,k}]}(T_{i+1} - T_{i-1}) \times 1_{(h_{n,k}, \infty)}(T_{p+1} - T_{p-1}).$$

which does not depend on T_p . So $T_p \rightarrow \pi_p$ is differentiable on (a_p, b_p) and $\partial_{T_p} \pi_p = 0$.

Moreover if $\pi_p > 0$ then $b_p = T_{p-2} + h_{n,k} < T_{p+1}$ (if not $T_{p+1} - T_{p-1} < T_{p+1} - T_{p-2} < h_{n,k}$ so $\pi_p = 0$). So for $T_{p-1} = a_p < v < b_p < T_{p+1}$ we have

$$p_p(\omega, v) = \frac{1}{T_{p+1} - T_{p-1}} \leq \frac{1}{h_{n,k}} \quad \text{and} \quad \bar{p}(\omega, v) = \partial_v \ln p_p(\omega, v) = 0.$$

We already know that $T_p \rightarrow X_t^N(T_1, \dots, T_m)$ is L times differentiable (see the appendix) so that we are in the framework given in Section 3.

Remark 3 Notice that $T_p \rightarrow \pi_p$ is discontinuous in T_{p-1} and T_{p+1} . It is tempting to try to employ the strategy given in [B] and to multiply by a dunction which is null in T_{p-1} and T_{p+1} . Then the border terms in the integration by parts formula cancel and the calculus simplifies drastically. But this is not possible. If we try to do it we have to construct a function $\phi : R \rightarrow R_+$ such that $\phi(0+) = \phi(1-) = 0$ and to define $\pi_p = \phi((T_p - T_{p-1}) / (T_{p+1} - T_{p-1}))$. We will see in a moment that $\partial_{T_p} \pi_p$ is invloved in $\delta(\xi)$ and $1/\pi_p$ is involved in the inverse of the Malliaivin covariance matrix. So we will need that $E |\partial_{T_p} \pi_p|^2 < \infty$ and $E |\pi_p|^{-2}$. And this amounts to $\int_0^1 |\phi'(x)|^2 dx < \infty$ and $\int_0^1 |\phi(x)|^{-2} dx < \infty$. Since $|\phi(x)| \leq \int_0^x |\phi'(x)| dx \leq x^{1/2} (\int_0^x |\phi'(x)|^2 dx)^{1/2} \leq Cx^{1/2}$ we will not have $\int_0^1 |\phi(x)|^{-2} dx < \infty$. In [B] we succed to get around this difficulty using a localization argument which allows to get rid of $E |\pi_p|^{-2}$. But this argument does not work here.

We come back to our proof. We will work with the direction

$$\xi_k^p = 1_{\{m_k \leq p < m_{k+1}\}} D_p X_t^N$$

and the coresponding Malliaivn covariance matrix is

$$\sigma_k := \sigma_{\xi_k}(X_t^N) = \sum_{p=1}^m \pi_p \xi_k^p D_p X_t^N = |D_{p_k} X_t^N|^2.$$

We also have

$$\delta(\xi_k) = - \sum_{p=1}^m (D_p(\pi_p \xi_k^p) + \pi_p \xi_k^p \bar{p}_p) = D_{p_k} D_{p_k} X_t^N.$$

We will now estimate the norms which used in the integration by parts formula. We denote $N_t(c_L) = \int_0^t \int_E c_L(a) N(du, da)$ and $e_t(c_L)$ is the solution of the linear equation

$e_t(c_L) = 1 + \int_0^t \int_E c_L(a) e_{u-}(c_L) N(du, da)$. Here c_L is the function which appear in our regularity hypothesis. We also denote

$$\Theta = N_t(c_L) e_t(c_L) e^{tg_L}.$$

One has $N_t(1_{E_N} c_L) \leq N_t(c_L)$ and $e_t(1_{E_N} c_L) \leq e_t(c_L)$ So using (37) we obtain

$$\left| \partial_{T_{q_1}} \dots \partial_{T_{q_l}} \partial_{T_q} X_t^N \right| + \left| \partial_{T_{q_1}} \dots \partial_{T_{q_l}} \partial_{T_q}^2 X_t^N \right| \leq C \Theta^{l+1}$$

for every $q_1 < \dots < q_l < q$. This gives (see the (16) and (18) for the notation)

$$d_L(\xi) + C_L(X_t^N) \leq C \Theta^{L+1}.$$

We also have $|\xi_l^p \pi_p p_p|_{l-1} \leq \Theta^{L+1} \times h_{n,k}^{-1} \times \delta_{p,p_k} =: \eta_p$ where δ_{p,p_k} is the Kronecker symbol. We conclude that the constant in the hypothesis $A_L(\xi)$ in Section 3 is

$$C_\xi = \Theta^{L+1} h_{n,k}^{-1}.$$

In order to compute the constant $C_{X_t^N}$ in $A_L(X_t^N)$ we have to estimate σ_k^{-1} . By (34)

$$\begin{aligned} D_p X_t^N &= \alpha(T_p^n, \Delta_p^n, X_{T_p-}^N) + \int_{T_p}^t \int_{E_N} \partial_x c(u, a, X_{u-}^N) D_p X_{u-}^N dN(du, da) \\ &\quad + \int_{T_p}^t \partial_x g(u, X_u^N) D_p X_{u-}^N du \end{aligned}$$

so the standard variance of constants method gives

$$D_p X_t^N = Y_t^N Z_{T_p}^N \alpha(T_p^n, \Delta_p^n, X_{T_p-}^N)$$

with Y_t^N and Z_t^N the solutions of the equations

$$\begin{aligned} Y_t^N &= 1 + \int_0^t \int_{E_N} \partial_x c(u, a, X_{u-}^N) Y_{u-}^N dN(du, da) + \int_0^t \partial_x g(u, X_u^N) Y_{u-}^N du, \\ Z_t^N &= 1 - \int_0^t \int_{E_N} \frac{\partial_x c}{1 + \partial_x c}(u, a, X_{u-}^N) Z_{u-}^N dN(du, da) - \int_0^t \partial_x g(u, X_u^N) Z_{u-}^N du. \end{aligned}$$

We have $Y_t^N \times Z_t^N = 1$ and it is easy to check that under the hypothesis **H3.3** (recall that $\hat{c} = c_L$) one has $(Y_t^N Z_{T_p}^N)^{-1} \leq \Theta^2$. So, using **H.3.2** we obtain

$$\sigma_k^{-1} = |D_{p_k} X_t^N|^{-2} = (Y_t^N Z_{T_{p_k}}^N)^{-2} \alpha^{-2}(T_{p_k}^n, \Delta_{p_k}^n, X_{T_{p_k}-}^N) \leq \frac{\Theta^4}{\underline{\alpha}^2(\Delta_{p_k}^n)}.$$

We conclude that we may take

$$C_{X_t^N} = \Theta^{4 \vee (L+1)} \times \max_{k=1, L} \underline{\alpha}^{-2}(\Delta_{p_k}^n).$$

We consider now a function $\phi \in C^\infty(R)$ and we use Proposition 6 (with $\psi = \|\phi\|_\infty$) and we obtain

$$\left| E_{\mathcal{G}}(\phi^{(L)}(X_t^N)) \right| \leq C \|\phi\|_\infty \Theta^{(L+1)^4} \times \max_{k=1, L} \left(\frac{1}{h_{n,k}^L} \alpha^{-4L^2}(\Delta_{p_k}^n) \right)$$

on the set $\{J_{t_1}^n = m_1, \dots, J_{t_L}^n = m_L\}$ (here C is an universal constants). This is true for all $1 \leq m_1 < \dots < m_L$ with $m_i = J_{t_i}^n$ so the above equality holds true on the set

$\Lambda_n := \{J_{t_i}^n - J_{t_{i-1}}^n \geq 1, i = 1, \dots, L\}$. It follows that

$$\left| E(\phi^{(L)}(X_t^N))1_{\Lambda_n} \right| \leq C \|\phi\|_\infty E(\Theta^{(L+1)^4} \times \max_{k=1,L} (\frac{1}{h_{n,k}^L} \underline{\alpha}^{-4L^2} (\Delta_{p_k}^n)1_{\Lambda_n})).$$

We have

$$\begin{aligned} E(\underline{\alpha}^{-8L^2} (\Delta_{p_k}^n)1_{\Lambda_n}) &= E\left(\sum_{p=J_{t_{k+1}}^n}^{J_{t_{k+1}}^n-1} \underline{\alpha}^{-8L^2} (\Delta_p^n) \pi_p 1_{\Lambda_n}\right) \\ &= \frac{1}{\mu(E_n)} \int_{E_n} \frac{1}{\underline{\alpha}^{-8L^2}(a)} \mu(da) E(1_{\Lambda_n} \sum_{p=J_{t_{k+1}}^n}^{J_{t_{k+1}}^n-1} \pi_p) \\ &= \frac{1}{\mu(E_n)} \int_{E_n} \frac{1}{\underline{\alpha}^{-8L^2}(a)} \mu(da) P(\Lambda_n). \end{aligned}$$

We also have $h_{n,k}^{-1} = 2(J_{t_{k+1}}^n - J_{t_k}^n)/Lt$ so $(E(h_{n,k}^{-4L}))^{1/4} \leq C\mu(E_n)^L$. And $E(\Theta^q) < \infty, \forall q \in \mathbb{N}$. So we obtain

$$\begin{aligned} \left| E(\phi^{(L)}(X_t^N)1_{\Lambda_n}) \right| &\leq C \|\phi\|_\infty \times A_{n,L} \quad \text{with} \\ A_{n,L} &:= \mu^{L-1}(E_n) \int_{E_n} \frac{1}{\underline{\alpha}^{-8L^2}(a)} \mu(da). \end{aligned}$$

Passing to the limit with $N \rightarrow \infty$ we get

$$\left| E(\phi^{(L)}(X_t)1_{\Lambda_n}) \right| \leq C \|\phi\|_\infty \times A_{n,L}.$$

We are now ready for the proof of the Theorem. We denote $e(\xi, x) = \exp(-i\xi x)$ and we consider the Fourier transforms $\widehat{p}(\xi) = E(e(\xi, X_t))$. Since $\partial_x^L e(\xi, x) = (i\xi)^L e(\xi, x)$ we employ the above inequality and we obtain

$$|\widehat{p}(\xi)| \leq P(\Lambda_n^c) + |\xi|^{-L} |E(\partial_x^L e(\xi, X_t))| \leq L e^{-\mu(E_n)t/L} + C |\xi|^{-L} A_{n,L}.$$

We use now the hypothesis **H.3.3**: for each $\eta > 0$ we may find n_η such that for any $n \geq n_\eta$ one has

$$e^{\mu(E_n)(\theta_{8L^2} + \eta) + (L-1) \ln \mu(E_n)} \geq A_{n,L} = \mu^{L-1}(E_n) \int_{E_n} \frac{1}{\underline{\alpha}^{-8L^2}(a)} \mu(da).$$

So if $\frac{t}{L} > \theta_{8L^2}$ we may find $n_{L,t}$ such that for $n \geq n_{L,t}$ one has

$$\varepsilon_n := e^{-\mu(E_n)t/L} \geq A_{n,L}$$

and consequently

$$|\widehat{p}(\xi)| \leq L\varepsilon_n + C |\xi|^{-L} \varepsilon_n^{-1} \quad \forall n \geq n_{L,t}. \quad (31)$$

Suppose that $|\xi|^{L/2} \geq \varepsilon_{n_{L,t}}^{-1}$. Isince $\varepsilon_n^{-1} \uparrow n \infty$ we may find some $n(\xi) \geq n_{L,t}$ such that

$$\varepsilon_{n(\xi)}^{-1} \leq |\xi|^{L/2} \leq \varepsilon_{n(\xi)+1,L}^{-1} \leq C \times \varepsilon_{n(\xi)}^{-1}.$$

The last inequality holds true because we ask that $\mu(E_{n+1}) \leq \mu(E_n) + c$ for some constant c . Since the inequality (31) holds true for any n we obtain $|\widehat{p}(\xi)| \leq C |\xi|^{-L/2}$ if

$|\xi|^{L/2} \geq \varepsilon_{nL,t}^{-1}$. We take now $q \in N$ and we write

$$\begin{aligned} \int_R |\xi|^q |\widehat{p}(\xi)| d\xi &= \int_{|\xi|^{L/2} < \varepsilon_{nL,t}^{-1}} |\xi|^q |\widehat{p}(\xi)| d\xi + \int_{|\xi|^{L/2} \geq \varepsilon_{nL,t}^{-1}} |\xi|^q |\widehat{p}(\xi)| d\xi \\ &\leq \varepsilon_{nL,t}^{-(q+1)} + C \int_{|\xi|^{L/2} \geq \varepsilon_{nL,t}^{-1}} |\xi|^{q-\frac{L}{2}} d\xi. \end{aligned}$$

And if we choose $L = 2q + 4$ the above quantity is finite. And a classical result says that the law of X_t is absolutely continuous and has a density of class C_b^q . In order to be allowed to choose $L = 2q + 4$ and to run the above argument we need that $\frac{t}{L} > \theta_{8L^2}$ and this amounts to $t > (2q + 4)\theta_{8(2q+4)^2}$. And this is our hypothesis. \square

Example 1. We take $E = R_+$ and $\mu_\lambda(da) = \sum_{n=1}^{\infty} \frac{1}{n^\lambda} \delta_{1/n}(da)$ with $0 < \lambda < 1$. Moreover we take $c(t, a, x) = a$ (so that $\int_0^t \int_E c(s, a, X_t) dN(s, a) = \int_0^t \int_E adN(s, a)$) is the Lévy process with intensity measure μ and a smooth function $g(t, x) = g(x)$ such that $|g'(x)| \geq \underline{g} > 0$. Then $|\alpha(t, a, x)| = |g(x) - g(x+a)| \geq \underline{g} \times a$. We also take $E_n = (\frac{1}{n}, 1)$ so that $\mu_\lambda(E_n) = c_\lambda n^{1-\lambda}$ for some $c_\lambda > 0$. It is easy to check that $\mu_\lambda(E_{n+1}) \leq \mu_\lambda(E_n) + c$ for some $c > 0$. So we are in the framework of the previous theorem. We have $\int_{E_n} a^{-q} \mu_\lambda(da) = c_{q+\lambda} n^{1-(q+\lambda)}$. It follows that

$$\frac{1}{\mu_\lambda(E_n)} \ln \left(\int_{E_n} a^{-q} \mu_\lambda(da) \right) = \frac{1 - (q + \lambda) + \ln c_{q+\lambda}}{c_\lambda} \times \frac{\ln n}{n^{1-\lambda}} \rightarrow 0.$$

We conclude that for every $t > 0$ the law of X_t has a density of class C_b^∞ .

We take now $\lambda = 1$. Then $\mu_\lambda(E_n) = c_0 \ln n$ and for $q \geq 1$

$$\frac{1}{\mu_\lambda(E_n)} \ln \left(\int_{E_n} a^{-q} \mu_\lambda(da) \right) = \frac{1 - q + \ln c_q}{c_0} =: \theta_q > 0.$$

So we will have to take a sufficiently large t in order to obtain a regular density. This is not just a technical matter: in [K] Theorem 1.4 the author proves that under appropriate conditions, for small t , the density is not continuous.

Example 2. We take the same intensity measure μ_λ as in the previous example but now we take the coefficients $g = 1$ and $c(u, a, x)$ such that $|\partial_x c(u, a, x)| \geq \underline{c}a$ for some $\underline{c} > 0$ (for example $c(u, a, x) = ax$). So we obtain $|\alpha(u, a, x)| = |\partial_x c(u, a, x)| \geq \underline{c}a$. The same discussion as above shows that for $0 < \lambda < 1$ one has a smooth density and for $\lambda = 1$ one has to wait in order to obtain smoothness. Let us compare this with the result of Ichikawa and Kunita [I.K] in the case $c(u, a, x) = ax$. The important point there is that they assume the so called "order condition":

$$\lim_{u \rightarrow 0} \frac{1}{u^h} \int_{|a| \leq u} a^2 \mu(da) > 0$$

for some $h \in (0, 2)$. In our case $\int_{|a| \leq u} a^2 \mu(da) = c_\lambda u^{1+\lambda}$ and so, if $\lambda < 1$ the order condition holds true. At the contrary, if $\lambda = 1$ the order condition fails and their result does not more apply. But in our approach also, we do not obtain a smooth density for small times.

4 Appendix. The deterministic equation

We fix some deterministic times $0 = u_0 < u_1 < \dots < u_m < T$ and we denote $u = (u_1, \dots, u_m)$ and $J_t = J_t(u) = k$ if $u_k \leq t < u_{k+1}$. They represent the jump times. We also consider an abstract set E and we denote $a = (a_1, \dots, a_m) \in E^m$, which will represent the amplitudes of the jumps. We consider some functions $c : R_+ \times E \times R \rightarrow R$ and $g : R_+ \times R \rightarrow R$ which are infinitely differentiable with respect to $(t, x) \in R_+ \times R$ and we associate to them the deterministic one-dimensional equation

$$X_t = x + \sum_{p=1}^{J_t} c(u_p, a_p, X_{u_p-}) + \int_0^t g(r, X_r) dr, \quad 0 \leq t \leq T. \quad (32)$$

We denote by $X_t(u, a)$ or simply by X_t the solution of this equation. In order to solve this equation we introduce the flow $\Phi = \Phi_u(t, x)$, $0 \leq u \leq t$, $x \in R$ solution of the ordinary integral equation $\Phi_u(t, x) = x + \int_u^t g(r, \Phi_u(r, x)) dr$, $t \geq u$. The solution X of the equation (32) is given by $X_0 = x$ and

$$\begin{aligned} X_t &= \Phi_{u_i}(t, X_{u_i}) \quad \text{for } u_i \leq t < u_{i+1} \\ X_{u_{i+1}} &= X_{u_{i+1}-} + c(u_{i+1}, a_{i+1}, X_{u_{i+1}-}) \\ &= \Phi_{u_i}(u_{i+1}, X_{u_i}) + c(u_{i+1}, a_{i+1}, \Phi_{u_i}(u_{i+1}, X_{u_i})). \end{aligned}$$

Since c and g are smooth, the function $(u_1, \dots, u_m) \rightarrow X_t(u_1, \dots, u_m)$ is of class C^∞ on the set $\{u_i \neq t, i = 1, \dots, m\}$. Our aim is to compute the derivatives of X_t with respect to $u_j, j = 1, \dots, m$.

We introduce first some notation. Since g is differentiable, the application $x \rightarrow \Phi_u(t, x)$ is differentiable and $\partial_x \Phi_u(t, x) =: e_{u,t}(x)$ verify the equation

$$e_{u,t}(x) = 1 + \int_u^t \partial_x g(r, \Phi_u(r, x)) e_{u,r}(x) dr.$$

We also have

$$\partial_u \Phi_u(t, x) = -g(u, x) + \int_u^t \partial_x g(r, \Phi_u(r, x)) \partial_u \Phi_u(r, x) dr$$

so that $\partial_u \Phi_u(t, x) = -\rho(u, x) \times e_{u,t}(x)$. Moreover we denote

$$\begin{aligned} \alpha(u, a, x) &= g(u, x) - g(u, x + c(u, a, x)) + (g \times \partial_x c + \partial_t c)(u, a, x), \\ \beta(u, a, x) &= (\partial_u \alpha)(u, a, x) + (\partial_x \alpha)(u, a, x) g(u, x). \end{aligned} \quad (33)$$

Finally we denote

$$V_p(t) := \partial_{u_p} X_{t-}(u, a), \quad W_p(t) := \partial_{u_p}^2 X_{t-}(u, a).$$

Lemma 8 *The functions $V_p(t), t > u_p$ and $W_p(t), t > u_p$ solve the equations*

$$V_p(t) = \alpha(u_p, a_p, X_{u_p-}) + \sum_{k=p+1}^{J_t-} \partial_x c(u_k, a_k, X_{u_k-}) V_p(u_k) + \int_{u_p}^t \partial_x g(u, X_u) V_p(u) du, \quad (34)$$

and

$$\begin{aligned} W_p(t) &= \beta(u_p, a_p, X_{u_p-}) + \sum_{k=p+1}^{J_t-} \partial_x^2 c(u_k, a_k, X_{u_k-}) V_p^2(u_k) + \int_{u_p}^t \partial_x^2 g(u, X_u) V_p^2(u) du \\ &\quad + \sum_{k=p+1}^{J_t-} \partial_x c(u_k, a_k, X_{u_k-}) W_p(u_k) + \int_{u_p}^t \partial_x g(u, X_u) W_p(u) du. \end{aligned} \tag{35}$$

with the convedntion $\sum_{k=p+1}^p = 0$.

Remark 4 Notice thate $V_p(u_p) = g(u_p, X_{u_p-}) \neq \alpha(u_p, a_p, X_{u_p-})$ and $V_p(t) = 0$ for $0 \leq t < u_p$.

Remark 5 Since $t \rightarrow V_p(t)$ is left continuous (and not right continuous) it is natural that we put in the sum $V_p(u_k)$ and not $V_p(u_k-)$. For example if $t = u_q$ then $J(u_q-) = q - 1$ so that the last term in the sum is $V_p(u_{q-1})$ which is known at time u_q .

Proof. We write $X_{u_p-} = \Phi_{u_{p-1}}(u_p, X_{u_{p-1}})$ and we notice that $X_{u_{p-1}}$ does not depend on u_p . We have $\partial_t \Phi_{u_{p-1}}(t, x) = g(t, \Phi_{u_{p-1}}(t, x))$ so that

$$V_p(u_p) = \partial_{u_p} X_{u_p-} = \partial_{u_p} \Phi_{u_{p-1}}(u_p, X_{u_{p-1}}) = g(u_p, \Phi_{u_{p-1}}(u_p, X_{u_{p-1}})) = g(u_p, X_{u_p-}).$$

Moreover

$$\begin{aligned} \partial_{u_p} X_{u_p} &= \partial_{u_p} (X_{u_p-} + c(u_p, a_p, X_{u_p-})) \\ &= \partial_t c(u_p, a_p, X_{u_p-}) + (1 + \partial_x c(u_p, a_p, X_{u_p-})) \partial_{u_p} X_{u_p-} \\ &= \partial_t c(u_p, a_p, X_{u_p-}) + (1 + \partial_x c(u_p, a_p, X_{u_p-})) g(u_p, X_{u_p-}). \end{aligned}$$

Consider now $u_p < t \leq u_{p+1}$. We have $X_{t-} = \Phi_{u_p}(t, X_{u_p})$ so that

$$\begin{aligned} \partial_{u_p} X_{t-} &= \partial_{u_p} \Phi_{u_p}(t, X_{u_p}) + \partial_x \Phi_{u_p}(t, X_{u_p}) \partial_{u_p} X_{u_p} \\ &= (-g(u_p, X_{u_p}) + \partial_t c(u_p, a_p, X_{u_p-}) + (1 + \partial_x c) \times g(u_p, X_{u_p-})) e_{u_p, t}(X_{u_p}) \\ &= \alpha(u_p, a_p, X_{u_p-}) e_{u_p, t}(X_{u_p}). \end{aligned}$$

In particular this gives

$$V_p(t) = \alpha(u_p, a_p, X_{u_p-}) + \int_{u_p}^t \partial_x g(u, X_u) V_p(u) du, \quad u_p < t \leq u_{p+1}.$$

We compute now

$$\partial_{u_p} X_{u_{p+1}} = \partial_{u_p} X_{u_{p+1}-} + \partial_{u_p} c(u_{p+1}, a_{p+1}, X_{u_{p+1}-}) = \partial_{u_p} X_{u_{p+1}-} (1 + \partial_x c)(u_{p+1}, a_{p+1}, X_{u_{p+1}-}).$$

For $u_{p+1} < t \leq u_{p+2}$ we have

$$\begin{aligned} V_p(t) &= \partial_{u_p} X_{t-} = \partial_{u_p} \Phi_{u_{p+1}}(t, X_{u_{p+1}}) = \partial_{u_p} X_{u_{p+1}} \times \partial_x \Phi_{u_{p+1}}(t, X_{u_{p+1}}) \\ &= \partial_{u_p} X_{u_{p+1}} \times e_{u_{p+1}, t}(X_{u_{p+1}}) = V_p(u_{p+1}) (1 + \partial_x c)(u_{p+1}, a_{p+1}, X_{u_{p+1}-}) \times e_{u_{p+1}, t}(X_{u_{p+1}}). \end{aligned}$$

This means that

$$V_p(t) = V_p(u_{p+1}) + \partial_x c(u_{p+1}, a_{p+1}, X_{u_{p+1}-}) V_p(u_{p+1}) + \int_{u_{p+1}}^t \partial_x g(u, X_u) V_p(u) du.$$

Notice that $J_{t-} = p + 1$ for $u_{p+1} < t \leq u_{p+2}$. We continue like this and we obtain the equation (34). And since $\beta(u_p, a_p, X_{u_p-}) = \partial_{u_p}(\alpha(u_p, a_p, X_{u_p-}))$ we take derivatives in (34) and we obtain (35). \square

We assume that there exists some bounded functions $c_L : E \rightarrow R, L \in N$ and some constants constant $g_L, L \in N$ such that

$$|c(u, a, x)| \leq c_1(a)(1 + |x|), \quad \sum_{k=1}^L |\partial_x^k c(u, a, x)| \leq c_L(a),$$

$$|g(u, x)| \leq g_1(1 + |x|), \quad \sum_{k=1}^L |\partial_x^k g(u, x)| \leq g_L.$$

We denote

$$e_t(c_L) = \prod_{i=1}^{J_t} (1 + c_L(a_i)), \quad N_t(c_L) = \sum_{i=1}^{J_t} c_L(a_i).$$

Lemma 9 *For every $L \in N$ there exists a universal constant C (depending on $\|c_L\|_\infty, g_L$ and $\|x\|$) such that for every $p_1 < p_2 < \dots < p_{L-1} < p$ with $u_p \leq t$ we have*

$$\left| \partial_{u_{p_1}} \dots \partial_{u_{p_{L-1}}} V_p \right| + \left| \partial_{u_{p_1}} \dots \partial_{u_{p_{L-1}}} W_p \right| \leq C(N_t(c_L) e_t(c_1) e^{tg_L})^{L+1}. \quad (36)$$

In particular, since $e_t(c_L)$ and $N_t(c_L)$ does not depend on u this gives

$$\sup_{u_{p_1} < \dots < u_{p_{L-1}} < u_p < t} \left(\left| \partial_{u_{p_1}} \dots \partial_{u_{p_{L-1}}} V_p \right| + \left| \partial_{u_{p_1}} \dots \partial_{u_{p_{L-1}}} W_p \right| \right) \leq C(N_t(c_L) e_t(c_L) e^{tg_L})^{L+1}. \quad (37)$$

Proof. Using a slight variant of Gronwall's lemma we obtain $|X_t| \leq |x| e_t(c_1) e^{tg_1}$. Notice that $|\alpha(u, a, x)| \leq (2g_1 + 1)c_1(a)(1 + |x|)$. Then, using once again Gronwall's lemma we have $|V_p(t)| \leq C e_t(c_1) e^{tg_1}$. We proceed now by recurrence. In order to shorten the notation we take $g = 0$. We take now $p_1 < p$ and we take derivatives in the equation (34) in order to obtain

$$\begin{aligned} \partial_{u_{p_1}} V_p(t) &= \partial_{u_{p_1}} \alpha(u_p, a_p, X_{u_p-}) + \sum_{k=p+1}^{J_{t-}} \partial_x^2 c(u_k, a_k, X_{u_k-}) V_{p_1}(u_k) V_p(u_k) \\ &\quad + \sum_{k=p+1}^{J_{t-}} \partial_x c(u_k, a_k, X_{u_k-}) \partial_{u_{p_1}} V_p(u_k). \end{aligned}$$

We have

$$\left| \partial_{u_{p_1}} \alpha(u_p, a_p, X_{u_p-}) \right| = \left| \partial_x \alpha(u_p, a_p, X_{u_p-}) \right| \left| \partial_{u_{p_1}} X_{u_p-} \right| \leq C e_t^2(c_L).$$

And using the recurrence hypothesis

$$|\partial_x^2 c(u_k, a_k, X_{u_k-}) V_{p_1}(u_k) V_p(u_k)| \leq C c_2(a_k) (N_t(c_1) e_t(c_1))^2$$

so that

$$\begin{aligned} |\partial_{u_{p_1}} V_p(t)| &\leq C e_t^2(c_L) e^{2tg_1} + C N_t(c_2) (N_t(c_1) e_t(c_1))^2 \\ &\quad + \sum_{k=p+1}^{J_t-} c_1(a_k) |\partial_{u_{p_1}} V_p(u_k)|. \end{aligned}$$

And so $|\partial_{u_{p_1}} V_p(t)| \leq C N_t^3(c_2) e_t^3(c_1)$. The proof is the same for higher order derivatives and for W_p . \square

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Vlad Bally,

Laboratoire Analyse et Mathématiques Appliquées
Université Paris-Est, Marne-La-Vallée
Cité Descartes 5, bd Descartes, Champs sur Marne
77454 MARNE LA VALLEE, CEDEX 2, France
bally@univ-mlv.fr