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Self-organized criticality via stochastic partial differential equations

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Models of self-organized criticality which can be described by stochastic partial differential equations with noncoercive monotone diffusivity function and multiplicative Wiener forcing term (as e.g. the Bak/Jang/Wiesenfeld- and Zhang-models) are analyzed. Existence and uniqueness of nonnegative strong solutions are proved. Previously numerically predicted transition to the critical state in 1-D is confirmed by a rigorous proof that this indeed happens in finite time with high probability.

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INTRODUCTION

The phenomenon of self-organized criticality is widely studied in Physics from different perspectives. (We refer to [1, 2], [3], [4], [5], [6], [7, 8], [9], [10], [11], [12], [13], [14], [15], [16] for various studies). In [3] it was proposed to describe this phenomenon, e.g. in the case of the avalanche dynamics in the BJW- (see [1, 2]) and Zhang- (see [16]) models, by a singular diffusion. In the absense of noise the density $\varrho(t,\xi)$, $t \geq 0$, $\xi \in \mathbb{R}^d$, of this diffusion is formally described by the evolution equation

$$\frac{\partial}{\partial t}\varrho(t,\xi) = \Delta\Psi(\varrho(t,\xi)), \tag{1}$$

where $\Psi(\varrho) := f(\varrho)H(\varrho-\varrho_c)$, H is the Heaviside function and $f(\varrho) = \text{const.}$ in the BJW-model and $f(\varrho) = \varrho$ in the Zhang-model. Setting $D(\varrho) = f'(\varrho)H(\varrho-\varrho_c)+f(\varrho)\delta(\varrho-\varrho_c)$, the equation turns into

$$\begin{aligned}\frac{\partial}{\partial t}\varrho(t,\xi) &= \nabla\cdot[D(\varrho(t,\xi))\,\nabla\varrho(t,\xi)]\\ &= D'(\varrho(t,\xi))\,|\nabla\varrho(t,\xi)|^2 + D(\varrho(t,\xi))\,\Delta\varrho(t,\xi).\end{aligned} \tag{2}$$

To discuss the problem in a heuristic way, let us consider a smooth version of H, for example

$$H_\varepsilon(\varrho) = \frac{1}{2} + \frac{1}{\pi}\arctan(\tfrac{\varrho}{\varepsilon})$$

(or the mathematical more convenient one in (7) below) and the corresponding function $D_\varepsilon = D_\varepsilon(\varrho)$. Since $H_\varepsilon(\varrho-\varrho_c)$ is convex left of ϱ_c and concave right of ϱ_c, we see that in the BJW-model $D'_\varepsilon(\varrho)|\nabla\varrho|^2$ is positive if ϱ is below ϱ_c and negative if ϱ is above ϱ_c, i.e. according to (2), ϱ is "pushed" towards the critical value ϱ_c. In the Zhang-model this does not seem to be true in general (see Remark 7 below). But if one adds a multiplicative noise term in (1), it turns out to be true in both models (cf. [11], [13], where the authors added an additive noise term).

The purpose of this note is to give a mathematically rigorous proof of this fact in the framework of stochastic partial differential equations (SPDE).

Let us introduce our framework, where we switch to common notation in SPDE, i.e. replace $\varrho(t,\xi)$ by $X(t,x)(\xi)$ with x being the density at $t=0$, $t \geq 0$, $\xi \in \mathcal{O}$, where $\mathcal{O}$ is an open bounded domain in $\mathbb{R}^d, d = 1,2,3$, with smooth boundary $\partial\mathcal{O}$. The appropriate class of SPDE is then of the form

$$\begin{cases} dX(t) - \Delta\Psi(X(t))dt \ni \sigma(X(t))dW(t), \\ \qquad\qquad\qquad\qquad \text{in } (0,\infty)\times\mathcal{O}, \\ \Psi(X(t)) \ni 0, \quad \text{on } (0,\infty)\times\partial\mathcal{O}, \\ X(0,x) = x \quad \text{on } \mathcal{O}, \end{cases} \tag{3}$$

where x is an initial datum, $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ a maximal monotone graph and

$$\sigma(X)dW = \sum_{k=1}^{N} \mu_k X d\beta_k\, e_k, \quad t \geq 0,$$

is a random forcing term, where $\{e_k\} \subset L^2(\mathcal{O})$ is the eigen basis of the Laplacian $-\Delta$ on $\mathcal{O}$ with Dirichlet boundary conditions, $N \in \mathbb{N}\cup\{+\infty\}$, μ_k are positive numbers and β_k independent standard Brownian motions on a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$.

Throughout this note we make the following assumptions:

Hypothesis 1 *(i)* $\Psi(r) = \rho\,\text{sign}\, r + \widetilde{\Psi}(r)$, *for* $r \in \mathbb{R}$, *where* $0 \in \Psi(0)$, $\rho > 0$, $\widetilde{\Psi} : \mathbb{R} \to \mathbb{R}$ *is Lipschitzian,*

$\widetilde{\Psi} \in C^1(\mathbb{R} \setminus \{0\})$ *and for some* $\delta > 0$ *it satisfies* $\widetilde{\Psi}'(r) \geq \delta$ *for all* $r \in \mathbb{R} \setminus \{0\}$.

(ii) If $N = \infty$, *the sequence* $\{\mu_k\}$ *is such that*

$$\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 < +\infty,$$

where λ_k *are the eigenvalues of* $-\Delta$.

A typical example is given by

$$\Psi(r) = \psi_0(r) + c, \ r \in \mathbb{R},$$

where

$$\psi_0(r) := \begin{cases} \alpha_1 r, & r > 0 \\ [-\varrho, \varrho], & r = 0 \\ \alpha_2 r, & r < 0 \end{cases}$$

and $\alpha_1, \alpha_2 > 0$, $\varrho \geq 0$, $c \in [-\varrho, \varrho]$ are constants.

In particular, we can treat continuous, stochastic versions of the BJW- and Zhang-models ([1, 2], [16]). On the basis of the above heuristics and also of numerical simulations in 1-D (see [3]), one expects that the system displays self-organized criticality, so that e.g. the supercritical region $\{X(t) > x_c\}$ is absorbed in time by the critical one $\{X(t) = x_c\}$. Here we shall rigorously prove that this is indeed the case in finite time with high probability arbitrarily close to 1 under appropriate assumptions on the parameters, at least if $N < \infty$. We emphasize that in some cases this is a purely stochastic phenomenon which does not occur when the noise is zero (cf. our final Remark 7 below).

The following notations will be used. $L^p(\mathcal{O})$, $p \geq 1$, is the usual space of p-integrable functions on $\mathcal{O}$ with norm $|\cdot|_p$. The scalar product in $L^2(\mathcal{O})$ and the duality induced by the pivot space $L^2(\mathcal{O})$ will be denoted by $\langle \cdot, \cdot \rangle_2$. $H_0^1(\mathcal{O}) \subset L^2(\mathcal{O})$ is the first order Sobolev space on $\mathcal{O}$ with zero trace on the boundary. For a fixed measure space $(E, \mathcal{E}, m)$, a Banach space B and $p \in [1, \infty]$ we denote the space of all with respect to the measure m p-integrable maps from E to B by $L^p(E; B)$.

In the following by H we shall denote the distribution space

$$H = H^{-1}(\mathcal{O}) = (H_0^1(\mathcal{O}))'$$

endowed with the scalar product and norm defined by

$$\langle u, v \rangle = \int_{\mathcal{O}} (-\Delta)^{-1} u(\xi) v(\xi) d\xi, \quad |u|_{-1} = \langle u, u \rangle^{1/2}.$$

We recall that the operator $x \to -\Delta \Psi(x)$ with the domain

$$\{x \in L^1(\mathcal{O}) \cap H : \ \Psi(x) \in H_0^1(\mathcal{O})\},$$

is maximal monotone in H (see e.g. [17]) and so the distribution space H is the natural functional setting for equation (3). However, the general existence theory of infinite dimensional stochastic equations in Hilbert space with nonlinear maximal monotone operators (see e.g. [18], [19]) is not directly applicable and so a direct approach must be used.

EXISTENCE, UNIQUENESS AND POSITIVITY

Definition 2 *Let* $x \in H$. *An* H*-valued continuous* $\mathcal{F}_t$*-adapted process* $X = X(t, x)$ *is called a solution to* (3) *on* $[0, T]$ *if for some* $p \in [1, \infty[$

$$X \in L^p(\Omega \times (0, T) \times \mathcal{O}) \cap L^2(0, T; L^2(\Omega, H)),$$

and there exists $\eta \in L^p(\Omega \times (0, T) \times \mathcal{O})$ *such that* $\mathbb{P}$*-a.s.*

$$\begin{array}{rl} & \langle X(t,x), e_j \rangle_2 \\ = & \langle x, e_j \rangle_2 + \int_0^t \int_{\mathcal{O}} \eta(s, \xi) \Delta e_j(\xi) d\xi ds \\ & + \sum_{k=1}^{\infty} \mu_k \int_0^t \langle X(s,x) e_k, e_j \rangle_2 d\beta_k(s), \\ & \qquad \forall \ j \in \mathbb{N}, \ t \in [0, T], \end{array} \tag{4}$$

$$\eta \in \Psi(X) \quad \text{a.e. in } \ \Omega \times (0, T) \times \mathcal{O}. \tag{5}$$

Below for simplicity we often write $X(t)$ instead of $X(t, x)$.

Theorem 3 *Under Hypothesis* 1 *for each* $x \in L^p(\mathcal{O})$, $p \geq 4$ *there is a unique solution* X *to* (3). *Moreover, if* x *is nonnegative a.e. in* $\mathcal{O}$ *then* $\mathbb{P}$*-a.s.*

$$X(t, x)(\xi) \geq 0, \quad \textit{for a.e. } (t, \xi) \in (0, \infty) \times \mathcal{O}.$$

Sketch of Proof. Consider the approximating equation

$$\begin{cases} dX_\lambda(t) - \Delta(\Psi_\lambda(X_\lambda(t))dt = \sigma(X_\lambda(t))dW(t), \\ X_\lambda(0, x) = x, \end{cases} \tag{6}$$

where $\lambda > 0$,

$$\Psi_\lambda(r) := \rho \ (\text{sign})_\lambda(r) + \widetilde{\Psi}(r), \quad r \in \mathbb{R}, \tag{7}$$

$$(\text{sign})_\lambda(r) := \begin{cases} 1 & \text{if } r > \lambda \\ \frac{r}{\lambda} & \text{if } r \in [-\lambda, \lambda] \\ -1 & \text{if } r < -\lambda. \end{cases}$$

By [20, Theorem 2.2] (applied with $m = 1$) equation (6) has a unique solution

$$X_\lambda \in L^2(\Omega \times (0, T) \times \mathcal{O}) \cap L^2(\Omega, C([0, T]; H))$$

in the sense of Definition 2 which is nonnegative, if so is x. Here as usual the space of continuous H-valued paths $C([0, T]; H)$ is equipped with the supremum norm.

By Ito's formula for $\alpha > 0$ large enough it follows that for all $\lambda, \mu \in (0,1)$ and $t \in [0,T]$

$$\begin{aligned}&\tfrac{1}{2}\,|X_\lambda(t) - X_\mu(t))|_{-1}^2 e^{-\alpha t}\\ &\leq C\max\{\lambda,\mu\}\int_0^t\int_{\mathcal{O}}\Big(|\Psi_\lambda(X_\lambda(s))|^2 + |X_\lambda(s)|^2\\ &+|\Psi_\mu(X_\mu(s))|^2 + |X_\mu(s)|^2\Big)e^{-\alpha s}d\xi\, ds\\ &+\int_0^t e^{-\alpha s}\langle X_\lambda(s) - X_\mu(s), \sigma(X_\lambda(s) - X_\mu(s))dW(s)\rangle_2.\end{aligned} \tag{8}$$

Hence by the Burkholder-Davis-Gundy inequality (for $p=1$) $\{X_\lambda\}$ is a Cauchy net in $L^2(\Omega; C([0,T],H))$ and by a standard technique from stochatic PDE one shows that the limit X is the desired solution to (3) (cf. [21] for details). □

Remark 4 *One can also show (see [21, Prop. 3.4]) that* X, $X_\lambda \in L^2(0,T; L^2(\Omega, H_0^1(\mathcal{O})))$, *that*

$$\lim_{\lambda\to 0}\mathbb{E}\int_0^T |X_\lambda - X|^2_{L^2(\mathcal{O})}dt = 0,$$

where $\mathbb{E}$ *denotes expectation with respect to* $\mathbb{P}$, *and that both* X *and* X_λ *have continuous paths in* $L^2(\mathcal{O})$. *Theorem 3 is true for more general not linear growing* Ψ *(see [21]).*

EXTINCTION IN FINITE TIME AND SELF-ORGANIZED CRITICALITY

In this section we assume $N < \infty$. Let τ be the stopping time

$$\tau = \inf\{t \geq 0 :\ |X(t,x)|_{-1} = 0\},$$

where $X(t,x), t \geq 0$, is the solution from Theorem 3.

Proposition 5

$$X(t,x) = 0 \quad \text{for } t \geq \tau,\ \ \mathbb{P}\text{-a.s.}.$$

Sketch of Proof. For simplicity we consider the case with $\varrho \equiv 1$. Define

$$\mu(t) := -\sum_{k=1}^N \mu_k e_k \beta_k(t), \quad t \in [0,T], \quad \tilde{\mu} := \sum_{k=1}^N \mu_k^2 e_k^2$$

and

$$Y(t) := e^{\mu(t)}X(t), \quad t \geq 0.$$

Then by Ito's product rule Y satisfies $\mathbb{P}$-a.s. the following ordinary PDE

$$\frac{dY(t)}{dt} = e^{\mu(t)}\Delta\eta(t) - \frac{1}{2}\tilde{\mu}Y(t), \quad t \geq 0.$$

Setting $Y_\lambda := e^\mu X_\lambda$ we consider the approximating equation

$$\frac{dY_\lambda(t)}{dt} = e^{\mu(t)}\Delta\eta_\lambda(t) - \frac{1}{2}\,\tilde{\mu}(t)Y_\lambda(t), \quad t \geq 0, \tag{9}$$

where

$$\eta_\lambda(t) = \Psi_\lambda(X_\lambda(t)) \in H_0^1(\mathcal{O}). \tag{10}$$

Hence

$$\begin{aligned}&\left\langle \frac{dY_\lambda(t)}{dt}, Y_\lambda(t)\right\rangle_2\\ &= \left\langle \eta_\lambda(t), \Delta(e^{\mu(t)}Y_\lambda(t))\right\rangle_2 - \frac{1}{2}\langle\tilde{\mu}(t)Y_\lambda(t), Y_\lambda(t)\rangle_2,\end{aligned} \tag{11}$$

where by (10), (7) and integrating by parts we have

$$\begin{aligned}&\langle\eta_\lambda(t), \Delta(e^{\mu(t)}Y_\lambda(t))\rangle_2\\ &= -\frac{1}{\lambda}\int_{\{e^{-\mu(t)}|Y_\lambda(t)|<\lambda\}}(|\nabla Y_\lambda(t)|^2\\ &\qquad -|Y_\lambda(t)|^2\,|\nabla\mu(t)|^2)d\xi\\ &\quad -\int_{\mathcal{O}}\widetilde{\Psi}'(e^{-\mu(t)}Y_\lambda(t))\\ &\qquad (|\nabla Y_\lambda(t)|^2 - |Y_\lambda(t)|^2\,|\nabla\mu(t)|^2)d\xi.\end{aligned}$$

This yields

$$\langle\eta_\lambda(t), \Delta(e^{\mu(t)}Y_\lambda(t))\rangle_2 \leq C\left(|Y_\lambda(t)|_2^2 + \lambda\right). \tag{12}$$

Hence (11) and Gronwall's lemma imply

$$|Y_\lambda(t)|_2^2 \leq e^{C(t-s)}\left(|Y_\lambda(s)|_2^2 + C\lambda T\right), \quad t \geq s.$$

Now letting $\lambda \to 0$ we get

$$|Y(t)|_2^2 \leq e^{C(t-s)}|Y(s)|_2^2, \quad t \geq s. \tag{13}$$

Taking in (13) $s = \tau$ we get $Y(t) = 0$ for all $t \geq \tau$ as claimed. □

For proving the extinction result we need $\mathcal{O} \subset \mathbb{R}$, i.e. $d = 1$. To be more specific let $\mathcal{O} = (0,\pi)$. Then $e_k(\xi) = \sqrt{\frac{2}{\pi}}\ \sin k\xi$, $\xi \in [0,\pi]$, $\lambda_k = k^2$ and $L^1(0,\pi) \subset H$ continuously, so

$$\gamma = \inf\left\{\frac{|x|_{L^1}}{|x|_{-1}} :\ x \in L^1(0,\pi)\right\} > 0. \tag{14}$$

Theorem 6 *Consider the equation*

$$\begin{cases} dX(t) - \Delta(\rho\ \text{sign}\ (X(t) - x_c) + \widetilde{\Psi}(X(t) - x_c))dt\\ \qquad\qquad \ni \sigma(X(t) - x_c)\displaystyle\sum_{k=1}^N \mu_k e_k d\beta_k, \quad t \geq 0,\\ \rho\ \text{sign}\ (X(t) - x_c) + \widetilde{\Psi}(X(t) - x_c) \ni 0, \text{ on } \partial(0,\pi),\\ X(0,x) = x.\end{cases} \tag{15}$$

where $x_c \in \mathbb{R}$.

Assume that

$$|x - x_c|_{-1} < \rho\gamma C_N^{-1},$$

where $C_N := \frac{\pi}{4}\ \sum_{k=1}^N (1+k)^2\mu_k^2$ *and* γ *is as in* (14). *Then*

$$\mathbb{P}(\tau_c \leq n) \geq 1 - \frac{C_N|x|_{-1}}{\rho\gamma\,(1 - e^{-C_N n})}, \tag{16}$$

where

$$\tau_c = \inf\{t \geq 0: \ |X(t) - x_c|_{-1} = 0\} \\ = \sup\{t \geq 0: \ |X(t) - x_c|_{-1} > 0\}$$

and $X - x_c$ is the solution from Theorem 3.

Sketch of Proof. For simplicity we assume $x_c = 0$. Since $\langle X(t), \Psi(X(t))\rangle_2 \geq 0$, $0 \in \Psi(0)$. An application of Ito's formula for $\varphi_\varepsilon(|X|_{-1}^2) = (|X|_{-1}^2 + \varepsilon^2)^{1/2}$ yields $\mathbb{P}$-a.s.

$$\begin{aligned} &\varphi_\varepsilon(|X(t)|_{-1}^2) + \gamma\rho \int_0^{\min(t\wedge\tau)} \frac{|X(s)|_{-1}}{(|X(s)|_{-1}^2 + \varepsilon^2)^{1/2}} ds \\ &\leq \varphi_\varepsilon(|x|_{-1}^2) + C_N \int_0^t \frac{|X(s)|_{-1}^2}{(|X(s)|_{-1}^2 + \varepsilon^2)^{1/2}} ds \\ &+ \int_0^{\min(t\wedge\tau)} \langle \sigma(X(s))dW(s), \varphi_\varepsilon'(|X(s)|_{-1}^2)X(s)\rangle. \end{aligned} \tag{17}$$

Now, letting ε tend to zero we get $\mathcal{P}$-a.s. for $t \geq 0$

$$\begin{aligned} &|X(t)|_{-1} + \gamma\rho \min(t \wedge \tau) \\ &\leq |x|_{-1} + C_N \int_0^t |X(s)|_{-1} ds \\ &+ \int_0^t 1_{[0,\tau]}(s) \, \langle \sigma(X(s))dW(s), X(s)|X(s)|_{-1}^{-1}\rangle_2. \end{aligned} \tag{18}$$

Hence by a standard comparison result

$$\begin{aligned} &|X(t)|_{-1} + \gamma\rho \int_0^t e^{C_N(t-s)} 1_{[0,\tau]}(s) ds \\ &\leq e^{C_N t}|x|_{-1} + \int_0^t e^{C_N(t-s)} 1_{[0,\tau]}(s) \\ &\quad \langle \sigma(X(s))dW(s), X(s)|X(s)|_{-1}^{-1}\rangle_2. \end{aligned}$$

Taking expectation we get

$$\int_0^t e^{-C_N s}\mathbb{P}(\tau > s) ds \leq \frac{|x|_{-1}}{\gamma\rho}.$$

Writing $\mathbb{P}(\tau > s) = 1 - \mathbb{P}(\tau \leq s)$ we deduce that

$$\mathbb{P}(\tau \leq t) \geq 1 - \frac{C_N|x|_{-1}}{\gamma\rho\,(1 - e^{-C_N t})}. \qquad \square$$

Remark 7 Let $\Psi(r) = \text{sign } r + r$, $r \in \mathbb{R}$, and $x \in L^4(0, \pi)$, $x \geq 0$, $x \not\equiv 0$. Let $X(t, x)$, $t \geq 0$, be the corresponding solution to (3). Then Theorem 6 applies. Suppose that $\mu_k = 0$, $k = 1, ..., N$. Then $C_N = 0$, so (16) gives no information and indeed $\tau = \infty$ in this case, since $X(t, x)$ is just the solution to the classical heat equation with initial datum x.

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