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A fractional Poisson equation: existence, regularity and approximations of the solution

by

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Abstract: We consider a stochastic boundary value elliptic problem on a bounded domain $D \subset \mathbb{R}^k$, driven by a fractional Brownian field with Hurst parameter $H = (H_1, \dots, H_k) \in [\frac{1}{2}, 1[^k$. First we define the stochastic convolution derived from the Green kernel and prove some properties. Using monotonicity methods, we prove existence and uniqueness of solution, along with regularity of the sample paths. Finally, we propose a sequence of lattice approximations and prove its convergence to the solution of the SPDE at a given rate.

Keywords: Stochastic partial differential equations. Fractional Brownian field. Finite differences. Rate of convergence.

AMS Subject Classification: 60H15, 60H35, 35J05.

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1 Introduction

This article deals with a stochastic Poisson equation on a bounded domain $D \subset \mathbb{R}^k$, with an arbitrary dimension $k \geq 1$ and driven by a fractional Brownian field B^H, with $H = (H_1, \ldots, H_k) \in [\frac{1}{2}, 1[^k$. We prove a theorem on existence and uniqueness of solution, we study the properties of its sample paths and finally, we give a numerical scheme based on lattice approximations and prove its convergence on a functional space with some explicit rate.
The equation is described as follows:

$$\begin{cases} \Delta u(x) - f(u(x)) & = g(x) + \dot{B}^H(x), \text{ for } x \in D, \\ \\ u(x) & = 0, \text{ for } x \in \partial D, \end{cases} \tag{1}$$

We assume that f has a decomposition $f = f_1 + f_2$, with $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ satisfying

(f1) f_1 is continuous, non-decreasing and $\sup_{x \in \mathbb{R}} |f_1(x)| \leq M$,

(f2) f_2 is Lipschitz with *small* Lipschitz constant L.

The function $g : D \to \mathbb{R}$ is measurable and satisfy some integrability conditions. The stochastic character of the equation comes from $\dot{B}^H(x)$, which denotes the formal derivative of a fractional Brownian field.
We give a rigorous meaning to (1) by means of a *mild* formulation, as it is pretty usual in the SPDEs literature. For this, we recall that if $k \geq 2$, the Green function of the deterministic Poisson equation on a bounded domain is given by

$$G_D^k(x, y) = G^k(x, y) - \mathbb{E}_x(G^k(B_\tau, y)), \tag{2}$$

with

$$G^k(x, y) = C_k \begin{cases} \log |x - y|, & k = 2 \\ \\ |x - y|^{2-k}, & k \geq 3 \end{cases}$$

where $C_2 = \frac{1}{2\pi}$, $C_k = \frac{1}{k(2-k)\omega_k}$ for $k \geq 3$, and B_τ is the random variable obtained by stopping a k-dimensional Brownian motion starting at x at its first exit time of D (see for instance [8] and also [6]). ω_k denotes the volume of the unit ball in $\mathbb{R}^k$. For $k = 1$, $G^1(x, y) = C_1|x - y|$ (see for instance [15], pg. 16). The particular form of $G_D^1(x, y)$ depends of the domain D. If $D =]0, 1[$ then $G_D^1(x, y) = x \wedge y - xy$ (see [6] pg. 258).
By a solution to (1) we mean a stochastic process $u = \{u(x), x \in D\}$ satisfying

$$u(x) = \int_D G_D^k(x, y) f(u(y)) dy + \int_D G_D^k(x, y) g(y) dy + \int_D G_D^k(x, y) dB^H(y). \tag{3}$$

For a similar SPDE in dimensions $k = 1, 2, 3$, but driven by a standard Wiener field W, different problems have been studied so far. For instance, existence and uniqueness of solution has been proved in [3] using the classical theory of equations defined by monotone operators (see [23]); the Markov field property has been investigated in [4] and [5] and numerical approximations have been given in [10]. Let us remark that for $k \le 3$ the stochastic convolution $\int_D G_D^k(x,y)dW(y)$ is well defined as a Wiener integral, because $G_D^k(x,\cdot)$ is square integrable. But $k=3$ is a threshold value for this issue.

For $k \ge 4$, a SPDE of the same type than (3) driven by a Gaussian stationary process F with an absolutely continuous covariance measure, but possibly having singularities, has been studied in [17], extending the results of [10] to higher dimensions. For this the authors combine conditions on deterministic functions and covariance densities derived from Young's type inequalities and provide a definition of an integral with respect to the random field F, and thereby a suitable meaning of the stochastic convolution $\int_D G_D^k(x,y)dF(y)$. As regards the approximation scheme, the approach for $k \le 3$ using a Fourier series expansion of $G_D^k(x,y)$ is not possible; instead, a more sophisticated procedure involving a smoothing of the Green function combined with its Fourier series expansion has been considered.

With the increasing attention devoted to fractional Brownian motion in the last years, the study of different type of problems on SPDEs driven by fractional noise is being more present in the mathematical literature. We refer the reader to [22] for an extensive list of references on the subject, including some motivating aspects from other disciplines. At the best of our knowledge, developments on this topic refer so far mainly to parabolic and hyperbolic SPDEs, the elliptic case being less explored.

A particular version of Equation (3) with null functions f and g appear in [13] (for $H_i \in]\frac{1}{2},1[^k$) and in [14] (for $H_i \in]0,1[^k$). In both references, the authors apply white noise analysis to give a meaning to the solution $u(x) = \int_D G_D^k(x,y)dB^H(y)$ in the sense of distributions. Conditions on H_i ensuring the existence of an $L^2(D)$-valued solution are given. In comparison with these references, our analysis of (3) allows a monotone nonlinearity $f(u)$ and a free term given by g, as in [10] and [17].

The content of the paper is as follows. In Section 2 we give some preliminaries on the fractional Brownian field B^H, when $H=(H_1,\dots,H_k) \in [\frac{1}{2},1[^k$. We combine ideas from [13] and [14] (see also [7]) with some results from [20] and [19] to give a moving average representation of B^H in terms of an standard Wiener field. We then identify a suitable L^p-space with mixed norm of deterministic functions which can be integrated against B^H. These spaces are related with the reproducing kernel Hilbert space of B^H by means of Hardy-Sobolev's inequality. Section 3 is devoted to the study of the stochastic convolution of the Poisson kernel (2). We give a sufficient condition on the Hurst parameter H ensuring the integrability of the Poisson kernel with respect to B^H, according to the result proved in Section 1. We also give some probabilistic properties of the stochastic convolution and prove the Hölder continuity of its sample paths (see Proposition 3.3 and Theorem 3.4). These ingredients though of own interest are meant to provide a rigorous meaning to Equation (3) in any dimension $k \ge 1$.

In Section 4 we give a theorem on existence and uniqueness of solution of Equation

(3) on the space of continuous functions vanishing at the boundary ∂D; we also prove Hölder continuity of the sample paths of the solution. Finally, Section 5 is devoted to numerical approximations of (3). We consider the domain $D=]0,1[^k$ and use the approach of [10] for $k \leq 3$ and that of [17] when $k \geq 4$. With an appropriate choice of the functional spaces we give the rate of convergence. For $k \leq 3$ we find the same as for the Brownian case, while in dimensions $k \geq 4$ it depends on the regularity of the noise and the rate of convergence, as may be expected.
Throughout the paper we shall denote by c_H any positive constant depending on the Hurst parameter $H=(H_1,\dots H_k)$, $k \geq 1$, independently of its particular value and by C any positive, finite constant.

2 Preliminaries

Let $H=(H_1,\dots,H_k) \in]0,1[^k$; a fractional Brownian field (fBf) on $\mathbb{R}^k$ with Hurst parameter H is a Gaussian stochastic processes $B^H=\{B^H(x),\ x\in\mathbb{R}^k\}$, with zero mean and covariance function given by

$$R_H(x,y)=\mathbb{E}\left(B^H(x)B^H(y)\right)=\prod_{i=1}^k R_{H_i}(x_i,y_i), \tag{4}$$

where

$$R_{H_i}(x_i,y_i)=\frac{1}{2^k}\left(|y_i|^{2H_i}+|x_i|^{2H_i}-|x_i-y_i|^{2H_i}\right).$$

Such a process has been introduced and considered in relation with different problems in [11], [12], [13], [14].
As has been mentioned in the introduction, in this article we restrict ourselves to values of the Hurst parameter $H=(H_1,\dots,H_k)\in[\frac{1}{2},1[^k$. Our goal is to define a stochastic convolution for the Poisson kernel with respect to B^H. For this, we shall identify a suitable class of deterministic functions $f:\mathbb{R}^k\to\mathbb{R}$ for which

$$\mathcal{I}^H(f)=\int_{\mathbb{R}^k} f(x)B^H(dx)$$

is a well defined random variable. As in the one parameter case, it will be useful to have a moving average type representation of the process B^H in terms of a standard Brownian field on $\mathbb{R}^k$. We shall prove such a representation owing ideas from [14] but considering the framework of [20] and [19] (see also Lemma 1.20.10 in [18]).
We start by introducing some notation. On $\mathbb{R}^k$ we consider the usual partial order defined coordinatewise and denote by $x=(x_1,\dots,x_k)$ a generic element in this space. For $x,y\in\mathbb{R}^k$ satisfying $x\leq y$, we set $1\!\!1_{[x,y]}(\eta)=\prod_{i=1}^k 1\!\!1_{[x_i,y_i]}(\eta_i)$. We shall denote by $\mathcal{E}$ the set of *elementary functions* on $\mathbb{R}^k$, that is functions of the form

$$\varphi(\eta)=\sum_{l=1}^{l_0}\varphi_l 1\!\!1_{[x^l,y^l]}(\eta),$$

with $\varphi_l \in \mathbb{R}$, and disjoint rectangles $[x^l, y^l]$, $l = 1, \ldots, l_0$. For $\varphi \in \mathcal{E}$, we define

$$\mathcal{I}(\varphi) = \sum_{l=1}^{l_0} \varphi_l B^H \left([x^l, y^l]\right),$$

where $B^H \left([x^l, y^l]\right)$ denotes the increment of B^H on the rectangle $[x^l, y^l]$ in the sense of k–dimensional distribution functions.
For any $x \in \mathbb{R}^k$ we set $1\!\!1_{[0,x]}(\eta) = \prod_{i=1}^k 1\!\!1_{[0,x_i]}(\eta_i)$ where by definition

$$1\!\!1_{[0,x_i]}(\eta_i) = \begin{cases} 1, & \text{if } \eta_i \in [0, x_i] \\ -1, & \text{if } \eta_i \in [x_i, 0] \\ 0, & \text{otherwise} \end{cases}$$

Then, on $\mathcal{E}$ we introduce an inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}^H}$ derived from the covariance structure of B^H as follows:

$$\langle 1\!\!1_{[0,x]}, 1\!\!1_{[0,y]} \rangle_{\mathcal{H}^H} = R_H(x, y). \tag{5}$$

By $\mathcal{H}^H$ we shall denote the closure of $\mathcal{E}$ with respect to the norm $\| \cdot \|_{\mathcal{H}^H}$.
For each $i = 1, \ldots, k$ we define the linear operator acting on functions $\varphi \in \mathcal{E}$ as follows:

$$\left(K^*_{H_i} \varphi\right)(\eta) = \begin{cases} \varphi(\eta), & \text{if } H_i = \frac{1}{2} \\ \\ c_{H_i} \int_{\mathbb{R}} \varphi(\eta_1, \ldots, \eta_{i-1}, u_i, \ldots, \eta_k)(u_i - \eta_i)_+^{H_i - \frac{3}{2}} du_i, & \text{if } H_i \in]\frac{1}{2}, 1[\end{cases}$$

where c_{H_i} are constants depending only on H_i (see (2.9) in [14] for its explicit value). For its further use, we introduce the sets $C^{(>)} = \{i = 1, \ldots, k : H_i > \frac{1}{2}\}$, $C^{(=)} = \{i = 1, \ldots, k : H_i = \frac{1}{2}\}$. With $c^{(>)}$, $c^{(=)}$, we denote the cardinals of $C^{(>)}$ and $C^{(=)}$, respectively. We notice that

$$\left(K^*_{H_i} \varphi\right)(\eta) = \int_{\mathbb{R}} \varphi(\eta_1 \ldots, \eta_{i-1}, u_i, \ldots, \eta_k) \mu_i^\eta(du_i), \tag{6}$$

where

$$\mu_i^\eta(du_i) = \begin{cases} c_{H_i}(u_i - \eta_i)_+^{H_i - \frac{3}{2}} du_i, & \text{if } i \in C^{(>)}, \\ \\ \delta_{\eta_i}(du_i), & \text{if } i \in C^{(=)}, \end{cases}$$

and δ_{η_i} denotes the Dirac measure at η_i.
By iteration, for $k \geq 2$ we define

$$\left(K_H^{*,(k)} \varphi\right) = \left(K^*_{H_k} \left(K^*_{H_{k-1}} \cdots \left(K^*_{H_1} \varphi\right) \cdots\right)\right). \tag{7}$$

Notice that for a function $\varphi = \otimes_{i=1}^k \varphi_i$,

$$\left(K_H^{*,(k)} \varphi\right)(\eta) = \prod_{i=1}^k \left(K^*_{H_i} \varphi_i\right)(\eta_i). \tag{8}$$

Consider the one-dimensional case ($k = 1$). In [21] pg. 320 it is proved that if $H > \frac{1}{2}$,

$$B^H(x) = \int_{\mathbb{R}} K_H(x,y)W(dy),$$

where

$$K_H(x,y) = c_H \left\{ (x-y)_+^{H-\frac{1}{2}} - (-y)_+^{H-\frac{1}{2}} \right\}, \tag{9}$$

and W is a standard Brownian motion on the real line. Thus,

$$\int_{\mathbb{R}} K_H(x,v)K_H(y,v)dv = R_H(x,y).$$

A simple computation yields $\left(K_H^* \, \mathbb{1}_{[0,x]}\right)(y) = K_H(x,y)$. Hence,

$$\mathcal{I}\left(\mathbb{1}_{[0,x]}\right) := B^H(x) = \int_{\mathbb{R}} \left(K_H^* \mathbb{1}_{[0,x]}\right)(y)W(dy).$$

With the definition (6) this representation also holds for $H = \frac{1}{2}$.
Then it is clear that the mapping $\mathbb{1}_{[0,x]} \mapsto \int_{\mathbb{R}} K_H^* \left(\mathbb{1}_{[0,x]}\right)(y)W(dy)$ is an isometry between $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{H}^H})$ and $L^2(\Omega)$. That means, the operator K_H^* is an isometry between $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{H}^H})$ and $L^2(\mathbb{R})$. Hence, K_H^* can be extended to the Hilbert space $\mathcal{H}^H$. Otherwise stated, we can define $\mathcal{I}(\varphi) = \int_{\mathbb{R}} (K_H^* \varphi)(y)W(dy)$ as an $L^2(\Omega)$–valued random variable and therefore to extend the definition of $\mathcal{I}$ from $\mathcal{E}$ to $\mathcal{H}^H$.
In the multidimensional case we prove similar results. We first recall a definition and introduce some notation.
A stochastic process $\{W(x), x \in \mathbb{R}^k\}$ is termed a *standard Wiener field* on $\mathbb{R}^k$ if it is Gaussian, with mean zero and covariance given by $\mathbb{E}\left(W(x)W(y)\right) = x \wedge y$, where $x \wedge y = \prod_{i=1}^k x_i \wedge y_i$, and

$$x_i \wedge y_i = \begin{cases} x_i \wedge y_i, & \text{if } x_i, y_i > 0 \\ (-x_i) \wedge (-y_i), & \text{if } x_i, y_i < 0 \\ 0, & \text{otherwise.} \end{cases}$$

For a function $\varphi : \mathbb{R}^k \to \mathbb{R}$, we define $\tilde{\varphi}(u_1, \ldots, u_k; v_1, \ldots, v_k) = \varphi(w_1, \ldots, w_k)$, with $w_i = u_i$, if $i \in C^{(=)}$ and $w_i = v_i$ if $i \in C^{(>)}$. Then we denote by $\left|\mathcal{H}^H\right|$ the set of functions $\varphi : \mathbb{R}^k \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^{c^{(=)}}} \prod_{i \in C^{(=)}} du_i \int_{\mathbb{R}^{2c^{(>)}}} \prod_{i \in C^{(>)}} \left(du_i dv_i H_i(2H_i - 1)|u_i - v_i|^{2H_i - 2} \right)$$
$$\times |\varphi(u_1, \ldots, u_k)||\tilde{\varphi}(u_1, \ldots, u_k; v_1, \ldots, v_k)| < +\infty.$$

Proposition 2.1 *Set* $K_H(x,y) = \prod_{i=1}^k K_{H_i}(x_i, y_i)$, *with* K_{H_i}, $i = 1, \ldots, k$, *defined in (9). Then,*

1. *There exists a standard Wiener field* W *on* $\mathbb{R}^k$ *such that*

$$B^H(x) = \int_{\mathbb{R}^k} K_H(x,y)W(dy) = \int_{\mathbb{R}^k} \left(K_H^{*,(k)} \mathbb{1}_{[0,x]}\right)(y)W(dy). \tag{10}$$

2. *For any* $\varphi \in \mathcal{H}^H$
$$\mathcal{I}(\varphi) = \int_{\mathbb{R}^k} \left(K_H^{*,(k)} \varphi \right)(y) dW(y) \tag{11}$$
defines a random variable in $L^2(\Omega)$.

3. *For any* $\varphi_1, \varphi_2 \in \left| \mathcal{H}^H \right|$ *the following isometry formula holds:*
$$\begin{aligned}\mathbb{E}\left(\mathcal{I}(\varphi_1)\mathcal{I}(\varphi_2)\right) &= \int_{\mathbb{R}^k} \left(K_H^{*,(k)} \varphi_1 \right)(y) \left(K_H^{*,(k)} \varphi_2 \right)(y) dy \\ &= \int_{\mathbb{R}^{c^{(=)}}} \prod_{i \in C^{(=)}} du_i \int_{\mathbb{R}^{2c^{(>)}}} \prod_{i \in C^{(>)}} \left(du_i dv_i H_i(2H_i - 1)|u_i - v_i|^{2H_i - 2} \right) \\ &\quad \times \varphi_1(u_1, \dots, u_k) \tilde{\varphi}_2(u_1, \dots, u_k; v_1, \dots, v_k). \end{aligned} \tag{12}$$

In particular, if $H_i \in]\frac{1}{2}, 1[$ *for any* $i = 1, \dots, k$,
$$\mathbb{E}\left(\mathcal{I}(\varphi_1)\mathcal{I}(\varphi_2)\right) = \int_{\mathbb{R}^{2k}} \varphi_1(u)\varphi_2(v) \prod_{i=1}^{k} \left(H_i(2H_i - 1)|u_i - v_i|^{2H_i - 2} \right) du dv. \tag{13}$$

Proof: To prove the existence of W we follow the arguments of [19], pg. 279, for $k = 1$ which extend easily to any $k \geq 1$, as follows.
For each $i = 1, \dots, k$ such that $H_i > \frac{1}{2}$, the action of the kernel $K_{H_i}^*$ on elementary functions can be expressed in terms of a fractional integral. More precisely, if $\varphi \in \mathcal{E}$, $K_{H_i}^* \varphi = c_{H_i} I_-^{H_i - \frac{1}{2}} \varphi$ (see section 3.2 in [20]). Then, for ψ in the image of $K_{H_i}^*$ and by considering fractional derivatives, we define $Q_{H_i}^* \psi = c_{H_i}^{-1} D_-^{H_i - \frac{1}{2}} \psi$, which by the rules of fractional calculus is seen to be the inverse operator of $K_{H_i}^*$. For $H_i = \frac{1}{2}$, $Q_{H_i}^*$ is defined to be the identity operator.
Set $W(x) = B^H \left(\prod_{i=1}^k Q_{H_i}^* \mathbb{1}_{[0,x_i]} \right)$. The process $W = \left\{ W(x),\ x \in \mathbb{R}^k \right\}$ is a standard Wiener field on $\mathbb{R}^k$, and the stochastic field B^H has the integral representation
$$B^H(x) = \int_{\mathbb{R}^k} K_H(x, y) dW(y). \tag{14}$$

Indeed, by defining $Q_H^{*,(k)} \mathbb{1}_{[0,x]} = \prod_{i=1}^k Q_{H_i}^* \mathbb{1}_{[0,x_i]}$ and by virtue of (8) we have $\left(K_H^{*,(k)} \circ Q_H^{*,(k)} \right) \mathbb{1}_{[0,x]} = \mathbb{1}_{[0,x]}$. Thus, for any $x, y \in \mathbb{R}^k$,
$$\begin{aligned}\mathbb{E}\left(W(x)W(y)\right) &= \left\langle Q_H^{*,(k)} \mathbb{1}_{[0,x]}, Q_H^{*,(k)} \mathbb{1}_{[0,y]} \right\rangle_{\mathcal{H}^H} \\ &= \left\langle \mathbb{1}_{[0,x]}, \mathbb{1}_{[0,y]} \right\rangle_{L^2(\mathbb{R}^k)} = x \wedge y. \end{aligned}$$

By construction, the operator $K_H^{*,(k)}$ is an isometry from $\mathcal{E}$ into $L^2(\mathbb{R}^k)$ that can be extended to the Hilbert space $\mathcal{H}^H$. Therefore, one can define $\mathcal{I}(\varphi)$ for any $\varphi \in \mathcal{H}^H$ by means of (11).

We now prove (12). By the very definition of $K_H^{*,(k)}$ (see (6)) and by applying Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^k} \left(K_H^{*,(k)}\varphi_1\right)(y) \left(K_H^{*,(k)}\varphi_2\right)(y)dy \\
&= \int_{\mathbb{R}^k} dy \left(\int_{\mathbb{R}^k} \varphi_1(u) \prod_{i\in C^{(=)}} \mu_i^y(du_i) \prod_{i\in C^{(>)}} \mu_i^y(du_i) \right) \\
&\quad \times \left(\int_{\mathbb{R}^k} \varphi_2(v) \prod_{i\in C^{(=)}} \mu_i^y(dv_i) \prod_{i\in C^{(>)}} \mu_i^y(dv_i) \right) \\
&= \int_{\mathbb{R}^{c^{(=)}}} \prod_{i\in C^{(=)}} du_i \\
&\quad \times \int_{\mathbb{R}^{2c^{(>)}}} \prod_{i\in C^{(>)}} du_i dv_i \left(\int_{\mathbb{R}^{c^{(>)}}} \prod_{i\in C^{(>)}} dy_i c_{H_i}^2 (u_i-y_i)_+^{H_i-\frac{3}{2}} (v_i-y_i)_+^{H_i-\frac{3}{2}} \right) \\
&\quad \times \varphi_1(u_1,\dots,u_k)\tilde{\varphi}_2(u_1,\dots,u_k;v_1,\dots,v_k).
\end{aligned}$$

From this and the identity

$$c_{H_i}^2 \int_{\mathbb{R}} dy_i (u_i-y_i)_+^{H_i-\frac{3}{2}} (v_i-y_i)_+^{H_i-\frac{3}{2}} = H_i(2H_i-1)|u_i-v_i|^{2H_i-2}$$

(see [9], page 404), (12) follows.
Finally, if $C^{(>)} = \{1,\dots,k\}$, (12) reads (13). This ends the proof of the Proposition.

$\blacksquare$

It is well known that for real functions φ, ψ and $H \in [\frac{1}{2}, 1[$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\varphi(\eta)||\psi(\theta)||\eta-\theta|^{2H-2} d\eta d\theta \le b_H \|\varphi\|_{L^{\frac{1}{H}}(\mathbb{R})} \|\psi\|_{L^{\frac{1}{H}}(\mathbb{R})}, \tag{15}$$

with some positive constant b_H. Indeed, for $H \in]\frac{1}{2}, 1[$ this follows from Hardy-Littlewood-Sobolev's inequality (see for instance inequality (1), page 321, in [2]); for $H = \frac{1}{2}$ it is simply Schwarz's inequality.
When applying this inequality recursively to functions $\varphi_1, \varphi_2 : \mathbb{R}^k \to \mathbb{R}$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{c^{(=)}}} \prod_{i\in C^{(=)}} du_i \int_{\mathbb{R}^{2c^{(>)}}} \prod_{i\in C^{(>)}} \left(du_i dv_i H_i(2H_i-1)|u_i-v_i|^{2H_i-2} \right) \\
&\quad \times |\varphi_1(u_1,\dots,u_k)||\tilde{\varphi}_2(u_1,\dots,u_k;v_1,\dots,v_k)| \\
&\le c_{H_1,\cdots,H_k} \|\varphi_1\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(\mathbb{R}^k)} \|\varphi_2\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(\mathbb{R}^k)},
\end{aligned} \tag{16}$$

where for $p_i \in [1,\infty]$, $i = 1,\dots,k$,

$$\|h\|_{L^{p_1,\dots,p_k}(\mathbb{R}^k)} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} |h(\eta_1,\dots,\eta_k)|^{p_1} d\eta_1 \right)^{\frac{p_2}{p_1}} \cdots \right)^{\frac{p_k}{p_{k-1}}} d\eta_k \right)^{\frac{1}{p_k}}.$$

Let us denote by $L^{p_1\cdots,p_k}(\mathbb{R}^k)$ the space of measurable functions h defined on $\mathbb{R}^k$ with $\|h\|_{L^{p_1\cdots,p_k}(\mathbb{R}^k)} < \infty$. Such spaces are termed *L^p spaces with mixed norm.* For details we refer the reader to [2] and also [1]. In particular, if $H_i \in]\frac{1}{2}, 1[$ for any $i = 1, \dots, k$, a proof of (16) is given in page 322 of [2], but it is easy to extend the result allowing $H_i = \frac{1}{2}$ for some indices i.

For its further use, we remark that for any $p \geq \sup_{i \in \{1,\dots,k\}} p_i$, and every measurable function h with bounded support $\mathcal{O}$ contained in $\mathbb{R}^k$,

$$\|h\|_{L^{p_1,\dots,p_k}(\mathcal{O})} \leq C \|h\|_{L^p(\mathcal{O})}, \tag{17}$$

with a constant C depending only on $\mathcal{O}$. Indeed, this follows by applying recursively Hölder's inequality with $pp_k^{-1}, \dots, pp_1^{-1}$.

The preceding discussion yields

$$L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(\mathbb{R}^k) \subset \left|\mathcal{H}^H\right| \subset \mathcal{H}^H. \tag{18}$$

3 The fractional stochastic convolution of the Poisson kernel

In this section we consider a bounded domain $D \subset \mathbb{R}^k$ if $k \geq 2$ and $D =]0, 1[$ if $k = 1$. We consider the Green function defined in (2) and $G_D^1(x, y) = x \wedge y - xy$, respectively. The purpose is to define the stochastic convolution $\int_D G_D^k(x, y) dB^H(y)$ with respect to the fractional Brownian field with parameters $H_i \in [\frac{1}{2}, 1[$, $i = 1, \dots, k$, introduced in the preceding section, and to study its sample paths.

Throughout the section we shall make use of the following remark:

Let $k \geq 2$ and assume that for some norm on $\mathbb{R}^k$, we have $\sup_{x \in D} \|G^k(x, y)\| < \infty$. Then, $\sup_{x \in D} \|\mathbb{E}_x \left(G^k(B_\tau, y)\right)\| < \infty$, and consequently $\sup_{z \in D} \|G_D^k(z, y)\| < \infty$. Indeed, since $\mathbb{E}_x$ is a convex operator, by denoting by P_x the probability law of B_τ, we obtain

$$\left\|\mathbb{E}_x \left(G^k(B_\tau, y)\right)\right\| \leq \mathbb{E}_x \left\|G^k(B_\tau, y)\right\| = \int_{\mathbb{R}} P_x(dz) \left\|G^k(z, y)\right\| \leq \sup_{x \in D} \left\|G^k(x, y)\right\|.$$

We are interested in the integrability properties of G_D^k. To start with, let us state a result that for dimensions $k \geq 3$ is Lemma 2 in [17]. Its extension to $k = 1, 2$ is trivial.

Lemma 3.1 *For any $p \in [1, \frac{k}{k \vee 2 - 2}[$, there exists a positive constant $\mathcal{K}_1$ depending on p and k, such that*

$$\sup_{x \in D} \|G_D^k(x, \cdot)\|_{L^p(\mathbb{R}^k)} \leq \mathcal{K}_1. \tag{19}$$

For the values $k = 1, 2$, (19) holds for any $p \in [1, \infty[$. Therefore by virtue of (17) we can choose $p_0 \geq \sup_{i \in \{1,\dots,k\}} \frac{1}{H_i}$ such that

$$\sup_{x \in D} \|G_D^k(x, \cdot)\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(\mathbb{R}^k)} \leq C \sup_{x \in D} \|G_D^k(x, \cdot)\|_{L^{p_0}(\mathbb{R}^k)} < \infty. \tag{20}$$

Consider now the case $k = 3$. Property (19) holds for any $p \in [1, 3[$. Hence there exist $p_0 \in [\sup_{i\in\{1,\dots,k\}} \frac{1}{H_i}, 3[$ such that (20) holds.
Then accordingly with the results stated in the preceding section, for $k = 1, 2, 3$, $\mathcal{I}(G_D^k(x, \cdot))$ is a well-defined random variable in $L^2(\Omega)$ for any $x \in D$.
A similar conclusion holds true for $k = 4$ under the additional assumption $H_i \in]\frac{1}{2}, 1[$, that is, excluding the possibility of having a standard Brownian motion in some of the components of B^H. By similar arguments, the existence of $\mathcal{I}(G_D^k(x, \cdot))$ for $k \geq 4$ is ensured by the stronger hypothesis $H_i \in]\frac{k-2}{k}, 1[$ for any $i = 1, \dots, k$. As we show in the next Lemma, one can relax this assumption by working with L^p spaces with mixed norm.
Throughout the section we suppose that $D \subset [-R, R]^k$ for some $R > 0$. We shall make use of the following inequality which follows from the trivial fact on Euclidean norms saying that $|x| \geq |x_i|$:
For any $\mu \geq 0$ and $\beta_i \geq 0$, $i = 1, \dots, k$, such that $\sum_{i=1}^k \beta_i = 1$

$$|x|^{-\mu} \leq \prod_{i=1}^k |x_i|^{-\beta_i \mu}. \tag{21}$$

Lemma 3.2 *Let $k \geq 4$ and assume that $\sum_{i=1}^k H_i > k - 2$. Then, there exists a positive constant $\mathcal{K}_2$ depending on H, D and k, such that*

$$\sup_{x\in D} \|G_D^k(x, \cdot)\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(\mathbb{R}^k)} \leq \mathcal{K}_2. \tag{22}$$

Proof: By applying the inequality (21) with $\mu = 2 - k$ and $\beta_i = \frac{H_i}{\sum_{i=1}^k H_i}$, and the remark at the beginning of the section, we obtain

$$\begin{aligned} \|G_D^k(x, \cdot)\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(\mathbb{R}^k)} &\leq 2\Big(\int_{-R}^R \cdots \Big(\int_{-R}^R |x - y|^{\frac{2-k}{H_1}} dy_1\Big)^{\frac{H_1}{H_2}} \cdots dy_k\Big)^{H_k} \\ &\leq 2 \prod_{i=1}^k \Big(\int_{-R}^R |x_i - y_i|^{\frac{\beta_i(2-k)}{H_i}} dy_i\Big)^{H_i}. \end{aligned}$$

The supremum in $x \in D$ of the last term is finite if and only if $\frac{\beta_i(2-k)}{H_i} > -1$. By the definition of β_i, this condition is equivalent to $\sum_{i=1}^k H_i > k - 2$. ■

In the sequel we will assume the hypothesis:
(H) $H_i \in [\frac{1}{2}, 1[$, $i = 1, \dots, k$, and for dimensions $k \geq 4$ we suppose that $\sum_{i=1}^k H_i > k - 2$.
Then for any dimension $k \geq 1$, Lemmas 3.1 and 3.2 yield

$$\sup_{x\in D} \|G_D^k(x, \cdot)\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(\mathbb{R}^k)} \leq \mathcal{K}, \tag{23}$$

with $\mathcal{K} = \max(\mathcal{K}_1, \mathcal{K}_2)$.
This proves the existence of a stochastic process

$$\mathcal{J} = \Big\{\mathcal{J}(x) = \mathcal{I}\left(G_D^k(x, \cdot)\right) = \int_D G_D^k(x, y) B^H(dy), x \in D\Big\}, \tag{24}$$

with values in $L^2(\Omega)$ and satisfying

$$\begin{aligned}\sup_{x\in D}\mathbb{E}\left|\int_{\mathbb{R}^k} G_D^k(x,y)dB^H(y)\right|^2 &= \sup_{x\in D}\int_{\mathbb{R}^k}\left(K_H^{*,(k)}G_D^k(x,\cdot)\right)^2(y)dy\\ &\le C_H\sup_{x\in D}\|G_D^k(x,\cdot)\|^2_{L^{\frac{1}{H_1},\cdots,\frac{1}{H_k}}(\mathbb{R}^k)} \le C_H\mathcal{K}^2<\infty. \quad (25)\end{aligned}$$

The next proposition gives additional properties of the process $\mathcal{J}$.

Proposition 3.3 *For $p\in[1,\infty[$, and $\bar p\in[2,\infty[$ we have*

$$\begin{aligned}\|\mathcal{J}\|_{L^p(\Omega;L^{\bar p}(D))} &\le C(p,\bar p)\left\|\left\|\int_D G_D^k(\cdot,y)dB^H(y)\right\|_{L^2(\Omega)}\right\|_{L^{\bar p}(D)}\\ &\le C(p,\bar p,H)\mathcal{K}, \qquad (26)\end{aligned}$$

where $C(p,\bar p)$, $C(p,\bar p,H)$ are some positive constants depending only on the specified parameters.

Proof: Assume first that $p\in[1,\bar p]$. By applying Hölder's inequality with $\widetilde{p}=\frac{\bar p}{p}\ge 1$ to the expectation operator, then Fubini's theorem and eventually the hypercontractivity property (see for instance [16]), we obtain

$$\begin{aligned}\|\mathcal{J}\|_{L^p(\Omega;L^{\bar p}(D))} &= \left(\mathbb{E}\left\|\int_D G_D^k(x,y)dB^H(y)\right\|^p_{L^{\bar p}(D)}\right)^{\frac{1}{p}}\\ &\le \left(\mathbb{E}\left\|\int_D G_D^k(x,y)dB^H(y)\right\|^{\bar p}_{L^{\bar p}(D)}\right)^{\frac{1}{\bar p}}\\ &\le (\bar p-1)^{\frac{1}{2}}\left(\int_D\left(\mathbb{E}\left|\int_D G_D^k(x,y)dB^H(y)\right|^2\right)^{\frac{\bar p}{2}}dx\right)^{\frac{1}{\bar p}}.\end{aligned}$$

Let $p>\bar p$. We apply first Minkowski's inequality with respect to the probability measure and the Lebesgue measure, then the hypercontractivity inequality. We obtain

$$\begin{aligned}\|\mathcal{J}\|_{L^p(\Omega;L^{\bar p}(D))} &= \left(\mathbb{E}\left(\int_D\left|\int_D G_D^k(x,y)dB^H(y)\right|^{\bar p}dx\right)^{\frac{p}{\bar p}}\right)^{\frac{1}{p}}\\ &= \left\|\int_D\left|\int_D G_D^k(x,y)dB^H(y)\right|^{\bar p}dx\right\|^{\frac{1}{\bar p}}_{L^{\frac{p}{\bar p}}(\Omega)}\\ &\le \left(\int_D\left\|\left(\int_D G_D^k(x,y)dB^H(y)\right)^{\bar p}\right\|_{L^{\frac{p}{\bar p}}(\Omega)}dx\right)^{\frac{1}{\bar p}}\\ &= \left(\int_D\left(\mathbb{E}\left|\int_D G_D^k(x,y)dB^H(y)\right|^p\right)^{\frac{\bar p}{p}}dx\right)^{\frac{1}{\bar p}}\\ &\le (p-1)^{\frac{1}{2}}\left(\int_D\left(\mathbb{E}\left|\int_D G_D^k(x,y)dB^H(y)\right|^2\right)^{\frac{\bar p}{2}}dx\right)^{\frac{1}{\bar p}}.\end{aligned}$$

We finally obtain (26) by applying (25). ■

Let us introduce a strengthening of the assumption **(H)**, as follows.
(H*) $H_i \in [\frac{1}{2}, 1[^k$, $i = 1, \ldots, k$. In addition, if $k \geq 4$ we assume that $\sum_{i=1}^k H_i > \frac{2(k-1)(k-2)}{2k-3}$.
Next we prove that the stochastic field $\mathcal{J}$ has a.s. Hölder continuous paths.

Theorem 3.4 *Under* **(H*)** *it holds that*

$$\left\| G_D^k(x,\cdot) - G_D^k(z,\cdot) \right\|_{L^{\frac{1}{H_1},\ldots,\frac{1}{H_k}}(\mathbb{R}^k)} \leq C|x-z|^\lambda, \tag{27}$$

for any $x, z \in D$, with

$$\begin{cases} \lambda = 1, & \text{for } k = 1, \\ \lambda \in \left] \left(1 - \frac{\sum_{i=1}^k H_i}{2(k-2)}\right) \vee 0, \frac{\sum_{i=1}^k H_i}{2(k-1)} \right[, & \text{for } k = 2, 3, \\ \lambda \in \left] 1 - \frac{\sum_{i=1}^k H_i}{2(k-2)}, \frac{\sum_{i=1}^k H_i}{2(k-1)} \right[, & \text{for } k \geq 4. \end{cases}$$

Therefore, the Gaussian random field $\mathcal{J}$ defined in (24) *satisfies*

$$\mathbb{E}\left(|\mathcal{J}(x) - \mathcal{J}(z)|^2\right) \leq C|x-z|^{2\lambda} \tag{28}$$

and a.s. the sample paths are Hölder continuous of order $\gamma \in]0, \lambda[$.

Proof: Fix $x, z \in D$. For $k = 1$, easy estimates yield

$$\left| G_D^k(x,y) - G_D^k(z,y) \right| \leq C|x-z|,$$

which implies the result.
Let $k \geq 2$ and set

$$T^{(k)} = \left\| G^k(x,\cdot) - G^k(z,\cdot) \right\|^2_{L^{\frac{1}{H_1},\ldots,\frac{1}{H_k}}(\mathbb{R}^k)} \tag{29}$$

With a slight modification of the remark at the beginning of the section, we see that to establish (27) it suffices to prove $T^{(k)} \leq C|x-z|^{2\lambda}$ for the values of λ given in the statement.
Consider $\lambda \in]0, 1[$ and apply Cauchy-Schwarz' inequality to obtain

$$\begin{aligned} T^{(k)} \leq & \Big(\int_{-R}^{R} \cdots \Big(\int_{-R}^{R} \left| G^k(x,y) - G^k(z,y) \right|^{\frac{2\lambda}{H_1}} dy_1 \Big)^{\frac{H_1}{H_2}} \cdots dy_k \Big)^{H_k} \\ & \times \Big(\int_{-R}^{R} \cdots \Big(\int_{-R}^{R} \left| G^k(x,y) - G^k(z,y) \right|^{\frac{2(1-\lambda)}{H_1}} dy_1 \Big)^{\frac{H_1}{H_2}} \cdots dy_k \Big)^{H_k}. \end{aligned} \tag{30}$$

We notice that by the very definition of G^k,

$$\begin{aligned} \left| G^k(x,y) - G^k(z,y) \right| & \leq C|x-z| \left| \int_0^1 (\theta|x-y| + (1-\theta)|z-y|)^{1-k} \, d\theta \right| \\ & \leq C|x-z| \Big(|x-y|^{1-k} + |z-y|^{1-k} \Big). \end{aligned} \tag{31}$$

Let $k = 2$. From (30) we have

$$\begin{aligned} T^{(2)} \le & C|x-z|^{2\lambda}\Big(\int_{-R}^{R}\Big(\int_{-R}^{R}\Big(|x-y|^{-1}+|z-y|^{-1}\Big)^{\frac{2\lambda}{H_1}}dy_1\Big)^{\frac{H_1}{H_2}}dy_2\Big)^{H_2} \\ & \times\Big(\int_{-R}^{R}\Big(\int_{-R}^{R}\Big|\log|x-y|-\log|z-y|\Big|^{\frac{2(1-\lambda)}{H_1}}dy_1\Big)^{\frac{H_1}{H_2}}dy_2\Big)^{H_2}. \end{aligned}$$

We next explore conditions on λ ensuring that the two integral factors in the preceding inequality are finite.
For the first factor, we can apply Minkowski's inequality for L^p spaces with mixed norm (see [2], page 302) and obtain

$$\begin{aligned} & \Big(\int_{-R}^{R}\Big(\int_{-R}^{R}\Big(|x-y|^{-1}+|z-y|^{-1}\Big)^{\frac{2\lambda}{H_1}}dy_1\Big)^{\frac{H_1}{H_2}}dy_2\Big)^{H_2} \\ & \le C(\lambda)\Big(\Big\||x-\cdot|^{-2\lambda}\Big\|_{L^{\frac{1}{H_1},\frac{1}{H_2}}([-R,R]^2)}+\Big\||z-\cdot|^{-2\lambda}\Big\|_{L^{\frac{1}{H_1},\frac{1}{H_2}}([-R,R]^2)}\Big). \end{aligned}$$

The last expression is finite provided that $H_1 + H_2 > 2\lambda$, as can be checked by using (21) with $\mu = 2\lambda$ and $\beta_i = \frac{H_i}{H_1+H_2}$, $i = 1, 2$.
As for the second integral, we first apply the above mentioned version of Minkowski's inequality and then, by taking $p > \frac{1}{H_1 \wedge H_2}$ and by virtue of (17) we obtain

$$\begin{aligned} & \Big(\int_{-R}^{R}\Big(\int_{-R}^{R}\Big|\log|x-y|-\log|z-y|\Big|^{\frac{2(1-\lambda)}{H_1}}dy_1\Big)^{\frac{H_1}{H_2}}dy_2\Big) \\ & \le C(\lambda,H_1,H_2,p)\Big(\Big(\int_{[-R,R]^2}\Big|\log|x-y|\Big|^{2(1-\lambda)p}dy\Big)^{\frac{1}{p}} \\ & \qquad +\Big(\int_{[-R,R]^2}\Big|\log|z-y|\Big|^{2(1-\lambda)p}dy\Big)^{\frac{1}{p}}\Big), \end{aligned}$$

which is finite.
Therefore we have proved that

$$T^{(2)} \le C(\lambda, H_1, H_2, p)|x-z|^{2\lambda},$$

with $\lambda \in]0, \frac{H_1+H_2}{2}[$.
We now consider dimensions $k \ge 3$. By using (30) and by applying Minkowski's inequality we obtain

$$\begin{aligned} & T^{(k)} \le C|x-z|^{2\lambda} \\ & \quad \times\Big(\Big\||x-\cdot|^{2\lambda(1-k)}\Big\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}([-R,R]^k)}+\Big\||z-\cdot|^{2\lambda(1-k)}\Big\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}([-R,R]^k)}\Big) \\ & \quad \times\Big(\Big\||x-\cdot|^{2(1-\lambda)(2-k)}\Big\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}([-R,R]^k)}+\Big\||z-\cdot|^{2(1-\lambda)(2-k)}\Big\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}([-R,R]^k)}\Big) \end{aligned}$$

Assume that the two conditions $\sum_{i=1}^{k} H_i > 2\lambda(k-1)$ and $\sum_{i=1}^{k} H_i > 2(1-\lambda)(k-2)$ hold; that is

$$\lambda \in \Big]1-\frac{\sum_{i=1}^{k} H_i}{2(k-2)}, \frac{\sum_{i=1}^{k} H_i}{2(k-1)}\Big[. \tag{32}$$

Then, the inequality (21) applied first to $\mu := 2\lambda(k-1)$ and then to $\mu := 2(1-\lambda)(k-2)$ yields as in the proof of Lemma 3.2 that all the norms in the last expression of the preceding inequalities are finite. Hence,

$$T^{(k)} \leq C|x-z|^{2\lambda},$$

for any λ satisfying (32).
In order to finish the proof, we have to analyze the constraints on λ imposed so far. For $k=2$, the condition $\lambda \in [0, \frac{H_1+H_2}{2}]$ is trivially equivalent to

$$\lambda \in \left]\left(1-\frac{\sum_{i=1}^k H_i}{2(k-2)}\right) \vee 0, \frac{\sum_{i=1}^k H_i}{2(k-1)}\right[.$$

For $k=3$ the interval $I := \left]1-\frac{\sum_{i=1}^k H_i}{2(k-2)}, \frac{\sum_{i=1}^k H_i}{2(k-1)}\right[$ is non empty, although it may contain negative numbers. Hence, the choice of λ should be restricted to the interval $\left]\left(1-\frac{\sum_{i=1}^k H_i}{2(k-2)}\right) \vee 0, \frac{\sum_{i=1}^k H_i}{2(k-1)}\right[$, which is clearly still non empty.
For $k \geq 4$, the interval I is non empty if and only if $\sum_{i=1}^k H_i > \frac{2(k-1)(k-2)}{2k-3}$. Under this assumption, $I \subset]0,\infty[$.
With this discussion we end the proof of (27). Then the inequality (28) is a consequence of the isometry property of the stochastic integral (see (25)). Finally, since the process $\mathcal{J}$ is Gaussian, the statement about the regularity of its sample paths follows from Kolmogorov's continuity criterion. ■

Remark 3.1 *Consider the following assumption*
$(\mathbf{H}^{**})$ $H_i \in [\frac{1}{2}, 1[^k$, $i=1,\dots,k$. *Moreover,* $\sum_{i=1}^k H_i > k-1$ *for any* $k \geq 2$, *which implies* **$(\mathbf{H}^{*})$**.
In this case, (27) holds true with $\lambda = 1$ *for any dimension* $k \geq 1$. *Indeed, this follows by estimating the* $L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}$ *norm of the right-hand side of (31) by means of (21).*

4 Existence and uniqueness of solution to the fractional Poisson equation

This section is devoted to establish the existence and uniqueness of solution to the equation (1). This result will be obtained by a pathwise argument; once it will be established, we will prove some probabilistic properties of the solution. We borrough the method of the proof from [17] (see also [3], [4] and [10]), which follows the classical monotonicity methods. We shall denote by R_D a k-dimensional rectangle $[-R,R]^k$ which contains D and by $\mathcal{S}$ the set $\{\omega : \omega \in \mathcal{C}(D), \omega|_{\partial D} = 0\}$.
For its further use we highlight some properties. The first one, denoted by **(P)** is a monotonicity property. The second one, named **(M)**, has been proved in [3] (Lemma 2.4); it is a consequence of the solvability of the Dirichlet problem on D and Poincaré's inequality (see [8] or [1]). They are formulated as follows:

(P) *f is a function of the form $f = f_1 + f_2$ with $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, f_1 non-decreasing and f_2 Lipschitz with Lipschitz constant L, if and only if for every $u, v \in \mathbb{R}$,*

$$(u-v)(f(u)-f(v)) \geq -L(u-v)^2. \tag{33}$$

(M) *There exists a constant $a > 0$ such that for any $\varphi \in L^2(D)$,*

$$\int_D \left(\int_D G_D^k(x,y)\varphi(y)dy \right) \varphi(x)dx \leq -a \int_D \left(\int_D G_D^k(x,y)\varphi(y)dy \right)^2 dx. \tag{34}$$

We begin with the existence and uniqueness result.

Theorem 4.1 *We assume* **(H*)***; we also suppose that $g \in L^{\frac{1}{1-H_1},\dots,\frac{1}{1-H_k}}(R_D)$, $f = f_1 + f_2$ and satisfies $(f1)$ and $(f2)$ with a Lipschitz constant $L < \min(a, \mathcal{K}^{-1})$. Then there exists a unique stochastic process solution to* (3) *with sample paths in $\mathcal{S}$, a.s.*

Proof: We follow the proof of Theorem 2 in [17] with slight changes on the functional spaces under use; in particular, some properties of L^p spaces with mixed norm are applied. For the sake of completeness we give some details.
Consider the operator $\mathcal{T} : \mathcal{S} \longrightarrow \mathcal{S}$, defined by

$$\mathcal{T}(w)(x) = w(x) - \int_D G_D^k(x,y)f(w(y))dy.$$

By Hölder's inequality for L^p spaces with mixed norm and (27) we have

$$\begin{aligned}
&\left| \int_D \left(G_D^k(x,y) - G_D^k(z,y) \right) g(y)dy \right| \\
&\leq \left\| G_D^k(x,\cdot) - G_D^k(z,\cdot) \right\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(R_D)} \|g\|_{L^{\frac{1}{1-H_1},\dots,\frac{1}{1-H_k}}(R_D)} \\
&\leq C|x-z|^{\lambda},
\end{aligned} \tag{35}$$

with $\lambda > 0$ given in Theorem 3.4. Together with the last conclusion of Theorem 3.4, we obtain a.s.

$$b(x) := \int_D G_D^k(x,y)g(y)dy + \int_D G_D^k(x,y)dB^H(y) \in \mathcal{S}. \tag{36}$$

We next show that the operator equation $\mathcal{T}w = b$ has a unique solution for any $b \in \mathcal{S}$, or equivalently that $\mathcal{T}$ is bijective. Uniqueness guarantees the measurability of the process $\{\omega(x),\ x \in D\}$.
The one to one property of $\mathcal{T}$ follows by applying **(P)** and **(M)**.
We next give a sketch of the steps of the proof that $\mathcal{T}$ is onto. In the next argument, we fix a sample path of the process B^H on a set of probability one, and $q \in [2, \infty[$.
Step 1: A solution for a regular problem. Let $b \in \mathcal{S}$ and $b_n \in C_c^{\infty}(D)$, $n \geq 1$, be such that $b_n \longrightarrow b$ in L^q. Then, one can construct a sequence of functions solving $\mathcal{T}u^{(n)} = b_n$ such that $u^{(n)} \longrightarrow u$ in $L^2(D)$; the limit u will be the candidate for a solution. (For details, see Lemma 3 in [17]).

The sequence $\{u^{(n)},\ n \geq 1\}$ satisfies

$$u^{(n)}(x) = \int_D G_D^k(x,y) f(u^{(n)}(y))dy + b_n, \quad \text{for} \quad x \in D, \quad u_n|_{\partial D} = 0. \tag{37}$$

By the properties **(P)**, **(M)** and since $u^{(n)} \in L^2(D)$, one can prove that $\{u^{(n)},\ n \geq 1\}$ is a Cauchy sequence in $L^2(D)$. Set $u = \lim_n u^{(n)}$ in $L^2(D)$.

Step 2: u is the solution. We would like to pass to the limit (37). For this, we choose subsequences $u^{(n)}$ and b_n (still denoted with the same subscript n) converging to u and b almost everywhere and we proceed in three steps.

Step 2.1. Suppose that f is bounded (and continuous). Then by bounded convergence the limit as n tends to infinity of each term in (37) exists and we obtain

$$u(x) = \int_D G_D^k(x,y) f(u(y))dy + b(x) \quad \text{for} \quad x \in D, \quad u|_{\partial D} = 0. \tag{38}$$

Moreover, $u \in \mathcal{S}$.

Step 2.2. Assume that f is bounded from below, that is, $f(x) \geq -N$ for every x and some $N > 0$. Set $\bar{f}_n(x) = f_1(x) + (f_2(x) \wedge n)$, $n \geq 0$. Observe that each $\bar{f}_n$ satisfies (f1) and (f2). Let

$$u_n(x) = \int_D G_D^k(x,y) \bar{f}_n(u_n(y))dy + b(x) \quad \text{for} \quad x \in D, \quad u_n|_{\partial D} = 0,$$

be the solution given in Step 2.1.

The sequence $\{\bar{f}_n,\ n \geq 0\}$ is increasing; hence by Lemma 4.2, the sequence of functions $\{u_n,\ n \geq 0\}$ satisfying

$$u_n(x) = \int_D G_D^k(x,y) f_1(u_n(y))dy + \int_D G_D^k(x,y)(f_2 \wedge n)(u_n(y))dy + b(x) \tag{39}$$

is decreasing. Set $u(x) = \inf_n u_n(x)$. Observe that it is an a.s. finite function. Since f_1 is bounded, we can permute the limit and the integral operator in the first integral of the left-hand side of (39). To perform a similar operation on the second integral, we apply Lemma 4.3 and Corollary 4.4. Indeed, since $u_n(x) \searrow u(x) > -\infty$, almost everywhere, one obtains $|u_n(x)|\mathbb{1}_{\{u>0\}} \searrow |u(x)|\mathbb{1}_{\{u>0\}}$, and $|u_n(x)|\mathbb{1}_{\{u<0\}} \nearrow |u(x)|\mathbb{1}_{\{u<0\}}$. Therefore,

$$|u_n(x)| = |u_n(x)|\mathbb{1}_{\{u>0\}} + |u_n(x)|\mathbb{1}_{\{u<0\}} \leq |u_0(x)|\mathbb{1}_{\{u>0\}} + |u(x)|\mathbb{1}_{\{u<0\}} = \Phi(x),$$

with $\Phi \in L^q(D)$. Thus,

$$|(f_2 \wedge n)(u_n(x))| \leq |f_2(u_n(x))| \leq |f_2(0)| + L\Phi(x) \in L^q(D),$$

and by Hölder's inequality,

$$|G_D^k(x,y)(f_2 \wedge n)(u_n(y))| \leq |G_D^k(x,y)|\{|f_2(0)| + L\Phi(x)\} \in L^1(D).$$

Hence, by the continuity of f_2 and the dominated convergence theorem,

$$\lim_{n\to\infty} \int_D G_D^k(x,y)(f_2 \wedge n)(u_n(y))dy = \int_D G_D^k(x,y) f_2(u(y))dy.$$

Summarizing, if f is bounded from below, there exists u satisfying (38) and $\|u\|_{L^q(D)} \leq \Upsilon$.

Step 2.3: f satisfies $(f1)$ *and* $(f2)$. Set $\bar{f}_n = f_1 + (f_2 \vee (-n))$, $n \geq 0$. By the results obtained in the previous step, there exists u_n such that

$$u_n(x) = \int_D G_D^k(x,y)\bar{f}_n(u_n(y))dy + b(x) \quad \text{for} \quad x \in D, \quad u|_{\partial D} = 0,$$

and $\sup_n \|u_n\|_{L^q(D)} \leq \Upsilon$.

The sequence $\{\bar{f}_n,\ n \geq 0\}$ is decreasing; hence, by Lemma 4.2, $\{u_n,\ n \geq 0\}$ is increasing. Set $u(x) = \sup_n u_n(x)$ for a.e. x. As in Step 2.2, it suffices to show that

$$\lim_{n\to\infty} \int_D G_D^k(x,y)(f_2 \vee (-n))(u_n(y))dy = \int_D G_D^k(x,y) f_2(u(y))dy.$$

Since $u_n(x) \nearrow u(x)$, one has $|u_n(x)|\,\mathbb{1}_{\{u>0\}} \nearrow |u(x)|\,\mathbb{1}_{\{u>0\}}$, and $|u_n(x)|\,\mathbb{1}_{\{u<0\}} \searrow |u(x)|\mathbb{1}_{\{u<0\}}$. Thus,

$$|u_n(x)| = |u_n(x)|\mathbb{1}_{\{u>0\}} + |u_n(x)|\mathbb{1}_{\{u<0\}} \leq |u(x)|\mathbb{1}_{\{u>0\}} + |u_0(x)|\mathbb{1}_{\{u<0\}} = \Psi(x),$$

with $\Psi \in L^q(D)$. Notice that $|f_2 \vee (-n)| \leq |f_2|$. Therefore

$$|(f_2 \vee (-n))(u_n(x))| \leq |f_2(u_n(x))| \leq |f_2(0)| + L\Psi(x) \in L^q(D).$$

As in Step 2.2, by Hölder's inequality,

$$|G_D^k(x,y)(f_2 \vee (-n))(u_n(y))| \leq |G_D^k(x,y)|\{|f_2(0)| + L\Psi(x)\} \in L^1(D).$$

Thus, by the continuity of f_2 and the dominated convergence theorem,

$$\lim_{n\to\infty} \int_D G_D^k(x,y)(f_2 \wedge n)(u_n(y))dy = \int_D G_D^k(x,y) f_2(u(y))dy.$$

Hence, we have proved the existence of $u \in L^q(D)$ satisfying (38). Observe that the terms in the right-hand side of (38) belong to $\mathcal{S}$; therefore, so does u. ■

We now state the lemmas used throughout the proof of Theorem 4.1. The first one is a comparison result. For details on its proof we refer the reader to Lemma 2.6 in [3], (and also Lemma 4 in [17]).

Lemma 4.2 *Let f and h satisfy $(f1)$, $(f2)$ and suppose that $f(x) \geq h(x)$ for every $x \in \mathbb{R}$. Let $b \in L^q(D)$ with $q \in [2, \infty[$, and*

$$u(x) - \int_D G_D^k(x,y) f(u(y))dy = b,$$
$$v(x) - \int_D G_D^k(x,y) h(v(y))dy = b.$$

Then $u(x) \leq v(x)$ for almost every $x \in D$.

The next one provides a priori estimates. For details on its proof, see Lemma 5 and Corollary 1, respectively, in [17].)

Lemma 4.3 *Assume that the Lipschitz constant of* f_2 *satisfies* $L < \mathcal{K}^{-1}$*, where the constant* $\mathcal{K}$ *is given in (23). Then the sequence* $\{u_n,\ n \geq 0\}$ *defined in* (39) *satisfies*

$$\sup_n \|u_n\|_{L^q(D)} \leq \Upsilon$$

for any $q \in [2, \infty[$ *with*

$$\Upsilon = \frac{(M + |f_2(0)|)\mathcal{K} + \|b\|_{L^q(D)}}{1 - L\mathcal{K}}.$$

From this result and Fatou's lemma, we obtain

Corollary 4.4 $\|u\|_{L^q(D)} \leq \Upsilon$*, with* q *and* Υ *as in Lemma* 4.3.

The last part of this section is devoted to a further analysis of the solution.

Lemma 4.5 *Assume the assumptions of Theorem* 4.1*. Then, for any* $p \in [1, \infty[$ *and any* $q \in [2, \infty[$*,*

$$\|u\|_{L^p(\Omega; L^q(D))} \leq C.$$

Proof: Owing to Corollary 4.4, the solution u to (1) satisfies $\|u\|_{L^q(D)} \leq \Upsilon$, where Υ is as in Lemma 4.3. Hence, it suffices to check that $b \in L^p(\Omega; L^q(D))$, for any $p \geq 1$, where b given in (36).
The function $x \rightarrow \int_D G_D^k(x,y)g(y)dy$ is continuous and deterministic and therefore belongs to the space $L^p(\Omega; L^q(D))$. As for $\int_D G_D^k(x,y)dB^H(y)$ this property has been proved in Proposition 3.3. ■
We finally state a result on the regularity of the sample paths of the solution.

Theorem 4.6 *With the same hypotheses as in Theorem* 4.1*, for any* $x, z \in D$ *and for any* $p \in [1, \infty[$*, the solution* u *to* (1) *satisfies*

$$\mathbb{E}(|u(x) - u(z)|)^p \leq C|x - z|^{p\lambda}, \tag{40}$$

with λ *defined in Theorem 3.4 (see also Remark 3.1).*
Consequently, a.s. the sample paths are γ*-Hölder continuous with* $\gamma \in]0, \lambda[$*.*

Proof: We write $\|u(x) - u(z)\|_{L^p(\Omega)} \leq \sum_{i=1}^3 I_i^k(x,z)$, with

$$I_1^k(x,z) = \left\|\int_D \left(G_D^k(x,y) - G_D^k(z,y)\right) f(u(y))dy\right\|_{L^p(\Omega)},$$
$$I_2^k(x,z) = \left\|\int_D \left(G_D^k(x,y) - G_D^k(z,y)\right) g(y)dy\right\|_{L^p(\Omega)},$$
$$I_3^k(x,z) = \left\|\int_D \left(G_D^k(x,y) - G_D^k(z,y)\right) dB^H(y)\right\|_{L^p(\Omega)}.$$

By Hölder's inequality and (27) we have

$$\begin{aligned}I_1^k(x,z) &\le \left\|G_D^k(x,\cdot)-G_D^k(z,\cdot)\right\|_{L^{\frac{1}{H_1},\cdots,\frac{1}{H_k}}(R_D)} \\ &\quad\times\left(M+|f_2(0)|+L\left(\mathbb{E}\|u\|^p_{L^{\frac{1}{1-H_1},\cdots,\frac{1}{1-H_k}}(R_D)}\right)^{\frac{1}{p}}\right) \\ &\le C\left(M+|f_2(0)|+L\left(\mathbb{E}\|u\|^p_{L^{\frac{1}{1-H_1},\cdots,\frac{1}{1-H_k}}(R_D)}\right)^{\frac{1}{p}}\right)|x-z|^{\lambda}.\end{aligned}$$

The factor $\mathbb{E}\|u\|^p_{L^{\frac{1}{1-H_1},\cdots,\frac{1}{1-H_k}}(R_D)}$ is finite. Indeed this follows from (17) and Lemma 4.5. Consequently,

$$I_1^k(x,z)\le C|x-z|^{\lambda}.$$

In a similar but easier way, we obtain a similar bound for $I_2^k(x,z)$. As for $I_3^k(x,z)$, the bound is obtained by first applying the hypercontractivity inequality and then Theorem 3.4.
This ends the proof of (40). The statement about the regularity of the sample paths follows from Kolmogorov's criterion. ■

5 Lattice approximations in L^2-spatial norm

This section is devoted to give finite difference approximation sequences for the SPDE (1) on the domain $D=]0,1[^k$, obtained by discretizing the Laplacian operator. We shall denote by I^k and I_n^k the sets of indices $\{1,2,\ldots\}^k$ and $\{1,2,\ldots,n-1\}^k$, respectively. To simplify the notation, we shall omit the superscript k when referring to G_D^k. Throughout the section, C denotes a positive constant not depending on n. The analysis of the speed of convergence is done by means of a general result given in the next theorem.

Theorem 5.1 *Assume the hypotheses of Theorem 4.1. Consider a sequence of step functions $\{g_n\}_{n\ge1}$ defined on D such that for n big enough,*

$$\|g-g_n\|_{L^{\frac{1}{1-H_1},\ldots,\frac{1}{1-H_k}}(D)}\le Cn^{-1}.$$

Let $\{\tilde u_n(x),x\in D\}$, $n\ge1$, be given by

$$\tilde u_n(x)=\int_D\tilde G_{D,n}(x,y)f(\tilde u_n(y))dy+\int_D\tilde G_{D,n}(x,y)g_n(y)dy+\int_D\tilde G_{D,n}(x,y)dB^H(y), \tag{41}$$

where $\tilde G_{D,n}$, $n\ge1$, are functions defined on $D\times D$ satisfying

$$\int_D\left\|\tilde G_{D,n}(x,.)-G_D(x,.)\right\|^2_{L^{\frac{1}{H_1},\cdots,\frac{1}{H_k}}(D)}dx\le Cn^{-\gamma}, \tag{42}$$

for some $\gamma > 0$.
Suppose that for any $p \in [1, \infty[$ *and* $q \in [2, \infty[$,

$$\sup_{n \geq 1} \|\tilde{u}_n\|_{L^p(\Omega; L^q(D))} \leq C. \tag{43}$$

Then,

$$\|u - \tilde{u}_n\|_{L^p(\Omega; L^2(D))} \leq C n^{-\frac{\gamma}{4}}, \tag{44}$$

where u *is the solution of (3).*

Proof: We shall follow the scheme of the proof of Theorem 2.4 of [10] (see also Theorem 4 in [17]). By defining

$$\begin{aligned} T(x) = &\int_D [G_D(x,y) - \tilde{G}_{D,n}(x,y)] f(\tilde{u}_n(y)) dy \\ &+ \int_D G_D(x,y)[g(y) - g_n(y)] dy + \int_D [G_D(x,y) - \tilde{G}_{D,n}(x,y)] g_n(y) dy \\ &+ \int_D [G_D(x,y) - \tilde{G}_{D,n}(x,y)] dB^H(y), \end{aligned}$$

we have $u(x) - \tilde{u}_n(x) = \int_D G_D(x,y)[f(u(y)) - f(\tilde{u}_n(y))] dy + T(x)$.
By virtue of (33) and (34), as in [17] we obtain

$$\begin{aligned} &(a - L)\|u - \tilde{u}_n\|^2_{L^2(D)} \\ &\leq 2a \int_D (u(x) - \tilde{u}_n(x)) T(x) dx + \int_D \left(f(u(x)) - f(\tilde{u}_n(x))\right) T(x) dx. \end{aligned} \tag{45}$$

Let $q \in [2, \infty[$ and let $\tilde{q} \in]1, 2]$ be its conjugate. By applying Hölder's inequality and by virtue of the assumptions on f, the right-hand side (45) is bounded by

$$(2a + L)\|u - \tilde{u}_n\|_{L^q(D)} \|T\|_{L^{\tilde{q}}(D)} + 2M\|T\|_{L^{\tilde{q}}(D)}. \tag{46}$$

Lemmas 4.5 and (43) yield for any $p \in [1, \infty[$, $q \in [2, \infty[$,

$$\sup_{n \geq 1} \|u - \tilde{u}_n\|_{L^p(\Omega; L^q(D))} \leq \widehat{C}. \tag{47}$$

From (45)–(47) and applying Schwarz's inequality we obtain

$$\|u - \tilde{u}_n\|_{L^p(\Omega; L^2(D))} \leq C \left(\|T\|_{L^p(\Omega; L^{\tilde{q}}(D))}\right)^{\frac{1}{2}} \tag{48}$$

for any $p \in [1, \infty[$.
Hölder's inequality for L^p spaces with mixed norm yields

$$\begin{aligned} |T(x)| \leq &\|G_D(x,\cdot) - \tilde{G}_{D,n}(x,\cdot)\|_{L^{\frac{1}{H_1}, \cdots, \frac{1}{H_k}}(D)} \|f(\tilde{u}_n)\|_{L^{\frac{1}{1-H_1}, \cdots, \frac{1}{1-H_k}}(D)} \\ &+ \|G_D(x,\cdot)\|_{L^{\frac{1}{H_1}, \cdots, \frac{1}{H_k}}(D)} \|g - g_n\|_{L^{\frac{1}{1-H_1}, \cdots, \frac{1}{1-H_k}}(D)} \\ &+ \|G_D(x,\cdot) - \tilde{G}_{D,n}(x,\cdot)\|_{L^{\frac{1}{H_1}, \cdots, \frac{1}{H_k}}(D)} \|g_n\|_{L^{\frac{1}{1-H_1}, \cdots, \frac{1}{1-H_k}}(D)} \\ &+ \left| \int_D [G_D(x,y) - \tilde{G}_{D,n}(x,y)] dB^H(y) \right|. \end{aligned}$$

By the assumptions on the function f we have

$$\|f(\tilde{u}_n)\|_{L^{\frac{1}{1-H_1},\dots,\frac{1}{1-H_k}}(D)} \leq M + |f_2(0)| + L\|\tilde{u}_n\|_{L^{\frac{1}{1-H_1},\dots,\frac{1}{1-H_k}}(D)}.$$

Thus, by (17) and (43)

$$\sup_{n\geq 1} \|f(\tilde{u}_n)\|_{L^p(\Omega;L^{\frac{1}{1-H_1},\dots,\frac{1}{1-H_k}}(D))} \leq M_1 < +\infty.$$

Consequently,

$$\begin{aligned}
\|T\|_{L^p(\Omega;L^{\tilde{q}}(D))} &\leq \|T\|_{L^p(\Omega;L^2(D))} \\
&\leq C_1 n^{-\frac{\gamma}{2}} \Big(M_1 + 1 + \|g\|_{L^{\frac{1}{1-H_1},\dots,\frac{1}{1-H_k}}(D)} \Big) \\
&\quad + C_2 n^{-1} + \left\| \int_D [G_D(\cdot,y) - \tilde{G}_{D,n}(\cdot,y)] dB^H(y) \right\|_{L^p(\Omega;L^2(D))}, \qquad (49)
\end{aligned}$$

where we have applied (23) and γ is given in (42).
Proceeding as in the proof of Proposition 3.3 by replacing the Green function G_D by $G_D - G_{D,n}$, we obtain

$$\begin{aligned}
&\left\| \int_D [G_D(\cdot,y) - \tilde{G}_{D,n}(\cdot,y)] dB^H(y) \right\|_{L^p(\Omega;L^2(D))} \\
&\leq C(p) \left(\int_D \left\| G_D(x,\cdot) - \tilde{G}_{D,n}(x,\cdot) \right\|^2_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(D)} dx \right)^{\frac{1}{2}} \\
&\leq C(p) n^{-\frac{\gamma}{2}},
\end{aligned}$$

where in the last inequality we have applied (42). Using this estimate in (49) we finish the proof of the theorem. ■

Consider the grid of $\overline{D}$ given by

$$\mathcal{G} = \left\{ \frac{j}{n} = \left(\frac{j_1}{n}, \dots, \frac{j_k}{n} \right) : \; j_l = 0, 1, \dots, n, \; l = 1, \dots, k \right\} \subset \overline{D}.$$

On the space $X = \{u : \; u = \{u_i\}_{i \in I_n^k}\} = \mathbb{R}^{(n-1)^k}$ endowed with the Hilbert-Schmidt norm, we define the second order difference operator $A : X \to X$,

$$(Au)_i = \sum_{j=1}^{k} n^2 \left(u_{i-e_j} - 2u_i + u_{i+e_j} \right),$$

where $\{e_j\}_{j=1}^k$ is the canonical basis of $\mathbb{R}^k$.
Consider the orthogonal complete system in $L^2(D)$ provided by the functions

$$v_\beta(x) = \sin(\beta_1 \pi x_1) \cdots \sin(\beta_k \pi x_k), \; \beta \in I^k, \; k \geq 1.$$

The set of vectors $\left\{ \left(\frac{2}{n}\right)^{k/2} U_\beta, \; \beta \in I_n^k \right\}$, $(U_\beta)_i = v_\beta\!\left(\frac{i}{n}\right)$, $i \in I_n^k$, is an orthonormal system in X of eigenvectors of A, with eigenvalues $\lambda_\beta = -\pi^2(\beta_1^2 c_{\beta_1} + \cdots + \beta_k^2 c_{\beta_k})$, where $c_l = \left(\frac{l\pi}{2n}\right)^{-2} \sin^2\left(\frac{l\pi}{2n}\right)$. Notice that $\frac{4}{\pi^2} \leq c_l \leq 1$ for every $1 \leq l \leq n-1$.

For any point $\frac{j}{n} \in \mathcal{G}$, we set $D_j = [\frac{j_1}{n}, \frac{j_1+1}{n}[\times \cdots \times [\frac{j_k}{n}, \frac{j_k+1}{n}[$; then, for each $x \in D_j$ we define $\kappa_n(x) = \frac{j}{n}$.
We begin by giving a first type of discrete approximations of u on points of $\mathcal{G}$, as follows. If $\frac{j}{n} \in \mathcal{G} \cap \partial D$, we set $u_n(\frac{j}{n}) = 0$ (boundary conditions), while for $\frac{j}{n}$ with $j \in I_n^k$, we define $u_n(\frac{j}{n})$ to be the solution of the system

$$Au_n = f(u_n) + g_n + n^k \mathbf{B}^H, \tag{50}$$

where $\mathbf{B}^H$ is the vector $\{B^H(D_i) = \int_{\mathbb{R}^k} 1\!\!1_{D_i}(y) dB^H(y),\ i \in I_n^k\}$, and $g_n(x) = g_n(\kappa_n(x))$, $n \geq 1$. Then, for any $x \in D$ we define $u_n(x) = u_n(\kappa_n(x))$. From [10] we know that $\{u_n(x), x \in D\}$ satisfies the evolution equation

$$u_n(x) = \int_D G_{D,n}(x,y) f(u_n(y)) dy + \int_D G_{D,n}(x,y) g_n(y) + \int_D G_{D,n}(x,y) dB^H(y), \tag{51}$$

where

$$G_{D,n}(x,y) = \sum_{\beta \in I_n^k} \frac{2^k}{\lambda_\beta} v_\beta(\kappa_n(x)) v_\beta(\kappa_n(y)). \tag{52}$$

In dimension $k = 1, 2, 3$ we shall consider $\{u_n(x), x \in D\}$, $n \geq 1$, as sequence of approximations of the process $\{u(x), x \in D\}$. We notice that in this case the kernel $G_{D,n}(x,.)$ is related with the truncation of the Fourier expansion of $G_D(x,.)$. For $k \geq 4$ we shall follow the more sophisticated approach of [17], which considers a smoothed version of $G_D(x,.)$. We remark that for such dimensions $G_D(x,.)$ is not square integrable.
For low dimensions, we have the following

Theorem 5.2 *Assume the assumptions of Theorem 4.1 with a Lipschitz constant for f_2 satisfying $L < \min(a, \mathcal{K}^{-1}, 4k, \widehat{\mathcal{K}}^{-1})$. Let $k \leq 3$ and $\{u_n(x), x \in D\}$, $n \geq 1$, be defined in (51), (52). Then for any $p \in [1, \infty[$,*

$$\|u - u_n\|_{L^p(\Omega; L^2(D))} \leq C n^{-\nu},$$

with $\nu = \frac{1}{2}$, $\nu \in]0, \frac{1}{2}[$, $\nu \in]0, \frac{1}{4}]$, for $k = 1$, $k = 2$ and $k = 3$, respectively.

Proof: It follows from Theorem 5.1. Indeed, Lemma 3.4 in [10] tell us that condition (42) holds for $\tilde{G}_{D,n} := G_{D,n}$ with $\gamma := 4\nu$; for $k = 1, 2$, the values of ν are those given in the statement, but for $k = 3$, $\nu \in]0, \frac{1}{4}[$. It is easy to check that for this dimension their result extends to $\nu = \frac{1}{4}$ as well. Moreover, condition (43) for $\tilde{u}_n := u_n$ is also satisfied, as can be checked by applying Lemma 3.3 in [10] and the arguments of the proof of Lemma 5.7 below.

■

We next deal with higher dimensions. The Fourier analysis techniques we shall use in the proofs require the identification of functions $f : [-1, 1[^k \longrightarrow \mathbb{R}$ with functions $F : \mathbb{T}^k \longrightarrow \mathbb{R}$ defined on the k-th dimensional torus through the exponential mapping $F(e^{i\pi x}) := F(e^{i\pi x_1}, \ldots, e^{i\pi x_k})$, which carries Lebesgue measure into the

Haar measure, that is $\int_{]-1,1[^k} f(x)dx = \int_{\mathbb{T}^k} F(e^{i\pi x})dx$. When dealing with the function $y \to G_D(x,y)$, we will consider its odd extension, that is, for any $y_j \in]0,1[$, $j = 1, \dots, k$, we define $G_D(y_1, \dots, -y_i, \dots, y_k) = -G_D(y_1, \dots, y_i, \dots, y_k)$. We still note $G_D(x,\cdot)$ the extension. Let $\mathbb{G}_D^x(e^{i\pi y}) = G_D(x,y)$ be its identification with a function defined on $\mathbb{T}^k$. Observe that $\mathbb{G}_D^x$ satisfies

$$\|\mathbb{G}_D^x(e^{i\pi\cdot})\|_{L^{p_1,\dots,p_k}(\mathbb{T}^k)} = \|G_D(x,\cdot)\|_{L^{p_1,\dots,p_k}(]-1,1[^k)} = 2^{\sum_{i=1}^k \frac{1}{p_i}} \|G_D(x,\cdot)\|_{L^{p_1,\dots,p_k}(D)},$$

for any $p_1, \dots, p_k$ such that the last norm is finite.
Let $\psi(x) \in \mathcal{C}_c^\infty(]-1,1[)$ be an even function, $0 \le \psi \le 1$, $\int_{-1}^1 \psi = 1$. Set $\Psi(x) = \prod_{i=1}^k \psi(x_i)$. Clearly, $\Psi(x) \in \mathcal{C}_c^\infty(]-1,1[^k)$ and it is an even function in each variable x_i. Define

$$\Phi(e^{i\pi x}) = \prod_{i=1}^k \phi(e^{i\pi x_i}) := \prod_{i=1}^k \psi(x_i) = \Psi(x).$$

The functions $\Phi_\varepsilon(e^{i\pi x}) = \frac{1}{\varepsilon^k}\Psi\left(\frac{x}{\varepsilon}\right) := \Psi_\varepsilon(x)$, $\varepsilon > 0$, provide an approximation of the identity in $\mathbb{T}^k$.
We shall denote by $\hat{\Psi}$ the Fourier transform of Ψ, which is a rapidly decreasing function, therefore for any $\theta \in [0,\infty[$ there is a constant $C(\theta)$ such that $\sup_\xi |\xi|^\theta |\hat{\Psi}(\xi)| \le C(\theta)$.

Let us now introduce a second kind of approximations of u. For this we start by writing $A = U^t D U$, with U the $(n-1)^k$ matrix whose rows are the vectors U_{β_j}, (here β_j, $j = 1, \cdots, (n-1)^k$ is the lexicographic enumeration of I_n^k) and D the square diagonal matrix with entries $D_{j,j} = \lambda_{\beta_j}$.
The smoothed version of A is defined as follows. Fix $\varepsilon > 0$ and define D^ε as the square diagonal matrix in dimension $(n-1)^k$ with diagonal entries

$$\lambda_{\beta_j}^\varepsilon = \frac{\lambda_{\beta_j}}{\hat{\Psi}(\varepsilon\beta_j)}.$$

In connection with D^ε we define a sequence $(u_n^\varepsilon, n \ge 1)$ of functions in the following way. If $\frac{j}{n} \in \mathcal{G} \cap \partial D$, set $u_n^\varepsilon(\frac{j}{n}) = 0$ (boundary conditions). For $\frac{j}{n}$, with $j \in I_n^k$, define $u_n^\varepsilon(\frac{j}{n})$ to be the solution of the system

$$(U^t D^\varepsilon U) u_n^\varepsilon = f(u_n^\varepsilon) + g_n + n^k \mathbf{B}^H. \tag{53}$$

Finally, for any $x \in D$ we define $u_n^\varepsilon(x) = u_n^\varepsilon(\kappa_n(x))$.
We shall prove later that an appropriate sequence $u_n := u_n^{\varepsilon(n)}$ of such functions converges to the solution of (1) in the space $L^p(\Omega; L^2(D))$, for any $p \ge 1$, with a rate of convergence which depends on the dimension k and on the driving noise.

The following result is proved by the same arguments as in Proposition 1 of [17].

Proposition 5.3 *With the same hypotheses as in Theorem 4.1 and assuming that the Lipschitz constant satisfies $L < 4k$, Equation (53) possesses a unique solution.*

Moreover, this solution satisfies the mild equation

$$u_n^\varepsilon(x) = \int_D G_{D,n}^\varepsilon(x,y) f(u_n^\varepsilon(y))dy + \int_D G_{D,n}^\varepsilon(x,y) g_n(y)dy + \int_D G_{D,n}^\varepsilon(x,y) dB^H(y), \tag{54}$$

where

$$G_{D,n}^\varepsilon(x,y) = \sum_{\beta \in I_n^k} \frac{\hat{\Psi}(\varepsilon\beta)2^k}{\lambda_\beta} v_\beta(\kappa_n(x)) v_\beta(\kappa_n(y)). \tag{55}$$

Both (52) and (55) correspond to discretized Fourier series expansions; the Fourier coefficients in (55) are smoothed by the factor $\hat{\Psi}(\varepsilon\beta)$.

Our aim is to apply Theorem 5.1 to $\tilde{u}_n := u_n^{\varepsilon(n)}$ defined in (54) for values of ε that depend on n, and $k \geq 4$. The next statements provide the ingredients for checking condition (42) for $\tilde{G}_{D,n} := G_{D,n}^{\varepsilon(n)}$.

Set $G_D^\varepsilon(x,y) = \mathbb{G}_D^{x,\varepsilon}(e^{i\pi y})$, where $\mathbb{G}_D^{x,\varepsilon}(e^{i\pi y}) = \int_{\mathbb{T}^k} \mathbb{G}_D^x(e^{i\pi(y-u)}) \Phi_\varepsilon(e^{i\pi u}) du$. The function $G_D^\varepsilon(x,y)$ is a smoothing of $G_D(x,\cdot)$.

Lemma 5.4 *For any $\varepsilon > 0$, we have*

$$G_D^\varepsilon(x,y) = \sum_{\beta \in I^k} \frac{-\hat{\Psi}(\varepsilon\beta)2^k}{\pi^2 |\beta|^2} v_\beta(x) v_\beta(y), \tag{56}$$

in $L^2(D \times D)$ and a.e.. In addition,

$$\|G_D^\varepsilon(x,\cdot)\|_{L^2(D)}^2 = \frac{2^k}{\pi^4} \sum_{\beta \in I^k} \frac{\hat{\Psi}^2(\varepsilon\beta)}{|\beta|^4} v_\beta^2(x), \tag{57}$$

and the series converges uniformly in $x \in D$ and $\varepsilon \in]0, \varepsilon_0]$.

Proof: We want to prove that $G_D^\varepsilon \in L^2(D \times D)$. Once this fact will be established, the expansion (56) follows from the computation of the Fourier coefficients carried out in Lemma 8 of [17].
Let H_i, $i = 1, \ldots, k$, satisfying hypothesis **(H)**. Young's inequality for convolutions and L^p spaces with mixed norm (see Theorem 1, page 319 of [2]) yields

$$\begin{aligned}
\|G_D^\varepsilon(x,\cdot)\|_{L^2(D)} &= 2^{-\frac{k}{2}} \|\mathbb{G}_D^{x,\varepsilon}(e^{i\pi\cdot})\|_{L^2(\mathbb{T}^k)} \\
&\leq 2^{-\frac{k}{2}} \|\mathbb{G}_D^x\|_{L^{\frac{1}{H_1},\cdots,\frac{1}{H_k}}(\mathbb{T}^k)} \|\Phi_\varepsilon\|_{L^{r_1,\cdots,r_k}(\mathbb{T}^k)} \\
&= 2^{\sum_{i=1}^k H_i - \frac{k}{2}} \|G_D(x,\cdot)\|_{L^{\frac{1}{H_1},\cdots,\frac{1}{H_k}}(D)} \|\Psi_\varepsilon\|_{L^{r_1,\cdots,r_k}(]-1,1[^k)},
\end{aligned}$$

with $H_i + \frac{1}{r_i} = 1 + \frac{1}{2}$, $i = 1, \ldots, k$.
Since for any $r_1, \ldots, r_k \geq 1$, $\sup_\varepsilon \|\Psi_\varepsilon\|_{L^{r_1,\cdots,r_k}(]-1,1[^k)} \leq C$ and by virtue of (23),

$$\sup_{x \in D} \sup_\varepsilon \|G_D^\varepsilon(x,\cdot)\|_{L^2(D)} \leq C. \tag{58}$$

Hence $G_D^\varepsilon \in L^2(D \times D)$ and the formula (56) holds true.
The identity (57) follows from the property $\|v_\beta\|_{L^2(D)} = 2^{-k/2}$ owned by the orthogonal complete system $(v_\beta, \beta \in I^k)$. Finally, (58) implies the uniform convergence of the series in (57). ■

The next result provides a bound for the discrepancy between G_D and G_D^ε.

Lemma 5.5 *Assume* **(H*)**. *Then for every* $\varepsilon > 0$,

$$\sup_{x\in D} \|G_D(x,\cdot) - G_D^\varepsilon(x,\cdot)\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(D)} \le C\varepsilon^\lambda, \tag{59}$$

with

$$\begin{cases} \lambda = 1, & \text{for } k = 1, \\ \lambda \in \left]\left(1 - \frac{\sum_{i=1}^k H_i}{2(k-2)}\right) \vee 0, \frac{\sum_{i=1}^k H_i}{2(k-1)}\right[, & \text{for } k = 2, 3, \\ \lambda \in \left]1 - \frac{\sum_{i=1}^k H_i}{2(k-2)}, \frac{\sum_{i=1}^k H_i}{2(k-1)}\right[, & \text{for } k \ge 4. \end{cases}$$

Under **(H**)** *(see Remark 3.1), (59) holds with* $\lambda = 1$ *for any dimension* $k \ge 1$.

Proof: Since $G_D(x,\cdot)$, $G_D^\varepsilon(x,\cdot)$ are odd in the y_i-variables, we have

$$\begin{aligned} \|G_D(x,\cdot) - G_D^\varepsilon(x,\cdot)\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(D)} &= C\|G_D(x,\cdot) - G_D^\varepsilon(x,\cdot)\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(]-1,1[^k)} \\ &= C\|\mathbb{G}_D^x(e^{i\pi\cdot}) - \mathbb{G}_D^{x,\varepsilon}(e^{i\pi\cdot})\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(\mathbb{T}^k)}, \end{aligned}$$

with $C = 2^{-\sum_{i=1}^k H_i}$. Since $\int_{\mathbb{T}^k} \Phi_\varepsilon(u)du = 1$, we can write

$$\begin{aligned} &\|\mathbb{G}_D^x(e^{i\pi\cdot}) - \mathbb{G}_D^{x,\varepsilon}(e^{i\pi\cdot})\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(\mathbb{T}^k)} \\ &\quad = \left\| \int_{\mathbb{T}^k} (\mathbb{G}_D^x(e^{i\pi\cdot}) - \mathbb{G}_D^x(e^{i\pi(\cdot-u)}))\Phi_\varepsilon(u)du \right\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(\mathbb{T}^k)} \\ &\quad \le \int_{\mathbb{T}^k} \|\mathbb{G}_D^x(e^{i\pi\cdot}) - \mathbb{G}_D^x(e^{i\pi(\cdot-u)})\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(\mathbb{T}^k)} \Phi_\varepsilon(u)du \\ &\quad = 2^{\sum_{i=1}^k H_i} \int_{\mathbb{T}^k} \|G_D(x,\cdot) - G_D(x,\cdot - u)\|_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(D)} \Phi_\varepsilon(u)du \\ &\quad \le C\int_{\mathbb{T}^k} |u|^\lambda \Phi_\varepsilon(u)du, \end{aligned}$$

where we have applied Minkowski's inequality with respect to the finite measure on $\mathbb{T}^k$ defined by $\Phi_\varepsilon(u)du$ and eventually (27). By virtue of the properties of the function Φ_ε, we obtain (59).
The last statement follows from Remark 3.1. ■

As an additional auxiliary result, we need *a priori estimates* for the solution of (54). An ingredient for this is provided by the following Lemma.

Lemma 5.6 *The smoothed and discretized Green function defined in (55) satisfies*

$$\sup_{n\geq 1, x\in D, \varepsilon\in]0,\varepsilon_0]} \|G^\varepsilon_{D,n}(x,\cdot)\|_{L^2(D)} \leq \widehat{\mathcal{K}}. \tag{60}$$

Proof: The system $\{v_\beta(\kappa_n(y))\}$ is orthogonal in $\mathbb{R}^{(n-1)^k}$, thus in $L^2(D)$ as well. Hence, using the lower bound $|\lambda_\beta| \geq 4|\beta|^2$ we have

$$\|G^\varepsilon_{D,n}(x,\cdot)\|^2_{L^2(D)} = \sum_{\beta\in I^k_n} \frac{\hat{\Psi}^2(\varepsilon\beta)2^{2k}}{\lambda^2_\beta} v^2_\beta(\kappa_n(x)) \leq \frac{2^{2k}}{16} \sum_{\beta\in I^k_n} \frac{\hat{\Psi}^2(\varepsilon\beta)}{|\beta|^4} v^2_\beta(\kappa_n(x)).$$

Consequently,

$$\sup_{n\geq 1, x\in D, \varepsilon\in]0,\varepsilon_0]} \|G^\varepsilon_{D,n}(x,\cdot)\|^2_{L^2(D)} \leq C \sup_{x\in D, \varepsilon\in]0,\varepsilon_0]} \left\{ \sum_{\beta\in I^k} \frac{\hat{\Psi}^2(\varepsilon\beta)}{|\beta|^4} v^2_\beta(x) \right\} =: \widehat{\mathcal{K}}^2,$$

with a finite constant $\widehat{\mathcal{K}}$ (see Lemma 5.4). ■

Lemma 9 of [17] gives a more particular statement than the previous Lemma 5.6. We have noticed an incorrect argument in the proof of the former that can be fixed using the proof of the later.

Let $\bar{q} \in]1,2]$; by (17), Hölder's inequality and Lemma 5.6 we have

$$\sup_{n\geq 1, x\in D, \varepsilon\in]0,\varepsilon_0]} \left(\|G^\varepsilon_{D,n}(x,\cdot)\|_{L^{\frac{1}{H_1},\cdots,\frac{1}{H_k}}(D)} \vee \|G^\varepsilon_{D,n}(x,\cdot)\|_{L^{\bar{q}}(D)} \right) \leq \widehat{\mathcal{K}}. \tag{61}$$

We can now prove an *a priori* estimate for the solution of (54).

Lemma 5.7 *Assume the same assumptions as in Proposition 5.3 with the Lipschitz constant satisfying the restriction* $L < \min(4k, \widehat{\mathcal{K}}^{-1})$, *where* $\widehat{\mathcal{K}}$ *is given in* (60). *Then, for any* $p \in [1,\infty[$ *and* $q \in [2,\infty[$,

$$\sup_{n\geq 1, \varepsilon\in]0,\varepsilon_0]} \left(\|u^\varepsilon_n\|_{L^p(\Omega;L^q(D))} \right) \leq C.$$

Proof: Fix $q \in [2,\infty[$ and denote by $\bar{q} \in]1,2]$ its conjugate. Hölder's inequality and the properties on f imply

$$\begin{aligned} |u^\varepsilon_n(x)| \leq & \sup_{n\geq 1, x\in D, \varepsilon\in]0,\varepsilon_0]} \|G^\varepsilon_{D,n}(x,\cdot)\|_{L^{\bar{q}}(D)} \Big(M + |f_2(0)| + L\|u^\varepsilon_n\|_{L^q(D)} \Big) \\ & + \sup_{n\geq 1, x\in D, \varepsilon\in]0,\varepsilon_0]} \|G^\varepsilon_{D,n}(x,\cdot)\|_{L^{\frac{1}{H_1},\cdots,\frac{1}{H_k}}(D)} \sup_{n\geq 1} \|g_n\|_{L^{\frac{1}{1-H_1},\dots,\frac{1}{1-H_k}}(D)} \\ & + \left| \int_D G^\varepsilon_{D,n}(x,y) dB^H(y) \right| \end{aligned} \tag{62}$$

Since u^ε_n is a step function, its L^q–norm is finite. Moreover, following the arguments of the proof of Proposition 3.3 we obtain

$$\left\| \int_D G^\varepsilon_{D,n}(\cdot,y) dB^H(y) \right\|_{L^p(\Omega;L^q(D))} \leq C(p,q,H)\widehat{\mathcal{K}}.$$

With this remarks, and assuming that $L < \widehat{\mathcal{K}}^{-1}$, from (62) we obtain the result by integration and applying (61). ■

The next lemma gives an estimate of the discrepancy between G^{ε}_{D} and $G^{\varepsilon}_{D,n}$.

Lemma 5.8 *Assume* $k \geq 4$. *Let* $\delta \in]0,2[$, $\mu \in \left]0, \frac{2-\delta}{k-2}\right[$ *and set* $\varepsilon(n) = n^{-\mu}$. *There exists a positive constant* C *such that*

$$\|G^{\varepsilon(n)}_{D} - G^{\varepsilon(n)}_{D,n}\|_{L^2(D\times D)} \leq C n^{-\frac{\delta}{2}}. \tag{63}$$

Proof: We follow the proofs of Lemmas 3.4 and 10 in [10] and [17], respectively. By the definitions of the kernels G^{ε}_{D} and $G^{\varepsilon}_{D,n}$ given in (56) and (55) respectively, we have

$$\|G^{\varepsilon(n)}_{D} - G^{\varepsilon(n)}_{D,n}\|^2_{L^2(D\times D)} \leq C \sum_{i=1}^{4} A_i(x,y),$$

with

$$\begin{aligned}
A_1 &= \int_{D\times D} \Big| \sum_{\beta \in I^k \setminus I^k_n} \frac{-2^k \hat{\Psi}(\varepsilon\beta)}{\pi^2 |\beta|^2} v_\beta(x) v_\beta(y) \Big|^2 dxdy,\\
A_2 &= \int_{D\times D} \Big| \sum_{\beta \in I^k_n} \left[\frac{-1}{\pi^2|\beta|^2} - \frac{1}{\lambda_\beta} \right] 2^k \hat{\Psi}(\varepsilon\beta) v_\beta(x) v_\beta(y) \Big|^2 dxdy,\\
A_3 &= \int_{D\times D} \Big| \sum_{\beta \in I^k_n} \frac{2^k \hat{\Psi}(\varepsilon\beta)}{\lambda_\beta} [v_\beta(x) - v_\beta(\kappa_n(x))] v_\beta(y) \Big|^2 dxdy,\\
A_4 &= \int_{D\times D} \Big| \sum_{\beta \in I^k_n} \frac{2^k \hat{\Psi}(\varepsilon\beta)}{\lambda_\beta} v_\beta(\kappa_n(x)) [v_\beta(y) - v_\beta(\kappa_n(y))] \Big|^2 dxdy.
\end{aligned}$$

In the sequel, we shall write ε instead of $\varepsilon(n)$ for simplicity, and we fix $\delta > 0$. Let $\theta > \frac{k-4}{2}$. Since $\hat{\Psi}$ is a rapidly decreasing function, and by the orthogonality of the functions v_β in $L^2(D)$ we have

$$\begin{aligned}
A_1 = \sum_{\beta \in I^k \setminus I^k_n} \frac{\hat{\Psi}^2(\varepsilon\beta)}{\pi^4|\beta|^4} &\leq \frac{C(\theta)}{\varepsilon^{2\theta}} \sum_{\beta \in I^k \setminus I^k_n} \frac{1}{|\beta|^{4+2\theta}}\\
&\leq C(\theta)\varepsilon^{-2\theta} n^{-4-2\theta+k} = C(\theta) n^{2\theta\mu-4-2\theta+k}.
\end{aligned}$$

Fix $\theta := \frac{\delta+k-4}{2-2\mu}$ in the last expression. We obtain $A_1 \leq C(\theta) n^{-\delta}$.

For the analysis of the term A_2 we apply the estimate $\left| \frac{-1}{\pi^2|\beta|^2} - \frac{1}{\lambda_\beta} \right| \leq \frac{C}{|\beta| n}$, valid for any $\beta \in I^k$. Taking $\theta < \frac{k-2}{2}$, we obtain

$$\begin{aligned}
A_2 = \sum_{\beta \in I^k_n} \left| \frac{-1}{\pi^2|\beta|^2} - \frac{1}{\lambda_\beta} \right|^2 \hat{\Psi}^2(\varepsilon\beta) &\leq \frac{C}{n^2} \sum_{\beta \in I^k_n} \frac{\hat{\Psi}^2(\varepsilon\beta)}{|\beta|^2}\\
&\leq \frac{C(\theta)}{\varepsilon^{2\theta} n^2} \sum_{\beta \in I^k_n} \frac{1}{|\beta|^{2+2\theta}} \leq C(\theta) n^{2\mu\theta-4+k-2\theta}.
\end{aligned}$$

Consider $\theta := \frac{k-4+\delta}{2(1-\mu)}$. The last estimates yield $A_2 \le C(\theta) n^{-\delta}$.
For the study of the remaining terms, we use that for any $\beta \in I^k$, $|v_\beta(x) - v_\beta(z)| \le C|\beta|\,|x-z|$, and $|\lambda_\beta| \ge 4|\beta|^2$, which ensure

$$A_3 \vee A_4 \le \frac{C}{n^2} \sum_{\beta \in I_n^k} \frac{\hat{\Psi}^2(\varepsilon\beta)|\beta|^2}{\lambda_\beta^2} \le \frac{C(\theta)}{\varepsilon^{2\theta} n^2} \sum_{\beta \in I_n^k} \frac{1}{|\beta|^{2+2\theta}}.$$

Hence, as for A_2, we obtain $A_3 \vee A_4 \le C(\theta) n^{-\delta}$.
The proof of the Lemma is now complete. ■
As a consequence of Lemma 5.5, (17) and Lemma 5.8 we obtain the following result.

Corollary 5.9 *With the same assumptions as in Lemmas 5.5 and 5.8, there exists a positive constant C not depending on n, such that*

$$\int_D \|G_D(x,\cdot) - G_{D,n}^{\varepsilon(n)}(x,\cdot)\|^2_{L^{\frac{1}{H_1},\dots,\frac{1}{H_k}}(D)}\, dx \le C n^{-\gamma}, \tag{64}$$

with $\gamma = (2\mu\lambda) \wedge \delta$.

Remark 5.1

1. *Assume* $(\mathbf{H}^{**})$. *Then in the preceding Corollary, $\lambda = 1$ and $\gamma = 2\mu \wedge \delta$. The biggest upper bound for γ corresponds to the value of δ such that $\frac{2(2-\delta)}{k-2} = \delta$, which is $\delta = \frac{4}{k}$. Thus, $\gamma \in \left]0, \frac{4}{k}\right[$.*

2. *Assume* $(\mathbf{H}^{*})$. *In this case $\mu\lambda \in]0, \frac{(2-\delta)\sum_{i=1}^k H_i}{2(k-2)(k-1)}[$. As before, the highest upper bound of γ is the solution to $\frac{(2-\delta)\sum_{i=1}^k H_i}{(k-2)(k-1)} = \delta$, which is $\delta = \frac{2\sum_{i=1}^k H_i}{(k-2)(k-1)+\sum_{i=1}^k H_i}$. Therefore $\gamma \in \left]0, \frac{2\sum_{i=1}^k H_i}{(k-2)(k-1)+\sum_{i=1}^k H_i}\right[$.*

We have now the conditions to establish the convergence of the discretized scheme defined in (54) to the solution of (1) when the parameters n and ε are related by the same constraints as in Lemma 5.8. The constants $\mathcal{K}$, $\widehat{\mathcal{K}}$ in the next statement are given in (23), (60), respectively.

Theorem 5.10 *We assume the assumptions of Theorem 4.1 with a Lipschitz constant for f_2 satisfying $L < \min(a, \mathcal{K}^{-1}, 4k, \widehat{\mathcal{K}}^{-1})$. Let $k \ge 4$ and $\varepsilon(n)$ be as in Lemma 5.8. Then for any $p \in [1, \infty[$,*

$$\|u - u_n^{\varepsilon(n)}\|_{L^p(\Omega; L^2(D))} \le C n^{-\nu},$$

with $\nu \in]0, \frac{\sum_{i=1}^k H_i}{2(k-2)(k-1)+2\sum_{i=1}^k H_i}[$.
Under the stronger assumption $(\mathbf{H}^{**})$, *we obtain $\nu \in]0, \frac{1}{k}[$.*

Proof: It follows from Theorem 5.1, Lemma 5.7, Corollary 5.9 and Remark 5.1. ■

Remark 5.2 *As in [10], by applying Borel-Cantelli's lemma we can prove the following statements.*

1. *With the same assumptions as in Theorem 5.2, there exists an a.s. finite random variable* ξ *such that*

   $$\|u - u_n\|_{L^2(D)} \leq \xi n^{-\nu},$$

   a.s., with $\nu \in]0, \frac{1}{2}[$ *for* $k = 1, 2$ *and* $\nu \in]0, \frac{1}{4}[$ *for* $k = 3$.

2. *Assume the hypotheses of Theorem 5.10 and let* ν *be as in this theorem. Then, there exists an a.s. finite random variable* ξ *such that a.s.*

   $$\|u - u_n^{\varepsilon(n)}\|_{L^2(D)} \leq \xi n^{-\nu}.$$

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