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FBM-driven integrals in Banach spaces

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FBM-DRIVEN INTEGRALS IN BANACH SPACES

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ABSTRACT. The L^p-theory of stochastic integration with respect to fractional Brownian motion is studied.

1. Introduction

The aim of this preprint is to study stochastic integrals of deterministic processes with respect to fractional Brownian motion with the Hurst parameter $H \in (0,1) \setminus \frac{1}{2}$ in L^p-spaces for $1 \le p < \infty$. This is a preliminary version of the first part of a forthcoming work aimed at developing a suitable framework for stochastic fractional convolution integrals in L^p-space. The results will be applied to the problem of existence and regularity of solutions to stochastic evolution equations driven by a fractional Brownian motion.

2. Notation and hypotheses

We assume that

- U is a real separable Hilbert space,
- B is a cylindrical FBM on U with the Hurst parameter H, i.e. $\{B_t(u) : t \ge 0, u \in U\}$ is a Gaussian family of centered random variables such that
$$B_t(\alpha u + v) = \alpha B_t(u) + B_t(v) \qquad \text{almost surely}$$
for every $t \ge 0$, $\alpha \in \mathbb{R}$ and $u, v \in U$, and
$$\mathbb{E}\{B_t(u)B_s(v)\} = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})\langle u, v\rangle_U, \qquad u, v \in U, \quad t, s \ge 0.$$
where the sum converges in $L^2(\Omega)$ and $\beta^1, \beta^2, \ldots$ are inependent FBM with Hurst parameter H and (e_k) is an orthonormal basis in U,
- (D, μ) is a separable σ-finite measure space (see Remark 2.1),
- $\mathscr{L}(U, X)$ denotes the space of linear continuous operators from U to X,
- we also define
$$K(t,s) = c_H \left\{ (t-s)^{H-\frac{1}{2}} + \left(\frac{1}{2} - H\right) \int_s^t (u-s)^{H-\frac{3}{2}} \left[1 - \left(\frac{u}{s}\right)^{H-\frac{1}{2}} \right] du \right\}$$

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for $0 < s < t$ and

$$C_H = \left[\frac{2H\Gamma\left(H + \frac{1}{2}\right)\Gamma\left(\frac{3}{2} - H\right)}{\Gamma(2-2H)} \right]^{1/2},$$

$$\phi(t,s) = H(2H-1)|t-s|^{2H-2}$$

and if $T > 0$ is given then

$$(\mathscr{K}_T^* \varphi)(s) = \varphi(s)K(T,s) + \int_s^T [\varphi(t) - \varphi(s)] \frac{\partial K}{\partial t}(t,s)\,dt, \qquad 0 < s < T$$

for every measurable φ such that the integral converges almost everywhere, e.g. for $\varphi \in W^{\theta,2}(0,T)$, $\theta > 1/2 - H$.

Remark 2.1. Separability of a measure space (D,μ) means that there exists a countable system (V_n) of measurable sets satisfying $\mu(V_n) < \infty$ such that, for every $\varepsilon > 0$ and every measurable set C satisfying $\mu(C) < \infty$, there is $n \in \mathbb{N}$ such that $\mu[(C \setminus V_n) \cup (V_n \setminus C)] < \varepsilon$. The following conditions are equivalent:

- the measure space (D,μ) is separable,
- there exists $1 \le p < \infty$ such that $L^p(D,\mu)$ is a separable Banach space,
- $L^p(D,\mu)$ is a separable Banach space for every $1 \le p < \infty$.

3. γ-Radonifying operators

It is known that if B is a cylindrical fractional Brownian motion on a Hilbert space U then the processes $(\varphi(t))_{t\ge 0}$ admissible for stochastic integration in a Hilbert space X must have values in the space of the Hilbert-Schmidt operators. In the scale of L^p-spaces, this holds only if $p = 2$. It turns out that if the target space is a Banach space, then the integrand φ must take values in the space of γ-radonifying operators.

For, let X be a separable Banach space. An operator $A \in \mathscr{L}(U,X)$ is γ-radonifying provided that there exists a centered Gaussian probability measure ν_A on X such that

$$\int_X \varphi^2(x)\,\nu_A(dx) = \|A^*\varphi\|_U^2, \qquad \varphi \in X^*.$$

Such a measure is at most one, therefore we set

$$\|A\|_\gamma^2 = \int_X \|x\|_X^2\,\nu_A(dx),$$

and denote by $R_\gamma(U,X)$ the space of γ-radonifying operators.

Proposition 3.1. *$R_\gamma(U,X)$ equipped with the norm $\|\cdot\|_\gamma$ is a separable Banach space.*

Proof. See [3] or [4]. □

Proposition 3.2. *Let $A : U \to X$ be a linear continuous operator, (e_k) an orthonormal series in U and (ε_k) independent standard centered Gaussian random variables. Denote*

$$S_n = \sum_{k=1}^{n} \varepsilon_k A e_k.$$

The following claims are equivalent:

- *A is γ-radonofying,*
- *the sequence (S_n) is convergent almost surely,*
- *the sequence (S_n) is convergent in probability,*
- *the sequence (S_n) is convergent in every $L^q(X)$, $1 \le q < \infty$.*

Proof. This is consequence of Theorem 2.3, Chapter V.2.4 and Theorem 5.3 in Chapter V.5.3 in [2]. The proof may be alternatively infered from the Ito-Nisio theorem and the Fernique theorem. □

Definition 3.3. *If A is γ-radonifying then we define the X-valued centered Gaussian random variable*

$$AB_t = \sum B_t(e_k) A e_k \tag{3.1}$$

where the sum is convergent almost surely and in every $L^q(X)$, $1 \le q < \infty$.

Remark 3.4. Notice that the definition (3.1) is independent of the choice of the orthonormal basis (e_k) since $\langle x^*, AB_t \rangle = B_t(A^* x^*)$ almost surely.

The γ-radonifying operators with range in L^p-spaces can be elegantly represented by point-wise kernels. We are going to use this property later on.

Definition 3.5. *We say that a measurable function $\varphi : [0,T] \to R_\gamma(U, L^p)$ is represented by a jointly-measurable mapping $J : [0,T] \times D \to U$ provided that there exists a measurable set $V \in [0,T]$ such that the Lebesgue measure of the set $[0,T] \setminus V$ is zero and $[\varphi(t)u](x) = \langle J(t,x), u \rangle_U$ holds for μ-almost every $x \in D$, for every $u \in U$ and for every $t \in V$.*

Proposition 3.6. *Let $1 \le p < \infty$, let $X = L^p(D, \mu)$ and let $A \in \mathscr{L}(U, X)$. Then $A \in R_\gamma(U, X)$ if and only if there exists a measurable function $a : D \to U$ satisfying*

$$\int_D \|a(x)\|_U^p \, \mu(dx) < \infty$$

and $(Au)(x) = \langle a(x), u \rangle_U$ holds μ-almost everywhere for every $u \in U$. There also exists a constant $c > 1$ independent of A such that

$$\frac{1}{c} \|A\|_\gamma \le \left[\int_D \|a(x)\|_U^p \, \mu(dx) \right]^{\frac{1}{p}} \le c \|A\|_\gamma.$$

Proof. See [1]. □

Corollary 3.7. *If $\varphi : [0,T] \to R_\gamma(U, L^p)$ is measurable then it is representable.*

See Corollary 1, Chapter V.5.5 in [2] for the proof of the following proposition:

Proposition 3.8. *Let $1 \le p < \infty$ then there exists a constant $1 < c_p < \infty$ such that*

$$\left(\int_X \|x\|_X^p \, \nu(dx)\right)^{1/p} \le c_p \int_X \|x\|_X \, \nu(dx)$$

holds for every centered Gaussian probability measure ν on X.

4. fBm-Preliminaries

Let Z be a separalble Hilbert space, φ a piecewise constant function in Z and β a standard fractional Brownian motion. Then the Stieltjes integral satisfies

$$\mathbb{E}\left\|\int_0^T \varphi \, d\beta\right\|_Z^2 = \int_0^T \|\mathscr{K}_T^* \varphi\|_Z^2 \, ds, \qquad H < \frac{1}{2} \tag{4.1}$$

$$\mathbb{E}\left\|\int_0^T \varphi \, d\beta\right\|_Z^2 = \int_0^T \int_0^T \langle \varphi(s), \varphi(t)\rangle_Z \phi(s,t) \, ds \, dt, \qquad H > \frac{1}{2}. \tag{4.2}$$

Remark 4.1. The Hardy-Littlewood-Sobolev inequality yields a constant c_H such that

$$\int_0^T \int_0^T |a(t)||a(s)|\phi(t,s) \, dt \, ds \le c_H \left[\int_0^T |a(t)|^{\frac{1}{H}} \, dt\right]^{2H}$$

holds for every measurable function a.

5. Sobolev spaces Preliminaries

In this Section, X is assumed to be an arbitrary Banach space.

Definition 5.1. *If $0 < \theta < 1$ then we introduce the space $W^{\theta,2}(0,T;X)$ of measurable functions satisfying*

$$\|\varphi\|_{W^{\theta,2}(0,T;X)}^2 = \|\varphi\|_{L^2(0,T;X)}^2 + \int_0^T \int_0^T \frac{|\varphi(t) - \varphi(s)|^2}{|t-s|^{1+2\theta}} \, ds \, dt < \infty.$$

Lemma 5.2. *Let $\theta < \frac{1}{2}$, $T > 0$ and $0 = t_0^n < \cdots < t_{N_n}^n = T$ a sequence of finite partitions. Define*

$$\varphi_n(t) = \sum_{i=0}^{N_n - 1} I_{[t_i^n, t_{i+i}^n)}(t) \frac{1}{t_{i+1}^n - t_i^n} \int_{t_i^n}^{t_{i+1}^n} \varphi(s) \, ds$$

for $\varphi \in W^{\theta,2}(0,T;X)$. If

$$\sup_n \frac{\Delta_n}{\delta_n} < \infty, \qquad \lim_{n\to\infty} \Delta_n = 0$$

where

$$\Delta_n = \max\{t_{i+1}^n - t_i^n : i < N_n\}, \qquad \delta_n = \min\{t_{i+1}^n - t_i^n : i < N_n\}$$

then there exists C independent of φ such that

$$\|\varphi_n\|_{W^{\theta,2}(0,T;X)} \le C \qquad \text{and} \qquad \lim_{n\to\infty} \|\varphi - \varphi_n\|_{W^{\theta,2}(0,T;X)} = 0.$$

Proof. The proof is straightforward. Notice that

$$\begin{aligned} |\varphi_n(t_i^n) - \varphi(s)| &\le \frac{1}{t_{i+1}^n - t_i^n} \int_{t_i^n}^{t_{i+1}^n} |\varphi(r) - \varphi(s)|\, dr \\ &\le c(|t_i^n - s|^\theta + |t_{i+1}^n - s|^\theta) \left(\int_{t_i^n}^{t_{i+1}^n} \frac{|\varphi(r) - \varphi(s)|^2}{|r-s|^{1+2\theta}}\, dr \right)^{\frac{1}{2}} \end{aligned}$$

by the Hölder inequality. We infer from this

$$|\varphi_n(t_n^i) - \varphi_n(t_{i+1}^n)| \le c\Delta_n^{\theta - \frac{1}{2}} \left(\int_{t_{i+1}^n}^{t_{i+2}^n} \int_{t_i^n}^{t_{i+1}^n} \frac{|\varphi(r) - \varphi(u)|^2}{|r-u|^{1+2\theta}}\, dr\, du \right)^{\frac{1}{2}}.$$

Now

$$\begin{aligned} \|\varphi - \varphi_n\|_{\theta,2}^2 &= \sum_i \int_{t_i^n}^{t_{i+1}^n} |\varphi(t) - \varphi_n(t_i^n)|^2\, dt \\ &+ \sum_i \sum_j \int_{t_i^n}^{t_{i+1}^n} \int_{t_j^n}^{t_{j+1}^n} \frac{|\varphi(t) - \varphi_n(t) - \varphi(s) + \varphi_n(s)|^2}{|t-s|^{1+2\theta}}\, ds\, dt \\ &\le c \sum_{i \le j \le i+1} \int_{t_i^n}^{t_{i+1}^n} \int_{t_j^n}^{t_{j+1}^n} \frac{|\varphi(t) - \varphi(s)|^2}{|t-s|^{1+2\theta}}\, ds\, dt \\ &+ c \sum_{j-i=1} \int_{t_i^n}^{t_{i+1}^n} \int_{t_j^n}^{t_{j+1}^n} \frac{|\varphi_n(t) - \varphi_n(s)|^2}{|t-s|^{1+2\theta}}\, ds\, dt \\ &+ c \sum_{i+2 \le j} \int_{t_i^n}^{t_{i+1}^n} \int_{t_j^n}^{t_{j+1}^n} \frac{|\varphi(t) - \varphi_n(t)|^2}{|t-s|^{1+2\theta}}\, ds\, dt \\ &+ c \sum_{i+2 \le j} \int_{t_i^n}^{t_{i+1}^n} \int_{t_j^n}^{t_{j+1}^n} \frac{|\varphi(s) - \varphi_n(s)|^2}{|t-s|^{1+2\theta}}\, ds\, dt \\ &\le C \sum_{i \le j \le i+1} \int_{t_i^n}^{t_{i+1}^n} \int_{t_j^n}^{t_{j+1}^n} \frac{|\varphi(t) - \varphi(s)|^2}{|t-s|^{1+2\theta}}\, ds\, dt. \end{aligned}$$

□

The following Sobolev embedding theorem is well known and can be found e.g. in [5].

Lemma 5.3. *Let $\theta < \frac{1}{2}$, $T > 0$ and $X = \mathbb{R}$. Then there exists a constant C such that*

$$\|\varphi\|_{L^{\frac{2}{1-2\theta}}(0,T)} \le C\|\varphi\|_{W^{\theta,2}(0,T)}$$

holds for every $\varphi \in W^{\theta,2}(0,T)$.

Lemma 5.4. *Let $H < \frac{1}{2}$, $\theta > \frac{1}{2} - H$, $T > 0$ and let X be a Hilbert space. Then $\mathscr{K}_T^*\varphi$ is well-defined for every $\varphi \in W^{\theta,2}(0,T;X)$ and there exists a constant c such that*

$$\|\mathscr{K}_T^*\varphi\|_{L^2(0,T;X)} \leq c\|\varphi\|_{W^{\theta,2}(0,T;X)}$$

holds for every $\varphi \in W^{\theta,2}(0,T;X)$.

Proof. Set $\alpha = \frac{1}{2} - H$ and realize that

$$K(t,s) \leq C\left[s^{-\alpha} + (t-s)^{-\alpha}\right], \qquad \left|\frac{\partial K}{\partial t}(t,s)\right| \leq C(t-s)^{-\alpha-1}.$$

Hence

$$\int_0^T \left[\int_s^T \frac{|\varphi(t)-\varphi(s)|}{|t-s|^{1+\alpha}}\,dt\right]^2 ds \leq c\int_0^T\int_0^T \frac{|\varphi(t)-\varphi(s)|^2}{|t-s|^{1+2\theta}}\,dt\,ds,$$

and

$$\begin{aligned}\int_0^T [s^{-2\alpha} + (T-s)^{-2\alpha}]|\varphi(s)|^2\,ds &= \sum_i \int_0^T [s^{-2\alpha} + (T-s)^{-2\alpha}]\langle\varphi(s), e_i\rangle_X^2\,ds \\ &\leq c\sum_i \|\langle\varphi, e_i\rangle_X\|^2_{L^{\frac{2}{1-2\theta}}(0,T)}\,ds \\ &\leq C\sum_i \|\langle\varphi, e_i\rangle_X\|^2_{W^{\theta,2}(0,T)}\,ds \\ &= C\|\varphi\|^2_{W^{\theta,2}(0,T;X)}\end{aligned}$$

where (e_i) is an orthonormal bases of the range of φ. □

6. Stochastic integral

Definition 6.1. *Let φ be piecewise constant function with values in $R_\gamma(U,X)$ and let B be a cylindrical fBm. Then*

$$t \mapsto \int_0^t \varphi\,dB$$

is defined pathwise as the indefinite Stieltjes integral.

Let $L^p = L^p(D,\mu)$, $1 \leq p < \infty$, $T > 0$ and let $\varphi : [0,T] \to R_\gamma(U,L^p)$ be a step-function, i.e. there exists a partition $0 = t_0 < t_1 < \cdots < t_k = T$,

$$[\varphi(t)u](x) = \langle J(t,x), u\rangle_U \tag{6.1}$$

holds for μ-almost every $x \in D$ and every $u \in U$, $0 \leq t \leq T$, and $J(t,\cdot) \in L^p(D,\mu;U)$ is constant on every $[t_i, t_{i+1})$, $i \in \{0, k-1\}$.

We are going to use the symbol $\sim$ for difference by a constant.

Lemma 6.2. *Let $1 \leq p < \infty$ and let φ be simple. If $1 \leq p < \frac{1}{H} < 2$ then suppose that*

$$\int_0^T\int_0^T \|J(t,x)\|_U\|J(s,x)\|_U\phi(s,t)\,ds\,dt < \infty \tag{6.2}$$

for μ-almost every $x \in D$. Then

$$\left\{\mathbb{E}\left\|\int_0^T \varphi\, dB\right\|_{L^p}^2\right\}^{\frac{1}{2}} \sim \left\{\int_D \left[\int_0^T \left\|\mathscr{K}_T^*\left[J(\cdot,x)\right](s)\right\|_U^2\, ds\right]^{\frac{p}{2}} \mu(dx)\right\}^{\frac{1}{p}} \tag{6.3}$$

if $H < \frac{1}{2}$, and

$$\left\{\mathbb{E}\left\|\int_0^T \varphi\, dB\right\|_{L^p}^2\right\}^{\frac{1}{2}} \sim \left\{\int_D \left[\int_0^T \int_0^T \langle J(t,x), J(s,x)\rangle_U \phi(s,t)\, ds\, dt\right]^{\frac{p}{2}} \mu(dx)\right\}^{\frac{1}{p}} \tag{6.4}$$

if $H > \frac{1}{2}$.

Proof. For,

$$\left\{\mathbb{E}\left\|\int_0^T \varphi\, dB\right\|_{L^p}^2\right\}^{\frac{1}{2}} = \left\{\mathbb{E}\left\|\sum_{n=1}^k \varphi(t_k)B_{t_{k+1}} - \varphi(t_k)B_{t_k}\right\|_{L^p}^2\right\}^{\frac{1}{2}}$$

$$= \lim_{m\to\infty}\left\{\mathbb{E}\left\|\sum_{i=1}^m\sum_{n=1}^k \varphi(t_k)e_i[B_{t_{k+1}}(e_i) - B_{t_k}(e_i)]\right\|_{L^p}^2\right\}^{\frac{1}{2}}$$

$$\sim \lim_{m\to\infty}\left\{\mathbb{E}\left\|\sum_{i=1}^m\sum_{n=1}^k \varphi(t_k)e_i[B_{t_{k+1}}(e_i) - B_{t_k}(e_i)]\right\|_{L^p}^p\right\}^{\frac{1}{p}}$$

$$= \lim_{m\to\infty}\left\{\mathbb{E}\int_D\left|\sum_{i=1}^m\sum_{n=1}^k \langle J(t_k,x), e_i\rangle_U[B_{t_{k+1}}(e_i) - B_{t_k}(e_i)]\right|^p \mu(dx)\right\}^{\frac{1}{p}}$$

$$\sim \lim_{m\to\infty}\left\{\int_D\left[\mathbb{E}\left|\sum_{i=1}^m\sum_{n=1}^k \langle J(t_k,x), e_i\rangle_U[B_{t_{k+1}}(e_i) - B_{t_k}(e_i)]\right|^2\right]^{\frac{p}{2}} \mu(dx)\right\}^{\frac{1}{p}}$$

$$= \lim_{m\to\infty}\left\{\int_D\left[\sum_{i=1}^m\mathbb{E}\left|\sum_{n=1}^k \langle J(t_k,x), e_i\rangle_U[B_{t_{k+1}}(e_i) - B_{t_k}(e_i)]\right|^2\right]^{\frac{p}{2}} \mu(dx)\right\}^{\frac{1}{p}}$$

$$= \lim_{m\to\infty}\left\{\int_D\left[\sum_{i=1}^m\mathbb{E}\left|\int_0^T \langle J(t,x), e_i\rangle_U\, dB_t(e_i)\right|^2\right]^{\frac{p}{2}} \mu(dx)\right\}^{\frac{1}{p}}$$

$$= \left\{\int_D\left[\sum_{i=1}^\infty\mathbb{E}\left|\int_0^T \langle J(t,x), e_i\rangle_U\, dB_t(e_i)\right|^2\right]^{\frac{p}{2}} \mu(dx)\right\}^{\frac{1}{p}}.$$

Now we first consider the case $H < \frac{1}{2}$. Then the chain continues as

$$= \left\{ \int_D \left[\sum_{i=1}^{\infty} \int_0^T \left| \mathscr{K}_T^* \left[\langle J(\cdot, x), e_i \rangle_U \right](s) \right|^2 \, ds \right]^{\frac{p}{2}} \mu(dx) \right\}^{\frac{1}{p}}$$

$$= \left\{ \int_D \left[\int_0^T \left\| \mathscr{K}_T^* \left[J(\cdot, x) \right](s) \right\|_U^2 \, ds \right]^{\frac{p}{2}} \mu(dx) \right\}^{\frac{1}{p}},$$

and if $H > \frac{1}{2}$ then

$$= \left\{ \int_D \left[\sum_{i=1}^{\infty} \int_0^T \int_0^T \langle J(t,x), e_i \rangle_U \langle J(s,x), e_i \rangle_U \phi(s,t) \, ds \, dt \right]^{\frac{p}{2}} \mu(dx) \right\}^{\frac{1}{p}}$$

$$= \left\{ \int_D \left[\int_0^T \int_0^T \langle J(t,x), J(s,x) \rangle_U \phi(s,t) \, ds \, dt \right]^{\frac{p}{2}} \mu(dx) \right\}^{\frac{1}{p}}.$$

We can exchange the sum and the integrals since

$$\int_0^T \int_0^T \|J(t,x)\|_U \|J(s,x)\|_U \phi(s,t) \, ds \, dt \leq C_H \left(\int_0^T \|J(t,x)\|_U^{\frac{1}{H}} \, dt \right)^{2H}$$

by the Hardy-Littlewood-Sobolev inequality, and

$$\begin{aligned} \int_D \left(\int_0^T \|J(t,x)\|_U^{\frac{1}{H}} \, dt \right)^{pH} \mu(dx) &\leq K_{T,p,H} \int_0^T \int_D \|J(t,x)\|_U^p \, \mu(dx) \, dt \\ &\leq k_{T,p,H} \int_0^T \|\varphi(t)\|_\gamma^p \, dt < \infty \end{aligned}$$

provided $pH \geq 1$. If $pH < 1$ then (6.2) must be assumed. □

Theorem 6.3. *Let $H < \frac{1}{2}$, $T > 0$, $\theta > \frac{1}{2} - H$, $1 \leq q < \infty$ and let $\varphi : [0,T] \to R_\gamma(U, L^p)$ be represented by a jointly-measurable mapping $J : [0,T] \times D \to U$ such that*

$$\|J\|_{L^p(D; W^{\theta,2}(0,T;U))} < \infty.$$

Then there exists a sequence $\varphi^n : [0,T] \to R_\gamma(U, L^p)$ of step-functions represented by kernels J^n such that $\|J - J^n\|_H \to 0$. In particular, the integral $\int_0^T \varphi \, dB$ is well-defined, (6.3) holds and

$$\left(\mathbb{E} \left\| \int_0^T \varphi \, dB \right\|_{L^p}^q \right)^{\frac{1}{q}} \leq C \|J\|_{L^p(D; W^{\theta,2}(0,T;U))}.$$

Remark 6.4. Notice that

$$\|J\|_{L^p(D; W^{\theta,2}(0,T;U))} \leq c \|\varphi\|_{W^{\theta,2}(0,T; R_\gamma(U, L^p))}$$

by the generalized Minkowski inequality, provided that $p \geq 2$.

Theorem 6.5. *Let $T > 0$, $H > \frac{1}{2}$, $1 \le q < \infty$ and let $\varphi : [0,T] \to R_\gamma(U, L^p)$ be a measurable function represented by a jointly-measurable mapping $J : [0,T] \times D \to U$ such that*

$$\|J\|_{L^p(D;L^{\frac{1}{H}}(0,T;U))} < \infty. \tag{6.5}$$

Then there exists a sequence $\varphi^n : [0,T] \to R_\gamma(U, L^p)$ of step-functions represented by kernels J^n such that $\|J - J^n\|_H \to 0$. In particular, the integral $\int_0^T \varphi \, dB$ is well-defined, (6.4) holds and

$$\left(\mathbb{E} \left\| \int_0^T \varphi \, dB \right\|_{L^p}^q \right)^{\frac{1}{q}} \le C \|J\|_{L^p(D;L^{\frac{1}{H}}(0,T;U))}.$$

Proof. Let $K_n \uparrow D$ satisfy $\mu(K_n) < \infty$ and set

$$J^n(t,x) = I_{K_n}(x) J(t,x), \qquad J^{n,m}(t,x) = I_{K_n}(x) J(t,x) I_{[\|J(t,x)\|_U \le m]},$$

$$J^{n,m,k}(t,x) = k \int_{(t-\frac{1}{k})_+}^t J^{n,m}(s,x) \, ds, \qquad J^{n,m,k,l}(t,x) = \sum_{i=0}^{l-1} I_{[\frac{iT}{l}, \frac{(i+1)T}{l})}(t) J^{n,m,k}\left(\frac{iT}{l}, x \right).$$

We have $\lim_{l\to\infty} \|J^{n,m,k,l} - J^{n,m,k}\|_H = 0$ for every $n, m, k \in \mathbb{N}$, $\lim_{k\to\infty} \|J^{n,m,k} - J^{n,m}\|_H = 0$ for every $n, m \in \mathbb{N}$, $\lim_{m\to\infty} \|J^{n,m} - J^n\|_H = 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|J^n - J\|_H = 0$. $\square$

Remark 6.6. Notice that

$$\|J\|_{L^p(D;L^{\frac{1}{H}}(0,T;U))} \le c \|\varphi\|_{L^{\frac{1}{H}}(0,T;R_\gamma(U,L^p))}$$

by the generalized Minkowski inequality, provided that $pH \ge 1$.

7. Integration in $W^{s,p}(D)$

Let D be a domain in $\mathbb{R}^d$, let $0 < s < 1$ and recall that

$$\|f\|_{W^{s,p}(D)}^p = \|f\|_{L^p(D)}^p + \int_D \int_D \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} \, dx \, dy.$$

Lemma 7.1. *Let $1 \le p < \infty$, $0 < s < 1$ and let φ be simple. If $1 \le p < \frac{1}{H} < 2$ then suppose that*

$$\int_0^T \int_0^T \|J(t,x)\|_U \|J(s,x)\|_U \phi(s,t) \, ds \, dt < \infty$$

and

$$\int_D \int_D \frac{\left[\int_0^T \int_0^T \|J(t,x) - J(t,y)\|_U \|J(t,x) - J(t,y)\|_U \phi(s,t) \, dt \, ds \right]^{\frac{p}{2}}}{|x-y|^{d+sp}} \, dx \, dy < \infty$$

for μ-almost every $x \in D$. Then

$$\left\{\mathbb{E}\left\|\int_0^T \varphi\, dB\right\|^2_{W^{s,p}(D)}\right\}^{\frac{1}{2}} \sim \left\{\int_D \left[\int_0^T \|\mathscr{K}_T^*\left[J(\cdot,x)\right](s)\|_U^2\, ds\right]^{\frac{p}{2}} dx\right\}^{\frac{1}{p}} + \left\{\int_D\int_D \frac{\left[\int_0^T \|\mathscr{K}_T^*\left[J(\cdot,x)-J(\cdot,y)\right](s)\|_U^2\, ds\right]^{\frac{p}{2}}}{|x-y|^{d+sp}}\, dx\, dy\right\}^{\frac{1}{p}}$$

if $H < \frac{1}{2}$, and

$$\left\{\mathbb{E}\left\|\int_0^T \varphi\, dB\right\|^2_{L^p}\right\}^{\frac{1}{2}} \sim \left\{\int_D \left[\int_0^T\int_0^T \langle J(t,x), J(s,x)\rangle_U \phi(s,t)\, ds\, dt\right]^{\frac{p}{2}} dx\right\}^{\frac{1}{p}} + \left\{\int_D\int_D \frac{\left[\int_0^T\int_0^T \langle J(t,x)-J(t,y), J(s,x)-J(s,y)\rangle_U \phi(s,t)\, ds\, dt\right]^{\frac{p}{2}}}{|x-y|^{d+sp}}\, dx\, dy\right\}^{\frac{1}{p}}$$

if $H > \frac{1}{2}$.

Now we have the following analogues of Theorems 6.3 and 6.5:

Theorem 7.2. *Let $H < \frac{1}{2}$, $T > 0$, $0 < s < 1$, $\theta > \frac{1}{2} - H$, $1 \le q < \infty$ and let $\varphi : [0,T] \to R_\gamma(U, W^{s,p}(D))$ be represented by a jointly-measurable mapping $J : [0,T]\times D \to U$ such that*

$$\|J\|_{W^{s,p}(D;W^{\theta,2}(0,T;U))} < \infty.$$

Then there exists a sequence $\varphi^n : [0,T] \to R_\gamma(U, W^{s,p}(D))$ of step-functions represented by kernels J^n such that $\|J - J^n\|_H \to 0$. In particular, the integral $\int_0^T \varphi\, dB$ is well-defined and

$$\left(\mathbb{E}\left\|\int_0^T \varphi\, dB\right\|^q_{W^{s,p}(D)}\right)^{\frac{1}{q}} \le C\|J\|_{W^{s,p}(D;W^{\theta,2}(0,T;U))}.$$

Remark 7.3. Notice that

$$\|J\|_{W^{s,p}(D;W^{\theta,2}(0,T;U))} \le c\|\varphi\|_{W^{\theta,2}(0,T;R_\gamma(U,W^{s,p}(D)))}$$

by the generalized Minkowski inequality, provided that $p \ge 2$.

Theorem 7.4. *Let $T > 0$, $H > \frac{1}{2}$, $0 < s < 1$, $1 \le q < \infty$ and let $\varphi : [0,T] \to R_\gamma(U, W^{s,p}(D))$ be a measurable function represented by a jointly-measurable mapping $J : [0,T]\times D \to U$ such that*

$$\|J\|_{W^{s,p}(D;L^{\frac{1}{H}}(0,T;U))} < \infty.$$

Then there exists a sequence $\varphi^n : [0,T] \to R_\gamma(U, W^{s,p}(D))$ of step-functions represented by kernels J^n such that $\|J - J^n\|_H \to 0$. In particular, the integral $\int_0^T \varphi \, dB$ is well-defined and

$$\left(\mathbb{E} \left\| \int_0^T \varphi \, dB \right\|^q_{W^{s,p}(D)} \right)^{\frac{1}{q}} \leq C \|J\|_{W^{s,p}(D; L^{\frac{1}{H}}(0,T;U))}.$$

Remark 7.5. Notice that

$$\|J\|_{W^{s,p}(D; L^{\frac{1}{H}}(0,T;U))} \leq c \|\varphi\|_{L^{\frac{1}{H}}(0,T; R_\gamma(U, W^{s,p}(D)))}$$

by the generalized Minkowski inequality, provided that $pH \geq 1$.

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