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**An Improper Arithmetically Closed Borel
Subalgebra of $P(\omega) \bmod \text{FIN}$**

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AN IMPROPER ARITHMETICALLY CLOSED BOREL SUBALGEBRA OF $\mathcal{P}(\omega)$ MOD FIN

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ABSTRACT. We show the existence of a subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ that satisfies the following three conditions.

- \mathcal{A} is Borel (when $\mathcal{P}(\omega)$ is identified with 2^ω).
- \mathcal{A} is arithmetically closed (i.e., \mathcal{A} is closed under the Turing jump, and Turing reducibility).
- The forcing notion (\mathcal{A}, \subseteq) modulo the ideal FIN of finite sets collapses the continuum to \aleph_0 .

1. INTRODUCTION

For a subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ let $\mathbb{P}_{\mathcal{A}}$ be the partial order obtained by reducing (\mathcal{A}, \subseteq) modulo the ideal FIN of finite sets. Gitman [G-1] made an advance towards the *Scott set problem*¹ by showing that, assuming the proper forcing axiom (PFA), if \mathcal{A} is arithmetically closed² and $\mathbb{P}_{\mathcal{A}}$ is a proper notion of forcing, then there is a model of Peano arithmetic whose standard system is \mathcal{A} . Gitman [G-2] also investigated proper posets of the form $\mathbb{P}_{\mathcal{A}}$ and showed that the existence of proper uncountable arithmetically closed algebras $\mathcal{A} \neq \mathcal{P}(\omega)$ is consistent with ZFC+PFA. These results naturally motivate the question whether there is an arithmetically closed \mathcal{A} for which $\mathbb{P}_{\mathcal{A}}$ is not proper. This question was answered in the affirmative by Enayat [E, Theorem D], using a highly nonconstructive reasoning that establishes the existence of an arithmetically closed \mathcal{A} of power \aleph_1 such that $\mathbb{P}_{\mathcal{A}}$ collapses \aleph_1 (and is therefore not proper). The nonconstructive feature of the proof prompted Question II(b) of [E], which asked whether $\mathbb{P}_{\mathcal{A}}$ is a proper poset if \mathcal{A} is both arithmetically closed and *Borel* (when $\mathcal{P}(\omega)$ is identified with the Cantor set).

The main result of this paper³, Theorem A below, provides a strong negative answer to the above question.

Theorem A. *There is an arithmetically closed Borel subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathbb{P}_{\mathcal{A}}$ is equivalent to $\text{LEVY}(\aleph_0, 2^{\aleph_0})$.*

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¹The Scott set problem [KS, Question 1] asks whether every Scott set \mathcal{A} can be realized as the standard system of a model of Peano arithmetic (a subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is a Scott set if \mathcal{A} is closed Turing reducibility, and every infinite subtree of ${}^{<\omega}2$ that is coded in \mathcal{A} has an infinite branch that is coded in \mathcal{A}). It is known that the answer to the Scott set problem is positive when $|\mathcal{A}| \leq \aleph_1$, and when $\mathcal{A} = \mathcal{P}(\omega)$.

²A subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is arithmetically closed if \mathcal{A} is closed under (1) Turing jump and (2) Turing reducibility. Note that if \mathcal{A} is arithmetically closed, then \mathcal{A} is a Scott set, but not vice versa.

³This is [En-Sh 936] in Shelah's master list of publications.

Theorem A is established in Section 3 using a rich toolkit from set theory and model theory. For this reason, Section 2 is devoted to the description of the machinery employed in the proof of Theorem A.

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2. PRELIMINARIES

2.1. FORCING

Given an infinite cardinal κ , $\text{LEVY}(\aleph_0, \kappa)$ is the usual partial order that collapses κ to \aleph_0 , i.e., $\text{LEVY}(\aleph_0, \kappa) = (\leq^\omega \kappa, \subseteq)$. The following result provides a structural characterization of $\text{LEVY}(\aleph_0, \kappa)$.⁴

2.1.1. Theorem (McAloon [K, Theorem 14.17]). *The following conditions are equivalent for a partial order \mathbb{P} of infinite cardinality κ .*

- (a) \mathbb{P} is equivalent⁵ to $\text{LEVY}(\aleph_0, \kappa)$.
- (b) \mathbb{P} is (\aleph_0, κ) nowhere distributive, i.e., there is a family $\{I_n : n \in \omega\}$ of maximal antichains of \mathbb{P} such that for every $p \in \mathbb{P}$, there is some $n < \omega$ such that there are κ elements of I_n that are compatible with p .

2.1.2. Corollary [J, Lemma 26.7]. *The following conditions are equivalent for a partial order \mathbb{P} of cardinality $\kappa \geq \aleph_0$.*

- (a) \mathbb{P} is equivalent to $\text{LEVY}(\aleph_0, \kappa)$.
- (b) $\mathbb{V}^{\mathbb{P}} \models$ “there is a surjective function $f : \omega \rightarrow \kappa$ ”.

The next result shows that one can use standard techniques to build a subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathbb{P}_{\mathcal{A}}$ is not proper.

2.1.3. Proposition⁶. *There is a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ of cardinality \aleph_1 such that $\mathbb{P}_{\mathcal{A}}$ is equivalent to $\text{LEVY}(\aleph_0, \aleph_1)$.*

Proof: By a theorem of Parovičenko [Pa] (see also [K, 5.28 and 5.29]) every Boolean algebra of cardinality $\leq \aleph_1$ can be embedded into $\mathcal{P}(\omega) \text{ mod FIN}$. On the other hand $\text{LEVY}(\aleph_0, \aleph_1)$ can be densely embedded into a Boolean algebra of power \aleph_1 since each s in $\text{LEVY}(\aleph_0, \aleph_1)$ determines a basic clopen X_s set in ${}^\omega \omega_1$, and the Boolean algebra \mathbb{B} of clopen sets generated by the family $\{X_s : s \in \text{LEVY}(\aleph_0, \aleph_1)\}$ is of size \aleph_1 . So by Parovičenko’s theorem there is an embedding f of \mathbb{B} into $\mathcal{P}(\omega) \text{ mod FIN}$. Let $\mathcal{A} := \{X \subseteq \omega : [X] \text{ is in the range of } f\}$. Since $\mathbb{P}_{\mathcal{A}}$ is isomorphic to \mathbb{B} , and \mathbb{B} is equivalent to $\text{LEVY}(\aleph_0, \aleph_1)$, $\mathbb{P}_{\mathcal{A}}$ collapses \aleph_1 . Therefore, by Corollary 2.1.2, $\mathbb{P}_{\mathcal{A}}$ is equivalent to $\text{LEVY}(\aleph_0, \aleph_1)$. \square

⁴See [BS, Cor.1.15] for a characterization of other collapsing algebras.

⁵For separative partial orders \mathbb{P}_1 and \mathbb{P}_2 , \mathbb{P}_1 is equivalent to \mathbb{P}_2 if $B(\mathbb{P}_1) \cong B(\mathbb{P}_2)$, where $B(\mathbb{P})$ is the complete Boolean algebra consisting of regular cuts subsets of \mathbb{P} [J, Thm 14.10].

⁶Thanks to K.P. Hart and Ken Kunen for (independently) drawing our attention to this consequence of Parovičenko’s theorem.

2.1.4. Remark⁷. Zapletal [Z, Lemma 2.3.1] used Woodin's Σ_1^2 -absoluteness theorem [L, Thm 3.2.1] to show that in the presence of the continuum hypothesis and large cardinals (more precisely: a measurable Woodin cardinal), a projective partial order \mathbb{P} preserves \aleph_1 iff \mathbb{P} is proper. Note that if \mathcal{A} is Borel, then $\mathbb{P}_{\mathcal{A}}$ is projective.

2.2. INFINITE COMBINATORICS

2.2.1. Definition. Suppose $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

(a) \mathcal{A} is *almost disjoint* (AD) if the intersection of any two distinct members of \mathcal{A} is finite.

(b) An AD family \mathcal{A} is *maximal almost disjoint* (MAD) if \mathcal{A} has no proper extension to another AD family.

(c) A MAD family \mathcal{A} has *true cardinality* \mathfrak{c} (where $\mathfrak{c} = 2^{\aleph_0}$) if for all $X \subseteq \omega$, $\{A \in \mathcal{A} : X \cap A \text{ is infinite}\}$ is either finite or of cardinality \mathfrak{c} .

(d) A MAD family \mathcal{A} is *completely separable* if every set $X \subseteq \omega$ either contains a member of \mathcal{A} or is a subset of the union of a finite subfamily of \mathcal{A} .

There is a close relationship between (c) and (d) above. It is easy to see that a completely separable MAD family has true cardinality \mathfrak{c} . Moreover, the existence of a MAD family of cardinality \mathfrak{c} implies the existence of a completely separable MAD family. Hechler [H, Theorem 8.2, Lemma 9.2] showed that Martin's axiom (MA) implies the existence of a completely separable family.⁸ A similar proof yields the following result.

2.2.2. Theorem. *The following statement (#) is provable within ZFC + MA.*

(#) *For every increasing sequence $\bar{n} = \langle n_i : i < \omega \rangle$ with $\lim \langle n_{i+1} - n_i : i < \omega \rangle = \infty$ there is a MAD family $\mathcal{A} = \mathcal{A}_{\bar{n}}$ that satisfies the following two conditions:*

(1) $\mathcal{A} \subseteq \{u \subseteq \omega : \forall i \in \omega \ |u \cap [n_i, n_{i+1}]| = 1\}$, and

(2) *If $X \subseteq \omega$ and $\limsup \langle |X \cap [n_i, n_{i+1}]| : i < \omega \rangle = \infty$, then $\{B \in \mathcal{A} : B \cap X \text{ is infinite}\}$ is either finite or of cardinality 2^{\aleph_0} .*

2.3. TREE INDISCERNIBLES

2.3.1. Definition. Suppose \mathcal{M} is a model with signature $\tau_{\mathcal{M}}$. An indexed family $\{a_{\eta} : \eta \in {}^\omega 2\}$ of pairwise distinct elements of \mathcal{M} is said to be a family of *tree indiscernibles in \mathcal{M}* if for every $\varphi(x_0, \dots, x_{m-1}) \in \mathbf{L}(\tau_{\mathcal{M}})$, there is some $n_{\varphi} < \omega$, such that for all natural numbers $n > n_{\varphi}$ and all infinite sequences $\eta_0, \dots, \eta_{m-1} \in {}^\omega 2$, $\nu_0, \dots, \nu_{m-1} \in {}^\omega 2$ the following implication is true

$$\left(\bigwedge_{i < m} \eta_i \upharpoonright n = \nu_i \upharpoonright n \right) \wedge \left(\bigwedge_{i < j < m} \eta_i \upharpoonright n \neq \eta_j \upharpoonright n \right) \implies \mathcal{M} \models \varphi[a_{\eta_0}, \dots, a_{\eta_{m-1}}] \leftrightarrow \varphi[a_{\nu_0}, \dots, a_{\nu_{m-1}}].$$

⁷We owe this remark to Paul Larson.

⁸The question of the existence of a completely separable MAD family in ZFC was posed by Erdős and Shelah [ES] and remains open. Shelah [Sh-7] has recently showed that (1) the existence of such families can be established within ZFC + $2^{\aleph_0} < \aleph_{\omega}$, and (2) the nonexistence of such families has very high large cardinal strength. See also [HS] for further open questions regarding completely separable families.

Tree indiscernibles were invented by Shelah⁹ ([Sh-1], [Sh-2]) to prove certain 2-cardinal theorems, including $(\aleph_\omega, \aleph_0) \rightarrow (2^{\aleph_0}, \aleph_0)$. More recently, Shelah [Sh-4] further developed the machinery of tree indiscernibles in his work on Borel structures. In particular, he isolated a cardinal $\lambda_{\omega_1}(\aleph_0)$ that satisfies the following three properties (note that the third property implies that $\lambda_{\omega_1}(\aleph_0)$ is the Hanf number of $\mathcal{L}_{\omega_1\omega}$ below the continuum when $\lambda_{\omega_1}(\aleph_0) < 2^{\aleph_0}$).

- $\lambda_{\omega_1}(\aleph_0) \leq \beth_{\omega_1}$ [Sh-4, Def. 1.1, Conclusion 1.8].
- $\lambda_{\omega_1}(\aleph_0)$ is preserved in c.c.c. extensions [Sh-4, Claim 1.10].
- If a sentence $\psi \in \mathcal{L}_{\omega_1,\omega}$ has a model \mathcal{M}_0 with $|R^{\mathcal{M}_0}| \geq \lambda_{\omega_1}(\aleph_0)$ (where R is a distinguished unary predicate of \mathcal{M}_0), then ψ has a model \mathcal{M} with Skolem functions that is generated by a family of tree indiscernibles in $R^{\mathcal{M}}$ (in particular $|R^{\mathcal{M}}| = 2^{\aleph_0}$) [Sh-4, Claim 2.1].

The above three facts immediately imply the following result.

2.3.2. Theorem. *Suppose \mathbf{V} satisfies $\aleph_{\omega_1} = \beth_{\omega_1}$ and \mathbb{P} is a c.c.c. notion of forcing. Then the following statement (\blacktriangle) holds in $\mathbf{V}^{\mathbb{P}}$.*

(\blacktriangle) *If a sentence ψ in $\mathcal{L}_{\omega_1,\omega}$ has a model \mathcal{M}_0 with $|R^{\mathcal{M}_0}| \geq \aleph_{\omega_1}$, then there is a countable first order theory T_1 with Skolem functions such that $\tau(T) \supseteq \tau(\psi)$, and $T + \psi$ has a model \mathcal{M} that is generated from a family of tree indiscernibles in $R^{\mathcal{M}}$.*

The next result shows that for a given $\mathcal{L}_{\omega_1,\omega}$ sentence ψ the existence of a model of ψ that is generated by tree indiscernibles is absolute.¹⁰

2.3.3. Theorem. *Let ψ be a sentence of $\mathcal{L}_{\omega_1,\omega}$ and consider the following statement S .*

$S :=$ “there is a Skolemized model \mathcal{M} with a countable signature $\tau(\mathcal{M}) \supseteq \tau(\psi)$ such that \mathcal{M} satisfies ψ and \mathcal{M} is generated from a family of tree indiscernibles”.

Then for any partial order \mathbb{P} , if $\mathbf{V}^{\mathbb{P}} \models S$, then $\mathbf{V} \models S$.

Proof: It is well known [B, Theorem 6.18] that there is countable Skolemized first order theory T_ψ in a countable signature τ^+ and a countable set Γ_ψ of 1-types of τ^+ such that (1) every model \mathcal{M} of ψ has an expansion to a model \mathcal{M}^+ of T_ψ which omits the types in Γ_ψ , and (2) every model of T_ψ that omits the types in Γ_ψ satisfies ψ . Suppose ψ has a model \mathcal{M} generated from a family $\{a_\eta : \eta \in {}^\omega 2\}$ of tree indiscernibles in $\mathbf{V}^{\mathbb{P}}$. Then in $\mathbf{V}^{\mathbb{P}}$ we can form the multi-sorted structure $(\mathcal{M}^+, \mathcal{N}, f)$, where \mathcal{N} is the standard model for second order number theory $(\omega, \mathcal{P}(\omega))$ (which is itself a two-sorted structure) and $f : \mathcal{P}(\omega) \rightarrow M$ by $f(A) = a_{\chi_A}$ (where χ_A is the characteristic function of A). In particular, the signature τ^* appropriate to $(\mathcal{M}^+, \mathcal{N}, f)$ has a sort U_M for the universe of \mathcal{M}^+ , a sort $U_{\mathcal{P}(\omega)}$ for $\mathcal{P}(\omega)$, and a

⁹Shelah also employed tree indiscernibles in his work on classification theory [Sh-3, VII, Sec.4] to show that for all $\lambda \geq \max\{|T|, \aleph_1\}$ T has 2^λ nonisomorphic models of cardinality λ for every complete theory T that is not superstable ([Pi] includes an expository account). Tree indiscernibles were also discovered by Paris and Mills ([PM], [KS, Theorem 3.5.3]) in the context of nonstandard models of Peano arithmetic to show, e.g., the existence of model \mathcal{M} of PA with a nonstandard integer m in \mathcal{M} such that the set of \mathcal{M} -predecessors of m is externally countable but the set of \mathcal{M} -predecessors of 2^m is of power 2^{\aleph_0} (this result is also an immediate corollary of [Sh-1, Theorem 1]).

¹⁰This result is stated for generic extensions, but the proof shows that this absoluteness result is true for any two ω -models \mathbf{V} and \mathbf{W} of ZF + DC with $\mathcal{P}^{\mathbf{V}}(\omega) \subseteq \mathcal{P}^{\mathbf{W}}(\omega)$.

sort U_ω for ω . Let θ be the conjunction of the following sentences $\theta_1, \dots, \theta_4$ of $\mathbf{L}_{\omega_1, \omega}(\tau^*)$. Note that θ_4 is the only finitary sentence in the list.

- θ_1 expresses: ψ holds in U_M .
- θ_2 expresses: the axioms of second order arithmetic¹¹ (\mathbf{Z}_2) hold in $(U_{\mathcal{P}(\omega)}, U_\omega)$.
- θ_3 expresses: U_ω is an ω -model.
- θ_4 expresses: f is an injection from $\mathcal{P}(\omega)$ into M .

Consider the subset B of $({}^\omega 2)^2$ that consists of elements of the form (r, s) , where r codes a countable model $(\mathcal{M}_r^+, \mathcal{N}_r, f_r)$ of θ such that \mathcal{M}_r^+ omits the types in Γ_ψ , and s codes a function $g_s : \omega \rightarrow \omega$ that witnesses the fact that the image of f_r forms a family of tree indiscernibles in the sense of \mathcal{N}_r , i.e., g_s has the property that for every formula $\varphi = \varphi(x_0, \dots, x_{m-1}) \in \mathbf{L}_{\omega, \omega}(\tau_\psi)$, if $n > g_s(\ulcorner \varphi \urcorner)$, then for all $x_0, \dots, x_{m-1} \in U_{\mathcal{P}(\omega)}$, and for all $y_0, \dots, y_{m-1} \in U_{\mathcal{P}(\omega)}$ the following implication is true (in what follows, φ^{U_M} is the relativization of φ to U_M)

$$\left(\bigwedge_{i < m} \chi_{x_i} \upharpoonright n = \chi_{y_i} \upharpoonright n \right) \wedge \left(\bigwedge_{i < j < m} \chi_{x_i} \upharpoonright n \neq \chi_{x_j} \upharpoonright n \right) \rightarrow \varphi^{U_M}[f(x_0), \dots, f(x_{m-1})] \leftrightarrow \varphi^{U_M}[f(y_0), \dots, f(y_{m-1})].$$

It is easy to see that B is a Borel set with a Borel code c in \mathbf{V} . Also, by downward Löwenheim-Skolem theorem for $\mathbf{L}_{\omega_1, \omega}$ sentences,

$$\mathbf{V}^{\mathbb{P}} \models \text{“the Borel set coded by } c \text{ is not empty”}.$$

On the other hand, the statement “the Borel set coded by c is empty” is provably equivalent (in $\mathbf{ZF} + \mathbf{DC}$) to a Π_1^1 -statement [J, Lemma 25.45] and therefore by Mostowski Π_1^1 -absoluteness theorem [J, Theorem 25.4], the Borel set coded by c is nonempty in the real world \mathbf{V} . This shows that in \mathbf{V} there is a countable model $(\mathcal{M}_0, \mathcal{N}_0, f_0)$ of ψ , and a function $g_0 : \omega \rightarrow \omega$ that witnesses the fact that the image of f forms a family of tree indiscernibles in the sense of \mathcal{N}_0 (in particular, \mathcal{N}_0 is an ω -model of second order arithmetic).

The countable model $(\mathcal{M}_0^+, \mathcal{N}_0, f_0)$ together with g_0 provides us with a blueprint to produce a model of ψ of cardinality continuum that is generated by tree indiscernibles. To do so, add new constants $\{c_\eta : \eta \in {}^\omega 2\}$ to the vocabulary τ^+ of \mathcal{M}_0^+ . Consider the theory Σ defined as follows. Given $\varphi(x_0, \dots, x_{m-1}) \in \mathbf{L}_{\omega, \omega}(\tau_{\mathcal{M}})$, fix any $n > g_s(\ulcorner \varphi \urcorner)$, and find $\nu_0, \dots, \nu_{m-1} \in {}^\omega 2$ such that each ν_i is coded in \mathcal{N}_0 (i.e., there is a some A_i in $U_{\mathcal{P}(\omega)}$ of \mathcal{N}_0 such that $\chi_{A_i} = \nu_i$) and $\eta_i \upharpoonright n = \nu_i \upharpoonright n$ for each $i < m$. Then put $\varphi[c_{\eta_0}, \dots, c_{\eta_{m-1}}]$ or its negation in Σ depending on whether \mathcal{M}_0^+ satisfies $\varphi[f_0(A_0), \dots, f_{m-1}(A_{m-1})]$ or its negation. Since \mathcal{M}_0^+ is Skolemized, Σ uniquely determines an elementary extension \mathcal{M}_2^+ of \mathcal{M}_0^+ that is generated by tree indiscernibles. In order to arrange an elementary extension of \mathcal{M}_0^+ which is generated by tree indiscernibles that also satisfies ψ we need to thin \mathcal{M}_2^+ as follows. Since \mathcal{M}_0^+ omits every type in Γ_ψ and g_0 provides a witness to the tree indiscernibility of the range of f_0 , we can easily construct a perfect subtree Δ of ${}^\omega 2$ such

¹¹We just need a workable theory of finite and infinite sequences, so it is more than sufficient to use \mathbf{RCA}_0 or \mathbf{ACA}_0 instead of \mathbf{Z}_2 here.

that the submodel \mathcal{M}_1^+ of \mathcal{M}_2^+ generated by $\{c_\eta : \eta \in \Delta\}$ omits every type in Γ_ψ . Therefore \mathcal{M}_1^+ is our desired model of ψ which is generated by indiscernibles. \square

2.4. BOREL STRUCTURES

Recall that a model \mathcal{M} is said to be *totally Borel* if the universe of \mathcal{M} is a Borel subset of \mathbb{R} , and every subset of X that is parametrically definable in \mathcal{M} is a Borel set. It is known that every countable theory has an uncountable totally Borel model. This result was established by H. Friedman [St-1] and also (later, but independently) by Malitz-Mycielski-Reinhardt [MMR]. The following results are included for those readers favoring a shorter but less self-contained proof of Theorem A.

2.4.1. Theorem (Steinhorn [St-2]). *If \mathcal{M} is a model generated by tree indiscernibles, then \mathcal{M} is isomorphic to a totally Borel model.*

2.4.2. Theorem (Harrington-Shelah [HMS]). *No analytic linear order contains an uncountable well-ordered set. In particular, the cofinality of every Borel linear order is \aleph_0 .*

3. PROOF OF THEOREM A

Before presenting the full technical details of the proof, let us describe a high-level summary of the three stages of the argument.

- **Stage 1 Outline.** Start with the constructible universe \mathbf{L} and a regular cardinal $\kappa > (\aleph_{\omega_1})^{\mathbf{L}} = (\beth_{\omega_1})^{\mathbf{L}}$. Then force $\text{MA} + 2^{\aleph_0} = \kappa$ with the usual c.c.c. partial order \mathbb{Q} of cardinality κ . By Theorem 2.2.2, in $\mathbf{L}^{\mathbb{Q}}$ there is a MAD family satisfying $(\#)$. In $\mathbf{L}^{\mathbb{Q}}$, use Theorem 2.3.2 to get hold of an ω -standard model \mathcal{M}' of $\text{ZFC}^- + "2^{\aleph_0}$ is the last cardinal" (here ZFC^- is ZFC without the powerset axiom) that is generated by tree indiscernibles and in which $(\#)$ holds.
- **Stage 2 Outline.** By Theorem 2.4.1 $\mathbf{L}^{\mathbb{Q}}$ believes that \mathcal{M}' is a totally Borel model. Combined with Theorem 2.3.3 this shows that there is also a totally Borel model \mathcal{M} in \mathbf{V} that shares the salient features of \mathcal{M}' . In particular, \mathcal{M} is an ω -standard model of ZFC^- that is generated by tree indiscernibles in which $(\#)$ of Theorem 2.2.2 holds. The family \mathcal{A} of Theorem A is the set of reals of \mathcal{M} . This family \mathcal{A} is both Borel and arithmetically closed.
- **Stage 3 Outline.** Let $\mathfrak{b}^{\mathcal{M}}$ be the *bounding number* \mathfrak{b} as computed in \mathcal{M} . By Theorem 2.4.2 $\text{cf}(\mathfrak{b}^{\mathcal{M}}) = \aleph_0$. Coupled with the fact that $(\#)$ is true in \mathcal{M} , one can then verify that $\mathbb{P}_{\mathcal{A}}$ is $(\aleph_0, 2^{\aleph_0})$ nowhere distributive. By Theorem 2.1.1, this completes the proof of Theorem A.

We now proceed to flesh out the above outline.

Stage 1. Let $\mu = (\aleph_{\omega_1})^{\mathbf{L}} = (\beth_{\omega_1})^{\mathbf{L}}$, and fix a regular cardinal $\kappa > \mu$. By GCH in \mathbf{L} , $\kappa = \kappa^{<\kappa}$ holds in \mathbf{L} . Let \mathbb{Q} be the usual notion of forcing $\text{MA} + 2^{\aleph_0} = \kappa$ [J, Theorem 16.13]. Note that $\mathbb{Q} \in \mathcal{H}^{\mathbf{L}}(\kappa^+)$, where $\mathcal{H}(\kappa^+)$ is the collection of sets whose transitive closure has cardinality at most κ , and $\mathcal{H}^{\mathbf{L}}(\kappa)$ is $\mathcal{H}(\kappa)$ as calculated in \mathbf{L} . In the forcing extension $\mathbf{L}^{\mathbb{Q}}$ let \mathcal{M}_0 be an expansion of the structure $(\mathcal{H}(\kappa^+), \in)$ by Skolem functions, a well-ordering of $\mathcal{H}(\kappa)$, and individual constants c_n and c_ω , where $c_n^{\mathcal{M}_0} = n$, and $c_\omega^{\mathcal{M}_0} = \omega$. Let $\tau = \tau_{\mathcal{M}_0}$ = the signature of \mathcal{M} . We may

assume that $\tau \in \mathbf{L}$ and τ is countable in \mathbf{L} , but note that $\text{Th}(\mathcal{M}_0)$ need not be in \mathbf{L} . Of course \mathcal{M}_0 is a model of $\text{ZFC}^- + “2^{\aleph_0}$ is the last cardinal”.

Since $\kappa > \mu$ we may invoke Theorem 2.3.2 to obtain a model \mathcal{M}' in $\mathbf{L}^{\mathbb{Q}}$ that satisfies the following five conditions:

- (a) \mathcal{M}' is a τ -model of $\text{Th}(\mathcal{M}_0)$.
- (b) \mathcal{M}' is an ω -model, i.e., \mathcal{M} omits $\{x \in c_\omega\} \cup \{x \neq c_n : n < \omega\}$.
- (c) There is a family $\{a_\eta : \eta \in {}^\omega 2\}$ of tree indiscernibles in \mathcal{M} .
- (d) For each $\eta \in {}^\omega 2$, $\mathcal{M}' \models “a_\eta \subseteq c_\omega”$ (i.e., each a_η is a real in the sense of \mathcal{M}').
- (e) \mathcal{M}' is the Skolem hull of $\{a_\eta : \eta \in {}^\omega 2\}$.

Stage 2. Let $T = \{\varphi \in \mathbf{L}_{\omega, \omega}(\tau) : 1 \Vdash_{\mathbb{Q}} \mathcal{M}' \models \varphi\}$. Note that since \mathcal{M}' is actually a \mathbb{Q} -name, $T \in \mathbf{L}$. By Theorem 2.3.3 there is a τ -model \mathcal{M} of T in \mathbf{V} and a family of tree indiscernibles $\langle a_\eta : \eta \in {}^\omega 2 \rangle$ that satisfy conditions (b), (c), (d) and (e) above. We may assume that the model \mathcal{M} is in “reduced form”, i.e., the well-founded part of \mathcal{M} is transitive. In particular, $\omega^{\mathcal{M}} = \omega$, and if $\mathcal{M} \models b \subseteq c_\omega$, then $b \in \mathcal{P}(\omega)$. Let $\mathcal{A} = \{b : \mathcal{M} \models b \subseteq c_\omega\}$. Obviously \mathcal{A} is arithmetically closed¹². By Theorem 2.4.1 \mathcal{A} is also Borel. This fact can also be established directly as follows. For any $\tau_{\mathcal{M}}$ -term $\sigma = \sigma(x_0, \dots, x_{m-1})$, $m < \omega$, $n^* < \omega$, and pairwise distinct $\nu_0, \dots, \nu_{m-1} \in n^* 2$, let $\bar{\nu} = \langle \nu_i : i < m \rangle$, and define $\mathcal{A}_{\sigma, \bar{\nu}}$ via

$$\mathcal{A}_{\sigma, \bar{\nu}} := \{\sigma^{\mathcal{M}}(a_{\eta_0}, \dots, a_{\eta_{m-1}}) : \bigwedge_{i < m} \nu_i \triangleleft \eta_i \in {}^\omega 2\}.$$

It is sufficient to prove that $\mathcal{A}_{\sigma, \bar{\nu}}$ is Borel for any $(\sigma, \bar{\nu})$ since \mathcal{A} is the union of the countable family of sets of the form $\mathcal{A}_{\sigma, \bar{\nu}}$. We can find an increasing $f : \omega \rightarrow \omega \setminus n^*$ and $\langle g_n : n < \omega \rangle$ such that

- (α) g_n is a function from ${}^m (f(n) 2)$ to $\{0, 1\}$.
- (β) If $\eta_0, \dots, \eta_{m-1} \in {}^\omega 2$ and $\bigwedge_{i < m} \nu_i \triangleleft \eta_i \in {}^\omega 2$ and $n < \omega$, then (using tree indiscernibility)

$$n \in \sigma^{\mathcal{M}}(a_{\eta_0}, \dots, a_{\eta_{m-1}}) \Leftrightarrow g_n(\eta_0, \dots, \eta_{m-1}) = 1.$$

By König’s lemma, for each $A \subseteq \omega$, we have:

- (γ) $A \in \mathcal{A}_{\sigma, \bar{\nu}}$ iff for every n there are $\rho_0, \dots, \rho_{m-1} \in f(n) 2$ such that

$$k < n \Rightarrow (k \in A \Leftrightarrow g_k(\rho_0 \upharpoonright f(k), \dots, \rho_{m-1} \upharpoonright f(k)) = 1).$$

This shows that each $\mathcal{A}_{\sigma, \bar{\nu}}$ is Borel.

Stage 3: In \mathcal{M} we can define the *bounding number*¹³ \mathfrak{b} . Let $\bar{f} = \langle f_\alpha : \alpha < \mathfrak{b} \rangle$ be an unbounded family. We may assume that each f_α is increasing. Clearly $(\mathfrak{b}, \in)^{\mathcal{M}}$ is a linear ordering as viewed in the real world.

¹²Indeed \mathcal{A} is even *hyperarithmetically* closed. This follows from the fact that any ω -model of $\Sigma_1^1\text{-AC}_0$ contains all hyperarithmetical sets [Si, Lemma VIII.4.15] (\mathcal{M} satisfies the axiom of choice, so the standard model of second order arithmetic in the sense of \mathcal{M} satisfies $\Sigma_n^1\text{-AC}_0$ for all $n < \omega$).

¹³Recall that \mathfrak{b} is defined as the least cardinality of an *unbounded* family $\mathcal{F} \subseteq {}^\omega \omega$. Here \mathcal{F} is an unbounded family if for all $g \in {}^\omega \omega$ there is some $f \in \mathcal{F}$ such that $g(n) < f(n)$ for infinitely many $n \in \omega$. See [J, Sec. 26] for more information.

Claim 3.1. *The external cofinality of $(\mathfrak{b}, \in)^{\mathcal{M}}$ is \aleph_0 .*

Proof: This follows from Theorems 2.4.1 and 2.4.2. Alternatively, one can argue directly as follows. Suppose to the contrary. Then for some regular uncountable cardinal κ , there is an increasing and unbounded subset $\{b_\alpha : \alpha < \kappa\}$ of $(\mathfrak{b}, \in)^{\mathcal{M}}$. Each b_α can be written as

$$b_\alpha = \sigma_\alpha(a_{\eta_0^\alpha}, \dots, a_{\eta_{m_\alpha-1}^\alpha}),$$

where, without loss of generality, $\sigma_\alpha = \sigma$, $m_\alpha = m$, and $\eta_0^\alpha <_{\text{lex}} \eta_1^\alpha <_{\text{lex}} \dots$, and $\{\{\eta_0^\alpha, \dots, \eta_{m_\alpha-1}^\alpha\} : \alpha < \kappa\}$ forms a Δ -system [J, Theorem 9.18]. So, we may assume that for some m ,

$$l < m \Rightarrow \eta_l^\alpha = \eta_l^0,$$

and

$$\eta_{l_1}^{\alpha_1} = \eta_{l_2}^{\alpha_2} \Rightarrow (l_1 = l_2 < m) \vee (\alpha_1, l_1) = (\alpha_2, l_2).$$

We can easily construct a countable $Y \subseteq \kappa$ such that if $\alpha < \kappa$ and $k < \omega$, then for some $\beta \in Y$ we have

$$\bigwedge_{l < m} \eta_l^\alpha \upharpoonright k = \eta_l^\beta \upharpoonright k.$$

We claim that $\{b_\alpha : \alpha \in Y\}$ is cofinal in $(\mathfrak{b}, \in)^{\mathcal{M}}$. To see this, let $\alpha < \kappa$, and choose k that satisfies the following two conditions (a) and (b).

- (a) $\langle \eta_l^\alpha : l < n \rangle \frown \langle \eta_l^{\alpha+1} : l \in [m, n] \rangle$ has no repetition.
- (b) If $\eta_l \in {}^\omega 2$ for all $l < n$, $\nu_l \in {}^\omega 2$ for all $l \in [m, n]$, and $(\eta_l \upharpoonright k = \eta_l^\alpha \upharpoonright k) \wedge (\nu_l \upharpoonright k = \eta_l^{\alpha+1} \upharpoonright k)$, then

$$\mathcal{M} \models \sigma(\dots, a_{\eta_l}, \dots) < \sigma(a_{\eta_0}, \dots, a_{\eta_{m-1}}, a_{\nu_m}, \dots).$$

Lastly, choose $\beta \in Y$ such that

$$\bigwedge_{l < m} \eta_l^\beta \upharpoonright k = \eta_l^{\beta+1} \upharpoonright k.$$

Hence

$$(b_\alpha <^{\mathcal{M}} b_{\alpha+1}) \Leftrightarrow (b_\beta <^{\mathcal{M}} b_{\beta+1}),$$

This completes the proof. □ (Claim 3.1)

We now complete the proof of Theorem A by showing that $\mathbb{P}_{\mathcal{A}}$ is equivalent to $\text{LEVY}(\aleph_0, 2^{\aleph_0})$. By Theorem 2.1.1, it suffices to prove the following claim.

Claim 3.2. *There is a family $\{I_n : n \in \omega\}$ of maximal antichains in $\mathbb{P}_{\mathcal{A}}$ such that for every $p \in \mathbb{P}_{\mathcal{A}}$, there is some $n < \omega$ such that $\{q \in I_n : p \text{ and } q \text{ are compatible}\}$ has cardinality 2^{\aleph_0} .*

Proof: By Claim 3.1 we may fix a countable family of functions $F = \{f_n : n < \omega\} \subseteq {}^\omega \omega$ with $X \in \mathcal{M}$, such that \mathcal{M} satisfies “there is no $g \in {}^\omega \omega$ such that every member of X is eventually dominated by g ”. We may assume that $f_n(i+1) - f_n(i) > i$ for all n and $i < \omega$. Since \mathcal{M} satisfies $(\#)$, given the increasing sequence $\langle f_n(i) : i < \omega \rangle$ there is for some $I_n \in \mathcal{M}$ such that \mathcal{M} satisfies following statement $(*)$

$(*)$ I_n is MAD and for each $A \in I_n$ and $|A \cap [f_n(i), f_n(i+1))| = 1$ for all $i < \omega$.

We shall prove that $\langle I_n : n < \omega \rangle$ exemplifies condition (b) of Theorem 2.1.1.

Given $p = [B] \in \mathbb{P}_{\mathcal{A}}$, we may assume that B is infinite. It is routine to construct a function $g \in \mathcal{M}$ by recursion such that $g(0) = 0$, and

$$\forall n (g(n) < g(n+1) \wedge B \cap [g(n), g(n+1)) \text{ has at least } g(n) \text{ elements}).$$

Now, by the choice of F there is some $f_n \in F$ such that the following set Y is infinite:

$$Y := \{i : \exists k (f_n(i) < g(k) < g(k+1) < f_n(i+1))\}.$$

Let $\mathcal{B} := \{A \in I_n : A \cap B \text{ is infinite}\}$. Note that $\mathcal{B} \in \mathcal{M}$. Therefore by (#) \mathcal{M} satisfies “ \mathcal{B} is finite or has cardinality 2^{\aleph_0} ”. But \mathcal{B} cannot be finite, since each $A \in \mathcal{B}$ has at most one element in each interval $[f_n(i), f_n(i+1))$, whereas A has more than $f_n(i)$ members for infinitely many i 's, and therefore so does \mathcal{B} . Hence $\{A : \mathcal{M} \models A \in \mathcal{B}\}$ has cardinality 2^{\aleph_0} in the sense of \mathcal{M} , which finishes the proof since \mathcal{M} has continuum-many reals. \square (Claim 3.2)

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