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# On Löwenheim-Skolem-Tarski numbers for extensions of first order logic\*

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## Abstract

We show that, assuming the consistency of a supercompact cardinal, the first (weakly) inaccessible cardinal can satisfy a strong form of a Löwenheim-Skolem-Tarski theorem for the equicardinality logic  $L(I)$ , a logic introduced in [4] strictly between first order logic and second order logic. On the other hand we show that in the light of present day inner model technology, nothing short of a supercompact cardinal suffices for this result. In particular, we show that the Löwenheim-Skolem-Tarski theorem for the equicardinality logic at  $\kappa$  implies the Singular Cardinals Hypothesis above  $\kappa$  as well as Projective Determinacy.

## 1 Introduction

The Löwenheim-Skolem Theorem is perhaps the most quoted result about first order logic. It shows the “local” character of first order formulas. The

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truth of a first order sentence depends only on a small part of the set theoretical universe. For many purposes first order logic is ideal, but there are also interesting and useful extensions of first order logic.

**Example 1** • *Second order logic  $L^2$  extends first order logic with quantifiers of the form  $\exists R\phi(R, x_0, \dots, x_{n-1})$ , where the second order variable  $R$  ranges over  $n$ -ary relations on the universe for some fixed  $n$ .*

- *The logic  $L(Q_1)$  extends first order logic with a new quantifier  $Q_1$  binding one variable. The formula  $Q_1x_0\phi(x_0, \dots, x_{n-1})$  has the meaning “there are uncountably many elements  $x_0$  satisfying  $\phi(x_0, \dots, x_{n-1})$ ”.*
- *The logic  $L(Q_1^{MM})$  extends first order logic with a new quantifier  $Q_1^{MM}$  binding two variables. The formula  $Q_1^{MM}x_0x_1\phi(x_0, \dots, x_{n-1})$  has the meaning “there is an uncountable set  $X$  such that any two elements  $x_0$  and  $x_1$  from  $X$  satisfy  $\phi(x_0, \dots, x_{n-1})$ ”.*

Second order logic is in a sense the opposite of first order logic. It is powerful enough to capture exactly a large part of the set theoretical universe. The logics  $L(Q_1)$  and  $L(Q_{\aleph_1}^{MM})$  are more close to first order logic. The first is axiomatizable and so is the second, if we assume  $\diamond$ . In this paper we study the following two, in a sense intermediate, extensions of first order logic:

**Example 2** • *Equicardinality logic  $L(I)$  [4]. This logic extends first order logic by formulas of the form*

$$Ix_0y_0\phi(x_0, \dots, x_{n-1})\psi(y_0, \dots, y_{n-1})$$

*with the meaning: “for given  $a_1, \dots, a_{n-1}$  and  $b_1, \dots, b_{n-1}$  the cardinality of the set of elements  $x_0$  satisfying  $\phi(x_0, a_1, \dots, a_{n-1})$  is the same as the cardinality of the set of elements  $y_0$  satisfying  $\psi(y_0, b_1, \dots, b_{n-1})$ ”.*

- *Equicofinality logic  $L(Q^{ec})$  [10]. This logic extends first order logic by formulas of the form*

$$Q^{ec}x_0x_1y_0y_1\phi(x_0, \dots, x_{n-1})\psi(y_0, \dots, y_{n-1})$$

*with the meaning: “for given  $a_2, \dots, a_{n-1}$  and  $b_2, \dots, b_{n-1}$ , both the set of pairs of elements  $x_0$  and  $x_1$  satisfying  $\phi(x_0, x_1, a_2, \dots, a_{n-1})$  and the set of pairs of elements  $y_0$  and  $y_1$  satisfying  $\psi(y_0, y_1, b_2, \dots, b_{n-1})$  are linear orders, and moreover these linear orders have the same cofinality.”*

The logics  $L(I)$  and  $L(Q^{ec})$  are in a clear sense between first order logic and second order logic. The results of this paper show that on the basis of ZFC alone there is mixed information as to whether  $L(I)$  and  $L(Q^{ec})$  are closer to first order logic or to second order logic.

Very little is known about the logic  $L(Q^{ec})$ . Shelah [10] conjectures that this logic is compact and axiomatizable. The hidden power of this logic is revealed in models with a wellordering. There the quantifier  $Q^{ec}$  can be used to pick elements of the well-ordering corresponding to regular cardinals. This puts severe limitations e.g. to the existence of small elementary submodels. In a sense, the stronger logic  $L(I, Q^{ec})$  is better understood. At least we know that this logic is very far from compact and axiomatizable, because  $L(I)$  is.

There is a quite general concept of a logic, that the above examples are special cases of. We define it as follows:<sup>1</sup>

**Definition 3** *Let  $\tau$  be a fixed vocabulary. A logic  $L$  consists of*

1. *A set, also denoted by  $L$ , of “formulas” of  $L$ . If  $\phi \in L$ , then there is a natural number  $n_\phi$ , called the length of the sequence of free variables,*
2. *A relation*

$$\mathcal{A} \models \phi[a_0, \dots, a_{n_\phi-1}]$$

*between models of vocabulary  $\tau$ , sequences  $(a_0, \dots, a_{n_\phi-1})$  of elements of  $A$  and formulas  $\phi \in L$ . It is assumed that this relation satisfies the isomorphism axiom, that is, if  $\pi : \mathcal{A} \cong \mathcal{B}$ , then  $\mathcal{A} \models \phi[a_0, \dots, a_{n_\phi-1}]$  and  $\mathcal{B} \models \phi[\pi a_0, \dots, \pi a_{n_\phi-1}]$  are equivalent.*

*We call  $\tau$  the vocabulary of the logic  $L$ .*

Note, that no syntax is a priori assumed of a logic. The meaning of “ $\phi$  has a model”, and “the theory  $T \subset L$  has a model” is obvious. We write  $\mathcal{A} \equiv_L \mathcal{B}$  if  $\mathcal{A} \models \phi$  and  $\mathcal{B} \models \phi$  are equivalent for all  $\phi \in L$  with  $n_\phi = 0$ . We write  $\mathcal{A} \prec_L \mathcal{B}$  if  $\mathcal{A} \models \phi[a_0, \dots, a_{n_\phi-1}]$  and  $\mathcal{B} \models \phi[a_0, \dots, a_{n_\phi-1}]$  are equivalent for all  $\phi \in L$  and all  $a_0, \dots, a_{n_\phi-1} \in A$ .

We now define two natural invariants for any logic  $L$ :

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<sup>1</sup>This is a little different than usual (e.g. [1, 6]) in that our logics have a fixed vocabulary.

**Definition 4** *The Löwenheim-Skolem number  $\text{LS}(L)$  of  $L$  is the smallest cardinal  $\kappa$  such that if a theory  $T \subset L$  has a model, it has a model of cardinality  $< \max(\kappa, |T|)$ . The Löwenheim-Skolem-Tarski number  $\text{LST}(L)$  of  $L$  is the smallest cardinal  $\kappa$  such that if  $\mathcal{A}$  is any  $\tau$ -structure, then there is a substructure  $\mathcal{A}'$  of  $\mathcal{A}$  of cardinality  $< \kappa$  such that  $\mathcal{A}' \prec_L \mathcal{A}$ .*

Note that  $\text{LS}(L)$  always exists, because  $L$  is a set. In general there is no guarantee that  $\text{LST}(L)$  exists, but if it exists, it is at least as big as  $\text{LS}(L)$ . We can think of the sizes of  $\text{LS}(L)$  and  $\text{LST}(L)$  as a “test” of how close the logic is to being first order. For first order logic these numbers are both  $\aleph_1$ , and for  $L(Q_1)$  and  $L(Q_1^{\text{MM}})$  they are  $\aleph_2$ . If  $\kappa$  is strongly inaccessible, then  $\text{LST}(L_{\kappa\kappa}) = \kappa$ .

The Löwenheim-Skolem numbers of  $L(I)$  and  $L(Q^{ec})$  are quite high in the hierarchy of cardinal numbers, certainly both cardinals are fixed points of the function  $\alpha \mapsto \aleph_\alpha$ . Whether the Löwenheim-Skolem number of  $L(I)$  can be below the first weakly inaccessible, was asked in [15] and has been an open question ever since, but will be settled positively in this paper. On the other hand, in the inner model  $L^\mu$  it is easy to see that  $\text{LS}(L(I))$  is above the measurable cardinal.

For second order logic,  $\text{LS}(L^2)$  is the supremum of  $\Pi_2$ -definable ordinals ([14]), which means that it exceeds the first measurable, the first  $\kappa^+$ -supercompact  $\kappa$ , and the first huge cardinal if they exist.

**Theorem 5** ([7]) *1. Suppose  $\kappa$  is strong, then  $\text{LS}(L^2) < \kappa$ .*

*2.  $\text{LST}(L^2)$  exists if and only if supercompact cardinals exist, and then  $\text{LST}(L^2)$  is the first of them.*

**Proof.** For the first claim, suppose  $T$  is a theory in  $L^2$  and  $T$  has a model  $\mathcal{A}$ . We may assume that the universe of  $\mathcal{A}$  is an ordinal  $\delta$ . Let  $i$  be an embedding into  $M$  with critical point  $\kappa$  such that  $T, \mathcal{A}, \mathcal{P}(\delta) \in M$ . It is easy to prove by induction on formulas  $\phi \in L^2$  that for all  $\vec{a} \in A^n$  and  $\vec{X} \in \mathcal{P}(A^{n_1}) \times \dots \times \mathcal{P}(A^{n_k})$  we have

$$\mathcal{A} \models \phi(\vec{a}, \vec{X}) \iff M \models \text{“}\mathcal{A} \models \phi(\vec{a}, \vec{X})\text{.”}$$

The point is that all subsets of  $A$  are in  $M$ . Thus  $M \models \exists x(x \models T \text{ and } |x| < i(\kappa))$ . Hence there is in  $V$  a model of  $T$  of cardinality  $< \kappa$ . For the second claim we refer to [7] but give the following argument for  $\text{LST}(L^2) \leq \kappa$  for

supercompact  $\kappa$  since we will use it later: Suppose  $\mathcal{A}$  is a model of cardinality  $\lambda$ . Let  $i$  be an elementary embedding of  $V$  into a transitive  $M$  so that  ${}^\lambda M \subseteq M$  and  $i(\kappa) > \lambda$ . Let  $\mathcal{B}$  be the pointwise image of  $\mathcal{A}$  under  $i$ . Since  ${}^\lambda M \subseteq M$ ,  $\mathcal{B} \in M$ . It is easy to prove by induction on formulas  $\phi \in L^2$  that for all  $\vec{a} \in A^n$  and  $\vec{X} \in \mathcal{P}(A^{n_1}) \times \dots \times \mathcal{P}(A^{n_k})$  we have

$$\begin{aligned} M \models \text{“}\mathcal{B} \models \phi(\vec{a}, \vec{X})\text{”} &\iff \mathcal{A} \models \phi(\vec{a}, \vec{X}) \\ &\iff M \models \text{“}i(\mathcal{A}) \models \phi(i(\vec{a}), i(\vec{X}))\text{”}. \end{aligned}$$

Thus  $M \models \exists \mathcal{B}(\mathcal{B} \prec \mathcal{A} \text{ and } |B| < i(\kappa))$ . Hence there is in  $V$  a model  $\mathcal{C} \prec \mathcal{A}$  of  $T$  of cardinality  $< \kappa$ .  $\square$

So second order logic meets the test of being very far from first order in terms of the size of its Löwenheim-Skolem numbers. We show that according to this test,  $L(I)$  and  $L(Q^{ec})$  can be close to second order logic but can also be, relatively speaking, close to first order logic.

The strongest large cardinal axiom from the point of view of Löwenheim-Skolem theorems is *Vopenka’s Principle*, which states that every proper class of structures of the same vocabulary has two members one of which is isomorphic to an elementary substructure of the other. In [12] an equivalent condition is given: Suppose  $A$  is a class. Let us call a cardinal  $\kappa$  *A-supercompact* if for all  $\eta > \kappa$  there is  $\alpha < \kappa$  and an elementary embedding

$$j : (V_\alpha, \in, A \cap V_\alpha) \rightarrow (V_\eta, \in, A \cap V_\eta)$$

with a critical point  $\gamma$  such that  $j(\gamma) = \kappa$ . It is proved in [12] that Vopenka’s Principle is equivalent to the statement that for every class  $A$  there is an *A-supercompact* cardinal. From this and the proof of Theorem 5 we get the following unpublished result of J. Stavi:

**Theorem 6** *Vopenka’s Principle holds if and only if every logic has a Löwenheim-Skolem-Tarski number.*

For the intermediate logics  $L(I)$  and  $L(I, Q^{ec})$  the analogue of Theorem 5 (2) is the substantially less conclusive:

**Theorem 7 ([13])** 1. *LST( $L(I)$ ) exists only if inaccessible cardinals exist, and then LST( $L(I)$ ) is at least as large as the first of them.*

2. *LST( $L(I, Q^{ec})$ ) exists only if Mahlo cardinals exist, and then the cardinal LST( $L(I, Q^{ec})$ ) is at least as large as the first of them.*

**Proof.** Let  $\mathcal{A} = (R(\kappa^+), \epsilon)$ , where  $\kappa = \text{LST}(L(I))$ . By the definition of  $\text{LST}(L(I))$  there is a transitive set  $M$  of power  $< \kappa$  and a monomorphism  $i : (M, \epsilon) \rightarrow \mathcal{A}$  which preserves  $L(I)$ -truth. Moreover, every  $M$ -cardinal is a real cardinal. Let  $\lambda$  be the largest cardinal in  $M$ . Clearly  $i(\lambda) = \kappa > \lambda$ . Let  $\gamma$  be the first ordinal moved by  $i$ . Trivially,  $\gamma$  is a limit cardinal. Suppose  $f \in M$  is a cofinal  $\delta$ -sequence in  $\gamma$  for some  $\delta < \gamma$ . Now  $i(f)$  is a cofinal  $\delta$ -sequence in  $i(\gamma)$  whence  $i(f)(\beta) > \gamma$  for some  $\beta < \delta$ . But  $i(f)(\beta) = i(f(\beta)) = f(\beta) < \gamma$ . Thus  $\gamma$  is weakly inaccessible in  $M$ , and therefore,  $i(\gamma)$  is weakly inaccessible in  $V$ . The second claim is proved similarly.  $\square$

The results of this paper explain why Theorem 7 is weaker than Theorem 5. The proof theoretic strength of the existence of either  $\text{LST}(L(I))$  or  $\text{LST}(L(I, Q^{ec}))$  exceeds substantially what follows from the mere size of these cardinals. Accordingly, and unlike  $\text{LST}(L^2)$ , the numbers  $\text{LST}(L(I))$  and  $\text{LST}(L(I, Q^{ec}))$  do not have to be very high in the scale of large cardinals. We will show in this paper that  $\text{LST}(L(I))$  can be the first weakly inaccessible cardinal and  $\text{LST}(L(I, Q^{ec}))$  can respectively be the first Mahlo cardinal. Also they can be of continuum size:

**Theorem 8 ([13])** *Suppose  $\kappa$  is a supercompact cardinal and  $P$  is the notion of forcing  $C_\kappa$ . Let  $L$  be a provably  $\mathcal{C}$ -absolute logic which is provably a sublogic of  $L^2$ . Then*

$$\Vdash_P \text{LST}(L(I, R)) \leq 2^\omega.$$

**Proof.** We give an outline of the proof for completeness. Suppose  $\mathcal{A}$  is a name for a finitary structure with universe  $\lambda$  in the  $P$ -forcing language. Let  $i : V \rightarrow M$  be an elementary embedding of the universe such that  $i(\kappa) > \lambda$ ,  ${}^\lambda M \subseteq M$  and  $i''\kappa = \kappa$ . Let  $\mathcal{B}$  be the point-wise image of  $\mathcal{A}$  under  $i$ . Using the fact that  $P$  preserve cardinals and cofinalities it is possible to show

$$M \models \text{“} \Vdash_{i(P)} i : \mathcal{B} \rightarrow_{L(I, R)} i(\mathcal{A}).\text{”} \tag{1}$$

It follows from this that

$$M \models \text{“} \Vdash_{i(P)} i(\mathcal{A}) \text{ has an } L\text{-elementary substructure of power } < i(\kappa).\text{”}$$

Therefore

$$\Vdash_P \mathcal{A} \text{ has an } L\text{-elementary substructure of power } < \kappa.$$

$\square$

To see some of the strength of the Löwenheim-Skolem-Tarski Theorem for the equicardinality quantifier, let us recall the following observation from [13]: Let  $\mathcal{A}$  be the structure  $(R(\kappa^+), \in)$ . Let  $\pi : (M, \in) \rightarrow (R(\kappa^+), \in)$  be an elementary embedding with  $M$  transitive and  $|M| < \kappa$ . If  $\delta = M \cap On$ , then  $\pi \upharpoonright L_\delta : (L_\delta, \in) \rightarrow (L_{\kappa^+}, \in)$ . Thus  $0^\#$  exists. Obviously this argument can be considerably strengthened. We show in this paper that the existence of  $LST(L(I))$  has enough combinatorial power to imply, when combined with current state of the inner model technology, Projective Determinacy.

## 2 The Failure of Squares

We have already alluded to the fact that the existence of  $LST(L(I))$  has non-trivial consistency strength, for example, it implies  $0^\#$ . In this section we show that the existence of  $LST(L(I))$  has a much stronger consistency strength, probably at the level of a supercompact cardinal.

We shall show that the existence of an  $LST(K(I))$  cardinal implies that the combinatorial principle  $\square_\lambda$  fails for *every*  $\lambda \geq \kappa$ . For  $\kappa$  singular of cofinality  $\omega$  we can do better than that and show that any reasonable version of  $\square_\lambda$  fails, in particular a consequence of any reasonable weakening of  $\square_\lambda$  (for  $\text{cof}(\lambda) = \omega$ ) fails globally above  $LST(L(I))$ . The consequence we allude to is the existence of “good” scales.

We shall conclude this section by showing that assuming the consistency of a supercompact cardinal it is consistent that the first  $LST(L(I))$  cardinal is the same as the first supercompact cardinal.

**Definition 9** *The square principle  $\square_\lambda$  says: There is a sequence  $\langle C_\alpha : \alpha < \lambda^+ \text{ a limit ordinal} \rangle$ , such that:*

1.  $C_\alpha$  is closed unbounded subset of  $\alpha$ .
2. The order type of  $C_\alpha$  is always  $\leq \lambda$ .
3. If  $\beta$  is a limit point of  $C_\alpha$ , then  $C_\beta = C_\alpha \cap \beta$ .

**Theorem 10** *If  $\kappa$  is an  $LST(L(I))$  number and  $\lambda \geq \kappa$ . Then  $\square_\lambda$  fails.*

**Proof.** Suppose  $\langle C_\alpha : \alpha < \lambda^+ \text{ a limit ordinal} \rangle$ , is a  $\square_\lambda$  sequence. Consider the structure

$$\mathcal{A} = \langle \lambda^+, \lambda, T, C \rangle,$$

where  $T$  is a function defined on the limit ordinals in  $\lambda^+$  such that  $T(\alpha)$  is the order-type of  $C_\alpha$ , and  $C$  is a ternary relation such that  $C(\alpha, \gamma, \eta)$  holds if and only if “ $\eta$  is the  $\gamma$ -th member of  $C_\alpha$ ”. Let  $\mathcal{B}$  be an  $L(I)$ -elementary substructure of  $\mathcal{A}$  of cardinality  $< \kappa$ . It is easily verified that the order-type of the universe  $B$  of  $\mathcal{B}$  is a successor cardinal  $\mu^+$ , where  $\mu$  is the order-type of  $B \cap \lambda$ . Let  $\mathcal{B}^*$  be the transitive collapse of  $\mathcal{B}$ . It is easily seen that  $\mathcal{B}^*$  has the form  $\langle \mu^+, \mu, T^*, C^* \rangle$ , where for some  $\square_\mu$ -sequence  $\langle C_\alpha^* : \alpha < \mu^+ \text{ a limit ordinal} \rangle$ ,  $T^*$  is a function defined on the limit ordinals in  $\mu^+$  such that  $T^*(\alpha)$  is the order-type of  $C_\alpha^*$ , and  $C^*$  is a ternary relation such that  $C^*(\alpha, \gamma, \eta)$  holds if and only if “ $\eta$  is the  $\gamma$ -th member of  $C_\alpha^*$ ”. Let  $\pi : B^* \rightarrow B$  be the inverse of the transitive collapse of  $\mathcal{B}$ . Let  $\delta = \sup(B) = \sup(\pi''B^*)$ . Note that  $\delta < \lambda^+$  and  $\text{cof}(\delta) = \mu^+$ .  $C_\delta$  is a closed unbounded subset of  $\delta$ ,  $B = \{\pi(\alpha) : \alpha < \mu^+\} = \pi''B^*$  is cofinal in  $\delta$ . So the set

$$A' = \{\eta < \delta : \eta \text{ limit point of } C_\delta \text{ and a limit point of } B\}$$

is closed unbounded in  $B_\delta$ .

For  $\eta \in A'$  let  $\bar{\eta}$  be the minimal element of  $B^*$  such that  $\pi(\bar{\eta}) \geq \eta$ . Obviously,  $\bar{\eta}$  is always defined because  $\sup(A') = \sup \pi''B^* = \delta$ . And if  $\eta_1 < \eta_2$  in  $A'$ , then  $\bar{\eta}_1 < \bar{\eta}_2$ .

**Claim:** For  $\eta \in A'$ ,  $\eta$  is a limit point of  $C_{\pi(\bar{\eta})}$ .

**Proof.** Otherwise, let  $\rho$  be  $\sup(C_{\pi(\bar{\eta})} \cap \eta)$ . So our assumption is  $\rho < \eta$ . As  $\eta \in A'$ , the range of  $\pi$  is cofinal in  $\eta$ , so there is  $\rho'$  such that

$$\rho < \pi(\rho') < \eta \leq \pi(\bar{\eta}).$$

By elementarity there is  $\rho''$  such that

$$\pi(\rho') < \pi(\rho'') \in C_{\pi(\bar{\eta})}.$$

But clearly  $\pi(\rho'') < \eta$ , so we get a contradiction.  $\square$  (Claim.)

**Claim:** If  $\eta_1 < \eta_2$  are in  $A'$ , then  $\bar{\eta}_1$  is a limit point of  $C_{\bar{\eta}_2}$ .

**Proof.** Otherwise, let  $\rho$  be  $\sup(C_{\pi(\bar{\eta}_2)}^* \cap \bar{\eta}_1)$ . We assume  $\rho < \bar{\eta}_1$ , which means by definition of  $\bar{\eta}_1$  that  $\pi(\rho) < \eta_1$ . By the previous claim  $\eta_2$  is a limit point of  $C_{\pi(\bar{\eta}_2)}$ . So by the definition of the square principle

$$C_{\eta_2} = C_{\pi(\bar{\eta}_2)} \cap \eta_2.$$

Note that  $\eta_2 \geq \pi(\bar{\eta}_1)$ . By elementarity of  $\pi$

$$\pi(\rho) = \sup(C_{\pi(\bar{\eta}_2)} \cap \pi(\bar{\eta}_1)) = \sup(C_{\eta_2} \cap \pi(\bar{\eta}_1)). \quad (2)$$

On the other hand  $\eta_1$  and  $\eta_2$  are in  $A'$ , hence they are limit points of  $C_\delta$ , so  $C_{\eta_2} = C_\delta \cap \eta_2$ , so  $\eta_1$  is a limit point of  $C_{\eta_2}$ . This contradicts (2).  $\square$  (Claim.)

It follows from the previous claim that if  $\eta_1, \eta_2 \in A'$ , then the order-type of  $C_{\bar{\eta}_2}^*$  exceeds the order-type of  $C_{\bar{\eta}_1}^*$ . So  $T^*(\bar{\eta}_2) > T^*(\bar{\eta}_1)$ . The set  $A'$  being cofinal in  $\delta$ , it has order-type at least  $\mu^+$ , so  $T^*$  is a monotone function from a set of ordinals of order-type  $\geq \mu^+$  into  $\mu$ , which is clearly a contradiction.  $\square$  (Theorem)

Weaker versions are the following list of weaker and weaker principles.

**Definition 11** *The weak square principle  $\square_{\kappa, \lambda}$  says: There is a sequence  $\langle \mathcal{C}_\alpha : \alpha < \kappa \text{ a limit ordinal} \rangle$ , such that:*

1.  $\mathcal{C}_\alpha$  is a set of closed unbounded subsets of  $\alpha$ .
2.  $|\mathcal{C}_\alpha| \leq \lambda$
3. The order type  $\text{otp}(C)$  of each member  $C$  of  $\mathcal{C}_\alpha$  is  $\leq \kappa$ .
4. If  $C \in \mathcal{C}_\alpha$  and  $\beta \in \lim(C)$ , i.e.  $C$  is a limit point of  $C$ , then  $C \cap \beta \in \mathcal{C}_\beta$ .

The principle  $\square_{\lambda, \lambda^+}$ , the so called ‘‘silly square’’ is actually provable (see the proof of Lemma 17), so the weakest reasonable principle is  $\square_{\lambda, \lambda}$ . Our goal is now to show that if  $\lambda$  is singular of cofinality  $\omega$  and above  $\text{LST}(L(I))$ , then  $\square_{\lambda, \lambda}$  fails. This fact by itself indicates that the assumption of the existence of a  $\text{LST}(L(I))$  cardinal has a large consistency strength. At the present it is not known how to get a model in which  $\square_{\lambda, \lambda}$  fails even for a single singular  $\lambda$  without assuming a supercompact cardinal.

The way we shall prove the failure of  $\square_{\lambda, \lambda}$  is by refuting an even weaker property: ‘‘The existence of a good sequence in  $\lambda^\omega / \text{FIN}$  of length  $\lambda^+$ .’’ The definitions and facts about ‘‘good sequences in  $\lambda^\omega / \text{FIN}$ ’’ are due to Shelah and based on his pcf theory ([11]). Since we shall need a much simpler version of the notions and the basic lemmas, we include them for the sake of completeness.

We consider elements of  $\text{On}^\omega$  ordered by eventual domination, i.e. for  $f, g \in \text{On}^\omega$

$$f <^* g \text{ if } f(n) < g(n) \text{ for all but finitely many } n < \omega.$$

**Definition 12** Suppose  $\langle f_\alpha : \alpha < \mu \rangle$  is a  $<^*$ -increasing sequence in  $\text{On}^\omega$ .

- (i) A point  $\delta \in \mu$  is called a good point for the sequence if there is a cofinal set  $C \subseteq \delta$  and a function  $\alpha \mapsto n_\alpha$  from  $C$  to  $\omega$  such that if  $\alpha < \beta$  in  $C$  and  $k > \max(n_\alpha, n_\beta)$ , then  $f_\alpha(k) < f_\beta(k)$ .
- (ii) The sequence is good, if there is a closed unbounded subset  $D$  of  $\mu$  such that  $\delta \in D$  implies that  $\delta$  is a good point of the sequence.

**Lemma 13** Suppose  $\delta$  is a good point for the sequence  $\langle f_\alpha : \alpha < \mu \rangle$  and  $D$  is any cofinal subset of  $\delta$ . Then there is  $E \subseteq D$  witnessing the goodness of  $\delta$ .

**Proof.** Let  $C$  and  $\alpha \mapsto n_\alpha$  witness the goodness of  $\delta$ . W.l.o.g.  $\text{otp}(C) = \text{otp}(D) = \text{cof}(\delta)$ . Let  $E \subseteq D$  be chosen so that for every  $\gamma \in E$  there are  $\gamma^-, \gamma^+ \in C$  in such a way that  $\gamma^- < \gamma < \gamma^+$  and if  $\gamma < \eta \in E$ , then  $\gamma^+ \leq \eta^-$ . Let  $m_\gamma \in \omega$  (for  $\gamma \in E$ ) be such that if  $i > m_\gamma$ , then  $f_{\gamma^-}(i) < f_\gamma(i) < f_{\gamma^+}(i)$ . Let  $n_\gamma^* = \max(n_{\gamma^-}, n_{\gamma^+}, m_\gamma)$ . Now, if  $i \geq \max(n_\gamma^*, n_\eta^*)$ , then

$$f_\gamma(i) < f_{\gamma^+}(i) \leq f_{\eta^-}(i) < f_\eta(i).$$

□ (Lemma)

**Theorem 14 (Shelah [11], see also [2] p.18)** If  $\text{cof}(\lambda) = \omega$  and  $\square_{\lambda, \lambda}$  holds, then there is a good sequence in  $\lambda^\omega$  of length  $\lambda^+$ .

**Proof.** Fix a sequence of regular cardinals  $\lambda_n$  cofinal in  $\lambda$ . We shall actually get our sequence in  $\prod_{n < \omega} \lambda_n \subseteq \lambda^\omega$ . Note that every sequence of functions in  $\prod_{n < \omega} \lambda_n$  of size  $\lambda$  has a  $<^*$ -upper bound in  $\prod_{n < \omega} \lambda_n$  (By taking  $g(n) =$  the supremum of  $f_n(n)$  for the first  $\lambda_{n-1}$  of our functions).

Fix a  $\square_{\lambda, \lambda}$ -sequence  $\langle \mathcal{C}_\alpha : \alpha < \kappa \text{ a limit ordinal} \rangle$ . Without loss of generality we can assume that  $\text{otp}(C) < \lambda$  for each  $C \in \mathcal{C}_\alpha$ . (Indeed, if  $\text{otp}(C) = \lambda$  when  $C \in \mathcal{C}_\alpha$ , then  $\text{cof}(\alpha) = \omega$  and we can replace  $C$  by an  $\omega$ -sequence cofinal in  $\alpha$ . Note that this  $C$  is never used as an initial segment of  $D \in \mathcal{C}_\beta$  for  $\alpha < \beta$  because it would imply  $\text{otp}(D) > \lambda$ ).

We define the  $<^*$ -increasing sequence  $\langle f_\alpha : \alpha < \lambda^+ \rangle$  in  $\prod_{n < \omega} \lambda_n$  by induction. The successor stage is trivial:  $f_{\alpha+1}(n) = f_\alpha(n)$ . Suppose then  $\alpha$  is limit. For each  $C \in \mathcal{C}_\alpha$  we define a function in  $\prod_n \lambda_n$  as follows:

$$g_C(i) = \begin{cases} \sup_{\beta \in C} g_\beta(i), & \text{if } \text{otp}(C) < \lambda_i, \\ 0, & \text{otherwise.} \end{cases}$$

Since  $\mathcal{C}_\alpha \leq \lambda$ , we can find  $f_\alpha \in \prod_n \lambda_n$  such that  $g_C <^* f_\alpha$  for every  $C \in \mathcal{C}_\alpha$ . Clearly  $f_\beta <^* f_\alpha$  for every  $\beta < \alpha$ . We claim that  $\langle f_\alpha : \alpha < \lambda^+ \rangle$  is a good sequence. Actually the claim is that every limit  $\delta < \lambda^+$  is a good point of the sequence. If  $\text{cof}(\delta) = \omega$ , then we pick a cofinal sequence  $\langle \delta_n : n < \omega \rangle$  in  $\delta$ . Let  $n(\delta_n)$  be such that for  $i \geq n(\delta_n)$  we have

$$f_{\delta_{n-1}}(i) < f_{\delta_n}(i) < f_{\delta_{n+1}}(i).$$

Clearly the set  $\{\delta_n : n < \omega\}$  and the map  $\delta_n \mapsto n(\delta_n)$  witness the goodness of  $\delta$ . If  $\text{cof}(\delta) > \omega$ , pick  $C \in \mathcal{C}_\delta$  and let  $C^*$  be the set of limit points of  $C$ . Let  $n$  be such that  $\text{otp}(C) < \lambda_n$  and also  $g_C(i) \leq f_\alpha(i)$  for  $i \geq n$ . If  $\beta < \beta' \in C^*$  and if  $i \geq \max(n_\beta, n_{\beta'})$ , we get  $f_\beta(i) < g_{C \cap \beta'}(i) < f_{\beta'}(i)$  (because  $\beta \in C \cap \beta'$ ). So the set  $C^*$  and the map  $\beta \mapsto n_\beta$  witnesses the goodness of  $\delta$ .  $\square$  (Theorem)

The result for the existence of  $\text{LST}(L(I))$  number follows from

**Theorem 15** *Suppose  $\kappa = \text{LST}(LI)$  and  $\lambda \geq \kappa$  with  $\text{cof}(\lambda) = \omega$ . Then there is no no good sequence in  $\lambda^\omega$  of length  $\lambda^+$ .*

**Proof.** Suppose  $\text{cof}(\lambda) = \omega$ . Suppose that  $\langle f_\alpha : \alpha < \lambda^+ \rangle$  is a good sequence in  $\lambda^\omega$ . Suppose  $D$  is a cub on  $\lambda^+$  such that all points of  $D$  of cofinality  $> \omega$  are good. Let

$$F = \{(\alpha, \beta, \gamma) \in \lambda^+ \times \omega \times \lambda : f_\alpha(\beta) = \gamma\},$$

and

$$\mathcal{A} = \langle \lambda^+, \lambda, <, F, D \rangle.$$

Since  $\kappa = \text{LST}(LI)$  there is

$$\mathcal{B} = \langle B, B \cap \lambda, <, F \cap B^3, D \cap B \rangle \prec_{LI} \mathcal{A}$$

such that  $|B| < \kappa$ . Of course,  $\omega \subset B$ . Since

$$\forall x \neg Iyz(y < x)(z = z)$$

$$\forall x(x < \lambda \rightarrow \neg Iyz(y < x)(z < \lambda))$$

$$\forall y(\lambda < y \rightarrow Iuv(u < \lambda)(v < y))$$

are true in  $\mathcal{A}$ , they are true in  $\mathcal{B}$  and it follows that for some cardinal  $\mu < \kappa$ ,  $\text{otp}(B)$  is  $\mu^+$  and  $\text{otp}(B \cap \lambda) = \mu$ . Let  $\delta = \text{sup}(B)$ . Note that  $\delta$  is a limit

point of  $D$ , and hence  $\delta \in D$ . Since  $B$  is cofinal in  $\delta$ ,  $\text{cof}(\delta) = \mu^+$ . By elementarity, each function  $f_\alpha$ ,  $\alpha \in B$ , maps  $\omega$  into  $B \cap \lambda$ . We now argue that  $\delta$  cannot be a good point of  $\langle f_\alpha : \alpha < \lambda^+ \rangle$ . Assume otherwise. Then there is a cofinal set  $C \subseteq \delta$  and a function  $\alpha \mapsto n_\alpha$  from  $C$  to  $\omega$  such that if  $\alpha < \beta$  in  $C$  and  $k \geq \max(n_\alpha, n_\beta)$ , then  $f_\alpha(k) < f_\beta(k)$ . By Lemma 13 we may assume  $C \subseteq B$ . Let  $C'$  be cofinal in  $C$  such that  $n_\alpha$  is a fixed integer  $N$  for all  $\alpha \in C'$ . Now  $\{f_\alpha(N) : \alpha \in C'\}$  is a subset of  $B \cap \lambda$  which is of order-type  $\mu^+$ , a contradiction.  $\square$  (Theorem)

**Corollary 16** *If  $\kappa = \text{LST}(L(I))$ , then  $\square_{\lambda, \lambda}$  fails for every singular  $\lambda \geq \kappa$  of cofinality  $\omega$ . Hence, in particular, PD holds.*

The existence of  $\text{LST}(L(I))$  also implies the Singular Cardinals Hypothesis above  $\kappa$ , i.e. if  $\lambda$  is singular  $\geq \kappa$ , then

$$\text{(SCH)} \quad \lambda^{\text{cof}(\lambda)} = \max(\lambda^+, 2^{\text{cof}(\lambda)}).$$

It follows from Silver's singular cardinals theorem that if  $\lambda$  violates the SCH and  $\text{cof}(\lambda) > \omega$ , then  $\lambda$  is a limit of cardinals that violate the SCH.

**Lemma 17 ([11])** *If  $\lambda$  is a singular cardinal of cofinality  $\omega$  and  $\lambda$  violates the SCH, then there is a good sequence in  $\lambda^\omega$  of length  $\lambda^+$ .*

**Proof.** By Shelah [11], if  $\lambda$  violates SCH and  $\text{cof}(\lambda) = \omega$ , then there is a sequence  $\langle \lambda_n : n < \omega \rangle$  cofinal in  $\lambda$  such that  $\prod_n \lambda_n / \text{FIN}$  has true cofinality  $\lambda^{++}$ , which implies that every set of functions in  $\prod_n \lambda_n$  of cardinality  $\lambda^+$  has a  $<^*$ -upper bound in  $\prod_n \lambda_n$ . Now one can repeat the proof Theorem 14 by replacing in that proof the  $\square_{\lambda, \lambda}$ -sequence by a  $\square_{\lambda, \lambda^+}$ -sequence (the ‘‘Silly Square’’) and getting the good sequence in  $\prod_n \lambda_n$ . The silly square is always true, for if  $C_\alpha$  is a cub subset of  $\alpha$  of order type  $\text{cof}(\alpha)$ , we can let  $\mathcal{C}_\alpha = \{C_\beta \cap \alpha : \beta < \lambda^+, \alpha \text{ limit point of } C_\beta\}$  and then  $\langle \mathcal{C}_\alpha : \alpha < \lambda^+ \rangle$  witnesses  $\square_{\lambda, \lambda^+}$ . The proof works as before using the fact that for every  $\alpha < \lambda^+$   $|\mathcal{C}_\alpha| \leq \lambda^+$  and that every set of functions in  $\prod_n \lambda_n$  of cardinality  $\lambda^+$  has a  $<^*$ -upper bound.  $\square$  (Lemma)

**Corollary 18** *If  $\kappa = \text{LST}(L(I))$ , then SCH holds above  $\kappa$ .*

**Theorem 19** *If it is consistent to assume the existence of a supercompact cardinal, then it is consistent to assume that  $\text{LST}(L(I))$  is the first supercompact cardinal.*

**Proof.** We refer to Magidor [8]. In this paper, assuming the existence of a supercompact cardinal, a model is constructed in which the first supercompact is the first strongly compact. It is achieved by forcing over a model in which  $\kappa$  is supercompact and arranging for SCH to fail for unboundedly many  $\lambda$ 's below  $\kappa$ , while preserving the supercompactness of  $\kappa$ . In the resulting model  $\kappa \geq \text{LST}(L(I))$  cardinal (even  $\kappa \geq \text{LST}(L^2)$ ). If  $\mu < \kappa$  and  $\mu \geq \text{LST}(L(I))$ , then pick  $\mu < \lambda < \kappa$  violating SCH to get a contradiction with Corollary 18. So  $\kappa = \text{LST}(L(I))$  cardinal.  $\square$  (Theorem)

In the next section we shall show that  $\text{LST}(L(I))$  can be much smaller than the first supercompact cardinal, namely it can be the first inaccessible, so we are in a true “identity crisis” situation.

### 3 The First Mahlo Cardinal

As we pointed out in Theorem 7,  $\text{LST}(L(I, Q^{ec}))$  is, if it exists at all, at least as big as the first Mahlo cardinal. We now prove the consistency of  $\text{LST}(L(I, Q^{ec}))$  being actually equal to the first Mahlo cardinal. As Corollary ?? shows, we have to start from a cardinal substantially larger than a Mahlo, even a strong cardinal is not enough. So we start from a supercompact cardinal.

**Theorem 20** *It is consistent, relative to the consistency of a supercompact cardinal, that  $\text{LST}(L(I, Q^{ec}))$  is the first Mahlo cardinal.*

**Proof.** Suppose  $\kappa$  is supercompact. We then make every  $\rho < \kappa$  non-Mahlo. Suppose  $\rho$  is Mahlo. Let  $P_\rho$  be the set of closed bounded sets of singular cardinals  $< \rho$  inversely ordered by end-extension, i.e. a weaker condition is an initial segment of a stronger condition. For every regular  $\lambda < \rho$  the forcing notions  $P_\rho$  contains a  $\lambda$ -closed dense set  $\{C : \max(C) > \lambda\}$ . Therefore  $P_\rho$  cannot collapse cardinals  $< \rho$  or change their cofinality. Moreover,  $P_\rho$  does not add new bounded subsets to  $\rho$ . On the other hand,  $|P_\rho| = \rho$ , so  $P_\rho$  preserves all cardinals and cofinalities. In particular  $P_\rho$  kills the Mahloness of  $\rho$  but preserves inaccessibility of  $\rho$ . Now we iterate this forcing. Suppose  $\mu_\alpha$ ,  $\alpha < \delta$ , is an increasing sequence of Mahlo cardinals. Let  $R_0 = P_{\mu_0}$ . Suppose  $P_\alpha$  has been defined. Let  $P_\alpha \Vdash \tilde{Q}_\alpha = P_{\mu_\alpha}$  and  $R_{\alpha+1} = R_\alpha \star \tilde{Q}_\alpha$ . For limit  $\alpha$  be let  $R_\alpha$  be the direct limit of the previous stages, if  $\alpha$  is inaccessible, and inverse limit otherwise. This will ensure that each  $R_\alpha$ ,  $\alpha$  inaccessible,

will have the  $\alpha$ -c.c. and will therefore preserve the Mahloness of each  $\mu_\beta$ ,  $\beta \geq \alpha$ . Let  $R = R_\kappa$ . Now  $V^R \models \kappa = \text{LST}(L(I, Q^{ec}))$ . To see why, suppose  $\mathcal{A}$  is a structure with universe  $\lambda$ , where  $\lambda \geq \kappa$ . Since  $\kappa$  is supercompact, there is  $j : V \rightarrow M$ ,  $M$  transitive, such that  ${}^\lambda M \subseteq M$  and  $j(\kappa) > \lambda$ . Note that  $j(R) = R \star P_\kappa \star R_{>\kappa}$ , where  $R_{>\kappa}$  contains a  $\kappa$ -closed set. Since  $R$  has the  $\kappa$ -c.c. and  $R_{>\kappa}$  is sufficiently closed,  $j$  can be extended to  $j^* : V^R \rightarrow M^{j(R)}$ . Now we can continue as above.  $\square$

## 4 The First Inaccessible Cardinal

In this section we prove the main result of this paper:

**Theorem 21** *If  $ZFC +$  "There is a supercompact cardinal" is consistent, so is  $ZFC +$  "There is an inaccessible cardinal" + " $\text{LST}(L(I))$  is the first inaccessible cardinal".*

The assumption of the consistency of a supercompact cardinal seems, on the basis of present technology, almost unavoidable. By Theorem 15 above, we know that the existence of  $\text{LST}(L(I))$  implies the negation of  $\square_{\lambda, \lambda}$  for every large enough  $\lambda$ . The only known way to get a model in which this holds is to start from a strongly compact cardinal. But the definition of strong compactness is not sufficient for getting reflection principles which seem to be necessary for getting the existence of  $\text{LST}(L(I))$ , so the assumption of a supercompact cardinal seems natural enough.

### 4.1 Outline of the Proof

We start with a supercompact cardinal  $\kappa$ . In our final model  $\kappa$  will be the first inaccessible cardinal, while preserving enough of the reflection properties of a supercompact cardinal, so that in the model  $\kappa$  will be  $\text{LST}(L(I))$ .

In the process of achieving this we force a closed unbounded set  $C$  of singular cardinals below  $\kappa$ . This will make  $\kappa$  non-Mahlo. We then collapse cardinals between consecutive elements of  $C$  so that none of them can be inaccessible. Thus  $\kappa$  has become the first inaccessible. But we have to be careful about the way in which we collapse cardinals in order to maintain enough reflection properties of  $\kappa$ , so that  $\kappa$  will be  $\text{LST}(L(I))$ . To argue that  $\text{LST}(L(I)) = \kappa$  in the final model is similar to the argument of Theorem

4. Namely, suppose  $P$  is the forcing used to get our final model and  $\mathcal{A}$  is a name for a finitary structure in  $V^P$  with domain  $\lambda$ . Let  $i : V \rightarrow M$  be an elementary embedding of the universe such that  $i(\kappa) > \lambda$  and  ${}^\lambda M \subseteq M$ , where  $\kappa$  is the critical point of  $i$  ( $V$  is our ground model).  $P$  will be a forcing such that  $P$  as a forcing notion is a regular subforcing of  $i(P)$ . In  $V^{i(P)}$  we can define an embedding  $i^* : V^P \rightarrow M^{i(P)}$  extending  $i$ . By our assumption  $i^* \upharpoonright \mathcal{A} \in V^P$ , and  $i^* \upharpoonright \mathcal{A}$  is an embedding of  $\mathcal{A}$  into  $i^*(\mathcal{A})$ . We would like  $i^*$  to preserve formulas of  $L(I)$ . Given that we are done because

$$M^{i(P)} \models \text{“}i^*(\mathcal{A}) \text{ has an } L(I)\text{-elementary substructure of cardinality } \lambda < i^*(\kappa)\text{”}.$$

By elementarity,

$$V^P \models \text{“}\mathcal{A} \text{ has an } L(I)\text{-elementary substructure of cardinality } < \kappa\text{”}.$$

To get  $i^*$  to preserve formulas of  $L(I)$  we need that  $i(P)/P$  collapses no cardinals  $\leq \lambda$ .

Suppose that when we collapsed cardinals between consecutive members of  $C$  we had some function  $f : \kappa \rightarrow \kappa$  such that for a member  $\delta$  of  $C$  no cardinal was collapsed between  $\delta$  and  $f(\delta)$ . Let us also assume that  $\lambda < i(f)(\kappa)$ . (Note that  $\kappa$  is a limit point of  $i^*(C)$ .) So no cardinal between  $\kappa$  and  $i(f)(\kappa)$  will be collapsed. In particular, all cardinals between  $\kappa$  and  $\lambda$  are preserved by  $i(P)/P$ .

Another issue is that  $\kappa$  is supposed to be a limit point of  $i^*(C)$  hence in  $M^{i(P)}$  it is supposed to be singular. In  $V^P$  it is supposed to be regular, indeed inaccessible. So we need  $i(P)/P$  to make some regular cardinals singular. Since  $i(P)$  “looks like  $P$ ”, we need  $P$  to make enough regular cardinals singular, so that  $M^{i(P)} \models \text{“}\kappa \text{ is singular”}$ .

The standard way of making a regular cardinal singular is by forcing with Prikry forcing on a measurable cardinal. Since we shall have to do it for many cardinals below  $\kappa$ , we shall have to iterate Prikry type forcings for measurable cardinals below  $\kappa$ .

So the forcing notion we shall use will be an iteration of several steps:

- (a) Iterated Prikry type forcing for every measurable  $\lambda < \kappa$ , where besides changing the cofinality of  $\lambda$  to  $\omega$  we do some preparatory forcing for the additional steps, which will be relevant only to  $\kappa$ . We denote this forcing by  $Q_\lambda$  and the iteration up to  $\kappa$  by  $P_\kappa$ .

- (b) On  $\kappa$  we force a closed unbounded set  $C$  such that every limit point of  $C$  is singular. We denote this forcing by  $\text{NM}(\kappa)$  (From “Non-Mahlo”).
- (c) We collapse cardinals between consecutive members of  $C$  making sure that if  $\beta \in C$ , then no cardinals are collapsed between  $\beta$  and  $f(\beta)$  for an appropriate function  $f : \kappa \rightarrow \kappa$ . (We denote this forcing  $\text{Col}(C)$ ).

The challenge will be to make sure that  $\text{NM}(\kappa) * \text{Col}(C)$  embeds nicely into  $\mathbb{Q}_\kappa$  so that if  $R = P_\kappa * \text{NM}(\kappa) * \text{Col}(C)$ , then  $R$  embeds nicely into  $i^*(R)$ . This will be achieved by embedding  $\text{NM}(\kappa) * \text{Col}(C)$  into  $\mathbb{Q}_\kappa$ , which is the  $\kappa$ -th stage in the iteration of  $i^*(P_\kappa) = P_{i^*(\kappa)}$ . We hope that these remarks make the following definition of the forcing notion somewhat less frightening.

## 4.2 The Forcing Construction

Our first step is to define the function  $f : \kappa \rightarrow \kappa$  that will determine intervals where all the cardinals will be preserved. We assume that our ground model satisfies G.C.H. and that there is no inaccessible above  $\kappa$ . A classical lemma of Laver [5] proves the following:

**Lemma 22** *Let  $\kappa$  be supercompact. Then there exists a function  $h : \kappa \rightarrow V_\kappa$  such that for every  $x$  and every  $\mu \geq \kappa$  there is a  $\mu$ -supercompact embedding  $j : V \rightarrow M$  (i.e.  $M^\mu \subseteq M$ ,  $j(\kappa) > \kappa$ ,  $j(\alpha) = \alpha$  for  $\alpha < \kappa$ ), such that  $j(h)(\kappa) = x$ .*

An easy corollary of Laver’s lemma is the following:

**Lemma 23** *Let  $\kappa$  be supercompact such that there is no inaccessible cardinal above  $\kappa$ . Then there is a function  $f : \kappa \rightarrow \kappa$  such that for all  $\alpha < \kappa$ ,  $\alpha < f(\alpha)$ , there is no inaccessible cardinal  $\lambda$  with  $\alpha < \lambda \leq f(\alpha)$ , and for all  $\mu \geq \kappa$  there is a  $\mu$ -supercompact embedding  $j : V \rightarrow M$  with  $\mu < j(f)(\kappa)$ .*

*Proof* Let  $h$  be the Laver-function from Lemma 22. Let  $f(\alpha) = (h(\alpha))^+$  if  $h(\alpha)$  is an ordinal  $> \alpha$  such that there is no inaccessible cardinal  $\lambda$  with  $\alpha < \lambda \leq h(\alpha)$ . Define  $f(\alpha) = \alpha^+$  otherwise. Apply Lemma 22 for  $\mu$  and  $x = \mu^+$ . Note that in  $M$  there is no inaccessible cardinal  $\lambda$  with  $\kappa < \lambda \leq j(h)(\kappa) = \mu^+$  so for  $j(f)(\kappa)$  the first possibility of the definition of  $f$  holds and hence  $j(f)(\kappa) = \mu^+ > \mu$ .  $\square$

So from now on we fix such a function  $f$ . Without loss of generality we can assume that the first inaccessible cardinal is closed under  $f$ . The cardinals that we shall be interested in will be cardinals  $\lambda \leq \kappa$  such that  $\lambda$  is measurable and  $\lambda$  is closed under the function  $f$ . For such  $\lambda$  we define the forcing notion  $\text{NM}(\lambda)$ , which is intended to make  $\lambda$  non-Mahlo by forcing a closed unbounded set  $C$  of cardinals such that every limit point of  $C$  is singular. For technical simplicity it will be convenient to assume that if  $\beta \in C$  and  $\beta'$  is the minimal member of  $C$  above  $\beta$ , then  $\beta'$  is inaccessible and  $f(\beta) < \beta'$ .

**Definition 24** *Suppose  $\lambda$  is a measurable cardinal, closed under  $f$ . Then  $\text{NM}(\lambda)$  is the set of all closed bounded subsets  $C$  of  $\lambda$  such that*

- (a) *Every member of  $C$  is a cardinal.*
- (b) *If  $\beta$  is a limit point of  $C$ , then  $\beta$  is singular.*
- (c) *If  $\beta \in C$ , and  $\beta'$  is the first point of  $C$  above  $\beta$ , then  $\beta'$  is inaccessible and  $f(\beta) < \beta'$ .*

*The partial order  $\leq$  on  $\text{NM}(\lambda)$  is defined by  $D \leq C$  iff  $D, C \in \text{NM}(\lambda)$  and  $D$  is an end-extension of  $C$ .*

So the successor members of  $C$  are all regular, and limit points of  $C$  are closed under  $f$ . It is easy to see that if  $C \in \text{NM}(\lambda)$ , and  $C$  contains a point above  $\mu < \lambda$ , then  $\{D : D \leq C\}$  is  $\mu$ -closed. Hence it follows that forcing with  $\text{NM}(\lambda)$  introduces no new  $\mu$ -sequences of ordinals when  $\mu < \lambda$ . So  $\lambda$  remains regular, and since no new bounded subsets of  $\lambda$  are introduced,  $\lambda$  remains strongly inaccessible. Also, it is easy to see that if  $G \subseteq \text{NM}(\lambda)$  is a generic filter, then  $\bigcup G$  is a closed unbounded subset of  $\lambda$ . Every limit point of  $\bigcup G$  is singular, so in the generic extension  $\lambda$  is a non-Mahlo inaccessible cardinal.

Since we are going to define many partial orders, we shall denote each of the relevant partial orders by  $\leq$ . Only in case of a possible confusion we shall add the subscript indicating the forcing notion ( $\leq_P$  for the partial order of  $P$ , etc). Also in some cases it will be convenient to define a preorder on the forcing notion (so we may write  $p \leq q$  and  $q \leq p$ ), so that we really mean the forcing notion is the equivalence classes of the relation “ $p \leq q$  and  $q \leq p$ ”.

Given two regular cardinals  $\mu < \rho$ ,  $\text{Col}(\mu, < \rho)$  is the usual Levy collapse of all the cardinals  $\delta$  with  $\mu < \delta < \rho$  to  $\mu$ . It is a  $\mu$ -closed forcing notion and

if  $\rho$  is inaccessible or the successor of a regular cardinal, the forcing notion  $\text{Col}(\mu, < \rho)$  satisfies the  $\rho$ -c.c.

Let  $C$  be a closed set of cardinals. For  $\beta \in C - \{\text{sup}(C)\}$  let  $\beta'$  be the next point of  $C$  after  $\beta$ . We assume that if  $\beta \in C$ , then  $\beta'$  is inaccessible and  $f(\beta) < \beta'$ . The forcing notion

$$\text{Col}(C)$$

is defined to be the Easton product of  $\text{Col}(f(\beta), < \beta')$  for  $\beta \in C - \{\text{sup}(C)\}$ . (Easton product means that it is the collection of all functions  $g$  defined on  $C - \{\text{sup}(C)\}$  such that  $g(\beta) \in \text{Col}(f(\beta), < \beta')$  and for regular  $\delta$  the set  $\{\beta < \delta : g(\delta) \neq \emptyset\}$  is bounded in  $\delta$ .) Note that for our case, if  $C \subseteq \lambda$  is the closed unbounded set introduced by  $\text{NM}(\lambda)$ , then the Easton condition simply means that if  $g \in \text{Col}(C)$  then the cardinality of  $\{\beta : g(\beta) \neq \emptyset\}$  is less than  $\lambda$ .

It is easy to see that if  $C$  is  $\text{NM}(\lambda)$ -generic and the first member of  $C$  is below the first inaccessible, then if we force with  $\text{Col}(C)$ , then  $\lambda$  will be the first inaccessible.

#### 4.2.1 Atomic Step

Let  $\lambda$  be a measurable cardinal  $\leq \kappa$  closed under  $f$ . We shall describe a variation  $Q_\lambda$  of Prikry forcing for making  $\lambda$  singular of cofinality  $\omega$ , while at the same time introducing a generic object over  $V$  to  $\text{NM}(\lambda)$ . Like Prikry forcing,  $Q_\lambda$  will introduce no new unbounded subsets of  $\lambda$ .  $Q_\lambda$  has an additional role which we shall explain below. Note that we have

$$V \subseteq V^{\text{NM}(\lambda)} \subseteq V^{Q_\lambda}.$$

(Note that in  $V^{\text{NM}(\lambda)}$  the cardinal  $\lambda$  is still regular.)

The forcing  $Q_\lambda$  has an additional role which we shall explain below. Like Prikry-forcing,  $Q_\lambda$  will introduce no new bounded subsets of  $\lambda$ . The final stage of our iteration will be forcing with  $\text{NM}(\kappa)$  followed by  $\text{Col}(D)$ , where  $D$  is the closed unbounded set introduced by  $\text{NM}(\kappa)$ . We shall need to embed the forcing  $\text{NM}(\kappa) * \text{Col}^{V^{\text{NM}(\kappa)}}(D)$  into  $Q_\kappa * \text{Col}^{V^{Q_\kappa}}(D)$ . Unfortunately,  $\text{Col}^{V^{Q_\kappa}}(D)$  in  $V^{Q_\kappa}$  is different from  $\text{Col}^{V^{\text{NM}(\kappa)}}$  (Because, for instance, in  $\text{NM}(\kappa)$  the cardinal  $\kappa$  is regular, but in  $V^{Q_\kappa}$  it is singular of cofinality  $\omega$ , so the meaning of the Easton Product used in the definition of  $\text{Col}(D)$  is very different.) But note that because  $V^{\text{NM}(\kappa)}$  and  $V^{Q_\kappa}$  have the same bounded

subsets of  $\kappa$ ,  $\text{Col}^{V^{\text{NM}(\kappa)}}(D)$  is a sub-partial order of  $\text{Col}^{V^{Q_\kappa}}(D)$ . Also, if  $G \subseteq \text{Col}^{V^{Q_\kappa}}(D)$  is a filter,  $G \cap V$  is a filter in  $\text{Col}^{V^{\text{NM}(\kappa)}}(D)$ . The issue is the genericity of  $G \cap V$  over  $V^{\text{NM}(\kappa)}$ . In general,  $G \cap V$  does not have to be generic over  $V^{\text{NM}(\kappa)}$ . The additional role of  $Q_\lambda$  (for all  $\lambda \leq \kappa$ ) will be to introduce a condition in  $h \in \text{Col}^{V^{Q_\kappa}}(D)$  such that if  $h \in G \subseteq \text{Col}^{V^{Q_\kappa}}(D)$  is a generic filter on  $V^{Q_\kappa}$ , then  $G \cap V$  is generic over  $V^{\text{NM}(\lambda)}$ .

For the next definition we fix a normal ultrafilter  $U$  on  $\lambda$ .

**Definition 25**  $Q_\lambda$  is the set of all objects  $p$  of the form

$$\langle \alpha_0, \dots, \alpha_{n-1}, A, c, h, C, H \rangle,$$

where

- (i)  $\alpha_0 < \alpha_1 < \dots < \alpha_{n-1} < \lambda$ , each  $\alpha_i$  is closed under  $f$ .
- (ii)  $A \subseteq \lambda$ ,  $A \in U$ .
- (iii)  $c \in \text{NM}(\lambda)$ ,  $c$  contains some point below the first inaccessible, and  $\alpha_i \in c$  for  $i < n$ .
- (iv)  $h \in \text{Col}(c)$
- (v)  $C$  is a function defined on  $A$  such that for all  $\beta \in A$  we have  $C(\beta) \in \text{NM}(\lambda)$  with  $\min(C(\beta)) = \beta$ .
- (vi)  $H$  is a function defined on  $A$  such that for  $\beta \in A$  we have  $H(\beta) \in \text{Col}(C(\beta))$ .
- (vii) For all  $\beta \in A$  we have  $\sup(c) < \beta$  and  $\alpha_{n-1} < \beta$ .
- (viii) For all  $\beta, \beta' \in A$  such that  $\beta < \beta'$  we have  $\sup(C(\beta)) < \beta'$ .

The intuitive meaning of the forcing is rather clear. The finite sequence  $\alpha_0, \dots, \alpha_{n-1}$  is an initial segment of the  $\omega$ -sequence that will be cofinal in  $\lambda$ . The set  $A$  is the set of possible candidates for extending the sequence  $\alpha_0, \dots, \alpha_{n-1}$ . As usual for Prikry type forcings, we require to have a large set of possible candidates to be members of the  $\omega$ -sequence leading up to  $\lambda$ . The set  $c$  is an initial segment of the generic object in  $\text{NM}(\lambda)$  that will be introduced by  $Q_\lambda$ ,  $h$  is a partial information about an object that will eventually be a condition in  $\text{Col}^{V^{Q_\lambda}}(D)$ , where  $D$  will be the club introduced

by the  $\text{NM}(\lambda)$  generic filter. For  $\beta \in A$  the set  $C(\beta)$  is a commitment, in case  $\beta$  will be added to the sequence, that the future generic object of  $\text{NM}(\lambda)$  will contain  $C(\beta)$ . (Not as initial segment because it will have to continue  $c$  and possibly other  $c(\gamma)$ 's for  $\gamma$ 's that were included in the sequence before  $\beta$ .) Similarly for  $H(\beta)$ . These remarks motivate the definition of the partial order in  $Q_\lambda$ :

**Definition 26** (*The partial order of  $Q_\lambda$* ). Suppose

$$p = \langle \alpha_0, \dots, \alpha_{n-1}, A, c, h, C, H \rangle$$

and

$$q = \langle \alpha_0^*, \dots, \alpha_{k-1}^*, A^*, c^*, h^*, C^*, H^* \rangle$$

are in  $Q_\lambda$ . We say that  $q$  extends  $p$ , in symbols  $q \leq p$ , if

- (i)  $n \leq k$ ,  $\alpha_i = \alpha_i^*$  for  $i < n$ , and  $n \leq i < k$  implies  $\alpha_i^* \in A$ .
- (ii)  $A^* \subseteq A$ ,
- (iii)  $c^*$  is an end extension of  $c$  (Hence it is an extension of  $c$  in the sense of  $\text{NM}(\lambda)$ .)
- (iv) If  $n = k$ , then  $c^* = c$ . If  $n < k$ , then  $c^* \cap \alpha_n^* = c$  and for all  $n \leq i < k-1$  the set  $c^* \cap [\alpha_i^*, \alpha_{i+1}^*]$  is an end extension of  $C(\alpha_i^*)$ , and  $c^* - \alpha_{k-1}^*$  is an end extension of  $C(\alpha_{k-1})$ . (Note that this condition together with clause (v) of Definition 25 implies that for  $n \leq i < k$  we have  $\alpha_i^* \in c^*$ . *Why need to say this? Isn't it true by definition?*)
- (v) If  $n = k$ , then  $h^* = h$ . If  $n < k$ , then  $h = h^* \upharpoonright c \cap \{\alpha_n^*\}$  (Namely, we do not add any additional information about the forcing condition we construct in  $\text{Col}(D)$  below  $\alpha_n^*$ .)
- (vi) For  $n \leq i < k-1$  the set  $h^* \upharpoonright c \cap [\alpha_i^*, \alpha_{i+1}^*]$  extends  $H(\alpha_i^*)$  and  $h^* \upharpoonright c \cap [\alpha_{k-1}^*, \kappa]$  extends  $H(\alpha_{k-1}^*)$
- (vii) For all  $\alpha \in A^*$  the condition  $C^*(\alpha)$  is an extension (as a member of  $\text{NM}(\lambda)$ ) of  $C(\alpha)$  and  $H^*(\beta)$  is an extension (as a member of  $\text{Col}(C^*(\alpha))$ ) of  $H(\alpha)$ . Note that  $H(\alpha)$ , being a member of  $\text{Col}(C(\alpha))$  can be considered also to be a member of  $C^*(\alpha)$  since  $C^*(\alpha)$  is an extensions of  $C(\alpha)$ .

If  $n = k$  above, we say that  $q$  is a direct extension of  $p$ , in symbols  $q \leq^* p$ .

**Notation:** If  $p = \langle \alpha_0, \dots, \alpha_{n-1}, A, c, h, C, H \rangle$  is in  $Q_\lambda$  we call  $n$  the *length* of  $p$  and denote it by  $n(p)$ . Similarly

$$\begin{array}{lll}
\alpha(p) & = & \langle \alpha_0, \dots, \alpha_{n-1} \rangle \text{ the } \alpha\text{-part of } p \\
A(p) & = & A \text{ the } A\text{-part of } p \\
c(p) & = & c \text{ the } c\text{-part of } p \\
h(p) & = & h \text{ the } h\text{-part of } p \\
C(p) & = & C \text{ the } C\text{-part of } p \\
H(p) & = & H \text{ the } H\text{-part of } p
\end{array}$$

Note that if  $\mu < \lambda$  then every decreasing sequence of direct extension of length  $\leq \mu$  has a lower bound which is a direct extension of all the members of the sequence. To see this one uses the fact that the  $c$  parts and  $h$  parts of all the members of the sequence are the same, and if  $\alpha$  is in the intersection of the  $A$  part of the member for the sequence and  $\alpha > \mu$ , then since  $C(\alpha)$  is assumed to contain  $\alpha$ , the sequence of the  $c$ -parts has a lower bound because  $\text{NM}(\lambda)$  is  $\mu^+$ -closed under a condition containing a point above  $\mu$ . A similar argument takes care of the  $H$  part.

The following lemma is a typical one for Prikry type forcing. Its proof is a straight-forward generalization of similar lemmas proved before (e.g. Magidor [9]):

**Lemma 27** *Let  $\Phi$  be a statement in the forcing language for  $Q_\lambda$  and  $p \in Q_\lambda$ . Then there exists a direct extension  $q$  of  $p$  such that  $q$  decides  $\Phi$ .*

As usual, it follows from the lemma that  $Q_\lambda$  introduces no new bounded subsets of  $\lambda$ .

**Lemma 28** *Let  $G \subseteq Q_\lambda$  be generic over  $V$ . Let  $D_G = \{c(p) : p \in G\}$ . Then  $D_G \subseteq \text{NM}(\lambda)$  generates a generic filter of  $\text{NM}(\lambda)$  over  $V$ .*

*Proof:* It is immediate that for  $p, p' \in G$  either  $c(p) \leq c(p')$  or  $c(p') \leq c(p)$ , so  $D_G$  generates a filter. We just have to prove its genericity. Let  $E \subseteq \text{NM}(\lambda)$ ,  $E \in V$ , be dense open in  $\text{NM}(\lambda)$ . We have to show that  $E \cap D_G \neq \emptyset$ . Let

$$p = \langle \alpha_0, \dots, \alpha_{n-1}, A, c, h, C, H \rangle \in Q_\lambda.$$

We show that an extension of  $p$  forces that  $E \cap D_G \neq \emptyset$ . For every  $\alpha \in A$  the set  $c \cup C(\alpha)$  is a condition in  $\text{NM}(\lambda)$ , so it has an extension in  $E$ . Denote this extension by  $C^*(\alpha)$ . Let

$$p^* = \langle \alpha_0, \dots, \alpha_{n-1}, A, c, h, C^*, H \rangle.$$

This is an extension of  $p$  (in fact a direct extension). Any extension of  $p^*$  of length  $> n$  forces  $E \cap D_G \neq \emptyset$ .  $\square$  (Lemma)

We abuse notation by denoting also  $\cup D_G$  by  $D_G$ . It is a club in  $\lambda$  in which limit points are all singulars. Its minimal point is below the first inaccessible cardinal. (Recall clause (iii) of Definition 25.) As usual,  $\bigcup\{\alpha(p) : p \in G\}$  is an  $\omega$ -sequence cofinal in  $\lambda$ , so  $\lambda$  has cofinality  $\omega$  in  $V[G]$ .

Now let us consider the  $h$ -parts of conditions in  $G$ . Let  $H_G = \bigcup\{h(p) : p \in G\}$

**Lemma 29**  $V[G] \models H_G \in \text{Col}(D_G)$ .

*Proof:*  $H_G$  is clearly a partial function defined on  $\beta \in D_G$ . Note that if  $p, q$  are in  $G$ , each of them of length  $> n$ , then if  $\alpha_n$  is the  $n$ -th coordinate (in both of them) of their  $\alpha$ -part, then  $h(p) \upharpoonright \alpha_n = h(q) \upharpoonright \alpha_n$ . It means that

$$H_G \upharpoonright \alpha_n = h(p) \text{ for any } p \text{ of length } > n \text{ belonging to } G. \quad (3)$$

Now  $h(p) \in \text{Col}(c(p))$ , so for every  $\beta' \in \text{dom}(H_G)$  we have  $H_G(\beta) \in \text{Col}(f(\beta), < \beta')$ , where  $\beta'$  is the first member of  $D_G$  above  $\beta$ . By (3) also  $H_G \upharpoonright \alpha_n$  has Easton support, but since  $\lambda = \sup_{n < \omega} \alpha_n$  is singular, it follows that the support constraint in  $\text{Col}^{V[G]}(D)$  is satisfied by  $H_G$  and hence  $H_G \in \text{Col}(D_G)$ .  $\square$  (Lemma 29)

The next lemma explains the role of the  $h$  part of the conditions in  $Q_\lambda$ .

**Lemma 30** *Let  $G \subseteq Q_\lambda$  be generic over  $V$  and  $D_G, H_G$  as above. Let  $G^* \subseteq \text{Col}^{V[G]}(D_G)$  be generic over  $V[G]$  such that  $H_G \in G^*$ . Let  $G^{**} = \text{Col}^{V[D_G]}(D_G) \cap G^*$ . Then  $G^{**}$  is a generic filter in  $\text{Col}^{V[D_G]}(D_G)$  over  $V[D_G]$ . (Note that  $\text{Col}^{V[D_G]}(D_G) \subseteq \text{Col}^{V[G]}(D_G)$ .)*

*Proof:*  $G^{**}$  is obviously a filter. We have to verify genericity. So let  $\mathring{E}$  be an  $\text{NM}(\lambda)$ -term, which is forced to be a dense open subset of  $\text{Col}^{V[D_G]}(D_G)$  and let  $E_G$  be its realization in  $V[D_G]$ . Let  $p \in Q_\lambda$ . We shall extend  $p$  to

a condition  $q$  such that  $q \Vdash G^* \cap \dot{E} \neq \emptyset$ . Then, since  $E_G \in V[D_G]$ ,  $q \in G$  implies

$$G^{**} \cap E_G = G^* \cap V[D_G] \cap E_G \neq \emptyset.$$

Assume

$$p = \langle \alpha_0, \dots, \alpha_{n-1}, A, c, h, C, H \rangle.$$

Without loss of generality we can assume that  $\alpha \in A$  implies  $\alpha$  is inaccessible. Fix  $\alpha \in A$  and consider  $c \cup C(\alpha)$  which is a condition in  $\text{NM}(\lambda)$  such that  $\alpha$  belongs to it. Assume for a while that  $c \cup C(\alpha) \in D_G$ . Note that in  $V[D_G]$ ,  $\text{Col}(D_G)$  can be represented as the cartesian product  $\text{Col}(c \cup \{\alpha\}) \times \text{Col}(D_G - \alpha)$ . Since  $\alpha$  is inaccessible,  $\text{Col}(c \cup \{\alpha\})$  has cardinality  $\alpha$ .  $\text{Col}(D_G - \alpha)$  is  $\alpha^+$ -closed. (Remember that we start collapsing cardinals above  $\alpha$  at  $f(\alpha) > \alpha$ .) So by standard arguments the following set is dense open subset of  $\text{Col}(D_G - \alpha)$  in  $V[D_G]$ :

$$\begin{aligned} E^* &= \{g : g \in \text{Col}(D_G - \alpha), \\ &\quad \{g' : g' \in \text{Col}(c \cup \{\alpha\}), (g, g') \in E\} \\ &\quad \text{is dense open in } \text{Col}(c \cup \{\alpha\})\} \end{aligned}$$

So there is an extension  $c_\alpha^*$  of  $c \cup C(\alpha)$  in  $\text{NM}(\lambda)$  such that some  $\dot{g}_\alpha$  satisfies

$$c_\alpha^* \Vdash \dot{g}_\alpha \in \text{Col}(D_G - \alpha) \wedge \dot{g}_\alpha \in E^* \wedge \dot{g}_\alpha \leq H(\alpha).$$

(Note that since  $c_\alpha^* \leq c \cup C(\alpha)$  and  $H(\alpha) \in \text{Col}(C(\alpha))$ ,  $c_\alpha^* \Vdash H(\alpha) \in \text{Col}(D_G - \alpha)$ .) Without loss of generality we can assume  $\dot{g}$  is essentially a bounded subset of  $\lambda$  and since forcing with  $\text{NM}(\lambda)$  does not add bounded subsets of  $\lambda$ , we can assume that we have a particular  $g_\alpha$  which we can assume to be in  $\text{Col}(c_\alpha^*)$  such that

$$c_\alpha^* \Vdash \check{g}_\alpha \in \text{Col}(D_G - \alpha) \wedge \check{g}_\alpha \in E^* \wedge \check{g}_\alpha \leq H(\alpha).$$

The condition  $c_\alpha^*$  must be of the form  $c \cup c_\alpha^{**}$  where  $c_\alpha^{**}$  is an end extension of  $C(\alpha)$ . So we define the extension of  $p$  to be

$$q = \langle \alpha_0, \dots, \alpha_{n-1}, A^*, c, h, C^*, H^* \rangle,$$

where  $C^*(\alpha) = c_\alpha^{**}$  and  $H^*(\alpha) = g_\alpha$ . The set  $A^*$  is a subset of  $A$  in  $U$  such that each  $\alpha \in A$  satisfies the conditions (v), (vi) and (viii) of Definition 25 with respect to the new  $C$  and  $H$  parts.

Now assume  $q \in G$ , let  $q'$  be an extension of  $q$  of length  $> n$ . Let  $\alpha$  be the  $n$ -th element of the  $\alpha$ -part of  $q'$ . Then  $\alpha \in A^*$ . The  $c$  part of  $q'$  extends  $c \cup C^*(\alpha) = c \cup c_\alpha^{**}$ . The  $h$  part extends  $h \cup H^*(\alpha) = h \cup g_\alpha$ . Now assume  $q' \in G$ . It means that  $D_G$  is an end extension of  $c \cup c_\alpha^{**}$ . In  $V[G]$  we can also represent  $\text{Col}(D_G)$  as  $\text{Col}(c \cup \{\alpha\}) \times \text{Col}(D_G - \alpha)$ . Note that  $H_G$  extends  $h \cup g_\alpha$  and  $H_G \in G^*$ , so  $g_\alpha \in G^* \cap V[D_G] = G^{**}$ . Hence in  $V[D_G]$  the set

$$E' = \{g' : g' \in \text{Col}(c \cup \{\alpha\}), (g', g_\alpha) \in E_G\}$$

is dense open.

But  $\text{Col}(c \cup \{\alpha\})$  is the same in  $V[D_G]$  and in  $V[G]$ . (Because it is a set of bounded subsets of  $\lambda$  and all bounded subsets of  $\lambda$  in  $V[D_G]$  and in  $V[G]$  are also in  $V$ .) The filter  $G^* \cap \text{Col}(c \cup \{\alpha\})$  is generic in  $\text{Col}(c \cup \{\alpha\})$  over  $V[G]$ , so  $G^* \cap E' \neq \emptyset$ . Let  $g' \in G^* \cap E'$ ,  $(g', g_\alpha) \in E_G$ ,  $(g', g_\alpha) \in G^*$ ,  $(g', g_\alpha) \in V[D_G]$ . So  $(g', g_\alpha) \in E_G \cap G^* \cap V[D_G] = E_G \cap G^{**} \neq \emptyset$ .  $\square$  (Lemma 30).

We also want to show that  $Q_\lambda$  collapses no cardinals. By Lemma 27 forcing with  $Q_\lambda$  introduces no new bounded subsets of  $\lambda$ , so no new cardinals  $\leq \kappa$  then  $\lambda$  is collapsed **Unclear meaning**.

We assume G.C.H.,  $Q_\lambda$  has cardinality  $2^\lambda = \lambda^+$  so no cardinals above  $\lambda^+$  are collapsed. So the only cardinal we have to consider is  $\lambda^+$ . Forcing with  $Q_\lambda$  makes  $\lambda$  singular of cofinality  $\omega$ , so if  $\lambda^+$  is not a cardinal it becomes singular of cofinality  $< \lambda$ . In order to handle this case we need arguments similar to Magidor [9]. First we need the following definition:

**Definition 31** Let  $q \leq p$  be as in Definition 26. Let

$$r = \langle \alpha_0^*, \dots, \alpha_{k-1}^*, A^{**}, c^*, h^*, C \upharpoonright A^{**}, H \upharpoonright A^{**} \rangle,$$

where  $A^{**}$  is the set  $\{\beta \in A : \alpha_{k-1}^* < \beta\}$ .

We call  $r$  the *Interpolant* of  $p, q$  ( $\text{Int}(p, q)$ ). The condition  $r$  is the maximal extension of  $p$  such that  $q$  is a direct extension of it. Note that knowing  $p$ , the condition  $r$  is determined by  $\alpha_n^*, \dots, \alpha_{k-1}^*, c^*$  and  $h^*$ . Since the cardinality of possible  $\alpha_n^*, \dots, \alpha_{k-1}^*, c^*, h^*$  is  $\lambda$ , we get that the set  $\{\text{Int}(q, p) : q \leq p\}$  is of size  $\lambda$ .

The following lemma is exactly like Theorem 2.6 in Magidor [9].

**Lemma 32** Let  $D$  be a dense open subset of  $Q_\lambda$  and let  $p \in Q_\lambda$  have length  $n$ . Then there exists a direct extension  $q$  of  $p$  ( $q \leq^* p$ ) such that if  $t \leq q$  and  $t \in D$ , then  $\text{Int}(t, q) \in D$ .

**Lemma 33** *In  $V^{Q_\lambda}$  the cardinal  $\lambda^+$  is still a cardinal.*

*Proof:* Suppose  $p$  forces that  $\tau$  is a function  $\mu \rightarrow \lambda^+$ ,  $\mu < \lambda$ , and  $\tau$  is cofinal in  $\lambda^+$ . For  $\alpha < \mu$ , let  $D_\alpha$  consist of such  $q$  that either  $q$  is incompatible with  $p$ , or  $q \leq p$  and  $q$  forces a value for  $\tau(\alpha)$ . Use Lemma 32  $\mu$  times to get a  $\leq^*$ -decreasing sequence  $\langle q_\alpha : \alpha \leq \mu \rangle$  where  $q_0 = q$ , and  $q_{\alpha+1}$  satisfies Lemma 32 with respect to  $D_\alpha$ . For every  $\alpha < \mu$  the possible values for  $\tau(\alpha)$  when we force below  $q_\mu$  are determined by a member of  $\{Int(t, q_\mu) : t \leq q_\mu\}$ , so it has cardinality  $\leq \mu$ . Hence the range of  $\tau$  is included in a set of  $V$  of cardinality  $\mu \cdot \lambda < \lambda^+$ . So the range of  $\tau$  is bounded in  $\lambda^+$ , a contradiction.  $\square$  (Lemma 33)

The forcing  $Col(D_G)$  of course collapses cardinals but the following lemma will be useful:

**Lemma 34** *Let  $G, G^*, G^{**}$  be as in Lemma 30. Then:*

- (a) *The only  $V$ -cardinals collapsed in  $V[G][G^*]$  are the cardinals in the interval  $(f(\beta), \beta')$ , where  $\beta \in D_G$  and  $\beta'$  is the next member of  $D_G$  above  $\beta$ .*
- (b) *The only  $V$ -cardinals collapsed in  $V[D_G][G^{**}]$  are the cardinals in the interval  $(f(\beta), \beta')$ , where  $\beta \in D_G$  and  $\beta'$  is the next member of  $D_G$  above  $\beta$ .*
- (c)  *$V[G][G^*]$  and  $V[D_G][G^{**}]$  have the same cardinals.*
- (d)  *$V[G][G^*]$  and  $V[D_G][G^*]$  have the same bounded subsets of  $\lambda$ .*

*Proof:* (a) is standard, after we have Lemma 33. The only possible problem is again  $\lambda^+$ , but if it collapsed, it becomes singular of cofinality  $< \lambda$ . The forcing  $Col(D_G)$  is such that every  $\mu$ -sequence of ordinals is introduced by  $Col(D_G \upharpoonright \rho)$  for some  $\rho < \lambda$ , so it is of cardinality  $< \lambda$ . So  $\lambda^+$  is not collapsed.

(b) follows from (a) for cardinals  $< \lambda$ .  $G^*$  is generic over  $V[D_G]$  with respect to a forcing notion of size  $\lambda$ , so no cardinal above  $\lambda$  is collapsed.

(c) follows immediately from (a) and (b).

(d) follows from the fact that a bounded subset of  $\lambda$  introduced by  $Col(D_G)$  is introduced by  $Col(D_G \cap \beta)$  for some  $\beta < \lambda$ . This is true for both  $V[D_G]$  and  $V[G]$ .  $Col(D_G \cap \beta)$  is the same in  $V[D_G]$  and  $V[G]$ , and  $G^{**} \cap Col(D_G \cap \beta) = G^* \cap Col(D_G \cap \beta)$ . So (d) follows.  $\square$  (Lemma 34)

## 4.2.2 Iteration

We would like to iterate the forcing  $Q_\lambda$  for all measurable  $\lambda < \kappa$ . The scheme of iteration we shall use is the scheme introduced by Gitik [3]. Our terminology follows (with minor changes) the terminology of [3].

**Definition 35** *Suppose  $\lambda$  is a regular cardinal. A forcing notion  $\langle P, \leq \rangle$  is said to be  $\lambda$ -Prikrý if there is a partial order  $\leq^* \subseteq \leq$  on  $P$  such that*

- (a) *Every  $\leq^*$ -decreasing sequence of length less than  $\lambda$  has a  $\leq^*$ -lower bound.*
- (b) *For every statement  $\Phi$  in the forcing language for  $P$  and for every  $p \in P$  there is  $q \in P$  such that  $q \leq^* p$  and  $q$  decides  $\Phi$ .*

*We call  $\leq^*$  the direct extension relation.*

Note that we do *not* assume that any two strict extensions of  $p$  are necessarily compatible. The remarks above show that if  $\lambda$  is measurable and  $U$  is a normal ultrafilter on  $\lambda$ , then  $Q_\lambda$  is a  $\lambda$ -Prikrý forcing notion.

When we refer to Prikrý forcing notions in the sequel we assume that they are given with  $\leq^*$ , that is, they are of the form  $\langle P, \leq, \leq^* \rangle$ . We shall also assume that each forcing notion  $P$  is given with its maximal element  $\mathbf{1}_P$ .

**Definition 36** *An iteration*

$$\langle \langle P_\alpha : \alpha \leq \mu \rangle, \langle Q_\alpha : \alpha < \mu \rangle \rangle$$

*is called a Gitik iteration of Prikrý forcings if the following holds: Each  $P_\alpha$  is a forcing notion of sequences of length  $\alpha$  such that*

- (i) *If  $p = \langle \tau_\beta : \beta < \alpha \rangle \in P_\alpha$  and  $\gamma < \alpha$ , then  $p \upharpoonright \gamma = \langle \tau_\beta : \beta < \gamma \rangle \in P_\gamma$ .*
- (ii) *If  $p = \langle \tau_\beta : \beta < \alpha \rangle \in P_\alpha$  and  $\gamma < \alpha$  then  $p \upharpoonright \gamma \Vdash_{P_\gamma} \tau_\gamma \in Q_\gamma$ , where  $Q_\gamma$  is a  $P_\gamma$ -name forced over  $P_\gamma$  to denote a  $\gamma$ -Prikrý forcing with the partial orders  $\leq_\gamma, \leq_\gamma^*$ .*
- (iii) *The sequence has Easton support, namely for every regular  $\gamma \leq \alpha$  the set  $\{\beta < \gamma : \tau_\beta \neq \mathbf{1}_{Q_\beta}\}$  has cardinality  $< \gamma$ .*
- (iv)  *$Q_\gamma$  is the trivial forcing notion unless both  $\gamma$  is Mahlo and  $\Vdash_{P_\beta} |Q_\beta| < \gamma$  for every  $\beta < \gamma$ .*

The partial order on  $P_\alpha$  is defined as follows: Suppose  $q = \langle \tau_\beta^* : \beta < \alpha \rangle$  and  $p = \langle \tau_\beta : \beta < \alpha \rangle$ . Then  $q \leq p$  if there is a finite set  $B \subseteq \{\beta < \alpha : \tau_\beta \neq \mathbf{1}_{Q_\beta}\}$  such that:

- (a) If  $\beta \notin B$  such that  $\tau_\beta \neq \mathbf{1}_{Q_\beta}$ , then  $q \restriction \beta \Vdash \tau_\beta^* \leq_\beta^* \tau_\beta$ .
- (b) If  $\beta \in B$  or  $\tau_\beta = \mathbf{1}_{Q_\beta}$ , then  $q \restriction \beta \Vdash \tau_\beta^* \leq_\beta \tau_\beta$ .

(Namely, we can take a non-direct extension of any point  $\beta$  in which  $\tau_\beta = \mathbf{1}_{Q_\beta}$  and only at finitely many points  $\beta$  in which  $\tau_\beta \neq \mathbf{1}_{Q_\beta}$ .) The direct extension for  $P_\alpha$  is defined as  $q \leq^* p$  if in the above definition we can take  $B = \emptyset$ .

Let us fix now a Gitik iteration  $\langle \langle P_\alpha : \alpha \leq \mu \rangle, \langle Q_\alpha : \alpha < \mu \rangle \rangle$  of Prikry forcings. Now Lemma 1.3 of Gitik [3] is essentially:

**Lemma 37** *Let  $\alpha$  be Mahlo such that  $\Vdash_{P_\gamma} |Q_\gamma| < \alpha$  for all  $\gamma < \alpha$ . Then  $P_\alpha$  has cardinality  $\leq \alpha$  and it satisfies the  $\alpha - c.c.$ .*

Lemma 1.4 of Gitik [3] is essentially:

**Lemma 38** *Let  $\Phi$  be a statement for the forcing language for  $P_\mu$  and  $p \in P_\mu$ . Then there is  $q \leq^* p$  such that  $q \Vdash \Phi$  or  $q \Vdash \neg \Phi$ .*

It follows that if  $\alpha$  is the first such that  $Q_\alpha$  is not the trivial forcing notion, then  $P_\mu$  is  $\alpha$ -Prikry. Also, if  $\alpha$  is a Mahlo cardinal such that  $\Vdash_{P_\gamma} |Q_\gamma| < \alpha$  for all  $\gamma < \alpha$ , then in  $V^{P_\alpha}$  we can consider

$$\langle \langle P_\beta/P_\alpha : \alpha \leq \beta \leq \mu \rangle, \langle Q_\beta : \alpha \leq \beta < \mu \rangle \rangle$$

to be a Gitik iteration of Prikry forcing notions, so in particular we get:

**Lemma 39** *If  $\alpha < \mu$  is a Mahlo cardinal such that  $\Vdash_{P_\gamma} |Q_\gamma| < \alpha$  for all  $\gamma < \alpha$ , then*

- (i)  $P_\mu/P_\alpha$  is an  $\alpha$ -Prikry forcing notion.
- (ii) Every bounded subset of  $\alpha$  in  $V^{P_\mu}$  belongs already to  $V^{P_\alpha}$ . (So, in particular, no  $\alpha$  satisfying the above requirement is collapsed.)

The next lemma is a variation of Lemma 1.5 in Gitik [3] and it deals with the preservation of measurable cardinals by the Gitik iterations:

**Lemma 40** *Let  $\alpha \leq \mu$  be measurable such that  $2^\alpha = \alpha^+$  and  $\Vdash_{P_\beta} |Q_\beta| < \alpha$  for all  $\beta < \alpha$ . Let  $A = \{\beta < \alpha : \Vdash_{P_\beta} \text{“}Q_\beta \text{ is the trivial forcing notion”}\}$ . Let  $U$  be a normal ultrafilter on  $\alpha$  such that  $A \in U$ . Then in  $V^{P_\alpha}$  the filter  $U$  can be extended to a normal ultrafilter. In particular,  $\alpha$  remains measurable.*

*Proof:* Let  $j$  be the ultrapower embedding  $j : V^\kappa/U \rightarrow M$ . Note that  $j(\langle\langle P_\beta : \beta \leq \alpha \rangle, \langle Q_\beta : \beta < \alpha \rangle\rangle)$  is in  $M$  a Gitik iteration of Prikry forcings of length  $j(\alpha)$ . Let us denote the new iteration  $\langle\langle P_\beta^* : \beta \leq j(\alpha) \rangle, \langle Q_\beta^* : \beta < j(\alpha) \rangle\rangle$ . Since  $\Vdash_{P_\beta} |Q_\beta| < \alpha$  for all  $\beta < \alpha$ , we get that for all  $\beta < \alpha$   $|P_\beta| < \alpha$ ,  $Q_\beta^* = Q_\beta$ ,  $P_\beta^* = P_\beta$  and also  $P_\alpha^* = P_\alpha$ . Our assumption that  $A \in U$  translates into

$$\Vdash_{P_\alpha} \text{“}Q_\alpha \text{ is the trivial forcing notion.”}$$

So

$$M^{P_\alpha} \Vdash \text{“}P_{j(\alpha)}/P_\alpha \text{ is an } \alpha^+ \text{-Prikry forcing notion.”}$$

The forcing  $P_\alpha$  satisfies the  $\alpha$ -c.c. and has cardinality  $\alpha$  with  $2^\alpha = \alpha^+$ , so we can enumerate in  $V$  in a sequence of length  $\alpha^+$  all  $P_\alpha$ -terms forced to denote subsets of  $\alpha$ . Let this list be  $\langle \dot{A}_\delta : \delta < \alpha^+ \rangle$ . Note that since  $M$  is closed under  $\alpha$ -sequences, initial segments of the sequence  $\langle j(\dot{A}_\delta) : \delta < \alpha^+ \rangle$  are in  $M$ . Now we argue in  $V^{P_\alpha}$ . By induction define a  $\leq^*$ -decreasing sequence  $\langle p_\delta : \delta < \alpha^+ \rangle$  in  $P_{j(\alpha)}/P_\alpha$  such that for each  $\delta < \alpha^+$  the condition  $p_{\delta+1}$  decides the statement ‘ $\alpha \in j(\dot{A}_\delta)$ ’. ( $j(\dot{A}_\delta)$  is a  $P_{j(\alpha)}^* = j(P_\alpha)$ -term, but in  $M^{P_\alpha}$  we can consider it to be a  $P_{j(\alpha)}^*/P_\alpha$ -term. By  $P_{j(\alpha)}^*/P_\alpha$  we can find such a condition  $p_{\delta+1} \leq^*$ -extending  $p_\delta$ . Every initial segment of the sequence  $\langle p_\delta : \delta < \alpha^+ \rangle$  is in  $M$ , so at limit stages  $\delta$  we can take  $p_\delta$  to be a  $\leq^*$ -lower bound for  $\langle p_\eta : \eta < \delta \rangle$ .) Now in  $V^{P_\alpha}$  define the ultrafilter  $U^*$  extending  $U$  by

$$A_\delta \in U^* \iff p_{\delta+1} \Vdash \kappa \in j(\dot{A}_\delta).$$

It is easy to check that  $U^*$  is well-defined and that it is a normal ultrafilter in  $V^{P_\alpha}$  extending  $U$ .  $\square$  (Lemma 40)

### 4.3 The Final Model

We are now ready to define our final model in which the first inaccessible will be a LST( $L(I)$ ). Let us fix a supercompact cardinal  $\kappa$ . Recall that we assume G.C.H. to hold in  $V$ . We start the construction by defining a Gitik iteration  $\langle\langle P_\alpha : \alpha \leq \kappa \rangle, \langle Q_\alpha : \alpha < \kappa \rangle\rangle$  of Prikry forcings. of length  $\kappa$ . The

iteration will be defined if we inductively define  $Q_\alpha$ . By induction it will be clear that for  $\alpha < \gamma$ ,  $\gamma$  Mahlo, we have  $\Vdash |Q_\alpha| < \gamma$ . So we define:

- (i) If  $\alpha$  is not measurable in  $V$  then  $Q_\alpha$  is the trivial forcing notion.
- (ii) If  $\alpha$  is measurable in  $V$ , then we pick a normal ultrafilter  $U$  on  $\alpha$  such that  $A = \{\beta < \alpha : \beta \text{ non-measurable}\} \in U$ .

Then  $\alpha$  and  $U$  satisfy the assumptions of Lemma 37. So in  $V^{P_\alpha}$  the cardinal  $\alpha$  is still measurable with a normal ultrafilter extending  $U$ . Define  $Q_\alpha$  to be the  $Q_\alpha$  as defined in Section 4.2.1. It is an  $\alpha$ -Prikry forcing and its cardinality is  $2^\alpha$  which is less than the next Mahlo cardinal. So the iteration is defined. Since  $P_\kappa$  is of cardinality  $\kappa$  and satisfies the  $\kappa$ -c.c., the cardinal  $\kappa$  is still Mahlo in  $V^{P_\kappa}$ . (In fact, by Lemma 37 it is still measurable.) So now we force over  $V^{P_\alpha}$  with  $\text{NM}(\kappa)$  to get a closed unbounded subset  $D$  of  $\kappa$  such that each of the limit points of  $D$  is singular, and then we force with  $\text{Col}(D)$ . So our final forcing notion is

$$P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D).$$

Forcing with  $\text{Col}(D)$  makes sure that there are no inaccessible cardinals  $< \kappa$ . Forcing with  $\text{NM}(\kappa)$  keeps  $\kappa$  inaccessible, similarly for  $\text{Col}(D)$ , so in  $V^{P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)}$  the cardinal  $\kappa$  is the first inaccessible. Our goal will be achieved when we show:

**Theorem 41** *In  $V^{P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)}$  the cardinal  $\kappa$  is  $\text{LST}(L(I))$ .*

*Proof:* Denote  $V^* = V^{P_\kappa}$ . In  $V^*$  the cardinal  $\kappa$  is still measurable, so by picking a normal ultrafilter  $U$  on  $\kappa$  we can define the forcing  $Q_\kappa$  and force with it over  $V^*$ . Let  $G$  be the generic filter in  $Q_\kappa$ . By Lemma 28 we can define from  $G$  an  $\text{NM}(\kappa)$  filter  $D_G$  which is going to be generic over  $V^*$ . We can force further with  $\text{Col}(D_G)$ . Let  $G^*$  be the generic filter, where we can assume that  $H_G \in G^*$ . By Lemma 30  $G^{**} = G^* \cap V^*[D_G]$  is generic over  $V^*[D_G]$  with respect to  $\text{Col}^{V^*[D_G]}(D_G)$ . So  $V^*[D_G][G^{**}]$  is a  $P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)$  generic extension of  $V$ . By Lemma 34  $V^*[D_G][G^{**}]$  has the same bounded sequences of elements of  $\kappa$  and the same cardinals as  $V^*[G][G^*]$ .

It is now easily seen that given any generic  $D \times H$  over  $V^*$  with respect to  $\text{NM}(\kappa) \star \text{Col}(D)$ , we can (by doing further forcing) assume that  $D = D_G$  and  $H = G^{**}$ , where  $G \subseteq Q_\kappa$  is generic over  $V^*$ . Assume otherwise. Let

$(c, h) \in \text{NM}(\kappa) \star \text{Col}(D)$  force the negation of our claim. Without loss of generality  $h \in \text{Col}(c)$ . Pick a condition  $p$  in  $Q_\lambda$  where the  $c$ -part is  $c$  and its  $h$ -part is  $h$ . Assume  $G$  is  $Q_\kappa$ -generic (over  $V^*$ ) such that  $p \in G$ . Note that  $c \in D_G$  and  $H_G$  extends  $h$ . Pick a generic  $G^*$  in  $\text{Col}(D_G)$  such that  $H_G \in G^*$ . Obviously  $h \in G^* \cap V^*[D_G] = G^{**}$ . So if we consider the generic pair  $D_G \star G^{**}$ ,  $(c, h)$  belongs to it, but this is a contradiction. So we proved (Using also Lemma 34):

**Lemma 42**  $V_1 = V^{P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)}$  has a forcing extension which is of the form  $V_2 = V^{P_\kappa \star Q_\kappa \star \text{Col}(D_G)}$ , where  $V_1$  and  $V_2$  have the same cardinals and the same bounded sequences of elements of  $\kappa$ .

Of course, the extension we describe in Lemma 42 does not preserve cofinalities because in  $V_1 = V^{P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)}$  the cardinal  $\kappa$  is regular while in  $V_2 = V^{P_\kappa \star Q_\kappa \star \text{Col}(D_G)}$  it has cofinality  $\omega$ .

We resume the proof of Theorem 41. So we are given in  $V_1 = V^{P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)}$  a structure  $\mathcal{A} = \langle \lambda, R_1, R_2, \dots \rangle$  (without loss of generality we may assume that the universe of the structure is an ordinal  $\lambda$ ). We have to get a substructure  $\mathcal{A}'$  of  $\mathcal{A}$  such that  $\mathcal{A}' \prec_{L(I)} \mathcal{A}$  and  $|A'| < \kappa$ . Without loss of generality we may assume that  $\kappa^+ \leq \lambda$ .

In  $V$  the cardinal  $\kappa$  is supercompact, so we fix in  $V$  an embedding  $j : V \rightarrow M$  such that  $\kappa$  is the critical point of  $j$ ,  $j(\kappa) > \lambda$ ,  $M^\lambda \subseteq M$  and by our definition of the function  $f$  we can assume that  $j(f)(\kappa) > \lambda$ . The structure  $\mathcal{A}$  is the realization of a term  $\dot{A}$  in the forcing language for  $P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)$ . We can assume  $\dot{A} \in M$  because  $|P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)| = \kappa$ .

Consider in  $M$  the forcing notion  $j(P_\kappa) \star j(\text{NM}(\kappa)) \star j(\text{Col}(D))$ . The forcing notion  $j(P_\kappa)$  is (in  $M$ ) an iteration of length  $j(\kappa)$ . Its first  $\kappa$  steps are exactly like in  $V$  (They are defined in  $V_\kappa$  which is the same as in  $M$ ). The  $\kappa$ -th step of the iteration is  $Q_\kappa$ , so  $j(P_\kappa) = P_\kappa \star Q_\kappa \star T$ , where  $T$  is the iteration in  $M$  between  $\kappa$  and  $j(\kappa)$ .

**Lemma 43** One can force over  $V_1$  to get a generic filter in the forcing  $j(P_\kappa) \star j(Q_\kappa) \star j(\text{Col}(D))$  such that in the resulting model there is an embedding

$$j^* : V_1 \rightarrow M^{j(P_\kappa) \star j(Q_\kappa) \star j(\text{Col}(D))}$$

extending  $j$  such that  $V_1$  and  $M^{j(P_\kappa) \star j(Q_\kappa) \star j(\text{Col}(D))}$  have the same cardinals  $\leq \lambda$ .

*Proof:* By Lemma 42, we can force over  $V_1$ , not collapsing any cardinals, to get a model  $V_2$  of the form  $V^{P_\kappa \star Q_\kappa \star \text{Col}(D)}$ , where the generic filter for  $\text{Col}(D)$  over  $V^{P_\kappa \star \text{NM}(\kappa)}$  is of the form  $G^{**} = G^* \cap V^{P_\kappa \star \text{NM}(\kappa)}$ , and where furthermore  $G^*$  is the generic filter in  $\text{Col}(D)$  over  $V^{Q_\kappa}$ . The generic filter for  $P_\kappa \star Q_\kappa$  provides the generic filter for the first  $\kappa + 1$  steps of the iteration of  $j(P_\kappa)$  over  $M$ . (Note that  $Q_\kappa$  is the same in the sense of  $M^{P_\kappa}$  and  $V^{P_\kappa}$ ). Follow this forcing by forcing with  $T$ . So we get a generic filter for  $P_\kappa \star Q_\kappa \star T = j(P_\kappa)$ . Recall that we assumed that  $\kappa$  is the last inaccessible in  $V$  so there are no inaccessibles (and hence no measurable) between  $\kappa$  and  $\lambda$ . Since  $M^\lambda \subseteq M$ , the same is true in  $M$ . So the iteration of  $j(P_\kappa)$  between  $\kappa$  and  $\lambda^+$  is the trivial iteration, so  $T$  is a Gitik iteration of  $\mu$ -Prikrý forcings for  $\lambda^+ \leq \mu$ . It means that

$$\mathcal{P}(\lambda) \cap V^{P_\kappa \star Q_\kappa} = \mathcal{P}(\lambda) \cap M^{P_\kappa \star Q_\kappa} = \mathcal{P}(\lambda) \cap V^{j(P_\kappa)}.$$

Hence no cardinals  $\leq \lambda$  are collapsed in  $M^{j(P_\kappa)}$ . We now have to force with  $j(\text{NM}(\kappa))$  which is  $\text{NM}(j(\kappa))$  in the sense of  $M^{j(P_\kappa)}$ . The club  $D$  introduced by  $\text{NM}(\kappa)$  is in  $V^{P_\kappa \star Q_\kappa}$  so it belongs to  $M^{P_\kappa \star Q_\kappa} \subseteq M^{j(P_\kappa)}$ . In  $M^{P_\kappa \star Q_\kappa}$  the cardinal  $\kappa$  is singular of cofinality  $\omega$ . So  $D \cup \{\kappa\}$  is a condition in  $\text{NM}(j(\kappa))$  (It is a closed subset of  $j(\kappa)$ , every limit point, including  $\kappa$ , is singular. The other conditions are easily verified.) So we can pick a generic filter  $D^*$  in  $\text{NM}(j(\kappa))$  such that  $D \cup \{\kappa\}$  is an initial segment of it. Forcing with  $\text{NM}(j(\kappa))$  over  $M^{j(P_\kappa)}$  does not add any bounded subsets of  $j(\kappa)$ , so again no cardinals  $\leq \lambda$  are collapsed.

Now we have to pick a generic filter for  $\text{Col}(D^*)$ . The set  $D \cup \{\kappa\}$  is an initial segment of  $D^*$  so

$$\text{Col}(D^*) = \text{Col}(D) \star \text{Col}(D^* - \kappa).$$

(The collapses are not in the sense of  $M^{j(P_\kappa)}$  but since  $M^{j(P_\kappa)}$  agrees with  $V^{P_\kappa \star Q_\kappa}$  on  $P_\kappa$ ,  $\text{Col}(D)$  is the same in the sense of  $V^{P_\kappa \star Q_\kappa}$  and  $M^{j(P_\kappa)}$ ). The filter  $G^*$  is generic in  $\text{Col}(D)$ , so we pick a generic filter for  $\text{Col}(D^*)$  such that its restriction to  $\text{Col}(D \cup \{\kappa\})$  is exactly  $G^*$ . Now we reach a crucial point for which we introduced the function  $f$ .

We defined  $\text{Col}(D)$  such that if  $\beta \in D$ , then  $\text{Col}(D)$  does not collapse any cardinals between  $\beta$  and  $f(\beta)$ . Since  $\kappa \in D^*$ , the forcing  $\text{Col}(D^*)$  does not collapse any cardinals between  $\kappa$  and  $j(f)(\kappa)$ . But  $j(f)(\kappa) > \lambda$ , so  $\text{Col}(D^*)$  collapses below  $\lambda$  only cardinals collapsed by  $\text{Col}(D^* \cap \kappa) = \text{Col}(D)$ . So it means that below  $\lambda$  the models  $M^{j(P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D))}$  and  $V^{P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)}$  have the same cardinals.

Denote by  $H$  the generic filter over  $V$  in  $P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)$ . We have defined a generic filter in  $j(P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D))$ . Let us denote this by  $H^*$ . We claim that the particular way we have defined  $H^*$  guarantees that  $H^*$  satisfies a condition which is known as “the master condition” i.e.

**Claim:** If  $p \in H$ , then  $j(p) \in H^*$ .

*Proof:* The condition  $p \in H$  is of the form  $(q, s, t)$  where  $q \in P_\kappa$ ,  $s$  is a term denoting an element of  $\text{NM}(\kappa)$  in  $V^{P_\kappa}$  and  $t$  is a term denoting a member of  $\text{Col}(D)$  in  $V^{P_\kappa \star \text{NM}(\kappa)}$ . The generic filter we picked for  $j(P_\kappa)$  extends the generic filter for  $P_\kappa$ , so  $q \in H$  implies that  $q = j(q) \in H^*$ .

The generic filter we picked for  $\text{NM}(j(\kappa))$  is an end extension of the generic filter picked for  $\text{NM}(\kappa)$ , so since  $s$  denotes a bounded subset of  $\kappa$  introduced by  $P_\kappa$ ,  $j(s) = s$  and  $s \in H$  implies  $j(s) \in H^*$ . The generic filter for  $\text{Col}(D)$  is  $G^{**} = G^* \cap V^{P_\kappa \star \text{NM}(\kappa)}$ . The way we picked the generic filter for  $\text{Col}(D^*)$  was such that  $G^{**} \subseteq H^*$ . Note that  $t$  denotes a subset of  $V_\kappa^{P_\kappa}$  of cardinality  $< \kappa$  in  $V^{P_\kappa}$ , so  $j(t) = t$ . But  $t \in G^*$  so  $t \in G^{**}$ , so  $j(t) = t \in H^*$ . (We abuse notation by denoting a term and its realization by the same symbol). So we have actually showed that for  $p \in H$  we have  $j(p) = (j(q), j(s), j(t)) \in H^*$ .  $\square$  (Claim)

Once we have the master condition we can as usual define the extension  $j^*$  of  $j$  by defining  $j^*$  in the realization  $[t]_H$  of an  $P_\kappa \star \text{NM}(\kappa) \star \text{Col}(\kappa)$ -term  $t$  as

$$j^*([t]_H) = [j(t)]_{H^*}.$$

It is a standard argument that given the assumptions of the claim,  $j^*$  is well-defined and it is elementary.

When we described the construction of  $H^*$  we argued that the cardinals  $\leq \lambda$  in  $V[H]$  are the same as in  $M[H^*]$ . So the lemma is verified.  $\square$  (Lemma 43)

So we have two universes  $V[H]$  and  $M[H^*]$  which agree on cardinals  $\leq \lambda$ . Moreover, we have  $j \subseteq j^*$  which is elementary

$$j^* : V[H] \rightarrow M[H^*].$$

The structure  $\mathcal{A}$  is in  $V[H]$  and because  $M^\lambda \subseteq M$  we have  $\mathcal{A} \in M[H^*]$  and  $j \upharpoonright \mathcal{A} = j \upharpoonright \lambda \in M \subseteq M[H^*]$ .

Suppose  $\Phi(x_1, \dots, x_n)$  is a formula in the logic  $L(I)$  and suppose  $a_1, \dots, a_n \in A$ . Now:

$$M[H^*] \models “\mathcal{A} \models \Phi(a_1, \dots, a_n)” \text{ iff } V[H] \models “\mathcal{A} \models \Phi(a_1, \dots, a_n)” \quad (4)$$

This is because  $M[H^*]$  agrees with  $V[H]$  on the cardinals  $\leq \lambda$  which are all the cardinals relevant for evaluating the truth of  $\Phi(a_1, \dots, a_n)$ . On the other hand, from  $j^*$  being elementary we get:

$$V[H] \models \text{“}\mathcal{A} \models \Phi(a_1, \dots, a_n)\text{” iff } M[H^*] \models \text{“}j^*(\mathcal{A}) \models \Phi(j^*(a_1), \dots, j^*(a_n))\text{”}.$$

Hence

$$M[H^*] \models \text{“}\mathcal{A} \models \Phi(a_1, \dots, a_n)\text{” iff } M[H^*] \models \text{“}j^*(\mathcal{A}) \models \Phi(j^*(a_1), \dots, j^*(a_n))\text{”}.$$

So

$$M[H^*] \models j^* \upharpoonright A = j \upharpoonright A \text{ is an } L(I)\text{-elementary embedding of } \mathcal{A} \text{ into } j^*(\mathcal{A}).$$

Since  $M[H^*] \models |A| \leq \lambda < j^*(\kappa)$ , we get

$$M[H^*] \models \text{“There is an } L(I)\text{-elementary substructure of } j^*(\mathcal{A}) \text{ of cardinality } < j^*(\kappa)\text{”}.$$

By  $j^*$  being elementary,

$$V[H] \models \text{“There is an } L(I)\text{-elementary substructure of } \mathcal{A} \text{ of cardinality } < \kappa\text{”}.$$

□ (Theorem 41)

This ends the proof of Theorem 21. □

We have shown that, assuming the consistency of a supercompact cardinal, it is consistent to assume that  $\text{LST}(L(I))$  exists and moreover, we can consistently assume that it is either the first supercompact cardinal, or something much smaller, namely the first (weakly) inaccessible cardinal. A fortiori, then  $\text{LST}(L(I))$  can be consistently equal to  $\text{LST}(L^2)$  or also consistently different from  $\text{LST}(L^2)$ . Moreover, we have shown that even the existence of  $\text{LST}(L(I))$  implies the consistency of large cardinals. In many respects the existence of  $\text{LST}(L(I))$  seems, in the light of present day knowledge, like Martin’s Maximum, and the cardinal  $\text{LST}(L(I))$  behaves – be it small or large - as  $\aleph_2$  in the presence of Martin’s Maximum. But  $\text{LST}(L(I))$  makes no claims about the size of the continuum: If it is consistent that there are supercompact cardinals, then it is consistent on the one hand that  $\text{LST}(L(I))$  exists and  $2^\omega = \aleph_1$  and on the other hand that  $\text{LST}(L(I)) = 2^\omega$  ([13]).

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