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## $C^{(n)}$ -Cardinals

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REPORT No. 26, 2009/2010, fall

ISSN 1103-467X

ISRN IML-R- -26-09/10- -SE+fall

# $C^{(n)}$ -CARDINALS

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ABSTRACT. For each natural number  $n$ , let  $C^{(n)}$  be the closed and unbounded proper class of ordinals  $\alpha$  such that  $V_\alpha$  is a  $\Sigma_n$ -elementary substructure of  $V$ . A  $C^{(n)}$ -cardinal is a cardinal  $\kappa$  that is the critical point of an elementary embedding  $j : V \rightarrow M$ ,  $M$  transitive, with  $j(\kappa) \in C^{(n)}$ . We show that the  $C^{(n)}$ -cardinals form a hierarchy that refines the usual large cardinal hierarchy, and we give characterizations of various natural reflection principles in terms of such cardinals. In particular, building on results of [1], we give sharper formulations of Vopěnka's Principle in terms of  $C^{(n)}$ -extendible cardinals.

For each natural number  $n$ , let  $C^{(n)}$  denote the *club* (i.e., closed and unbounded) proper class of ordinals  $\alpha$  that are  $\Sigma_n$ -correct in the universe  $V$  of all sets, meaning that  $V_\alpha$  is a  $\Sigma_n$ -elementary substructure of  $V$ , and written  $V_\alpha \preceq_n V$ . Thus,  $C^{(0)}$  is the class of all ordinals. But if  $V_\alpha \preceq_1 V$ , then  $\alpha$  is already an uncountable strong limit cardinal, for if  $\beta < \alpha$ , then the sentence

$$\exists \gamma \exists f (\gamma \text{ an ordinal} \wedge f : \gamma \rightarrow V_\beta \text{ is onto})$$

is  $\Sigma_1$  in the parameter  $V_\beta$ , and therefore it must hold in  $V_\alpha$ . Further, if  $\alpha \in C^{(1)}$ , then  $V_\alpha = H(\alpha)$ . Thus  $C^{(1)}$  is precisely the class of all uncountable cardinals  $\alpha$  such that  $V_\alpha = H(\alpha)$ . It follows that  $C^{(1)}$  is  $\Pi_1$  definable, for  $\alpha \in C^{(1)}$  iff  $\alpha$  is a cardinal and

$$\forall M (M \text{ a transitive model of } ZFC^* \wedge \alpha \in M \rightarrow M \models V_\alpha = H(\alpha)).$$

(Here  $ZFC^*$  denotes a sufficiently large finite fragment of  $ZFC$ .) The point is that if  $\alpha \in C^{(1)}$  and  $M$  is a transitive model of  $ZFC^*$  that contains  $\alpha$ , then if in  $M$  we could find some transitive  $x \in V_\alpha \setminus H(\alpha)$ , we would have that  $|x| \geq \alpha$ . But since in  $V$  the cardinality of  $x$  is less than  $\alpha$ , because  $V_\alpha = H(\alpha)$ , this would mean that, in  $V$ ,  $\alpha$  is not a cardinal, which is absurd.

More generally, since the truth predicate  $\models_n$  for  $\Sigma_n$  sentences (for  $n \geq 1$ ) is  $\Sigma_n$  definable (see [3], section 0.2), and since the relation  $x = V_y$  is  $\Pi_1$ , the class  $C^{(n)}$  is  $\Pi_n$  definable:  $\alpha \in C^{(n+1)}$  iff

$$\alpha \in C^{(n)} \wedge \forall \varphi (x \in \Sigma_{n+1} \forall a \in V_\alpha (\models_{n+1} \varphi(a) \rightarrow V_\alpha \models \varphi(a))).$$

Let us remark that  $C^{(n)}$ , for  $n \geq 1$ , cannot be  $\Sigma_n$  definable. Otherwise, if  $\alpha$  is the least ordinal in  $C^{(n)}$ , then the sentence “there is some ordinal in  $C^{(n)}$ ” would be  $\Sigma_n$  and so it would hold in  $V_\alpha$ , yielding an ordinal in  $C^{(n)}$  smaller than  $\alpha$ , which is impossible.

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*Date:* March 16, 2010.

The author was supported by the Spanish Ministry of Education and Science under grant MTM2008-03389/MTM, and by the Generalitat de Catalunya under grant 2009SGR-00187. Part of this work was done in November 2009 during the author's stay at the Mittag-Leffler Institut, whose support and hospitality is gratefully acknowledged.

The classes  $C^{(n)}$ ,  $n \geq 1$ , form a *basis* for definable club proper classes of ordinals, in the sense that every  $\Sigma_n$  club proper class of ordinals contains  $C^{(n)}$ . More generally, every club proper class  $C$  of ordinals that is  $\Sigma_n$  (i.e.,  $\Sigma_n$ -definable with parameters) contains all  $\alpha \in C^{(n)}$  that are greater than the rank of the parameters involved in any  $\Sigma_n$  definition of  $C$ .

Finally, note that since the least ordinal in  $C^{(n)}$  does not belong to  $C^{(n+1)}$ ,  $C^{(n+1)} \subset C^{(n)}$ , all  $n$ .

When considering non-trivial elementary embeddings  $j : V \rightarrow M$ , with  $M$  transitive, one would like to have some control over where the image  $j(\kappa)$  of the critical point  $\kappa$  goes. A especially interesting case is when one wants  $V_{j(\kappa)}$  to reflect some specific property of  $V$  or, more generally, when one wants  $j(\kappa)$  to belong to a particular definable club proper class of ordinals. Now, since the  $C^{(n)}$ ,  $n \in \omega$ , form a basis for such classes, the problem can be reformulated as follows: when can we have  $j(\kappa) \in C^{(n)}$ , for a given  $n \in \omega$ ? This prompts the following definition.

Let us say that a cardinal  $\kappa$  is  $C^{(n)}$ -*measurable* if there is an elementary embedding  $j : V \rightarrow M$ , some transitive class  $M$ , with critical point  $\text{crit}(j) = \kappa$  and with  $j(\kappa) \in C^{(n)}$ .

Observe that if  $j : V \rightarrow M \cong \text{Ult}(V, \mathcal{U})$ ,  $M$  transitive, is the ultrapower elementary embedding obtained from a non-principal  $\kappa$ -complete ultrafilter  $\mathcal{U}$  on  $\kappa$ , then  $2^\kappa < j(\kappa) < (2^\kappa)^+$  (see [3]). Hence, since  $V_{j(\kappa)} \preceq_1 V$  implies that  $j(\kappa)$  is a (strong limit) cardinal,  $j$  cannot witness the  $C^{(1)}$ -measurability of  $\kappa$ . Nonetheless, by using iterated ultrapowers (see [2], 19.15), one has that for every cardinal  $\alpha > 2^\kappa$ , the  $\alpha$ -th iterated ultrapower embedding  $j_\alpha : V \rightarrow M_\alpha \cong \text{Ult}(V, \mathcal{U}_\alpha)$ , where  $\mathcal{U}_\alpha$  is the  $\alpha$ -th iterate of  $\mathcal{U}$ , has critical point  $\kappa$  and  $j_\alpha(\kappa) = \alpha$ . So, if  $\kappa$  is measurable, then for each  $n$  we can always find an elementary embedding  $j : V \rightarrow M$ ,  $M$  transitive, with  $j(\kappa) \in C^{(n)}$ . We have thus shown the following.

**Proposition 0.1.** *Every measurable cardinal is  $C^{(n)}$ -measurable, for all  $n$ .*

A similar situation occurs in the case of strong cardinals.

Let us say that a cardinal  $\kappa$  is  $C^{(n)}$ -*strong* if for every  $\lambda > \kappa$ ,  $\kappa$  is  $\lambda$ - $C^{(n)}$ -*strong*, that is, there exists an elementary embedding  $j : V \rightarrow M$ ,  $M$  transitive, with critical point  $\kappa$ , and such that  $j(\kappa) > \lambda$ ,  $V_\lambda \subseteq M$ , and  $j(\kappa) \in C^{(n)}$ . Equivalently (see [3] 26.7),  $\kappa$  is  $\lambda$ - $C^{(n)}$ -strong iff there exists a  $(\kappa, \beta)$ -extender  $E$ , for some  $\beta > |V_\lambda|$ , with  $V_\lambda \subseteq M_E$  and  $\lambda < j_E(\kappa) \in C^{(n)}$ .

Suppose now that  $j : V \rightarrow M$  witnesses the  $\lambda$ -strongness of  $\kappa$ , with  $j(\kappa)$  not necessarily in  $C^{(n)}$ . Let  $E$  be the  $(\kappa, j(\kappa))$ -extender obtained from  $j$ , and let  $j_E : V \rightarrow M_E$  be the corresponding  $\lambda$ -strong embedding (see [3]). Then in  $M_E$ ,  $E' := j_E(E)$  is a  $(j_E(\kappa), j_E(j(\kappa)))$ -extender, which gives rise to an elementary embedding  $j_{E'} : M_E \rightarrow M_{E'}$  with critical point  $j_E(\kappa)$ . Still in  $M_E$ , let  $\mathcal{U}$  be the  $j_E(\kappa)$ -complete ultrafilter on  $j_E(\kappa)$  derived from  $j_{E'}$ , i.e.,

$$\mathcal{U} = \{X \subseteq j_E(\kappa) : j_E(\kappa) \in j_{E'}(X)\}$$

and let  $j_{\mathcal{U}} : M_E \rightarrow M$  be the corresponding elementary embedding. Then we can iterate  $j_{\mathcal{U}}$   $\alpha$ -times, for some  $\alpha \in C^{(n)}$  greater than  $2^{j_E(\kappa)}$ , so that if  $j_\alpha : M_E \rightarrow M_\alpha$  is the resulting elementary embedding, then  $j_\alpha(j_E(\kappa)) = \alpha$ .

Letting  $k := j_\alpha \circ j_E$ , we have that  $k : V \rightarrow M_\alpha$  is a  $\lambda$ -strong elementary embedding with critical point  $\kappa$  and with  $k(\kappa) \in C^{(n)}$ . Thus we have shown the following.

**Proposition 0.2.** *Every  $\lambda$ -strong cardinal is  $\lambda$ - $C^{(n)}$ -strong, for all  $n$ . Hence, every strong cardinal is  $C^{(n)}$ -strong, for every  $n$ .*

Thus for measurable or strong cardinals  $\kappa$ , the requirement that  $j(\kappa) \in C^{(n)}$  for the corresponding elementary embeddings  $j : V \rightarrow M$  does not yield stronger large cardinal notions. But, as we shall see next, the situation changes completely in the case of superstrong embeddings, that is, when  $j$  is such that  $V_{j(\kappa)} \subseteq M$ .

### 1. $C^{(n)}$ -SUPERSTRONG CARDINALS

In the case of superstrong cardinals  $\kappa$ , the requirement that  $j(\kappa) \in C^{(n)}$ , for  $n > 1$ , produces a hierarchy of ever stronger large cardinal principles.

**Definition 1.1.** *A cardinal  $\kappa$  is  $C^{(n)}$ -superstrong if there exists an elementary embedding  $j : V \rightarrow M$ ,  $M$  transitive, with critical point  $\kappa$ ,  $V_{j(\kappa)} \subseteq M$ , and  $j(\kappa) \in C^{(n)}$ .*

Note that in the definition above, from the fact that  $j(\kappa) \in C^{(n)}$  and  $V_{j(\kappa)} \subseteq M$  it already follows that  $\kappa \in C^{(n)}$ . Thus, every  $C^{(n)}$ -superstrong cardinal belongs to  $C^{(n)}$ .

**Proposition 1.2.** *If  $j : V \rightarrow M$  is an elementary embedding such that  $V_{j(\kappa)} \subseteq M$ , where  $\kappa = \text{crit}(j)$ , then  $j(\kappa) \in C^{(1)}$ . Hence, every superstrong cardinal is  $C^{(1)}$ -superstrong.*

*Proof.* Since  $\kappa \in C^{(1)}$ ,  $M$  thinks that  $j(\kappa) \in C^{(1)}$ , i.e.,  $M$  thinks  $j(\kappa)$  is a strong limit cardinal and  $V_{j(\kappa)} = H(j(\kappa))$ . But then, since  $(V_{j(\kappa)})^M = V_{j(\kappa)}$ ,  $j(\kappa)$  is, in  $V$ , a strong limit cardinal with  $V_{j(\kappa)} = H(j(\kappa))$ , and therefore  $j(\kappa) \in C^{(1)}$ .  $\square$

Observe that for  $n \geq 1$ , the sentence “ $\kappa$  is  $C^{(n)}$ -superstrong” is  $\Sigma_{n+1}$ , for  $\kappa$  is  $C^{(n)}$ -superstrong iff

$$\begin{aligned} \exists \beta \exists \mu \exists E (\beta < \mu \wedge \beta, \mu \in C^{(n)} \wedge E \text{ is a } (\kappa, \beta)\text{-extender} \wedge E \in V_\mu \wedge \\ V_\mu \models “V_{j_E(\kappa)} \subseteq M_E”). \end{aligned}$$

**Proposition 1.3.** *For every  $n \geq 1$ , if  $\kappa$  is  $C^{(n+1)}$ -superstrong, then there is a  $\kappa$ -complete normal ultrafilter  $\mathcal{U}$  over  $\kappa$  such that*

$$\{\alpha < \kappa : \alpha \text{ is } C^{(n)}\text{-superstrong}\} \in \mathcal{U}.$$

*Hence, the first  $C^{(n)}$ -superstrong cardinal  $\kappa$ , if it exists, is not  $C^{(n+1)}$ -superstrong.*

*Proof.* Suppose  $\kappa$  is  $C^{(n+1)}$ -superstrong, witnessed by a normal  $(\kappa, \beta)$ -extender  $E$  with associated elementary embedding  $j_E = j : V \rightarrow M$  such that  $\beta = j(\kappa)$  and  $V_{j(\kappa)} \subseteq M$ . Since  $j(\kappa) \in C^{(n+1)}$ ,

$$V_{j(\kappa)} \models “\kappa \text{ is } C^{(n)}\text{-superstrong”}.$$

And since  $\kappa \in C^{(n)}$ ,  $M \models "j(\kappa) \in C^{(n)}"$ . Hence, since  $V_{j(\kappa)} = (V_{j(\kappa)})^M$ , and since " $\kappa$  is  $C^{(n)}$ -superstrong" is a  $\Sigma_{n+1}$  statement, we have:

$$M \models "\kappa \text{ is } C^{(n)}\text{-superstrong}."$$

Therefore, by normality of  $E$ , the set  $\{\{\alpha\} : \alpha < \kappa \wedge \alpha \text{ is } C^{(n)}\text{-superstrong}\}$  belongs to  $E_{\{\kappa\}}$ . Hence,  $\{\alpha < \kappa : \alpha \text{ is } C^{(n)}\text{-superstrong}\}$  belongs to the standard  $\kappa$ -complete normal ultrafilter  $\mathcal{U} := \{X \subseteq \kappa : \kappa \in j(X)\}$ .  $\square$

The following Proposition gives an upper bound on the relative position of  $C^{(n)}$ -superstrong cardinals in the large cardinal hierarchy.

**Proposition 1.4.** *If  $\kappa$  is  $2^\kappa$ -supercompact and belongs to  $C^{(n)}$ , then there is a  $\kappa$ -complete normal ultrafilter  $\mathcal{U}$  over  $\kappa$  such that the set of  $C^{(n)}$ -superstrong cardinals smaller than  $\kappa$  belongs to  $\mathcal{U}$ .*

*Proof.* Let  $j : V \rightarrow M$  be an elementary embedding coming from a normal ultrafilter  $\mathcal{V}$  on  $\mathcal{P}_\kappa(2^\kappa)$ . Let  $\bar{j} := j \upharpoonright V_{\kappa+1}$ . So,  $\bar{j} : V_{\kappa+1} \rightarrow M_{j(\kappa)+1}$  is elementary and  $\bar{j} \in M$ . Hence  $M \models "\bar{j} : V_{\kappa+1} \rightarrow V_{j(\kappa)+1} \text{ is elementary}"$ . Since  $\kappa \in C^{(n)}$ , also  $M \models "j(\kappa) \in C^{(n)}"$ . Thus,  $M \models "\kappa \text{ is } \kappa+1\text{-}C^{(n)}\text{-extendible}"$ . Hence,  $\{x \in \mathcal{P}_\kappa(2^\kappa) : ot(x \cap \kappa) \text{ is } ot(x \cap \kappa) + 1\text{-}C^{(n)}\text{-extendible}\} \in \mathcal{V}$ . Letting  $\mathcal{U}$  be the projection of  $\mathcal{V}$  on  $\kappa$ , we have

$$\{\alpha < \kappa : \alpha \text{ is } \alpha + 1\text{-}C^{(n)}\text{-extendible}\} \in \mathcal{U}.$$

Now as in [3], Proposition 26.11 (a), one can show that if  $\alpha$  is  $\alpha + 1\text{-}C^{(n)}$ -extendible, then  $\alpha$  is  $C^{(n)}$ -superstrong.  $\square$

## 2. $C^{(n)}$ -EXTENDIBLE CARDINALS

Recall that a cardinal  $\kappa$  is  $\lambda$ -*extendible* if there is an elementary embedding  $j : V_\lambda \rightarrow V_\mu$ , some  $\mu$ , with critical point  $\kappa$  and such that  $j(\kappa) > \lambda$ . And  $\kappa$  is *extendible* if it is  $\lambda$ -extendible for all  $\lambda > \kappa$ .

The following lemma implies that every extendible cardinal is supercompact (see also [3], 23.6).

**Lemma 2.1** (M. Magidor [4]). *Suppose  $j : V_\lambda \rightarrow V_\mu$  is elementary,  $\lambda$  is a limit ordinal, and  $\kappa$  is the critical point of  $j$ . Then  $\kappa$  is  $< \lambda$ -supercompact.*

**Definition 2.2.** *For a cardinal  $\kappa$  and  $\lambda > \kappa$ , we say that  $\kappa$  is  $\lambda\text{-}C^{(n)}$ -extendible if there is an elementary embedding  $j : V_\lambda \rightarrow V_\mu$ , some  $\mu$ , with critical point  $\kappa$ , and such that  $j(\kappa) > \lambda$  and  $j(\kappa) \in C^{(n)}$ .*

*We say that  $\kappa$  is  $C^{(n)}$ -extendible if it is  $\lambda\text{-}C^{(n)}$ -extendible for all  $\lambda > \kappa$ .*

It follows from the next Proposition that a cardinal is extendible if and only if it is  $C^{(1)}$ -extendible.

**Proposition 2.3.** *Every extendible cardinal is  $C^{(1)}$ -extendible.*

*Proof.* Suppose  $\kappa$  is extendible and  $\lambda$  is greater than  $\kappa$ . Pick  $\lambda' \geq \lambda$  such that  $\lambda' \in C^{(1)}$ . Let  $j : V_{\lambda'} \rightarrow V_\mu$  be an elementary embedding with  $\text{crit}(j) = \kappa$  and  $j(\kappa) > \lambda'$ . Since  $\lambda'$  is a cardinal and  $V_{\lambda'} = H(\lambda')$ , by elementarity we must also have that  $\mu$  is a cardinal and  $V_\mu = H(\mu)$ . Hence  $\mu \in C^{(1)}$ . And since, again by elementarity,  $V_\mu \models j(\kappa) \in C^{(1)}$ , we have that  $j(\kappa) \in C^{(1)}$ .  $\square$

In the definition above, notice that if  $\kappa, \lambda, \mu \in C^{(n)}$ , then the requirement  $j(\kappa) \in C^{(n)}$  follows automatically. Also, if  $\kappa$  is  $C^{(n)}$ -extendible, then it follows easily from the fact that  $j(\kappa) \in C^{(n)}$  and  $j$  is elementary that  $\kappa \in C^{(n)}$ . But more is true.

**Proposition 2.4.** *If  $\kappa$  is  $C^{(n)}$ -extendible, then  $\kappa \in C^{(n+2)}$ .*

*Proof.* By induction on  $n$ . For  $n = 0$ , since  $\kappa \in C^{(1)}$ , we only need to show that if  $\exists x\varphi(x)$  is a  $\Sigma_2$  sentence, where  $\varphi$  is  $\Pi_1$  and has parameters in  $V_\kappa$ , that holds in  $V$ , then it holds in  $V_\kappa$ . So suppose  $a$  is such that  $\varphi(a)$  holds in  $V$ . Let  $\lambda > \kappa$  be such that  $a \in V_\lambda$ , and let  $j : V_\lambda \rightarrow V_\mu$  be elementary, with critical point  $\kappa$  and with  $j(\kappa) > \lambda$ . Then  $V_{j(\kappa)} \models \varphi(a)$ , and so by elementarity  $V_\kappa \models \exists x\varphi(x)$ .

Now suppose  $\kappa$  is  $C^{(n)}$ -extendible and  $\exists x\varphi(x)$  is a  $\Sigma_{n+2}$  sentence, where  $\varphi$  is  $\Pi_{n+1}$  and has parameters in  $V_\kappa$ . If  $\exists x\varphi(x)$  holds in  $V_\kappa$ , then since by induction hypothesis  $\kappa \in C^{(n+1)}$ , we have that  $\exists x\varphi(x)$  holds in  $V$ . Now suppose  $a$  is such that  $\varphi(a)$  holds in  $V$ . Let  $\lambda > \kappa$  be such that  $a \in V_\lambda$ , and let  $j : V_\lambda \rightarrow V_\mu$  be elementary, with critical point  $\kappa$  and with  $j(\kappa) > \lambda$ . Then since  $j(\kappa) \in C^{(n)}$ , we have  $V_{j(\kappa)} \models \varphi(a)$ , and so by elementarity  $V_\kappa \models \exists x\varphi(x)$ .  $\square$

Let us observe that for any  $\alpha < \lambda$ , the relation “ $\alpha$  is  $\lambda$ - $C^{(n)}$ -extendible” is  $\Sigma_{n+1}$  (for  $n \geq 1$ ), for it holds if and only if

$$\exists \mu \exists j (j : V_\lambda \rightarrow V_\mu \wedge j \text{ elementary} \wedge \text{crit}(j) = \alpha \wedge j(\alpha) > \lambda \wedge j(\alpha) \in C^{(n)}).$$

Hence, “ $x$  is a  $C^{(n)}$ -extendible cardinal” is a  $\Pi_{n+2}$  property of  $x$ .

**Proposition 2.5.** *For every  $n \geq 1$ , if  $\kappa$  is  $C^{(n)}$ -extendible and  $\kappa + 1$ - $C^{(n+1)}$ -extendible, then the set of  $C^{(n)}$ -extendible cardinals is unbounded below  $\kappa$ . Hence, the first  $C^{(n)}$ -extendible cardinal  $\kappa$ , if it exists, is not  $\kappa + 1$ - $C^{(n+1)}$ -extendible. In particular, the first extendible cardinal  $\kappa$  is not  $\kappa + 1$ - $C^{(2)}$ -extendible.*

*Proof.* Suppose  $\kappa$  is  $C^{(n)}$ -extendible and  $\kappa + 1$ - $C^{(n+1)}$ -extendible, witnessed by  $j : V_{\kappa+1} \rightarrow V_{j(\kappa)+1}$ . Since  $j(\kappa) \in C^{(n+1)}$ ,

$$V_{j(\kappa)} \models \text{“}\kappa \text{ is } C^{(n)}\text{-extendible”}.$$

Hence, for every  $\alpha < \kappa$ ,

$$V_{j(\kappa)} \models \text{“}\exists \beta > \alpha (\beta \text{ is } C^{(n)}\text{-extendible)”},$$

since this is witnessed by  $\kappa$ . By elementarity of  $j$ , for every fixed  $\alpha < \kappa$ , there is  $\beta > \alpha$  such that,

$$V_\kappa \models \text{“}\beta > \alpha \wedge \beta \text{ is } C^{(n)}\text{-extendible”}.$$

And since, by Proposition 2.4,  $\kappa \in C^{(n+2)}$ ,  $\beta$  is  $C^{(n)}$ -extendible in  $V$ .  $\square$

**Proposition 2.6.** *For every  $n$ , if there exists a  $C^{(n+2)}$ -extendible cardinal, then there exist a proper class of  $C^{(n)}$ -extendible cardinals.*

*Proof.* By the last Proposition, if  $\kappa$  is  $C^{(n+2)}$ -extendible, then the set of  $C^{(n)}$ -extendible cardinals is unbounded below  $\kappa$ . Now the Proposition follows easily from the fact that if  $\kappa$  is  $C^{(n+2)}$ -extendible, then  $\kappa \in C^{(n+4)}$  (Proposition 2.4), and the fact that being  $C^{(n)}$ -extendible is a  $\Pi_{n+2}$ -property.  $\square$

Note that the existence of a  $C^{(n+1)}$ -extendible cardinal  $\kappa$  (for  $n \geq 1$ ) does not imply the existence of a  $C^{(n)}$ -extendible cardinal greater than  $\kappa$ . For if  $\lambda$  is such a cardinal, then  $V_\lambda \models$  “ $\kappa$  is  $C^{(n+1)}$ -extendible”.

The following Proposition shows that  $C^{(n)}$ -extendible cardinals are much larger than  $C^{(n)}$ -superstrong cardinals.

**Proposition 2.7.** *If  $\kappa$  is  $\kappa + 1$ - $C^{(n)}$ -extendible and belongs to  $C^{(n)}$ , then  $\kappa$  is  $C^{(n)}$ -superstrong, and there is a  $\kappa$ -complete normal ultrafilter  $\mathcal{U}$  over  $\kappa$  such that the set of  $C^{(n)}$ -superstrong cardinals smaller than  $\kappa$  belongs to  $\mathcal{U}$ .*

*Proof.* As in [3], Proposition 26.11 (a). □

### 3. $C^{(n)}$ -EXTENDIBLE CARDINALS AND VOPĚNKA’S PRINCIPLE

This section builds on results from [1], giving sharper characterizations of Vopěnka’s Principle in terms of  $C^{(n)}$ -extendible cardinals.

Recall that *Vopěnka’s Principle (VP)* states that for every proper class  $\mathcal{C}$  of structures of the same type, there exist  $A \neq B$  in  $\mathcal{C}$  such that  $A$  is elementarily embeddable into  $B$ .

VP can be formulated in the first-order language of set theory as an axiom schema, i.e., as an infinite set of axioms, one for each formula with two free variables. Formally, for each such formula  $\varphi(x, y)$  one has the axiom:

$$\forall x[(\forall y \forall z(\varphi(x, y) \wedge \varphi(x, z) \rightarrow y \text{ and } z \text{ are structures of the same type}) \wedge \\ \forall \alpha \in OR \exists y(\text{rank}(y) > \alpha \wedge \varphi(x, y)) \rightarrow \\ \exists y \exists z(\varphi(x, y) \wedge \varphi(x, z) \wedge y \neq z \wedge \exists e(e : y \rightarrow z \text{ is elementary}))].$$

Henceforth, *VP* will be understood as this axiom schema.

The theory  $ZFC + VP$  implies, for instance, that the class of extendible cardinals is stationary, i.e., every definable club proper class contains an extendible cardinal ([4]). And its consistency is known to follow from the consistency of  $ZFC$  plus the existence of an almost-huge cardinal (see [3], or [2]). We will give the exact consistency strength below.

Let us consider the following variants of VP, the first apparently much stronger than the second.

**Definition 3.1.** *If  $\Gamma$  is one of  $\Sigma_n, \Pi_n, \Delta_n$ , for  $n \in \omega$ , and  $\kappa$  is an infinite cardinal, then we write  $VP(\kappa, \Gamma)$  for the following assertion:*

For every  $\Gamma$  proper class  $\mathcal{C}$  of structures of the same type  $\tau$  such that both  $\tau$  and the parameters of some  $\Gamma$ -definition of  $\mathcal{C}$ , if any, belong to  $H(\kappa)$ ,  $\mathcal{C}$  reflects below  $\kappa$ , i.e., for every  $B \in \mathcal{C}$ , there exists  $A \in \mathcal{C} \cap H(\kappa)$  that is elementarily embeddable into  $B$ .

*If  $\Gamma$  is one of  $\Sigma_n, \Pi_n, \Delta_n$ , or  $\Sigma_n, \Pi_n, \Delta_n$ , for  $n \in \omega$ , we write  $VP(\Gamma)$  for the following statement:*

For every  $\Gamma$  proper class  $\mathcal{C}$  of structures of the language of set theory with one (equivalently, finitely-many) additional 1-ary relation symbol, there exist distinct  $A$  and  $B$  in  $\mathcal{C}$  with an elementary embedding of  $A$  into  $B$ .

Clearly, for every  $\Gamma$ ,  $VP(\kappa, \Gamma)$  for some  $\kappa$  implies  $VP(\Gamma)$ .

VP for  $\Sigma_1$  classes follows from ZFC. In fact, the following holds.

**Theorem 3.2.**  *$VP(\kappa, \Sigma_1)$  holds for every uncountable cardinal  $\kappa$ .*

*Proof.* Fix an uncountable cardinal  $\kappa$  and a class  $\mathcal{C}$  of structures of the same type  $\tau \in H(\kappa)$ , definable by a  $\Sigma_1$  formula with parameters in  $H(\kappa)$ .

Given  $B \in \mathcal{C}$ , let  $\lambda$  be a regular cardinal greater than  $\kappa$ , with  $B \in H(\lambda)$ , and let  $N$  be an elementary substructure of  $H(\lambda)$ , of cardinality less than  $\kappa$ , which contains  $B$  and the transitive closure of  $\{\tau\}$  together with the parameters involved in some  $\Sigma_1$  definition of  $\mathcal{C}$ .

Let  $A$  and  $M$  be the transitive collapses of  $B$  and  $N$ , respectively, and let  $j : M \rightarrow N$  be the collapsing isomorphism. Then  $A \in H(\kappa)$ , and  $j \upharpoonright A : A \rightarrow B$  is an elementary embedding. Observe that  $j(\tau) = \tau$ . So, since  $\Sigma_1$  formulas are upwards absolute for transitive models, and since  $M \models A \in \mathcal{C}$ , we have that  $A \in \mathcal{C}$ .  $\square$

In contrast to the last Theorem, Vopěnka's Principle for  $\Pi_1$  proper classes implies the existence of very large cardinals.

**Theorem 3.3.**

- (1) *If  $VP(\Pi_1)$  holds, then there exists a supercompact cardinal.*
- (2) *If  $VP(\mathbf{\Pi}_1)$  holds, then there is a proper class of supercompact cardinals.*

*Proof.* (1). Let  $\mathcal{C}$  be the class of structures of the form  $\langle V_{\lambda+2}, \in, \alpha, \lambda \rangle$ , where  $\lambda$  is the least limit ordinal greater than  $\alpha$  such that no  $\kappa \leq \alpha$  is  $< \lambda$ -supercompact.

We claim that  $\mathcal{C}$  is  $\Pi_1$  definable without parameters. For  $X \in \mathcal{C}$  if and only if  $X = \langle X_0, X_1, X_2, X_3 \rangle$ , where

- (1)  $X_2$  is an ordinal.
- (2)  $X_3$  is a limit ordinal greater than  $X_2$ .
- (3)  $X_0 = V_{X_3+2}$
- (4)  $X_1 = \in \upharpoonright X_0$
- (5) And the following holds in  $\langle X_0, X_1 \rangle$ :
  - (a)  $\forall \kappa \leq X_2 (\kappa \text{ is not } < X_3\text{-supercompact})$
  - (b)  $\forall \mu (\mu \text{ limit} \wedge X_2 < \mu < X_3 \rightarrow \exists \kappa \leq X_2 (\kappa \text{ is } < \mu\text{-supercompact}))$ .

If there is no supercompact cardinal, then  $\mathcal{C}$  is a proper class. So by  $VP(\Pi_1)$ , there exist  $\langle V_{\lambda+2}, \in, \alpha, \lambda \rangle \neq \langle V_{\mu+2}, \in, \beta, \mu \rangle$  and an elementary embedding

$$j : \langle V_{\lambda+2}, \in, \alpha, \lambda \rangle \rightarrow \langle V_{\mu+2}, \in, \beta, \mu \rangle.$$

Since  $j$  must send  $\alpha$  to  $\beta$  and  $\lambda$  to  $\mu$ ,  $j$  is not the identity. Hence by Kunen's theorem (see [3]) we must have  $\lambda < \mu$ , and therefore also  $\alpha < \beta$ . So,  $j$  has critical point some  $\kappa \leq \alpha$ . It now follows by Lemma 2.1 that  $\kappa$  is  $< \lambda$ -supercompact. But this is impossible because  $\langle V_{\lambda+2}, \in, \alpha, \lambda \rangle \in \mathcal{C}$ .

(2). Fixing an ordinal  $\xi$ , to show that there is a supercompact cardinal greater than  $\xi$ , we argue as above. The only difficulty now is to ensure that  $\kappa > \xi$ . But this can be achieved by letting  $\mathcal{C}$  be the class of structures of the form  $\langle V_{\lambda+2}, \in, \alpha, \lambda, \{\gamma\}_{\gamma \leq \xi} \rangle$ , where  $\alpha > \xi$  and  $\lambda$  is the least limit ordinal greater than  $\alpha$  such that no  $\kappa \leq \alpha$  is  $< \lambda$ -supercompact. The class  $\mathcal{C}$  is now  $\Pi_1$  definable with  $\xi$  as an additional parameter.  $\square$

Let us say that a limit ordinal  $\lambda$  *captures* a proper class  $\mathcal{C}$  if the ranks of elements of  $\mathcal{C}$  are unbounded below  $\lambda$ .

Note that if  $\mathcal{C}$  is  $\Pi_n$ , then every  $\lambda$  in  $C^{(n+1)}$  greater than the rank of the parameters involved in a  $\Pi_n$  definition of  $\mathcal{C}$  captures  $\mathcal{C}$ . Notice also that every cardinal in  $C^{(2)}$  belongs to  $Lim(C^{(1)})$  – i.e., is a limit point of  $C^{(1)}$  – and captures all  $\Pi_1$  proper classes. However, the least ordinal  $\lambda$  in  $Lim(C^{(1)})$  that captures all  $\Pi_1$  proper classes is strictly less than the least ordinal in  $C^{(2)}$ . The point is that, fixing an enumeration  $\langle \varphi_n(x) : n < \omega \rangle$  of all  $\Pi_1$  formulas that define proper classes, the sentence

$$\exists \lambda \exists x (\lambda \in Lim(C^{(1)}) \wedge x = V_\lambda \wedge \forall n (V_\lambda \models \forall \alpha \exists \beta > \alpha \exists a (rk(a) > \beta \wedge \varphi_n(a))))$$

is  $\Sigma_2$  in the parameter  $\langle \varphi_n(x) : n < \omega \rangle$ .

**Proposition 3.4.** *Suppose  $\mathcal{C}$  is a  $\Pi_1$  proper class of structures of the same type, and  $\kappa$  is  $< \lambda$ -supercompact, for some  $\lambda \in Lim(C^{(1)})$  greater than  $\kappa$  that captures  $\mathcal{C}$ . Then VP holds for  $\mathcal{C}$ .*

*Proof.* Since  $\lambda$  captures  $\mathcal{C}$ , in  $V_\lambda$  there exist elements of  $\mathcal{C}$  of arbitrarily high rank. So, since  $\lambda \in Lim(C^{(1)})$ , we can find  $\delta < \lambda$  such that  $V_\delta = H(\delta)$ , and  $B \in \mathcal{C} \cap V_\delta$  of rank greater than  $\kappa$ . Let  $j : V \rightarrow M$  be an elementary embedding with critical point  $\kappa$ , with  $j(\kappa) > \delta$ , and  $M$  closed under  $\delta$ -sequences. Since  $B \in M$  and  $\mathcal{C}$  is  $\Pi_1$ -definable,  $M \models "B \in \mathcal{C}"$ . And since  $M$  is closed under  $\delta$ -sequences, the elementary embedding  $j \upharpoonright B : B \rightarrow j(B)$  belongs to  $M$ . Thus,

$$M \models "\exists A \in \mathcal{C} \exists e (rank(A) < j(\kappa) \wedge e : A \rightarrow j(B) \text{ is elementary}),"$$

since this is witnessed by  $B$  and  $j \upharpoonright B$ .

By elementarity, the same must hold in  $V$ , namely,

$$\exists A \in \mathcal{C} \exists e (rank(A) < \kappa \wedge e : A \rightarrow B \text{ is elementary}),$$

which is what we wanted.  $\square$

**Theorem 3.5** ([1]). *Suppose that  $\mathcal{C}$  is a  $\Sigma_2$  class of structures of the same type, and suppose that there exists a supercompact cardinal  $\kappa$  larger than the rank of the parameters that appear in some  $\Sigma_2$  definition of  $\mathcal{C}$ . Then for every  $B \in \mathcal{C}$  there exists  $A \in \mathcal{C} \cap V_\kappa$  that is elementarily embeddable into  $B$ .*

*Proof.* Fix a  $\Sigma_2$  formula  $\varphi(x, y)$  and a set  $b$  such that  $\mathcal{C} = \{a : \varphi(a, b)\}$ , and suppose that  $\kappa$  is a supercompact cardinal with  $b \in V_\kappa$ . Fix  $B \in \mathcal{C}$ , and let  $\lambda \in C^{(2)}$  be greater than  $rank(B)$ . Let  $j : V \rightarrow M$  be an elementary embedding with  $M$  transitive and critical point  $\kappa$ , such that  $j(\kappa) > \lambda$  and  $M$  is closed under  $\lambda$ -sequences. Thus,  $B$  and  $j \upharpoonright B : B \rightarrow j(B)$  are in  $M$ , and also  $V_\lambda \in M$ . Hence  $V_\lambda \preceq_1 M$ .

Since  $V_\lambda \preceq_2 V$ ,  $V_\lambda \models \varphi(B, b)$ . And since  $\Sigma_2$  formulas are upwards absolute between  $V_\lambda$  and  $M$ ,  $M \models \varphi(B, b)$ .

Thus, in  $M$  it is true that there exists  $X \in M_{j(\kappa)}$  such that  $\varphi(X, b)$ , namely  $B$ , and there exists an elementary embedding  $e : X \rightarrow j(B)$ , namely  $j \upharpoonright B$ . Therefore, by elementarity, the same holds in  $V$ ; that is, there exists  $X \in V_\kappa$  such that  $\varphi(X, b)$ , and there exists an elementary embedding  $e : X \rightarrow B$ .  $\square$

The following corollaries give characterizations of Vopěnka’s principle for  $\Pi_1$  and  $\Sigma_2$  classes (and therefore also for  $\Delta_2$  classes) in terms of supercompactness. The equivalence of (2) and (3) was already proved in [1].

**Corollary 3.6.** *The following are equivalent:*

- (1)  $VP(\Pi_1)$ .
- (2)  $VP(\kappa, \Sigma_2)$ , for some  $\kappa$ .
- (3) *There exists a supercompact cardinal.*

**Corollary 3.7.** *The following are equivalent:*

- (1)  $VP(\mathbf{\Pi}_1)$ .
- (2)  $VP(\kappa, \Sigma_2)$ , for a proper class of cardinals  $\kappa$ .
- (3) *There exists a proper class of supercompact cardinals.*

Recall from Definition 3.1 that a cardinal  $\kappa$  *reflects* a class of structures  $\mathcal{C}$  if for every  $B \in \mathcal{C}$  there exists  $A \in \mathcal{C} \cap V_\kappa$  which is elementary embeddable into  $B$ . The following theorem is due to Magidor [4], which follows from his characterization of supercompact cardinals, namely:  $\kappa$  is supercompact if and only if for a proper class of  $\lambda$  greater than  $\kappa$  there exists  $\alpha < \kappa$  and an elementary embedding  $j : V_\alpha \rightarrow V_\lambda$  such that  $\kappa$  is the image under  $j$  of the critical point (see [4, 3]).

**Theorem 3.8** (Magidor [4]). *If  $\kappa$  is the least cardinal that reflects the  $\Pi_1$  proper class  $\mathcal{C}$  of structures of the form  $\langle V_\lambda, \in \rangle$ , then  $\kappa$  is supercompact.*

*Proof.* For each  $\lambda$  greater than  $\kappa$  there is  $\alpha < \kappa$  and an elementary embedding

$$j_\lambda : \langle V_\alpha, \in \rangle \rightarrow \langle V_\lambda, \in \rangle.$$

Let  $\alpha$  be the least ordinal for which there is such an embedding for a proper class of  $\lambda$ s. We may assume that the  $j_\lambda$  are not the identity (otherwise,  $V_\alpha$  would be an elementary substructure of  $V$ ) and that the critical point of all these embeddings is the same, say  $\beta$ . Moreover, we may assume that the image of  $\beta$  is always the same, for otherwise for a proper class of  $\lambda$ s,  $j_\lambda \upharpoonright \beta$  would witness that  $V_\beta$  is an elementary substructure of  $V_{j_\lambda(\beta)}$ , with the  $j_\lambda(\beta)$ s forming a proper class, which in turn would imply that  $V_\beta$  is an elementary substructure of  $V$ .

So let  $\delta$  be least such that for a proper class of  $\lambda$ s the  $\alpha$  is the same,  $j_\lambda$  is not the identity, and the image of the critical point is  $\delta$ . By Magidor's characterization of supercompactness,  $\delta$  is supercompact. Thus  $\delta \geq \kappa$ , because  $\delta$  reflects  $\mathcal{C}$ , by Theorem 3.5, and  $\kappa$  is the least cardinal that does this. So suppose, aiming for a contradiction, that  $\delta > \kappa$ . Then by Theorem 3.5,  $\delta$  reflects the proper class of structures of the form  $\langle V_\lambda, \in, \gamma \rangle$ , where  $\gamma$  is some ordinal less than  $\lambda$ , which is  $\Pi_1$ . Arguing similarly as above we have that for a proper class of  $\lambda$ s there are fixed  $\gamma < \alpha < \kappa$  with an embedding from  $\langle V_\alpha, \in, \gamma \rangle$  into  $\langle V_\lambda, \in, \kappa \rangle$  whose image of the critical point is less or equal than  $\kappa$ , contradicting the minimality of  $\delta$ .  $\square$

From the last two Theorems we obtain a characterization of the first supercompact cardinal in terms of a natural form of reflection.

**Corollary 3.9.** *A cardinal  $\kappa$  reflects all  $\Pi_1$  proper classes of structures of the same type if and only if it is greater or equal than the first supercompact cardinal.*

David Asperó pointed out the following parameterized form of the Corollary above. One direction follows from Theorem 3.5 and by observing that

the property of reflecting  $\Pi_1$  classes is closed under limits. The other direction can be proved similarly as in the Theorem above, but working, for any fixed  $\xi < \kappa$ , with the class of structures of the form  $\langle V_\lambda, \in, \{\eta\}_{\eta < \xi} \rangle$ , where  $\beta$  is some ordinal less than  $\lambda$ , which is  $\Pi_1$  definable in the parameter  $\alpha$ .

**Corollary 3.10.** *A cardinal  $\kappa$  reflects all  $\Pi_1$  proper classes of structures of the same type if and only if either  $\kappa$  is a supercompact cardinal or a limit of supercompact cardinals.*

We will prove similar results for classes of higher complexity, for which we shall need  $C^{(n)}$ -extendible cardinals.

**Theorem 3.11.** *For every  $n \geq 1$ , if  $\kappa$  is a  $C^{(n)}$ -extendible cardinal, then  $VP(\kappa, \Sigma_{n+2})$  holds.*

*Proof.* Fix a  $\Sigma_{n+2}$  formula  $\exists x\varphi(x, y, z)$ , where  $\varphi$  is  $\Pi_{n+1}$ , such that for some set  $b \in V_\kappa$ ,

$$\mathcal{C} := \{B : \exists x\varphi(x, B, b)\}$$

is a proper class of structures of the same type.

Fix  $B \in \mathcal{C}$  and let  $\lambda \in C^{(n+2)}$  be greater than  $\kappa$  and the ranks of  $b$  and  $B$ . Thus,

$$V_\lambda \models \exists x\varphi(x, B, b).$$

Let  $j : V_\lambda \rightarrow V_\mu$  be an elementary embedding with critical point  $\kappa$ , with  $j(\kappa) > \lambda$ , and  $j(\kappa) \in C^{(n)}$ . Note that  $B$  and  $j \upharpoonright B : B \rightarrow j(B)$  are in  $V_\mu$ .

As  $\kappa, \lambda \in C^{(n+2)}$  (see Proposition 2.4), it follows that  $V_\kappa \preceq_{n+2} V_\lambda$ . And this implies that  $V_{j(\kappa)} \preceq_{n+2} V_\mu$ . Indeed, we have that

$$V_\lambda \models \text{“}\forall x \in V_\kappa \forall \theta \in \Sigma_{n+2}(V_\kappa \models \theta(x) \leftrightarrow \models_{n+2} \theta(x))\text{”}.$$

Hence, by elementarity,

$$V_\mu \models \text{“}\forall x \in V_{j(\kappa)} \forall \theta \in \Sigma_{n+2}(V_{j(\kappa)} \models \theta(x) \leftrightarrow \models_{n+2} \theta(x))\text{”},$$

which implies  $V_{j(\kappa)} \preceq_{n+2} V_\mu$ .

Since  $j(\kappa) \in C^{(n)}$ , we also have  $V_\lambda \preceq_{n+1} V_{j(\kappa)}$ , and therefore  $V_\lambda \preceq_{n+1} V_\mu$ . It follows that  $V_\mu \models \exists x\varphi(x, B, b)$ .

Thus, in  $V_\mu$  it is true that there exists  $X \in V_{j(\kappa)}$  such that  $X \in \mathcal{C}$ , namely  $B$ , and there exists an elementary embedding  $e : X \rightarrow j(B)$ , namely  $j \upharpoonright B$ . Therefore, by elementarity of  $j$ , the same is true in  $V_\lambda$ , that is, there exists  $X \in V_\kappa$  such that  $X \in \mathcal{C}$ , and there exists an elementary embedding  $e : X \rightarrow B$ . Let  $A \in V_\kappa$  be such an  $X$ , and let  $e : A \rightarrow B$  be an elementary embedding. Since  $\lambda \in C^{(n+2)}$ ,  $A \in \mathcal{C}$ , and we are done.  $\square$

The next Theorem will yield a strong converse to Theorem 3.11.

The notion of  $C^{(n)}$ -extendibility used in [1] has the following (apparently) stronger form – let us call it  $C^{(n)+}$ -extendibility: For  $\lambda \in C^{(n)}$ , a cardinal  $\kappa$  is  $\lambda$ - $C^{(n)+}$ -extendible if it is  $\lambda$ - $C^{(n)}$ -extendible, witnessed by some  $j : V_\lambda \rightarrow V_\mu$  which, in addition to satisfying  $j(\kappa) > \lambda$  and  $j(\kappa) \in C^{(n)}$ , it also satisfies that  $\mu \in C^{(n)}$ .

$\kappa$  is  $C^{(n)+}$ -extendible if it is  $\lambda$ - $C^{(n)+}$ -extendible for every  $\lambda > \kappa$  with  $\lambda \in C^{(n)}$ .

Every extendible cardinal is  $C^{(1)+}$ -extendible (see [1], or the proof of Proposition 2.3 above).

**Theorem 3.12.** *Suppose  $n \geq 1$ . If  $VP(\Pi_{n+1})$  holds, then there exists a  $C^{(n)+}$ -extendible cardinal.*

*Proof.* Suppose there are no  $C^{(n)+}$ -extendible cardinals. Then the class function  $F$  on the ordinals given by:

$F(\alpha) =$  the least  $\lambda \in C^{(n+1)}$  greater than  $\alpha$  such that  $\alpha$  is not  $\lambda$ - $C^{(n)+}$ -extendible,

is defined for all ordinals  $\alpha$ .

Let  $C = \{\eta > 0 : \forall \alpha < \eta F(\alpha) < \eta\}$ . So  $C$  is a closed unbounded proper class of ordinals contained in  $C^{(n+1)}$ .

We claim that  $C$  is  $\Pi_{n+1}$  definable, without parameters. First note that  $F$  is  $\Pi_{n+1}$  definable, for  $\lambda = F(\alpha)$  iff

- (1)  $\lambda \in C^{(n+1)}$
- (2)  $\alpha < \lambda$
- (3)  $\forall \beta > \lambda (\beta \in C^{(n)} \rightarrow V_\beta \models (\alpha \text{ is not } \lambda\text{-}C^{(n)+}\text{-extendible}))$ , and
- (4)  $V_\lambda \models \forall \lambda' > \alpha (\lambda' \in C^{(n+1)} \rightarrow (\alpha \text{ is } \lambda'\text{-}C^{(n)+}\text{-extendible}))$ .

The point is that, for any  $\alpha < \lambda'$ , the relation “ $\alpha$  is  $\lambda'$ - $C^{(n)+}$ -extendible” is  $\Sigma_{n+1}$ , for it holds iff

$$\exists \mu \exists j (j : V_{\lambda'} \rightarrow V_\mu \wedge j \text{ elementary} \wedge \text{crit}(j) = \alpha \wedge j(\alpha) > \lambda' \wedge j(\alpha), \mu \in C^{(n)}).$$

So it holds in  $V$  if and only if it holds in  $V_\lambda$ , for any  $\lambda \in C^{(n+1)}$  greater than  $\lambda'$ . And if it holds in  $V_\beta$ , with  $\beta \in C^{(n)}$ , then it holds in  $V$ . Moreover, since  $\lambda \in C^{(n+1)}$ , for every  $\lambda' < \lambda$  we have  $\lambda' \in C^{(n+1)}$  if and only if  $V_\lambda \models \lambda' \in C^{(n+1)}$ .

It follows that  $C$  is also  $\Pi_{n+1}$  definable.

For each ordinal  $\alpha$ , let  $\lambda_\alpha$  be the least limit point of  $C$  greater than  $\alpha$ . We have that  $x = \lambda_\alpha$  iff  $x$  is an ordinal greater than  $\alpha$  that belongs to  $C$  and such that

- (1)  $V_x \models \forall \beta \exists \gamma (\gamma > \beta \wedge \gamma \in C)$
- (2)  $V_x \models \forall \beta (\beta > \alpha \rightarrow \exists \gamma < \beta \forall \eta (\gamma < \eta < \beta \rightarrow \eta \notin C))$ ,

which shows that the function  $\alpha \mapsto \lambda_\alpha$  is  $\Pi_{n+1}$  definable.

Consider now the proper class  $\mathcal{C}$  of structures  $\mathcal{A}_\alpha$  of the form

$$\langle V_{\lambda_\alpha}, \in, \alpha, \lambda_\alpha, C \cap \alpha + 1 \rangle,$$

where  $\alpha \in C$ .

We claim that  $\mathcal{C}$  is  $\Pi_{n+1}$  definable. We have:  $X \in \mathcal{C}$  if and only if  $X = \langle X_0, X_1, X_2, X_3, X_4 \rangle$ , where

- (1)  $X_2 \in C$
- (2)  $X_3 = \lambda_{X_2}$
- (3)  $X_0 = V_{X_3}$
- (4)  $X_1 = \in \upharpoonright X_0$
- (5)  $X_4 = C \cap X_2 + 1$

We have already seen that (1) and (2) are  $\Pi_{n+1}$  expressible. And the same is true for (3) and (4), as one can easily see. As for (5), note that  $X_4 = C \cap \alpha + 1$  holds in  $V$  iff it holds in  $V_{X_3}$ .

So by  $VP(\Pi_{n+1})$  there exist  $\alpha \neq \beta$  and an elementary embedding

$$j : \mathcal{A}_\alpha \rightarrow \mathcal{A}_\beta.$$

Since  $j$  must send  $\alpha$  to  $\beta$ ,  $j$  is not the identity. So  $j$  has critical point some  $\kappa \leq \alpha$ .

We claim that  $\kappa \in C$ . Otherwise,  $\gamma := \sup(C \cap \kappa) < \kappa$ . Let  $\delta$  be the least ordinal in  $C$  greater than  $\gamma$  such that  $\delta < \lambda_\alpha$ . So  $\kappa < \delta \leq \alpha$ . Since  $\delta$  is definable from  $\gamma$  in  $\mathcal{A}_\alpha$ , and since  $j(\gamma) = \gamma$ , we must also have  $j(\delta) = \delta$ . But then  $j \upharpoonright V_{\delta+2} : V_{\delta+2} \rightarrow V_{\delta+2}$  is a nontrivial elementary embedding, contradicting Kunen's Theorem (see [3]).

Hence by elementarity,  $j(\kappa) \in C$ .

Since  $\alpha \in C$ , we have  $\kappa < F(\kappa) < \alpha$ . Thus,

$$j \upharpoonright V_{F(\kappa)} : V_{F(\kappa)} \rightarrow V_{j(F(\kappa))}$$

is elementary, with critical point  $\kappa$ .

And since  $j(\kappa) \in C$ ,  $F(\kappa) < j(\kappa)$ . Moreover, by elementarity of  $j$ ,  $V_{\lambda_\beta}$  satisfies that  $j(F(\kappa))$  belongs to  $C^{(n)}$ , so since  $\lambda_\beta \in C^{(n+1)}$  this is true in  $V$ . This shows that  $j \upharpoonright V_{F(\kappa)}$  witnesses that  $\kappa$  is  $F(\kappa)$ - $C^{(n)+}$ -extendible. But this is impossible, by definition of  $F$ .  $\square$

The proof of the last theorem can easily be adapted to prove the parameterized version: if  $VP(\Pi_{n+1})$  holds, then there is a proper class of  $C^{(n)}$ -extendible cardinals. (See the proof of Proposition 3.12 (2).)

Note that from the last two theorems it follows that there exists a  $C^{(n)}$ -extendible cardinal if and only if there exists a  $C^{(n)+}$ -extendible cardinal. The following corollaries summarize the results above (one may replace  $C^{(n)}$ -extendible by  $C^{(n)+}$ -extendible).

**Corollary 3.13.** *The following are equivalent:*

- (1)  $VP(\Pi_2)$ .
- (2)  $VP(\kappa, \Sigma_3)$ , for some  $\kappa$ .
- (3) *There exists an extendible cardinal.*

**Corollary 3.14.** *The following are equivalent for  $n \geq 1$ :*

- (1)  $VP(\Pi_{n+1})$ .
- (2)  $VP(\kappa, \Sigma_{n+2})$ , for some  $\kappa$ .
- (3) *There exists a  $C^{(n)}$ -extendible cardinal.*

**Corollary 3.15.** *The following are equivalent:*

- (1)  $VP(\Pi_n)$ , for every  $n$ .
- (2)  $VP(\kappa, \Sigma_n)$ , for a proper class of cardinals  $\kappa$ , and for every  $n$ .
- (3)  $VP$
- (4) *For every  $n$ , there exists a  $C^{(n)}$ -extendible cardinal.*

We shall next obtain characterizations of  $C^{(n)}$ -extendible cardinals in terms of natural reflection principles.

**Theorem 3.16.** *If  $\kappa$  is the least cardinal that reflects all  $\Pi_{n+1}$ -classes of structures of the same type, then  $\kappa$  is  $C^{(n)}$ -extendible.*

*Proof.* Suppose otherwise. Then by 3.11 there is no  $C^{(n)}$ -extendible cardinal less or equal than  $\kappa$ .

Consider the class  $\mathcal{C}$  of structures of the form  $\langle V_\xi, \in, \lambda, \alpha, C^{(n)} \cap \xi \rangle$ , where  $\alpha < \lambda < \xi$ , and

- (1)  $\lambda \in C^{(n)}$ ,
- (2)  $\xi \in \text{Lim}(C^{(n)})$ ,
- (3) the cofinality of  $\xi$  is uncountable,
- (4)  $\forall \beta < \xi \forall \mu (\exists j (j : V_\lambda \rightarrow V_\mu \wedge \text{crit}(j) = \alpha \wedge j(\alpha) = \beta) \rightarrow \exists j' \exists \mu' (j' : V_\lambda \rightarrow V_{\mu'} \wedge \mu' < \xi \wedge \text{crit}(j') = \alpha \wedge j'(\alpha) = \beta))$ , and
- (5)  $\lambda$  witnesses that no ordinal less or equal than  $\alpha$  is  $\lambda$ - $C^{(n)}$ -extendible.

Clearly,  $\mathcal{C}$  is a  $\Pi_{n+1}$  definable proper class. So let

$$j : \langle V_{\xi'}, \in, \lambda', \alpha' \rangle \rightarrow \langle V_\xi, \in, \lambda, \kappa \rangle$$

with  $\langle V_{\xi'}, \in, \lambda', \alpha' \rangle, \langle V_\xi, \in, \lambda, \kappa \rangle \in \mathcal{C}$  and  $\xi' < \kappa$ .

Let  $\alpha = \text{crit}(j)$ . We claim that  $\alpha \in C^{(n)}$ . Otherwise, let  $\gamma := \text{sup}(C^{(n)} \cap \alpha)$ , so  $\gamma < \alpha$ . Let  $\delta \in C^{(n)}$  be the least such that  $\gamma < \delta < \xi'$ . Since  $\delta$  is definable from  $\gamma$  in  $\langle V_{\xi'}, \in, \lambda', \alpha', C^{(n)} \cap \xi' \rangle$  and  $j(\gamma) = \gamma$ , also  $j(\delta) = \delta$ . Hence  $j \upharpoonright V_{\delta+s} \rightarrow V_{\delta+2}$ , contradicting Kunen's Theorem.

If  $j^m(\alpha) < \xi'$  for all  $m$ , then  $\{j^m(\alpha)\}_{m \in \omega} \in V_{\xi'}$ , because  $\xi'$  has uncountable cofinality, contradicting Kunen's Theorem. So suppose for some  $m$  we have  $j^m(\alpha) < \xi' \leq j^{m+1}(\alpha)$ .

We claim that there exists an elementary embedding  $k : V_{\lambda'} \rightarrow V_\mu$ , some  $\mu$ , with  $\text{crit}(k) = \alpha$  and  $k(\alpha) = j^{m+1}(\alpha)$ . We prove this by induction on  $i \leq m$ . For  $i = 0$  take  $k = j \upharpoonright V_{\lambda'}$ . So suppose it true for  $i < m$ . Since  $j^{i+1}(\alpha) < \xi'$ , by (3) above, there exist  $j'$  and  $\mu'$  such that  $j' : V_{\lambda'} \rightarrow V_{\mu'}$  is elementary,  $\mu' < \xi'$ ,  $\text{crit}(j') = \alpha$ , and  $j'(\alpha) = j^{i+1}(\alpha)$ . Applying  $j$  to  $j'$  we have  $j(j') : V_{j(\lambda')} \rightarrow V_{j(\mu')}$  has critical point  $j(\alpha)$  and  $j(j')(\alpha) = j^{i+2}(\alpha)$ . Let

$$k = j(j') \upharpoonright V_{\lambda'} : V_{\lambda'} \rightarrow V_{j(j')(\lambda')}.$$

Thus  $k$  has critical point  $\alpha$  and  $k(\alpha) = j^{i+2}(\alpha)$ , which proves our claim.

Note that since  $\alpha, \xi', \xi \in C^{(n)}$ , we have  $j^{m+1}(\alpha) \in C^{(n)}$ . Thus,  $k$  witnesses that  $\alpha$  is  $\lambda'$ - $C^{(n)}$ -extendible, contradicting (4) above.  $\square$

**Corollary 3.17.** *A cardinal  $\kappa$  reflects all  $\Pi_{n+1}$ -classes of structures of the same type (where  $n \geq 1$ ) if and only if it is greater or equal than the first  $C^{(n)}$ -extendible cardinal.*

The parameterized version also follows.

**Theorem 3.18.** *A cardinal  $\kappa$  reflects all  $\Pi_{n+1}$  proper classes of structures of the same type if and only if either  $\kappa$  is a  $C^{(n)}$ -extendible cardinal or a limit of  $C^{(n)}$ -extendible cardinals.*

#### 4. $C^{(n)}$ -HUGE CARDINALS

Recall that a cardinal  $\kappa$  is *m-huge* if it is the critical point of an elementary embedding  $j : V \rightarrow M$  with  $M$  transitive and closed under  $j^m(\kappa)$ -sequences, where  $j^m$  is the  $m$ -th iterate of  $j$  (i.e.,  $j^1 = j$  and  $j^{m+1} = j \circ j^m$ ). A cardinal is called *huge* if it is 1-huge. Let us say that  $\kappa$  is  $C^{(n)}$ -*m-huge* if it is  $m$ -huge and  $j(\kappa) \in C^{(n)}$ . And let us say that  $\kappa$  is  $C^{(n)}$ -*huge* if it is huge and  $j(\kappa) \in C^{(n)}$ .

As with  $m$ -huge cardinals,  $C^{(n)}$ - $m$ -huge cardinals can be characterized in terms of normal measures. To wit:  $\kappa$  is  $C^{(n)}$ - $m$ -huge if and only if it is uncountable and there is a  $\kappa$ -complete normal ultrafilter  $\mathcal{U}$  over some  $\mathcal{P}(\lambda)$

and cardinals  $\kappa = \lambda_0 < \lambda_1 < \dots < \lambda_m = \lambda$ , with  $\lambda_1 \in C^{(n)}$ , and such that for each  $i < m$ ,

$$\{x \in \mathcal{P}(\lambda) : ot(x \cap \lambda_{i+1}) = \lambda_i\} \in \mathcal{U}.$$

(See [3] 24.8 for a proof of the case  $n = 1$ , which also works for arbitrary  $n$ .) It follows that “ $\kappa$  is  $C^{(n)}$ - $m$ -huge” is  $\Sigma_{n+1}$  expressible.

Clearly, every huge cardinal is  $C^{(1)}$ -huge. But the first huge cardinal is not  $C^{(2)}$ -huge. For suppose  $\kappa$  is the least huge cardinal and  $j : V \rightarrow M$  witnesses that  $\kappa$  is  $C^{(2)}$ -huge. Then since “ $x$  is huge” is  $\Sigma_2$  expressible, we have

$$V_{j(\kappa)} \models \text{“}\kappa \text{ is huge”}.$$

Hence, since  $(V_{j(\kappa)})^M = V_{j(\kappa)}$ ,

$$M \models \text{“}\exists \delta < j(\kappa)(V_{j(\kappa)} \models \text{“}\delta \text{ is huge”})\text{”}.$$

By elementarity, there is a huge cardinal less than  $\kappa$  in  $V$ , which is absurd.

A similar argument, using that “ $\kappa$  is  $C^{(n)}$ - $m$ -huge” is  $\Sigma_{n+1}$  expressible, for all  $m$ , shows that the first  $C^{(n)}$ - $m$ -huge cardinal is not  $C^{(n+1)}$ - $m$ -huge, for all  $m$  and  $n$  greater or equal than 1.

## 5. ON ELEMENTARY EMBEDDINGS OF A RANK INTO ITSELF

Consider the large cardinal principle, known as I3, which asserts the existence of a non-trivial elementary embedding  $j : V_\delta \rightarrow V_\delta$  (see [3], 24). Let us call the critical point of such an embedding an *I3 cardinal*.

If  $j : V_\delta \rightarrow V_\delta$  witnesses that  $\kappa$  is I3, then Kunen’s Theorem implies that either  $\delta$  is a limit ordinal or is of the form  $\gamma + 1$  with  $\gamma$  a limit ordinal (see [3], 24). And if  $\delta$  is a limit ordinal, then  $\delta = \sup\{j^m(\kappa) : m \in \omega\}$ , where  $j^m$  is the  $m$ -th iterate of  $j$ . It follows that  $\delta \in C^{(1)}$ , because all the  $j^m(\kappa)$  are measurable cardinals and therefore they all belong to  $C^{(1)}$ .

Now suppose  $j : V_\delta \rightarrow V_\delta$  witnesses that  $\kappa$  is I3, with  $\delta$  a limit ordinal. Then  $V_\kappa$  and  $V_{j^m(\kappa)}$ , all  $m$ , are elementary substructures of  $V_\delta$ . In particular,  $V_\delta$  is a model of ZFC. Moreover, it is easily seen that in  $V_\delta$  the cardinal  $\kappa$ , and all the  $j^m(\kappa)$ , are  $C^{(n)}$ -supercompact,  $C^{(n)}$ -extendible, and  $C^{(n)}$ - $m$ -huge, for all  $n$  and  $m$ . In particular,  $V_\delta$  satisfies VP. Thus the consistency of the existence of an I3 cardinal implies the consistency of the existence of all  $C^{(n)}$ -cardinals considered in previous sections.

Let us now say that  $\kappa$  is a  $C^{(n)}$ -I3 cardinal if it is an I3 cardinal, witnessed by some embedding  $j : V_\delta \rightarrow V_\delta$ , with  $j(\kappa) \in C^{(n)}$ .

Observe that if  $\kappa$  is  $C^{(n)}$ -I3, then  $\kappa \in C^{(n)}$ . For if  $\varphi$  is a  $\Sigma_n$  sentence with parameters in  $V_\kappa$  that holds in  $V$ , then it must hold in  $V_{j(\kappa)}$  as well, for some  $j : V_\delta \rightarrow V_\delta$  that witnesses the fact that  $\kappa$  is  $C^{(n)}$ -I3. But then, since  $j$  fixes the parameters of  $\varphi$ , by elementarity  $\varphi$  also holds in  $V_\kappa$ .

If  $\kappa = \text{crit}(j)$ , then  $\kappa$  and  $j(\kappa)$  are measurable cardinals. Thus every I3-cardinal is  $C^{(1)}$ -I3. However, a simple reflection argument shows that the least  $C^{(n)}$ -I3 cardinal is not  $C^{(n+1)}$ -I3, for  $n \geq 1$ . For suppose  $\kappa$  is the least  $C^{(n)}$ -I3 cardinal ( $n \geq 1$ ), and suppose, towards a contradiction, that there is some  $j : V_\delta \rightarrow V_\delta$  with  $\text{crit}(j) = \kappa$ , and  $j(\kappa) \in C^{(n+1)}$ . Then  $V_{j(\kappa)}$  satisfies the following  $\Sigma_{n+1}$  statement

$$\exists \beta \exists k \exists \lambda (k : V_\beta \rightarrow V_\beta \text{ is elementary} \wedge \text{crit}(k) = \lambda \wedge k(\lambda) \in C^{(n)})$$

because  $\delta$ ,  $j$ , and  $\kappa$  are such  $\beta$ ,  $k$ , and  $\lambda$ , respectively. By elementarity the same holds in  $V_{\kappa}$ . Hence, since  $\kappa \in C^{(n)}$ , it also holds in  $V$ , contradicting our assumption that  $\kappa$  was the least  $C^{(n)}$ -I3 cardinal.

**Proposition 5.1.** *Suppose  $j : V_{\delta} \rightarrow V_{\delta}$  witnesses that  $\kappa$  is I3, and  $\delta$  is a limit ordinal. Then the following are equivalent for every  $n \geq 1$ :*

- (1)  $j^m(\kappa) \in C^{(n)}$ , all  $m < \omega$ .
- (2)  $\delta \in C^{(n)}$ .

*Proof.* For  $n = 1$ , (1) and (2) are true.

(1) implies (2) is immediate, since  $\delta = \sup\{j^m(\kappa) : m < \omega\}$ .

Let's prove (2) implies (1) for  $n+1$ , assuming it holds for  $n$ . Fix  $m < \omega$ . Let  $\exists x\varphi(x)$  be a  $\Sigma_{n+1}$  formula whose parameters, if any, are in  $V_{j^m(\kappa)}$ , and suppose the formula is true in  $V$ . Then, by (2), it is also true in  $V_{\delta}$ . Let  $\ell \geq m$  be big enough so that  $V_{j^{\ell}(\kappa)}$  contains a witness to the formula. Then  $V_{\delta}$  satisfies:

$$\exists x \in V_{j^{\ell}(\kappa)} \varphi(x).$$

Hence by elementarity of  $j^{\ell-m}$ ,  $V_{\delta}$  satisfies that there exists  $x \in V_{j^m(\kappa)}$  such that  $\varphi(x)$  holds. If  $a$  is such an  $x$ , then since by inductive hypothesis  $V_{j^m(\kappa)} \preceq_n V_{\delta}$ , we have that  $\varphi(a)$  holds in  $V_{j^m(\kappa)}$ .  $\square$

So let us say that  $\kappa$  is  $C^{(n)+}$ -I3 if it is I3, witnessed by  $j : V_{\delta} \rightarrow V_{\delta}$ , with  $\delta \in C^{(n)}$ . Clearly,  $\kappa$  is I3 if and only if it is  $C^{(1)}$ -I3 if and only if it is  $C^{(1)+}$ -I3.

**Proposition 5.2.** *If  $\kappa$  is  $C^{(n)}$ -I3, then it is  $C^{(n)}$ - $m$ -huge, for all  $m$ , and there is a normal ultrafilter  $\mathcal{U}$  over  $\kappa$  such that*

$$\{\alpha < \kappa : \alpha \text{ is } C^{(n)}\text{-}m\text{-huge for every } m\} \in \mathcal{U}.$$

*Proof.* Let  $j : V_{\delta} \rightarrow V_{\delta}$  witness that  $\kappa$  is  $C^{(n)}$ -I3, with  $\delta$  limit. Then as in [3] 24.8 one can show that the ultrafilter  $\mathcal{V}$  over  $\mathcal{P}(\lambda)$ , where  $\lambda = j^m(\kappa)$ , defined by

$$X \in \mathcal{V} \text{ if and only if } j''\lambda \in j(X)$$

witnesses that  $\kappa$  is  $C^{(n)}$ - $m$ -huge. Now let  $\mathcal{U}$  be the usual normal ultrafilter over  $\kappa$  obtained from  $j$ . Since  $V_{\delta}$  satisfies that  $\kappa$  is  $C^{(n)}$ - $m$ -huge, by normality of  $\mathcal{U}$  the Proposition follows.  $\square$

Let us call a cardinal  $\kappa$  a *I1 cardinal* if there exists an elementary embedding  $j : V_{\delta+1} \rightarrow V_{\delta+1}$  with  $\text{crit}(j) = \kappa$ . By Kunen's Theorem,  $\delta$  must be a limit ordinal. So let us say that  $\kappa$  is  $C^{(n)}$ -I1 if it is I1, witnessed by  $j : V_{\delta+1} \rightarrow V_{\delta+1}$ , with  $j(\kappa) \in C^{(n)}$ . Clearly, if  $\kappa$  is  $C^{(n)}$ -I1, then it is also  $C^{(n)}$ -I3.

As with  $C^{(n)}$ -I3 cardinals, a simple reflection argument shows that the first  $C^{(n)}$ -I1 cardinal is not  $C^{(n+1)}$ -I1, for all  $n \geq 1$ .

Let us also observe that if  $\kappa$  is  $C^{(n)}$ -I1, then the least  $\delta$  for which there is an elementary embedding  $j : V_{\delta+1} \rightarrow V_{\delta+1}$  with  $\text{crit}(j) = \kappa$  and  $j(\kappa) \in C^{(n)}$  is smaller than the first ordinal in  $C^{(n+1)}$  greater than  $\kappa$ . Moreover, the least  $C^{(n)}$ -I1 cardinal, if it exists, is smaller than the first ordinal in  $C^{(n+1)}$ , for all  $n \geq 1$ .

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