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Remarks on Gaps in Dense(Q)/nwd

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Remarks on gaps in Dense(Q)/nwd

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Abstract

The structure Dense($\mathbb{Q}$)/**nwd** and gaps in analytic quotients of $\mathscr{P}(\omega)$ have been studied in the literature. We prove a that in ZFC you can prove that structures Dense($\mathbb{Q}$)/**nwd** and $\mathscr{P}(\mathbb{Q})$/**nwd** have gaps of type $(\mathrm{add}(\mathscr{M}), \omega)$ and that there are no (λ, ω)-gaps for $\lambda < \mathrm{add}(\mathscr{M})$, where $\mathrm{add}(\mathscr{M})$ is the additivity of the meager ideal. We also give a direct proof of the existence of (ω_1, ω_1)-gaps.

In this paper we study some aspects of the structure of the dense subsets of rational numbers Dense($\mathbb{Q}$) ordered by almost inclusion with respect to nowhere dense sets of rationals. The interest in this structure was raised by Blass [Bla] and it was later investigated by Cichoń [Cic01] and Balcar, Hernández-Hernández and Hrušák [BHHH04]. It can be viewed as a counterpart to the well-studied structure of subsets of natural numbers ordered by almost inclusion with respect to finite sets, and the same kinds of concepts can be used to investigate its structure.

This paper concentrates on the concept of a gap, which is well known for $\mathscr{P}(\omega)$/**fin** (see eg. [Sch93]). For $\mathscr{P}(\mathbb{Q})$/**nwd** gaps have been studied in a more general setting of analytic ideals of $\mathbb{N}$ by Todorčević [Tod98] and Farah [Far00]. First we answer positively a question in [Tod98] if other cardinal

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invariants than $\mathfrak{b}$ show up in structures $\mathscr{P}(\omega)/I$, where I is an analytic ideal. Todorčević [Tod98] showed that Hausdorff gaps are preserved by any Baire embedding from $\mathscr{P}(\omega)/\mathbf{fin}$ to $\mathscr{P}(\omega)/I$, and moreover, we can get a Hausdorff gap on Dense($\mathbb{Q}$)/**nwd** from the construction in the proof. In the second result of this paper we give a direct proof of the existence of Hausdorff gaps in the case of Dense($\mathbb{Q}$)/**nwd**. The third result of this paper draws an analogy for gaps in $\mathscr{P}(\omega)/\mathbf{fin}$ and the dominating number $\mathfrak{d}$ and proves that the analogous cardinal invariant for Dense($\mathbb{Q}$)/**nwd** is the cofinality of the meager ideal cof($\mathscr{M}$).

1 Preliminaries

Instead of considering $\mathbb{Q} \subset \mathbb{R}$ on the real line, we consider real numbers as functions in 2^ω, and rational numbers as functions in $2^{<\omega}$. The topology on the former is as usual, the basic open sets are of the form

$$U_s = [s] = \{f \in 2^\omega : s \subseteq f\},$$

where $s \in 2^{<\omega}$. The topology on $2^{<\omega}$ is defined similarly with basic open sets

$$U'_s = [s] = \{f \in 2^{<\omega} : s \subseteq f\},$$

where again $s \in 2^{<\omega}$. We use $[s]$ to mean two different things, usually the latter, but it will be clear from context which is intended.

We use the function $|\cdot|$ to denote the cardinality of a subset and the length of a sequence in $2^{<\omega}$. For the rest of this paper, let $(q_n)_{n<\omega}$ enumerate $\mathbb{Q}$ so that the sequence $|q_n|$ is increasing. We also assume that the tree $2^{<\omega}$ grows upwards, when speaking of being above or below in the tree.

Even though $2^{<\omega}$ is not a subset of 2^ω and there are some differences in the topologies, it is not central to our arguments, and $2^{<\omega}$ could be identified with eventually zero infinite sequences. We follow the convention in [BHHH04] and do not do this.

Dense sets of reals $X \subseteq 2^\omega$ are sets with the property

$$\forall s \in 2^{<\omega} \exists x \in X : s \subseteq x.$$

The definition for dense sets of rationals is verbatim to the above, except that now $X \subseteq 2^{<\omega}$.

Definition 1. A set $X \subseteq 2^\omega$ is *nowhere dense*, if $\forall s \in 2^{<\omega} \exists s' \in 2^{<\omega} (U_{s'} \cap X = \emptyset)$.

Similarly a set $X \subseteq 2^{<\omega}$ is *nowhere dense*, if $\forall s \in 2^{<\omega} \exists s' \in 2^{<\omega} (U'_{s'} \cap X = \emptyset)$.

Denote the collection of nowhere dense sets of reals by $\mathbf{nwd}(\mathbb{R})$ and the collection of nowhere dense sets of rationals by $\mathbf{nwd}(\mathbb{Q}) = \mathbf{nwd}$. If $D \subseteq 2^{<\omega}$, we let $\mathbf{nwd}(D) = \{A \subseteq D : A \in \mathbf{nwd}\}$.

The structure under study is now the partial ordering $\subseteq_{\mathbf{nwd}}$ on dense sets of rationals where

$$A \subseteq_{\mathbf{nwd}} B \leftrightarrow A \setminus B \in \mathbf{nwd}(\mathbb{Q}).$$

We could also say that sets A and B are equivalent if their symmetric difference is nowhere dense, and look at the induced relation on these equivalence classes, but it is more convenient to study the relation $\subseteq_{\mathbf{nwd}}$. We use the shorthand Dense$(\mathbb{Q})/\mathbf{nwd}$ to denote either one of these structures.

Later we will need to consider both nowhere dense sets on reals and on rationals. They are quite similar, but they differ so much that we must define a few ways to transform nowhere dense sets between these two structures.

Definition 2. If $T \subseteq 2^{<\omega}$ then let $[T] = \{x \in 2^\omega : \exists^\infty n (x \restriction n \in T)\}$. If $X \subseteq 2^\omega$, let $\bar{X} = \{x \restriction n \in 2^{<\omega} : x \in X, n < \omega\}$.

Lemma 3. *If $T \subseteq 2^{<\omega}$ is nowhere dense, then $[T]$ is nowhere dense. If $X \subseteq 2^\omega$ is nowhere dense, then $\bar{X}$ is nowhere dense and $X \subseteq [\bar{X}]$.*

This is not sufficient for some of our uses, because if T in the previous lemma is for example finite, then $[T]$ is empty. The following functions help with this problem.

Lemma 4. *There is a function $\Phi^{\mathbb{Q}} : \mathbf{nwd}(\mathbb{Q}) \to \mathbf{nwd}(\mathbb{R})$ and a function $\Phi^{\mathbb{R}} : \mathbf{nwd}(\mathbb{R}) \to \mathbf{nwd}(\mathbb{Q})$ such that for all nowhere dense $T \subseteq \mathbb{Q}$ it holds that $T \subseteq \Phi^{\mathbb{R}}(\Phi^{\mathbb{Q}}(T))$.*

Proof. For $T \subseteq \mathbb{Q}$ be nowhere dense, let $\Phi^{\mathbb{Q}}(T) = \{y \frown \bar{0} : y \in T\}$. Then $\Phi^{\mathbb{Q}}(T)$ is clearly nowhere dense. For $X \subseteq \mathbb{R}$ nowhere dense, let $\Phi^{\mathbb{R}}(X) = \bar{X}$. Let then $T \subseteq \mathbb{Q}$ and $s \in T$. Then $s \frown \bar{0} \in \Phi^{\mathbb{Q}}(T)$ and so $(s \frown \bar{0}) \restriction n \in \Phi^{\mathbb{R}}(\Phi^{\mathbb{Q}}(T))$ for all n, thus also with $n = |s|$. □

There is still one more way to transform nowhere dense sets of one type to the other. This will be used in chapter 4.

Lemma 5. *Let D be a somewhere dense set of rationals. There are functions $\Phi_D^{\mathbb{Q}} : \mathbf{nwd}(D) \to \mathbf{nwd}(\mathbb{R})$ and $\Phi_D^{\mathbb{R}} : \mathbf{nwd}(\mathbb{R}) \to \mathbf{nwd}(D)$ so that if $X \in \mathbf{nwd}(\mathbb{R})$, $A \in \mathbf{nwd}(D)$ and $\Phi_D^{\mathbb{R}}(X) \subseteq A$ then $X \subseteq \Phi_D^{\mathbb{Q}}(A)$.*

Proof. Fix a somewhere dense set D of rationals. We form a suitable injective map $f : \mathbb{Q} \to D$. First let $t_{()}$ be some element of D so that D is dense above $t_{()}$, and let $f(()) = t_{()}$. Define t_s for $s \in 2^{<\omega}$ by induction so that if t_s is defined, let $t_{s\frown 0}$ and $t_{s\frown 1}$ be two incomparable elements of D above t_s. Let $f(s) = t_s$. Now let $\Phi_D^{\mathbb{R}}(X) = \{s \in 2^{<\omega} : \exists x \in X \exists n < \omega(s = f(x \upharpoonright n))\}$ and let $\Phi_D^{\mathbb{Q}}(A) = \{x \in 2^\omega : \exists^\infty n \exists s \in A(s = f(x \upharpoonright n))\}$.

It is clear that these functions are well defined on their domain and their range is as desired. For the claim, let $X \in \mathbf{nwd}(\mathbb{R})$, $A \subseteq D$, $A \in \mathbf{nwd}(\mathbb{Q})$, $\Phi_D^{\mathbb{R}}(X) \subseteq A$, and $x \in X$. Then $f(x \upharpoonright n) \in \Phi_D^{\mathbb{R}}(X)$ for all $n < \omega$, so by assumption $f(x \upharpoonright n) \in A$. Thus $x \in \Phi_D^{\mathbb{Q}}(A)$. □

For the rest of this paper, for every somewhere dense $D \subseteq \mathbb{Q}$, fix a function $f = f_D$ that is used for $\phi_D^{\mathbb{R}}$ and $\phi_D^{\mathbb{Q}}$.

2 A Rothberger gap

The gap structure for $\mathscr{P}(\omega)/\mathbf{fin}$ is well known, and it is known that the only gaps provable from ZFC are Rothberger gaps $(\omega, \mathfrak{b})$ and Hausdorff gaps (ω_1, ω_1). It is easy to define gaps for the structure $\mathscr{P}(\mathbb{Q})/\mathbf{nwd}$, and it turns out that this notion also works for Dense$(\mathbb{Q})/\mathbf{nwd}$ and the gaps for these structures are essentially the same. In this part we prove that there is a $(\omega, \mathrm{add}(\mathscr{M}))$ gap in Dense$(\mathbb{Q})/\mathbf{nwd}$ (thus also in $\mathscr{P}(\mathbb{Q})/\mathbf{nwd}$) and there are no (ω, λ)-gaps for $\lambda < \mathrm{add}(\mathscr{M})$. This behaviour is similar to $\mathfrak{b}$ for $\mathscr{P}(\omega)/\mathbf{fin}$.

Let us first recall the definitions for gaps in $\mathscr{P}(\omega)/\mathbf{fin}$. We let $A \subseteq^* B \Leftrightarrow A \setminus B \in \mathbf{fin}$ and $A \perp^* B \Leftrightarrow B \cap A \in \mathbf{fin}$.

Definition 6. Let $\mathscr{A} = \{A_\alpha \in [\omega]^\omega : \alpha < \kappa\}$ and $\mathscr{B} = \{B_\beta \in [\omega]^\omega : \beta < \lambda\}$. The pair $(\mathscr{A}, \mathscr{B})$ is a *pre-gap* on $\mathscr{P}(\omega)/\mathbf{fin}$ if both $\mathscr{A}$ and $\mathscr{B}$ are strictly increasing and for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$ it holds that $A \perp^* B$.

A set $C \in [\omega]^\omega$ *fills* a pre-gap $(\mathscr{A}, \mathscr{B})$ if for all $A \in \mathscr{A}$ it holds that $A \subseteq^* C$ and for all $B \in \mathscr{B}$ it holds that $B \perp^* C$.

A pre-gap $(\mathscr{A}, \mathscr{B})$ is a *gap* if there is no set C that fills the pre-gap. We also say that this is a (κ, λ)-gap.

The following is well known.

Proposition 7. *The cardinal* $\mathfrak{b}$ *is the smallest cardinal* λ *for which there exists an* (ω, λ)*-gap in* $\mathscr{P}(\omega)/\mathbf{fin}$.

For reference, here are the definitions for three other cardinal invariants we shall use.

Definition 8. The *additivity* $\mathrm{add}(\mathscr{M})$ of a meager ideal is the least cardinal κ such that there is a family $\mathscr{A}$ of size κ of meager sets for which $\bigcup \mathscr{A}$ is not meager.

The *covering number* $\mathrm{cov}(\mathscr{M})$ of a meager ideal is the least cardinal κ such that there is a family $\mathscr{A}$ of size κ of meager sets for which $\bigcup \mathscr{A} = 2^\omega$.

The *cofinality* $\mathrm{cof}(\mathscr{M})$ of a meager ideal is the least cardinal κ such that there is a family $\mathscr{A}$ of size κ of meager sets so that for any meager set B there is $A \in \mathscr{A}$ so that $B \subseteq A$.

A definition for gaps for $\mathscr{P}(\mathbb{Q})/\mathbf{nwd}$ is simply the above one with the ideal $\mathbf{fin}$ replaced with the ideal $\mathbf{nwd}(\mathbb{Q})$. However, a definition for $\mathrm{Dense}(\mathbb{Q})/\mathbf{nwd}$ needs some consideration. Instead of a dichotomy of sets in the ideal and outside of the ideal, we get a trichotomy of nowhere dense, everywhere dense and somewhere (but not everywhere) dense sets. We could get alternative definitions of gaps where orthogonality of two sets would not mean that the intersection is small, in $\mathbf{nwd}(\mathbb{Q})$, but that it's not large, ie. not in $\mathrm{Dense}(\mathbb{Q})$. However, it looks like that the most natural way to define it, by just restricting the possible sets for the definition of gaps in $\mathscr{P}(\mathbb{Q})/\mathbf{nwd}$ to $\mathrm{Dense}(\mathbb{Q})$, is a good one. For many other definitions there are (ω, ω)-gaps or no gaps at all. Thus let $A \perp_{\mathbf{nwd}} B \Leftrightarrow A \cap B \in \mathbf{nwd}$ and $A \subseteq_{\mathbf{nwd}} B \Leftrightarrow A \setminus B \in \mathbf{nwd}$.

Definition 9. Let $\mathscr{A} = \{A_\alpha \in \mathrm{Dense}(\mathbb{Q}) : \alpha < \kappa\}$ and $\mathscr{B} = \{B_\beta \in \mathrm{Dense}(\mathbb{Q}) : \beta < \lambda\}$. The pair $(\mathscr{A}, \mathscr{B})$ is a *pre-gap* on $\mathrm{Dense}(\mathbb{Q})/\mathbf{nwd}$ if both $\mathscr{A}$ and $\mathscr{B}$ are strictly increasing and for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$ it holds that $A \perp_{\mathbf{nwd}} B$.

A set $C \in \mathrm{Dense}(\mathbb{Q})$ *fills* a pre-gap $(\mathscr{A}, \mathscr{B})$ if for all $A \in \mathscr{A}$ it holds that $A \subseteq_{\mathbf{nwd}} C$ and for all $B \in \mathscr{B}$ it holds that $B \perp_{\mathbf{nwd}} C$.

A pre-gap $(\mathscr{A}, \mathscr{B})$ is a *gap* if there is no set C that fills the pre-gap. We also say that this is a (κ, λ)-gap.

Gaps with this definition coincide with gaps on $\mathscr{P}(\mathbb{Q})/\mathbf{nwd}$.

Lemma 10. *If* $((A_\alpha)_{\alpha<\kappa}, (B_\beta)_{\beta<\lambda})$ *is a gap in* $\mathscr{P}(\mathbb{Q})/\mathbf{nwd}$*, then there is a* (κ, λ)*-gap in* $\mathrm{Dense}(\mathbb{Q})/\mathbf{nwd}$*. If* $((C_\alpha)_{\alpha<\kappa}, (D_\beta)_{\beta<\lambda})$ *is a gap in* $\mathrm{Dense}(\mathbb{Q})/\mathbf{nwd}$*, then it is a gap in* $\mathscr{P}(\mathbb{Q})/\mathbf{nwd}$*.*

Proof. For the first claim we can get a gap on $\mathrm{Dense}(\mathbb{Q})/\mathbf{nwd}$ by the following construction. If $a \in 2^{<\omega} \setminus \{()\}$, let $\tilde{a} \in 2^{<\omega}$ be such that $|\tilde{a}| = 3|a| - 2$, $\forall n < 3|a|-2, \tilde{a}(3n) = a(n)$ and $\forall n < 3|a|-3, \tilde{a}(3n+1) = \tilde{a}(3n+2) = 0$. For $\alpha < \kappa$, let $A'_\alpha = \{s \in 2^{<\omega} : \text{either } \exists t \in A_\alpha \setminus \{()\}, s = \tilde{t}, \text{ or } |s| = 3n+1\}$ and for $\beta < \lambda$ let $B'_\beta = \{s \in 2^{<\omega} : \text{either } \exists t \in B_\beta \setminus \{()\}, s = \tilde{t}, \text{ or } |s| = 3n+2\}$. Then clearly for all $\alpha < \kappa$ and $\beta < \lambda$, every A'_α and B'_β are dense, and still $A'_\alpha \cap B'_\beta \in \mathbf{nwd}$. Moreover, if D would be a filling set for the gap on $\mathrm{Dense}(\mathbb{Q})/\mathbf{nwd}$, its restriction to levels $3n$, $n < \omega$ would be a filling set for the gap on $\mathscr{P}(\mathbb{Q})/\mathbf{nwd}$.

The second claim is clear, as a filling set in $\mathscr{P}(\mathbb{Q})/\mathbf{nwd}$ would also have to be dense, as it almost contains a dense set A_0. ∎

The following lemma can probably be found in the literature, but the original author is unknown to the current author. The proof here is for completeness. We shall use this lemma many times in other proofs.

Lemma 11. *Assume* $\kappa < \mathrm{add}(\mathscr{M})$ *and for all* $\alpha < \kappa$*,* A_α *are nowhere dense sets of reals. Then there is an increasing sequence of nowhere dense sets* $(A'_n)_{n<\omega}$ *so that* $\forall \alpha \exists n A_\alpha \subset A'_n$*.*

Theorem 12 (Truss-Miller)**.**

$$\mathrm{add}(\mathscr{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathscr{M})).$$

Proof of lemma 11. Let $\kappa < \mathrm{add}(\mathscr{M})$ and $(A_\alpha)_{\alpha<\kappa}$ be nowhere dense. Because $\kappa < \mathrm{cov}(\mathscr{M})$, for each q_n there is $x_n \in 2^\omega$, $q_n \subseteq x_n$, such that $x_n \notin \bigcup A_\alpha$. For each $\alpha < \kappa$ and $n < \omega$, let

$$f_\alpha(n) = \min\{|s| : s \in 2^{<\omega}, s \subseteq x_n, [s] \cap A_\alpha = \emptyset\}.$$

Thus we have a family of $\kappa < \mathfrak{b}$ many functions, so there is a function $g < \omega^\omega$ for which $f_\alpha <^* g$ for all α. Without loss of generality, g is strictly increasing.

Let $A'_n = 2^\omega \setminus \bigcup_{n\leq i<\omega}[x_i \restriction g(i)]$. Let us check that these sets are as required. First of all, they are nowhere dense. Let $s \in 2^{<\omega}$. Then $s \subseteq q_i$ for some $i \geq n$. If $g(i) < |q_i|$ then $[q_i] \subseteq [x_i \restriction g(i)]$ and so $[q_i] \cap A'_n = \emptyset$. If $g(i) \geq |q_i|$, then s can be extended to $s' = x_i \restriction g(i)$, and now $[s'] \cap A'_n = \emptyset$. For all $\alpha < \kappa$, we have to show $A_\alpha \subseteq A'_k$ for some k. Thus let $\alpha < \kappa$ and let $k = \max\{i : f_\alpha(i) > g(i)\} + 1$. Let $y \in A_\alpha$. Now for all $i \geq k$ it holds that $g(i) \geq f_\alpha(i)$. Thus for all $i \geq k$, $[x_i \restriction g(i)] \subseteq [x_i \restriction f_\alpha(i)]$. Now for all $i \geq k$, $y \notin [x_i \restriction f_\alpha(i)]$, so $y \notin [x_i \restriction g(i)]$, and we get that $y \in A'_k$ as required. □

Next we prove the main theorem of this paper. This is done in the next two lemmas.

Lemma 13. *There are sequences* $\mathscr{A} \subseteq \mathrm{Dense}(\mathbb{Q})$ *of size* $\mathrm{add}(\mathscr{M})$ *and* $\mathscr{B} \subseteq \mathrm{Dense}(\mathbb{Q})$ *of size* ω *such that* $(\mathscr{A}, \mathscr{B})$ *is a gap on* $\mathrm{Dense}(\mathbb{Q})/\mathbf{nwd}$.

In other words, there is a $(\mathrm{add}(\mathscr{M}), \omega)$*-gap.*

Proof. Let $\kappa = \mathrm{add}(\mathscr{M})$ and choose meager sets $G_\alpha, \alpha < \kappa$, so that $\bigcup_{\alpha<\kappa} G_\alpha$ is not meager. Using these, we construct sequences $\mathscr{A}$ and $\mathscr{B}$ that form a pre-gap and there is no set filling the gap. If there was such a set, we show that there would also be a meager set containing each G_α, a contradiction.

First we can assume that the sequence $(G_\alpha)_{\alpha<\kappa}$ is strictly increasing and by thinning out the sequence we can arrange that $G_\alpha \setminus \bigcup_{\beta<\alpha} G_\beta \notin \mathbf{nwd}(\mathbb{R})$ for each $\alpha < \kappa$. For each $\alpha < \kappa$ the set G_α is meager, so let $(G^n_\alpha)_{n<\omega}$ be an increasing sequence of nowhere dense sets (in $\mathbb{R}$) so that $G_\alpha = \bigcup_{n<\omega} G^n_\alpha$.

Claim. *We can arrange that for each* α *and* $\beta < \alpha$ *it holds that there is* n_0 *such that* $\forall k \geq n_0 : G^k_\beta \subseteq G^k_\alpha$.

Proof of claim. If the condition is not met by the original sequence, we can form another sequence $(H_\alpha)_{\alpha<\kappa}$ with the following construction. If $\alpha = 0$, let $H^n_\alpha = G^n_\alpha$ for each n. If $\alpha < \kappa$ is a successor ordinal, $\alpha = \gamma + 1$, let $H^n_\alpha = H^n_\gamma \cup G^n_\alpha$ for each n. Let $H_\alpha = \bigcup_{n<\omega} H^n_\alpha$.

If $\alpha < \kappa$ is a limit ordinal, we can fix n and apply Lemma 11 to sequence $(G^n_\beta)_{\beta<\alpha}$ to get an increasing sequence A^n_k of nowhere dense sets so that for each β there is i such that $H^n_\beta \subseteq A^n_k$ for each $k \geq i$. Let $f_\beta(n)$ to be that i. Because $\alpha < \mathfrak{b}$, there is a function g_α eventually dominating all f_β for $\beta < \alpha$. Let $H^n_\alpha = A^n_{g_\alpha(n)} \cup G^n_\alpha$ and let $H_\alpha = \bigcup_{n<\omega} H^n_\alpha$. Clearly each H_α is meager and $G_\alpha \subseteq H_\alpha$.

We check that the condition required in the claim holds by induction on α. Let $\beta < \alpha < \kappa$. If α is a successor ordinal $\alpha = \gamma + 1$ and the condition holds for γ, then clearly the condition holds, even if we choose $n_0 = 0$. Let α then be a limit ordinal. Consider functions f_β and g_α defined above. Let n_0 be such that $\forall i \geq n_0 : f_\beta(i) < g_\alpha(i)$. We show that this is the required n_0 for the claim. Let $j \geq n_0$ and $x \in H^j_\beta$. We have to show that $H^j_\beta \subseteq H^j_\alpha$. For all $i \geq n_0$ the sets $(A^i_k)_{k<\omega}$ are increasing, so it holds that $A^i_{f_\beta(i)} \subseteq A^i_{g_\alpha(i)}$. From the definition of f_β we get that $H^j_\beta \subseteq A^j_{f_\beta(j)} \subseteq A^j_{g_\alpha(j)}$ and so $x \in H^j_\alpha$.
□ Claim

We may then assume that the sequence G is constructed as in the previous claim. Choose $\mathscr{J} = \{J_n \subseteq \omega : n < \omega, |J_n| = \omega\}$ any family of infinite disjoint sets whose union is ω. For each n, let

$$B_n = \{s : |s| = k \text{ for some k } \in J_n\},$$

so each B_n consists of full levels and they are pairwise disjoint. Let $b_n = \min J_n$. Let $B'_n = \bigcup_{i \leq n} B_n$ and $\mathscr{B} = \{B'_n : n < \omega\}$. This family is strictly increasing. For convenience we use sets B_n instead of B'_n in definitions.

Now the sets $\Phi^{\mathbb{R}}(G^n_\alpha)$ are nowhere dense in $\mathbb{Q}$. For each α, let

$$A_\alpha = \{s \frown t : \exists n \text{ such that } |s| = b_n \text{ and } t \in \Phi^{\mathbb{R}}(G^n_\alpha) \text{ and } s \frown t \in B_n\}.$$

In other words, for each n we take every element s on the first level of B_n, and attach a copy of the tree $\Phi^{\mathbb{R}}(G^n_\alpha)$ as a subtree above it, but so that only B_n intersects with it.

We have to check that the family $(A_\alpha)_{\alpha<\kappa}$ is increasing and does it strictly. Let $\beta < \alpha < \kappa$. First we check that $A_\beta \subseteq_{\mathbf{nwd}} A_\alpha$. Let i be such that for all

$n \geq i$ it holds that $G^n_\beta \subseteq G^n_\alpha$. Let $x \in A_\beta \cap \bigcup_{k>i} B_k$ and let k be the specific one for which $x \in B_k$. Then there is s such that $|s| = b_k$ and $t \in \Phi(G^k_\beta)$ and $x = s \frown t$. Because $G^k_\beta \subseteq G^k_\alpha$, we get that $t \in \Phi(G^k_\alpha)$, and so there is an element $s \frown t$ contained also in A_α.

Next we check that for $\beta < \alpha < \kappa$, $A_\alpha \setminus A_\beta \notin \mathbf{nwd}(\mathbb{Q})$. Because the sets G_α are strictly increasing, there must be infinite number of indices n for which $G^n_\alpha \setminus G^n_\beta \neq \emptyset$. Let $s \in 2^{<\omega}$ and let k be such that the previous inequality holds and $b_k > |s|$. Then we can extend s to any s' of length b_k, we can choose any element from $G^k_\alpha \setminus G^k_\beta$, and we get some $t \in (\Phi^{\mathbb{R}}(G^k_\alpha) \setminus \Phi^{\mathbb{R}}(G^k_\beta))$ so that $s' \frown t \in B_n$. This shows that the set of new elements in A_α is actually not only somewhere dense but dense.

It also holds that for every $\alpha < \kappa$, A_α is almost disjoint from each B_n. Let $n < \omega$. If we choose s on or below the first level of B_n, we can choose any such $t \in 2^{<\omega}$ that $[t] \cap \Phi(G^n_\alpha) = \emptyset$, and choose any s' such that $s \subset s'$ and $|s'| = b_n$. Now $[s' \frown t] \cap A_\alpha \cap B_n = \emptyset$. If $|s| > b_n$, then let $t(i) = s(i + b_n)$, $0 \leq i < |s| - b_n$. Choose an element t' above t such that $[t'] \cap \Phi(G^n_\alpha) = \emptyset$. It follows that $[s \restriction b_n \frown t']$ is a basic open set above s which is disjoint from $A_\alpha \cap B_n$.

We finally claim that the sequences $(A_\alpha)_{\alpha<\kappa}$ and $(B_n)_{n<\omega}$ form a gap. Towards contradiction, assume that they don't, and there is a set D filling this gap. Thus, $D \cap B_n \in \mathbf{nwd}(\mathbb{Q})$ for each B_n and $A_\alpha \subseteq_{nwd} D$ for each A_α.

Let $G = \bigcup_{n<\omega} \bigcup_{|s|=b_n} \Phi^{\mathbb{Q}}(\{t : s \frown t \in D \cap B_n\})$. Because $D \cap B_n$ is nowhere dense, clearly the function $\Phi^{\mathbb{Q}}$ is defined for the set above, and the result is nowhere dense in $\mathbb{R}$. Thus G is well-defined and meager.

Let $\alpha < \kappa$ and $x \in G_\alpha$. Fix n so that $x \in G^n_\alpha$. From $A_\alpha \subseteq_{\mathbf{nwd}} D$ we infer that there must be some s for which $[s] \cap A_\alpha \subseteq D$. Further, there must be some $b_{n'} \geq |s|$, for $n' \geq n$. Choose any s' so that $s \subseteq s'$ and $|s'| = b_{n'}$. Now consider how we have built the set $A_\alpha \cap B_{n'}$. Clearly $x \restriction i \in \Phi(A^{n'}_\alpha)$ for all i, so $s' \frown x \restriction i \in A_\alpha$ whenever $s' \frown x \restriction i \in B_{n'}$. The elements $s' \frown x \restriction i$ belong to D for infinitely many i, and we get that $x \in \Phi^{\mathbb{Q}}(\{t : s \frown t \in D \cap B_{n'}\})$ with the chosen s and n', thus $x \in G$. $\square$

Lemma 14. *Let $\kappa < \mathrm{add}(\mathscr{M})$ and $\mathscr{A} = \{A_\alpha \in \mathrm{Dense}(\mathbb{Q}) : \alpha < \kappa\}$ and $\mathscr{B} = \{B_n \in \mathrm{Dense}(\mathbb{Q}) : n < \omega\}$ be any sequences so that $(\mathscr{A}, \mathscr{B})$ is a pre-gap on $\mathrm{Dense}(\mathbb{Q})/\mathbf{nwd}$. Then there is a set $C \in \mathrm{Dense}(\mathbb{Q})$ which fills the pre-gap.*

In other words, there are no (κ, ω)-gaps for $\kappa < \mathrm{add}(\mathscr{M})$.

Proof. Let $B'_n = B_n \setminus \bigcup_{i<n} B_i$ and note that they are pairwise disjoint. If a set is orthogonal to some B_n, then it's also orthogonal to B'_n, and if a set is orthogonal to all B'_i for $i \leq n$, then it's also orthogonal to B_n. Some of the B'_n's might be nowhere dense or empty.

Let $A^n_\alpha = A_\alpha \cap B'_n$. It follows that A^n_α is nowhere dense for all α and n. For each n, apply lemma 11 to the sequence $(\Phi^{\mathbb{Q}}(A^n_\alpha))_{\alpha<\kappa}$ to get sequences of nowhere dense sets of reals $\mathscr{F}_n = \{F^n_i : i < \omega\}$. Also for each $\alpha < \kappa$ let f_α be a function defined by

$$f_\alpha(n) = \min\{k : \Phi^{\mathbb{Q}}(A^n_\alpha) \subset F^n_k\}.$$

Because we have $\kappa < \mathrm{add}(\mathscr{M}) \leq b$ many functions, let g be a function that eventually dominates all functions f_α. Finally, let $D_n = B'_n \cap \Phi^{\mathbb{R}}(F^n_{g(n)})$ and $D = (\cup_{n<\omega} D_n) \cup (2^{<\omega} \setminus \bigcup_{n<\omega} B_n)$.

We are left to check that this is a set that fills the gap. First we check that $D \perp_{\mathbf{nwd}} B_n$ for each n. Because $B_n = \bigcup_{i\leq n} B'_i$, we can write $D \cap B_n = \bigcup_{i\leq n}(D \cap B'_i)$. But $D \cap B'_i = (\bigcup_{n<\omega} D_n) \cap B'_i = \bigcup_{n<\omega}(D_n \cap B'_i) = D_i \cap B'_i = D_i$, which is nowhere dense.

Next we check that $A_\alpha \subseteq_{\mathbf{nwd}} D$ for each α. For each α let k_α be some k so that $\forall i \geq k$, $f_\alpha(i) < g(i)$. It must be shown that for each $s \in 2^{<\omega}$ there is t extending s such that $A_\alpha \cap [t] \subseteq D$. Thus, let s be arbitrary. Let $A = \bigcup\{A^i_\alpha : i < k_\alpha\}$. As a union of finitely many nowhere dense sets, A is itself nowhere dense. Choose t extending s so that $[t] \cap A = \emptyset$. We show that this t satisfies the condition above.

Let t' be an element of $A_\alpha \cap [t]$. If t' is not contained in any B_i, then it is contained in the last part of the union that defines D, thus it's contained in D. Assume it's contained in B_i for some i, so $t' \in A^i_\alpha$. Because $[t] \cap \bigcup\{A_\alpha \cap B'_i : i < k_\alpha\} = \emptyset$, t' can't be contained in any of the B'_i for $i < k_\alpha$. Thus $t' \in B'_i$ for some $i \geq k_\alpha$. Because $i \geq k_\alpha$, we know $g(i) \geq f_\alpha(i)$, so we get $\Phi^{\mathbb{Q}}(A^i_\alpha) \subset F^i_{f_\alpha(i)} \subset F^i_{g(i)}$. Because $t' \in A^i_\alpha$, $t' \frown \bar{0} \in \Phi^{\mathbb{Q}}(A^i_\alpha)$ and thus $t' \frown \bar{0} \in F^i_{g(i)}$ and thus $t' \in B'_i \cap \Phi^{\mathbb{R}}(F^i_{g(i)}) \subseteq D$. □

Theorem 15. *The least κ for which there is a (κ, ω)-gap in* Dense($\mathbb{Q}$)/**nwd** *or $\mathscr{P}(\mathbb{Q})$/**nwd** is* add($\mathscr{M}$).

Proof. Lemmas 13, 14 and 10. □

Corollary 16. *There are no (ω, ω)-gaps in* Dense($\mathbb{Q}$)/**nwd**.

3 A Hausdorff gap

Using similar techniques as used to prove the existence of Hausdorff gaps in $\mathscr{P}(\omega)/\mathbf{fin}$, that is, certain gaps of type (ω_1, ω_1), we can now also prove the existence of these gaps in $\mathrm{Dense}(\mathbb{Q})/\mathbf{nwd}$. This proof vaguely follows such proof in Jech [Jec03], theorem 29.7. As was previously mentioned, this result is already known by [Tod98].

Theorem 17. *There are sequences $\mathscr{A}$ and $\mathscr{B}$ both of length $\aleph_1$ such that they form a pre-gap and no set $D \in \mathrm{Dense}(\mathbb{Q})$ fills the pre-gap. In other words, there is a (ω_1, ω_1)-gap in $\mathrm{Dense}(\mathbb{Q})$.*

Proof. We construct sets A_α and B_α by induction on $\alpha < \omega_1$ so that the following conditions hold for every $\alpha < \omega_1$.

1. $2^{<\omega} \setminus (A_\alpha \cup B_\alpha)$ is dense.
2. For each $n < \omega$ the set $\{\beta < \alpha : [q_n] \cap B_\beta \cap A_\alpha = \emptyset\}$ is finite.
3. If α is a successor ordinal $\alpha = \beta + 1$, then $B_\alpha \setminus B_\beta$ is dense.

Let us first check that if condition 2 holds for all $\alpha < \omega_1$ and $(A_\alpha)_{\alpha<\omega_1}$ and $(A_\alpha)_{\alpha<\omega_1}$ form a pre-gap, then no set D fills the pre-gap. Assume otherwise, so that there is $D \in \mathrm{Dense}(\mathbb{Q})$ so that $A_\alpha \subseteq_{nwd} D$ and $B_\alpha \perp_{nwd} D$ for each α. Thus there is some $s \in 2^{<\omega}$ and $Z_0 \subseteq \omega_1$ so that $A_\alpha \cap [s] \subseteq D$ for $\alpha \in Z_0$ and $|Z_0| = \aleph_1$. There is also some $s' \in 2^{<\omega}$ extending s and $Z_1 \subseteq Z_0$ so that $B_\alpha \cap [s'] \cap D = \emptyset$ for $\alpha \in Z_1$, and $|Z_1| = \aleph_1$. Let α be the ω'th member of Z_1. Then for each $\beta < \alpha$, $\beta \in Z_1$ it holds that $[s'] \cap B_\beta \cap A_\alpha = \emptyset$, which contradicts condition 2.

Let us construct the required sets. Choose disjoint dense sets C_0, C_1 and C_2 such that $2^{<\omega} = C_0 \cup C_1 \cup C_2$. Take $A_0 = C_0$ and $B_0 = C_2$. If α is a successor ordinal $\alpha = \beta + 1$, then using condition 1 we can find disjoint dense sets C_0', C_1' and C_2' so that

$$2^{<\omega} \setminus (A_\beta \cup B_\beta) = C_0' \cup C_1' \cup C_2'.$$

Let $A_\alpha = A_\beta \cup C_0'$ and $B_\alpha = B_\beta \cup C_2'$.

The construction is more involved when α is a limit ordinal. The ordinal α has countable cofinality, so let δ_i, $i < \omega$ be a sequence converging to α. We can assume that all δ_i are successor ordinals. Using corollary 16 on families $(A_{\delta_i})_{i<\omega}$ and $(B_{\delta_i})_{i<\omega}$ we can find a set D so that $A_{\delta_i} \subseteq_{\mathbf{nwd}} D$ and $B_{\delta_i} \perp_{\mathbf{nwd}} D$ for each i. Note that if $\beta < \alpha$, then also $B_\beta \perp_{\mathbf{nwd}} D$, as there is some i' so that $\beta < \delta_{i'}$, and $B_\beta \subseteq_{\mathbf{nwd}} B_{\delta_{i'}}$, and $B_\beta \cap D \subseteq (B_\beta \setminus B_{\delta_{i'}}) \cup (B_{\delta_{i'}} \cap D)$.

Let us first make sure condition 2 holds by choosing a suitable A_α, and then choose B_α so that condition 1 holds as well. Condition 2 might not hold if we took A_α to be set D, but we can arrange it by changing D slightly. For each n, let $C_n = \{\beta < \alpha : [q_n] \cap D \cap B_\beta = \emptyset\}$. If each C_n is finite, then condition 2 holds already for the set D. Thus assume some of the C_n are infinite. We construct an increasing sequence $(E_n)_{n<\omega}$, $E_n \in \mathrm{Dense}(\mathbb{Q})$, so let $D = E_0$ to start the induction.

Let $n = k + 1$ and assume we have constructed E_k. Assume C_k is infinite, otherwise let $E_n = E_k$. Because condition 2 holds for all $\beta < \alpha$, $C_k \cap \gamma$ is finite for each γ, and so C_k is cofinal in α and of cofinality ω. Let $(\gamma_i^k)_{i<\omega}$ enumerate C_k. For each i, choose a maximal antichain $(s_i^{k,j})_{j<\omega}$, so that $s_i^{k,j} \in B_{\gamma_i} \setminus (D \cup \bigcup_{l<i} B_{\gamma_l})$ and $|s_i^{k,j}| \geq i$ for all $i < \omega$. We can choose these using conditions 1 and 3 for each γ_i. Now let $E_{k+1} = E_k \cup \{s_i^{k,j} : i, j < \omega\}$.

Each E_n is clearly dense. We check that if $\beta < \alpha$, then $E_n \cap B_\beta \in \mathbf{nwd}$. For E_0 this holds, so assume $n = k+1$ and it holds for E_k. Of course we only have to check that $\{s_i^{k,j} : i, j < \omega\} \cap B_\beta \in \mathbf{nwd}$. But let l be such that $\gamma_l^k > \beta$. Now $B_\beta \subseteq_{\mathbf{nwd}} B_{\gamma_l}$ and $\{s_i^j : i, j < \omega\} \cap B_{\gamma_l} = \{s_i^j : i < l, j < \omega\} \in \mathbf{nwd}$ as a finite union of antichains.

We can use corollary 16 to get a set D' that fills the pre-gap $((E_n)_{n<\omega}, (B_{\delta_i})_{i<\omega})$. We can assume $D \subseteq D'$. We show that this set satisfies condition 2, if we set $A_\alpha = D'$. Thus let $n < \omega$ and we show that $C = \{\beta < \alpha : [q_n] \cap B_\beta \cap D' = \emptyset\}$ is finite. First note that $C \subseteq C_n$. As $E_{n+1} \subseteq_{\mathbf{nwd}} D'$, we can find some $s \in 2^{<\omega}$, $q_n \subseteq s$, so that $E_{n+1} \cap [s] \subseteq D'$.

We claim that if $i \geq |s|$, then $[q_n] \cap B_{\gamma_i} \cap D' \neq \emptyset$. This holds, because if $i \geq |s|$, then the antichain $(s_i^{n,j})_{j<\omega} \subseteq B_{\gamma_i}$ must intersect $[s]$ at some point s'. But $s' \in E_{n+1}$ and thus $s \in D'$. To summarize, $s \in [q_n] \cap B_{\gamma_i} \cap D'$, which is what was required. Thus if $\gamma \in C$, we get that $\gamma = \gamma_i$ for some i using the above enumeration, and it follows that C must be finite, and so we can set $A_\alpha = D'$.

Assume then that condition 2 holds for A_α and $A_\alpha \perp_{\mathbf{nwd}} B_\gamma$ for all $\gamma < \alpha$, on top of the other induction assumptions. Then we can choose B_α so that also condition 1 holds for this pair. To show this, choose for each n elements $s_n \supseteq q_n$ and $t_n \supseteq q_n$ such that $s_n \neq t_n$ and $s_n \in B_{\delta_n} \setminus (\bigcup_{\beta<\delta_n} B_\beta \cup \{t_i : i \leq n\}$ and $s_n \notin A_\alpha$. Then let $B_\alpha = 2^{<\omega} \setminus (A_\alpha \cup \{s_n : n < \omega\})$. We can choose these elements, because condition 3 holds for B_β with $\beta < \alpha$.

The set B_α is a valid extension of the sequence, because if $s \in B_\beta$ for some $\beta < \alpha$, then from some n on, $s_n \notin B_\beta$, so we can extend s to $s' \in B_\beta$ so that $[s'] \cap B_\beta \cap A_\alpha \cap \{s_n : n < \omega\} = \emptyset$ and thus $[s'] \cap B_\beta \subseteq B_\alpha$.

Now clearly the elements s_n show that condition 1 holds, and condition 2 clearly still holds. □

4 Dominating number

As mentioned above, the unbounding number $\mathfrak{b}$ can be characterized using gaps on $\mathscr{P}(\omega)/\mathbf{fin}$. It is well known that there is a similar characterization for the dominating number $\mathfrak{d}$ with respect to the structure of $\mathscr{P}(\omega)/\mathbf{fin}$, namely

Proposition 18.

$$\mathfrak{d} = \min\{|\mathscr{A}| : \mathscr{A} \subseteq [\omega]^\omega \wedge \exists \mathscr{B} \subseteq [\omega]^\omega \ s.t. \\ |\mathscr{B}| = \omega, \mathscr{B} \perp \mathscr{A} \ and \ \forall X \in [\omega]^\omega : \ if \ \forall B \in \mathscr{B}(B \perp X) \\ then \ \exists A \in \mathscr{A}(X \subseteq^* A)\}$$

Proof. Easy. □

As before, we can replace the underlying structure with Dense($\mathbb{Q}$)/**nwd**, which results in a viable definition. However, we need to ignore too weakly increasing sequences in the definition, as the following definition and lemma shows.

Definition 19. Let $s \in \mathbb{Q}$. The sequence $(B_n)_{n<\omega}$ is *uniformly strictly increasing above* s, if for each $t \in \mathbb{Q}$ so that $s \subseteq t$ there is $i \in \mathbb{N}$ so that $B_i \setminus \bigcup_{k<i} B_k$ is somewhere dense below t.

We say that the sequence is *uniformly strictly increasing* if there is some $s \in \mathbb{Q}$ above which it does so.

Lemma 20. *If a strictly increasing $(B_n)_{n<\omega}$ is not a uniformly strictly increasing above any s, then there is a $\subseteq_{\mathbf{nwd}}$-maximal set $A \in \mathscr{P}(\mathbb{Q})$ orthogonal to each B_n.*

Proof. As $(B_n)_{n<\omega}$ is not uniformly strictly increasing above any s, for each $s \in \mathbb{Q}$ we can choose $t_s \supseteq s$ so that for all i, $B_i \setminus \bigcup_{k<i} B_k$ is nowhere dense below t_s. Let t'_n enumerate the antichain $\{t_s : \nexists s' \in \mathbb{Q} t_{s'} \subsetneq t_s\}$ Let A' be the downwards closure of t'_n's and let

$$A_n = \bigcup_{1<i<\omega} ([t'_n] \cap (B_i \setminus B_0)).$$

Let $A = A' \cup \bigcup_{n<\omega} A_n \cup (2^{<\omega} \setminus \bigcup_{i<\omega} B_i)$. A is orthogonal to each B_n as $A \cap B_n \subseteq A' \cup (B_n \setminus B_0)$ which is nowhere dense.

Let us check that this A is the required set. Let $C \in \mathscr{P}(\mathbb{Q})$ be orthogonal to each B_n. We have to show that for each $s \in \mathbb{Q}$ there is $u \supseteq s$ so that $[u] \cap C \subseteq A$. Thus let $s \in \mathbb{Q}$ and let t be an element in the maximal antichain $(t'_n)_{n<\omega}$ that is compatible with it. Because $C \cap B_0$ is nowhere dense, let u be a common extension of s and t so that $[u] \cap C \cap B_0 = \emptyset$. Now $[u] \cap C \cap B_i \subseteq A$ for each $i > 0$. To show this, let $v \in [u] \cap C \cap B_i$ with some $i > 0$. Now $v \in [t'_n]$ with some n and as $v \notin B_0$, we get that $v \in B_i \setminus B_0$. Thus $v \in A_n$ and so $v \in A$.

This u is as required, because if $v \in [u] \cap C$, then either $v \in \bigcup_{i<\omega} B_i$ holds or not. If it holds, $v \in B_k$ with some $k > 0$, so we get $v \in [u] \cap C \cap B_k \subseteq A$, and if it doesn't hold, clearly $v \in A$. ∎

As expected from the duality of the cardinal invariants involved, the following holds.

Proposition 21.

$$\begin{aligned}\operatorname{cof}(\mathscr{M}) = \min\{|\mathscr{A}| : \mathscr{A} \subseteq \operatorname{Dense}(\mathbb{Q}) \wedge \exists \mathscr{B} \subseteq \operatorname{Dense}(\mathbb{Q}) \textit{ s.t.} \\ |\mathscr{B}| = \omega, \mathscr{B} \textit{ is uniformly strictly increasing}, \\ \mathscr{B} \perp_{\mathbf{nwd}} \mathscr{A} \textit{ and } \forall X \in \operatorname{Dense}(\mathbb{Q}): \textit{ if } \forall B \in \mathscr{B} (B \perp_{\mathbf{nwd}} X) \\ \textit{then } \exists A \in \mathscr{A} (X \subseteq_{\mathbf{nwd}} A)\}\end{aligned}$$

Proof. Let κ be the cardinal on the right side of the equation, if it exists.

Claim. *κ exists and $\kappa \leq \mathrm{cof}(\mathscr{M})$.*

Proof of claim. First let us fix a cofinal family $\mathscr{G} = \{G_\alpha \subseteq \mathbb{R} : \alpha < \mathrm{cof}(\mathscr{M})\}$ of meager sets of reals and a dominating family $\mathscr{F} = \{f_\alpha \in \omega^\omega : \alpha < \mathfrak{d}\}$ of functions. Using these we construct families $\mathscr{A}$ and $\mathscr{B}$. As in proof of lemma 13, we let $\mathscr{B} = \{B_n : n < \omega\}$ be the disjoint dense sets of full levels with respect to some collection of disjoint infinite sets $\mathscr{B}'$, and we let b_n be the minimum of each of these sets.

First fix $G \in \mathscr{G}$. We shall construct a family of ω-sequences $\{(C_n^{G,\alpha})_{n<\omega} : \alpha < \mathfrak{d}\}$, of nowhere dense sets of reals so that if $(G_n)_{n<\omega}$ is any sequence of nowhere dense sets contained in G, then there are α and n_0 so that $n \geq n_0$ implies $G_n \subseteq C_n^{G,\alpha}$.

Because G is meager, we can for each n choose $x_n \in \mathbb{R}$ so that $q_n \subseteq x_n$ and $x_n \notin G$. We can assume all x_n are different. Now if $D \subseteq G$ is any nowhere dense set, also $x_n \notin D$ holds, so for each n there is k so that $D \cap [x_n \restriction k] = \emptyset$. This induces a function $g_D : \omega \to \omega$. Also if $g : \omega \to \omega$ is any function, we can define a nowhere dense set $D_g = 2^\omega \setminus \bigcup_{n<\omega} [x_n \restriction k]$. For some g this set might be empty, but if $g \geq g_D$, then $D \subseteq D_g$.

For each $\alpha < \mathrm{cof}(\mathscr{M})$ and $n < \omega$ define functions f_α^n as follows:

$$f_\alpha^n(k) = \begin{cases} f_\alpha(n) & \text{if } k \leq n, \\ f_\alpha(k) & \text{if } k > n. \end{cases}$$

Then for each $\alpha < \mathrm{cof}(\mathscr{M})$ and n let

$$C_n^{G,\alpha} = D_{f_\alpha^n}.$$

Let us check that this family satisfies the required condition. Let $(G_n)_{n<\omega}$ be a sequence of nowhere dense sets contained in G. For each n we can find a function g_{G_n}. Without loss of generality, we can assume that each function g_{G_n} is increasing, and further, that for any $i < j < \omega$, $g_{G_i} \leq g_{G_j}$. Let g be the diagonal function of that sequence.

Because $\mathscr{F}$ is a dominating family of functions, there must be some γ and k_0 for which $f_\gamma(k) > g(k)$ whenever $k > k_0$. By the choice of g, $g(k) \geq g_{G_k}(k)$ for all $k < \omega$. Because g_{G_n} were chosen to be increasing, it also follows that $f_\gamma^k(n) > g_{G_k}(n)$ for every n and $k > k_0$. From this it follows that for every $k > k_0$ it holds that $G_k \subseteq C_k^{G,\alpha}$, as required.

Now we can finally construct the family $\mathscr{A}$. For each γ and G let $A^{G,\gamma} = \bigcup_{n<\omega}(C_n^{G,\alpha} \cap B_n)$ and let $\mathscr{A} = \{A^{G,\gamma} : G \in \mathscr{F}, \gamma < \mathfrak{d}\}$.

Finally we have to check that this family satisfies what is needed. Because each B_n is disjoint, clearly $A^{G,\gamma}$ is orthogonal to each B_n as their intersection is exactly $C_n^{G,\alpha}$. Each $A^{G,\gamma}$ is dense, or can easily changed to be dense by including the complete first level b_n of each B_n in the set.

Now let X be any dense set of rationals which is orthogonal to each B_n. Let $X_n = X \cap B_n$ and using function $\Phi^{\mathbb{Q}}$ from Lemma 4, we let $X'_n = \Phi^{\mathbb{Q}}(X_n)$. Finally let $X' = \bigcup_{n<\omega} X'_n$. This set is clearly meager, so there is some $G \in \mathscr{G}$ such that $X' \subseteq G$. Now $(X'_n)_{n<\omega}$ is a sequence of nowhere dense sets contained in G, so there is some γ and n_0 so that $n \geq n_0$ implies $X'_n \subseteq C_n^{G,\alpha}$. But this means that the set $A^{G,\gamma}$ almost includes X. All elements in X not contained in $A^{G,\gamma}$ must be in some B_k for $k < n_0$, but the set of such elements is nowhere dense. □ Claim

Claim. $\kappa \geq \mathrm{cof}(\mathscr{M})$.

Proof of claim. Towards contradiction assume $\kappa < \mathrm{cof}(\mathscr{M})$. Let $\mathscr{A} = \{A^\alpha : \alpha < \kappa\}$ and $\mathscr{B} = \{B_n : n < \omega\}$ be the families that witness this minimal κ. From these we construct a cofinal family of meager sets of reals of size κ leading to contradiction.

Let $t \in \mathbb{Q}$ be the element above which the sequence $\mathscr{B}$ is uniformly strictly increasing. Without loss of generality we may assume that for each n, if $t \subseteq q_n$, then $B_n \setminus \bigcup_{k<n} B_k$ is somewhere dense below q_n. Otherwise we can form a new sequence $(C_n)_{n<\omega}$ so that $C_n = \bigcup_{i<k_n} B_i$ for some sequence $(k_n)_{n<\omega}$ and that C_n is somewhere dense below q_n. This new sequence still witnesses the same properties as the original sequence $\mathscr{B}$.

Let $B'_n = B_n \setminus \bigcup_{i<n} B_i$. Let $A_n^\alpha = A^\alpha \cap B'_n$. We use functions $\Phi_D^{\mathbb{Q}}$ and $\Phi_D^{\mathbb{R}}$ from lemma 5. Let

$$G^\alpha = \bigcup_{n<\omega} \Phi^{\mathbb{Q}}_{B'_n \cap [q_n]}(A_n^\alpha).$$

These sets are meager. Let $\mathscr{G} = \{G^\alpha : \alpha < \kappa\}$. We claim this is a cofinal family. Let G be any meager set of reals, and $G = \bigcup_{n<\omega} G_n$ its decomposition to any increasing nowhere dense sets G_n. Let $X = \bigcup_{n<\omega} \Phi^{\mathbb{R}}_{B'_n \cap [q_n]}(G_n)$. Clearly X itself is dense, or without loss of generality we can add finitely many elements to each intersection with B_n to make it dense. Also $X \cap B_n$ is nowhere dense. Thus we get α so that $X \subseteq_{\mathbf{nwd}} A^\alpha$.

We have to show that $G \subseteq G^\alpha$. Let $x \in G$. There is n_0 so that $x \in G_n$ if $n \geq n_0$. Let $s \in \mathbb{Q}$ be such that $t \subseteq s$ and $X \cap [s] \subseteq A^\alpha$. So if $n \geq n_0$ and $q_n \supseteq s$, there are increasing $s_k^n \in B'_n \cap [q_n]$ so that $s_k^n = f_{B'_n \cap [q_n]}(x \restriction k)$ for all k. Choose any $n_1 \geq n_0$ so that $q_{n_1} \supseteq s$. Now it must hold that $s_k^{n_1}$ is an increasing sequence contained in $B'_{n_1} \cap [q_{n_1}]$. This sequence is then also a sequence of A^α, so $x \in \Phi^{\mathbb{Q}}_{B'_n \cap [q_n]}(A_n^\alpha) \subseteq G^\alpha$. $\square$ Claim, Proposition

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