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Dimension and measure of baker-like skew-products of β -transformations

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Abstract

We consider a generalisation of the baker's transformation, consisting of a skew-product of contractions and a β -transformation. The Hausdorff dimension and Lebesgue measure of the attractor is calculated for a set of parameters with positive measure. The proofs use a new transversality lemma similar to Solomyak's [5]. This transversality, which is applicable to the considered class of maps holds for a larger set of parameters than Solomyak's transversality.

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1 Introduction

In [1], Alexander and Yorke considered fat baker's transformations. These are maps on the square $[0, 1) \times [0, 1)$, defined by

$$(x, y) \mapsto \begin{cases} (\lambda x, 2y) & \text{if } y < 1/2 \\ (\lambda x + 1 - \lambda, 2y - 1) & \text{if } y \geq 1/2 \end{cases} ,$$

where $\frac{1}{2} < \lambda < 1$ is a parameter. They showed that the SRB-measure of this map is a product of Lebesgue-measure and the distribution of the

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corresponding Bernoulli convolution $\sum_{k=1}^{\infty} \pm \lambda^k$. Together with Erdős' result [2], this implies that if λ is an inverse of a Pisot-number, then the SRB-measure is singular with respect to the Lebesgue measure on $[0, 1) \times [0, 1)$.

In [2], Solomyak proved that for almost all $\lambda \in (\frac{1}{2}, 1)$, the distribution of the corresponding Bernoulli convolution $\sum_{k=1}^{\infty} \pm \lambda^k$ is absolutely continuous with respect to Lebesgue measure. Hence this implies that the SRB-measure of the fat baker's transformation is absolutely continuous for almost all $\lambda \in (\frac{1}{2}, 1)$. Solomyak's proof used a transversality property of power series of the form $g(x) = 1 + \sum_{k=1}^{\infty} a_k x^k$, where $a_k \in \{-1, 0, 1\}$. More precisely, Solomyak proved that there exists a $\delta > 0$ such that if $x \in (0, 0.64)$ then

$$|g(x)| < \delta \implies g'(x) < -\delta. \quad (1)$$

The constant 0.64 is an approximation of a root to a power series and cannot be improved to something larger than this root.

In this paper we consider maps of the form

$$(x, y) \mapsto \begin{cases} (\lambda x, \beta y) & \text{if } y < 1/\beta \\ (\lambda x + 1 - \lambda, \beta y - 1) & \text{if } y \geq 1/\beta \end{cases},$$

where $0 < \lambda < 1$ and $1 < \beta < 2$. Using the above mentioned transversality of Solomyak one can prove that for almost all $\lambda \in (0, 0.64)$ and $\beta \in (1, 2)$ the SRB-measure is absolutely continuous with respect to Lebesgue measure provided $\lambda\beta > 1$, and the Hausdorff dimension of the SRB-measure is $1 + \frac{\log \beta}{\log 1/\lambda}$ provided $\lambda\beta < 1$.

A problem with this approach is that the condition $\lambda < 0.64$ is very restrictive when β is close to 1. Then the above method yields no λ for which the SRB-measure is absolutely continuous, and it does not give the dimension of the SRB-measure for any $\lambda \in (0.64, 1/\beta)$.

We prove that these results about absolute continuity and dimension of the SRB-measure hold for sets of (β, λ) of positive Lebesgue measure, even when $\lambda > 0.64$. This is done by extending the interval on which the transversality property (1) holds. This can be done in our setting, since in our class of maps, not every sequence $(a_k)_{k=1}^{\infty}$ with $a_k \in \{-1, 0, 1\}$ occurs in the power series $g(x) = 1 + \sum_{k=1}^{\infty} a_k x^k$ that we need to consider in the proof. To control which sequences that occur, we will use some results of Brown and Yin [6] and Kwon [7] on natural extensions of β -shifts.

The paper is organised as follows. In Section 2 we recall some facts about β -transformations and β -shifts. We then present the results of Brown and Yin, and Kwon in Sections 3 and 4. In Section 5 we state our results, and give the proofs in Section 7. The transversality property is stated and proved in Section 6.

2 β -shifts

Let $\beta > 1$ and define $f_\beta: [0, 1] \rightarrow [0, 1]$ by $f_\beta(x) = \beta x \bmod 1$. For $x \in [0, 1]$ we associate a sequence $d(x, \beta) = (d_k(x, \beta))_{k=1}^\infty$ defined by $d_k(x, \beta) = [\beta f_\beta^{k-1}(x)]$ where $[x]$ denotes the integer part of x . If $x \in [0, 1]$, then $x = \phi_\beta(d(x, \beta))$, where

$$\phi_\beta(i_1, i_2, \dots) = \sum_{k=1}^{\infty} \frac{i_k}{\beta^k}$$

This representation, among others, of real numbers was studied by Rényi [4]. He proved that there is a unique probability measure μ_β on $[0, 1]$ invariant under f_β and equivalent to Lebesgue measure. We will use this measure in Section 7.

We let S_β^+ denote the closure in the product topology of the set $\{d(x, \beta) : x \in [0, 1]\}$. The compact symbolic space S_β^+ together with the left shift σ is called a β -shift.

If we define $d_-(1, \beta)$ to be the limit in the product topology of $d(x, \beta)$ as x approaches 1, we have the equality

$$S_\beta^+ = \{(a_1, a_2, \dots) \in \{0, 1, \dots, [\beta]\}^\mathbb{N} : \sigma^k(a_1, a_2, \dots) \leq d_-(1, \beta) \forall k \geq 0\},$$

where σ is the left-shift. This was proved by Parry in [3], where he studied the β -shifts and their invariant measures. Note that $d_-(1, \beta) = d(1, \beta)$ if and only if $d(1, \beta)$ contains infinitely many non-zero digits.

The map $\phi_\beta: S_\beta^+ \rightarrow [0, 1]$ is not necessarily injective, but we have $d(\cdot, \beta) \circ f_\beta = \sigma \circ d(\cdot, \beta)$.

3 Symmetric β -shifts

Let $\beta > 1$ and consider S_β^+ . The natural extension of (S_β^+, σ) can be realised as (S_β, σ) , with

$$S_\beta = \{(\dots, a_{-1}, a_0, a_1, \dots) : (a_n, a_{n+1}, \dots) \in S_\beta^+ \forall n \in \mathbb{Z}\},$$

where σ is the left shift on bi-infinite sequences.

We will use the concept of cylinder sets only in S_β . A cylinder set is a subset of S_β of the form

$$[a_{-n}, a_{-n+1}, \dots, a_0] = \{(\dots, b_{-1}, b_0, b_1, \dots) \in S_\beta : a_k = b_k \forall k = -n, \dots, 0\}.$$

We define S_β^- to be the set

$$\begin{aligned} S_\beta^- &= \{(b_1, b_2, \dots) : \exists (a_1, a_2, \dots) \in S_\beta^+ \text{ s.t. } (\dots, b_2, b_1, a_1, a_2, \dots) \in S_\beta\} \\ &= \{(b_1, b_2, \dots) : (\dots, b_2, b_1, 0, 0, \dots) \in S_\beta\}. \end{aligned}$$

We will be interested in the set S of β for which $S_\beta^+ = S_\beta^-$. This set was considered by Brown and Yin in [6]. We now describe the properties of S that we will use later on.

Consider a sequence of the digits a and b . Any such sequence can be written in the form

$$(a^{n_1}, b, a^{n_2}, b, \dots),$$

where each n_k is a non-negative integer or ∞ . We say that such a sequence is allowable if $a \in \mathbb{N}$, $b = a - 1$, and $n_1 \geq 1$. If the sequence (n_1, n_2, \dots) is also allowable, we say that $(a^{n_1}, b, a^{n_2}, b, \dots)$ is derivable, and we call (n_1, n_2, \dots) the derived sequence of $(a^{n_1}, b, a^{n_2}, b, \dots)$. For some sequences, this operation can be carried out over and over again, generating derived sequences out of derived sequences. We have the following theorem.

Theorem 1 (Brown–Yin [6], Kwon [7]). *$\beta \in S$ if and only if $d(1, \beta)$ is derivable infinitely many times.*

The “only if”-part was proved by Brown and Yin in [6] and the “if”-part was proved by Kwon in [7]. Using this characterisation of S , Brown and Yin proved that S has the cardinality of the continuum, but its Hausdorff dimension is zero.

4 Examples

Let us find some numbers in S to see better what the theorems in Section 5 say. We do this by first finding some sequences that are infinitely derivable, and then we find the corresponding β by solving the equation $1 = \phi_\beta(d(1, \beta))$. Let us first remark that the sequence $(1, 0, 0, \dots)$ is its own derived sequence.

The sequence $d(1, \beta) = (1, 1, 0, (1, 0)^\infty)$ is clearly derivable infinitely many times. Its derived sequence is $(2, 1, 1, \dots)$, and the derived sequence of this sequence is $(1, 0, 0, \dots)$. One finds numerically that the corresponding β is given by $\beta = 1.801938\dots$ and that $1/\beta = 0.554958\dots$

There are however smaller numbers in the set S . Consider the sequence $d(1, \beta) = (1, 0, (1, 0, 0)^\infty)$. Its derived sequence is $(1, 1, 0, (1, 0)^\infty)$, which derives to $(2, 1, 1, \dots)$, and so on. Solving for β we find that $\beta = 1.558980\dots$ and $1/\beta = 0.641445\dots$

Let us now show that $\inf S = 1$. For all natural n , let β_n be such that

$$d(1, \beta_n) = (1, 0^n, (1, 0^{n+1})^\infty).$$

Then, for $n \geq 2$, the derived sequence of $d(1, \beta_n)$ is the sequence $d(1, \beta_{n-1})$. Hence all sequences $d(1, \beta_n)$ are infinitely derivable, and so $\beta_n \in S$. Moreover it is clear that $\beta_n \rightarrow 1$ as $n \rightarrow \infty$. See Table 1.

n	β_n	$1/\beta_n$
1	1.558980...	0.641445...
2	1.438417...	0.695209...
3	1.365039...	0.732580...
4	1.315114...	0.760390...
5	1.278665...	0.782066...

Table 1: Some numerical values of β_n .

5 Results

Let $0 < \lambda < 1$ and $1 < \beta < 2$. Put $Q = [0, 1) \times [0, 1)$ and define $T: Q \rightarrow Q$ by

$$T_{\beta,\lambda}(x, y) = \begin{cases} (\lambda x, \beta y) & \text{if } y < 1/\beta \\ (\lambda x + 1 - \lambda, \beta y - 1) & \text{if } y \geq 1/\beta \end{cases}.$$

Denote by ν the 2-dimensional Lebesgue measure on Q . For any $n \in \mathbb{N}$ we define the measure

$$\nu_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu \circ T_{\beta,\lambda}^{-k}.$$

The unique SRB-measure of $T_{\beta,\lambda}$ is the weak limit of ν_n as $n \rightarrow \infty$.

The SRB-measures are characterised by the property that their conditional measures along unstable manifolds are equivalent to Lebesgue measure. The existence of such measures was established for invertible maps by Pesin [8] and extended to non-invertible maps by Schmeling and Troubetzkoy [9]. We denote the SRB-measure of $T_{\beta,\lambda}$ by μ_{SRB} . It is clearly unique by a simple Hopf-argument. The support of μ_{SRB} is the set

$$\Lambda = \text{closure} \bigcap_{n=0}^{\infty} T_{\beta,\lambda}^n(Q).$$

If $\lambda\beta < 1$, it is easy to see that the Hausdorff dimension of Λ is at most $1 + \frac{\log \beta}{\log 1/\lambda}$. The following theorem states that for a set of parameters of positive Lebesgue measure, this bound is optimal.

Theorem 2. *Let $1 < \beta < 2$ and $\gamma = \inf\{\beta' \in S : \beta' \geq \beta\}$. Then for Lebesgue almost every $\lambda \in (0, 1/\gamma)$ the Hausdorff dimension of the SRB-measure of $T_{\beta,\lambda}$ is $1 + \frac{\log \beta}{\log 1/\lambda}$.*

Recall from Section 4 that $\inf S = 1$. This implies that when β gets close to 1, Theorem 2 gives the dimension of the SRB-measure for a large set of $\lambda > 0.64$, which is not obtainable using Solomyaks transversality from [5], described in the introduction.

In the area-expanding case, when $\lambda\beta > 1$, we have the following theorem.

Theorem 3. For any $\gamma \in S$, there is an $\varepsilon > 0$ such that for all β with $1/\beta \in (1/\gamma, 1/\gamma + \varepsilon)$, and Lebesgue almost every $\lambda \in (1/\beta, 1/\gamma + \varepsilon)$ the SRB-measure of $T_{\beta, \lambda}$ is absolutely continuous with respect to Lebesgue measure.

Since $\inf S = 1$ there are β arbitrarily close to 1 for which we have a set of λ of positive Lebesgue measure, where the SRB-measure is absolutely continuous. In particular, this means that for these parameters, the set Λ has positive 2-dimensional Lebesgue measure.

6 Transversality

Fix $\beta > 1$. Consider the set of power series of the form

$$g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k) x^k, \quad (2)$$

where $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$ are sequences in S_{β}^+ .

Lemma 1. There exist $\varepsilon > 0$ and $\delta > 0$ such that for any g of the form (2), $x \in [0, 1/\beta + \varepsilon]$ and $|g(x)| < \delta$ implies that $g'(x) < -\delta$.

Proof. Let us first prove the transversality property for $x \in [0, 1/\beta]$. Assume that no such δ exists. Then there is a sequence g_n of power series and a sequence of numbers $x_n \in [0, 1/\beta]$, such that $\lim_{n \rightarrow \infty} g_n(x_n) = 0$ and $\liminf_{n \rightarrow \infty} g_n'(x_n) \geq 0$.

We can take a subsequence such that g_n converges term-wise to a series $g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k) x^k$, with $\mathbf{a}, \mathbf{b} \in S_{\beta}^+$, and x_n converges to some number x_0 . Clearly, $g(x_0) = 0$ and $g'(x_0) \geq 0$, so $x_0 \neq 0$.

Let $\beta_0 = 1/x_0 \geq \beta$. Then $\mathbf{a}, \mathbf{b} \in S_{\beta_0}^+$ and $g(x_0) = 0$ implies that

$$\phi_{\beta_0}(a_1, a_2, \dots) - \phi_{\beta_0}(b_1, b_2, \dots) = \sum_{k=1}^{\infty} \frac{a_k}{\beta_0^k} - \sum_{k=1}^{\infty} \frac{b_k}{\beta_0^k} = -1.$$

Since both sums are in $[0, 1]$, we conclude that

$$\sum_{k=1}^{\infty} \frac{a_k}{\beta_0^k} = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{b_k}{\beta_0^k} = 1.$$

We must therefore have $(a_1, a_2, \dots) = (0, 0, \dots)$. This implies that $g'(x) < 0$ for all $x \in (0, 1/\beta]$, contradicting the fact that $g'(x_0) \geq 0$.

We have now proved the transversality for $x \in [0, 1/\beta]$. Let us now prove transversality for $x \in [0, 1/\beta + \varepsilon]$, where ε is to be determined. Arguing exactly as above, we get a series $g(x)$ and an x_0 such that $g(x_0) = 0$ and $g'(x_0) \geq 0$. However, this time we have $x_0 \in (0, 1/\beta + \varepsilon]$.

Let

$$h_1(x) = \sum_{k=1}^{\infty} a_k x^k \quad \text{and} \quad h_2(x) = \sum_{k=1}^{\infty} b_k x^k.$$

Then $h_2(1/\beta) \leq 1$. Moreover we have $0 \leq h_2'(x) \leq \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$. Therefore we have

$$h_2(x_0) \leq 1 + \int_{1/\beta}^{1/\beta+\varepsilon} \frac{dx}{(1-x)^2} = 1 + \frac{\varepsilon}{(1-1/\beta-\varepsilon)(1-1/\beta)}.$$

Since $h_1(x_0) - h_2(x_0) = -1$, we must have $0 \leq h_1(x_0) \leq \frac{\varepsilon}{(1-1/\beta-\varepsilon)(1-1/\beta)}$.

Since all terms in $h_1(x)$ are non-negative we must have $a_i = 0$ for $i \leq k$, where k is the largest number such that $(1/\beta + \varepsilon)^k \geq \frac{\varepsilon}{(1-1/\beta-\varepsilon)(1-1/\beta)}$. This implies that $h_1'(x) \leq \sum_{i=k+1}^{\infty} ix^{i-1} = \frac{(k+1)x^k + kx^{k+1}}{(1-x)^2}$. Moreover we have $x_0^{k+1} \leq (1/\beta + \varepsilon)^{k+1} < \frac{\varepsilon}{(1-1/\beta-\varepsilon)(1-1/\beta)}$, so that

$$h_1'(x_0) \leq \frac{\varepsilon}{(1-1/\beta+\varepsilon)(1-1/\beta)} \frac{k+1+kx_0}{x_0(1-x_0)^2} \leq \frac{\varepsilon(2k+1)}{(1-1/\beta+\varepsilon)^4 x_0}.$$

To estimate $h_2'(x_0)$ from below we argue as follows. It is clear that h_2 is strictly convex and $h_2(0) = 0$. This implies that $h_2'(x_0) > \frac{h_2(x_0)}{x_0} \geq \frac{1}{x_0}$.

Now we want to choose ε so that $h_1'(x_0) - h_2'(x_0) < 0$, to get a contradiction. Using the estimates from above, we see that it is enough to choose ε so small that

$$\frac{\varepsilon(2k+1)}{(1-1/\beta+\varepsilon)^4 x_0} - \frac{1}{x_0} < 0 \quad \iff \quad \varepsilon < \frac{(1-1/\beta+\varepsilon)^4}{2k+1}.$$

Hence, it is sufficient to choose $\varepsilon < \frac{(1-1/\beta)^4}{2k+1}$. Assume that $\varepsilon \leq \frac{1-1/\beta}{2}$. For k we have the estimate $k \leq \frac{\log \varepsilon}{\log \frac{1+1/\beta}{2}}$, so we get the sufficient condition

$$\varepsilon < \frac{(1-1/\beta)^4}{\frac{2 \log \varepsilon}{\log \frac{1+1/\beta}{2}} + 1},$$

or equivalently,

$$\frac{2}{\log \frac{1+1/\beta}{2}} \varepsilon \log \varepsilon + \varepsilon < (1-1/\beta)^4.$$

Using that $\varepsilon \leq \frac{-\varepsilon \log \varepsilon}{\log \frac{1-1/\beta}{2}}$, we see that this inequality is satisfied if

$$-\varepsilon \log \varepsilon \left(\frac{2}{\log \frac{1+1/\beta}{2}} + \frac{1}{\log \frac{1-1/\beta}{2}} \right) < (1-1/\beta)^4.$$

Finally we use that $-\varepsilon \log \varepsilon < \frac{3}{4}\sqrt{\varepsilon}$, and conclude that any

$$\varepsilon \leq \frac{16}{9} \frac{(1 - 1/\beta)^8}{\left(\frac{2}{\log \frac{2}{1+1/\beta}} + \frac{1}{\log \frac{2}{1-1/\beta}}\right)^2}$$

works. □

7 Proofs

It is not hard to see that Λ satisfies

$$\Lambda = \{ (x, y) : \exists \mathbf{a} \in S_\beta \text{ such that } x = \pi_1(\mathbf{a}, \lambda), y = \pi_2(\mathbf{a}, \beta) \},$$

where

$$\begin{aligned} \pi_1(\mathbf{a}, \lambda) &= (1 - \lambda) \sum_{k=0}^{\infty} a_{-k} \lambda^k, \\ \pi_2(\mathbf{a}, \beta) &= \sum_{k=1}^{\infty} a_k \beta^{-k}. \end{aligned}$$

Let $\pi(\mathbf{a}, \beta, \lambda) = (\pi_1(\mathbf{a}, \lambda), \pi_2(\mathbf{a}, \beta))$ and transfer the measure μ_{SRB} to a measure η on S_β by $\eta = \mu_{\text{SRB}} \circ \pi(\cdot, \beta, \lambda)$. We take a closer look at this measure before we start the proofs.

Recall, from Section 2, the probability measure μ_β on $[0, 1]$ that is invariant under f_β and equivalent to Lebesgue measure. We get a shift-invariant measure on S_β^+ by taking $\mu_\beta \circ \phi_\beta$ and it can be extended in the natural way to a shift-invariant measure η_β on S_β .

It is easy to see that the conditional measures of $\eta_\beta \circ \pi(\cdot, \beta, \lambda)^{-1}$ along the unstable manifolds are equivalent to Lebesgue measure, so by the uniqueness of μ_{SRB} , η_β must coincide with η . Thus, η is independent of λ and satisfies

$$\begin{aligned} \eta([a_{-n} \dots a_0]) &= \mu_\beta \left(\phi_\beta \left(\{ (x_i)_{i=1}^{\infty} \in S_\beta^+ : x_1 \dots x_{n+1} = a_{-n} \dots a_0 \} \right) \right) \\ &\leq K \text{diamater} \left(\phi_\beta \left(\{ (x_i)_{i=1}^{\infty} \in S_\beta^+ : x_1 \dots x_{n+1} = a_{-n} \dots a_0 \} \right) \right) \\ &\leq K \beta^{-n+1}, \end{aligned} \tag{3}$$

where $K < \infty$ is a constant.

Proof of Theorem 2. Let $\beta > 1$ and pick any $\beta' > \beta$ such that $\beta' \in S$.

Consider a sequence $\mathbf{a} \in S_\beta$ and the corresponding point $p = \pi(\mathbf{a}, \beta, \lambda)$. The local unstable manifold of \mathbf{a} is the set of sequences \mathbf{b} such that $a_k = b_k$ for $k \leq 0$. For $\lambda < 1/\beta$, π is injective on S_β so the local unstable manifold of p is unique. If $\lambda \geq 1/\beta$, then π is not injective on S_β , so the local unstable

manifold of p is not unique. In both cases the local unstable manifold of p corresponding to \mathbf{a} is the set of points $q = \pi(\mathbf{b}, \beta, \lambda)$ such that \mathbf{b} is in the unstable manifold of \mathbf{a} . It consists of vertical line segments and in some cases single points.

For η -almost every sequence \mathbf{a} the local unstable manifold of $\pi(\mathbf{a}, \beta, \lambda)$ corresponding to \mathbf{a} , contains a line segment of positive length. Note that the length does not depend on λ .

Let ω_δ be the set of sequences \mathbf{a} , such that the corresponding local unstable manifold of $\pi(\mathbf{a}, \beta, \lambda)$ has a length of at least $\delta > 0$. Take $\delta > 0$ so that ω_δ has positive η -measure. Then the set $\Omega_\delta = \pi(\omega_\delta, \beta, \lambda)$ has the same positive μ_{SRB} -measure.

Consider the restriction of μ_{SRB} to Ω_δ and project this measure to $[0, 1) \times \{0\}$. Let μ_{SRB}^s denote this projection.

Take an interval $I = (c, d)$ with $0 < c < d < 1/\beta'$. Let t be a number in $(0, 1)$. We estimate the quantity

$$J(t) = \int_I \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{1}{|x_1 - x_2|^t} d\mu_{\text{SRB}}^s(x_1) d\mu_{\text{SRB}}^s(x_2) d\lambda,$$

If this integral converges, then for Lebesgue almost every $\lambda \in I$, the dimension of μ_{SRB}^s is at least t , and so the dimension of μ_{SRB} is at least $1 + t$. Writing $J(t)$ as an integral over the symbolic space we have that

$$J(t) = \int_I \int_{\omega_\delta} \int_{\omega_\delta} \frac{1}{|\pi_1(\mathbf{a}, \lambda) - \pi_1(\mathbf{b}, \lambda)|^t} d\eta(\mathbf{a}) d\eta(\mathbf{b}) d\lambda.$$

Since η does not depend on λ we can change order of integration and write

$$J(t) = \int_{\omega_\delta} \int_{\omega_\delta} \int_I \frac{1}{|\pi_1(\mathbf{a}, \lambda) - \pi_1(\mathbf{b}, \lambda)|^t} d\lambda d\eta(\mathbf{a}) d\eta(\mathbf{b}).$$

By making use of the transversality from Lemma 1 it is apparent that for \mathbf{a} and \mathbf{b} with $a_j = b_j$ for $j = -k + 1, \dots, 0$ and $a_{-k} \neq b_{-k}$, we have

$$\begin{aligned} & \int_I \frac{1}{|\pi_1(\mathbf{a}, \lambda) - \pi_1(\mathbf{b}, \lambda)|^t} d\lambda \\ &= \int_I \lambda^{-kt} \frac{1}{|\pi_1(\sigma^{-k}\mathbf{a}, \lambda) - \pi_1(\sigma^{-k}\mathbf{b}, \lambda)|^t} d\lambda \leq Cc^{-kt}, \end{aligned}$$

where C only depends on t and β . We can write $S_\beta \times S_\beta = A \cup B$, where

$$\begin{aligned} A &= \bigcup_{k=1}^{\infty} \bigcup_{[a_{-k+1}, \dots, a_0]} [0, a_{-k+1}, \dots, a_0] \times [1, a_{-k+1}, \dots, a_0] \\ &\cup \bigcup_{k=1}^{\infty} \bigcup_{[a_{-k+1}, \dots, a_0]} [1, a_{-k+1}, \dots, a_0] \times [0, a_{-k+1}, \dots, a_0], \end{aligned}$$

and

$$B = \bigcup_{\mathbf{a} \in S_\beta} \{\mathbf{a}\} \times \{\mathbf{a}\}.$$

Since $\eta(\mathbf{a}) = 0$ for all $\mathbf{a} \in S_\beta$, we can replace $\omega_\delta \times \omega_\delta$ by A in the estimates, and we get

$$\begin{aligned} J(t) &\leq \sum_{k=1}^{\infty} \sum_{[a_{-k+1}, \dots, a_0]} 2C c^{-kt} \int_{[0, a_{-k+1}, \dots, a_0]} \int_{[1, a_{-k+1}, \dots, a_0]} d\eta d\eta \\ &\leq \sum_{k=1}^{\infty} \sum_{[a_{-k+1}, \dots, a_0]} 2CK c^{-kt} \beta^{-k} \int_{[1, a_{-k+1}, \dots, a_0]} d\eta \\ &\leq 2CK \sum_{k=0}^{\infty} c^{-kt} \beta^{-k}, \end{aligned}$$

by (3) and the fact that η is a probability measure. This series converges provided that $t < \frac{\log \beta}{\log 1/c}$.

We have now proved that for a.e. λ in $I = (c, d)$, the dimension of the SRB-measure is at least $1 + \frac{\log \beta}{\log 1/c}$. To get the result of the theorem, we let $\varepsilon > 0$ and write $I = (0, 1/\beta')$ as a union of intervals $I_n = (c_n, d_n)$ such that $\frac{\log \beta}{\log 1/c_n} > \frac{\log \beta}{\log 1/d_n} - \varepsilon$. Then the dimension is at least $1 + \frac{\log \beta}{\log c_n} \geq 1 + \frac{\log \beta}{\log 1/\lambda} - \varepsilon$ for a.e. $\lambda \in I$. Since ε and β' was arbitrary this proves the theorem. \square

Sketch of the proof of Theorem 3. The proof is analogous to the proof of Theorem 2, using the technique of Peres and Solomyak [10]. The only difference from that paper is that we have β -shifts instead of full shifts and we handle this as in the proof of Theorem 2. In particular Lemma 1 gives an interval $I = (1/\gamma, 1/\gamma + \varepsilon)$ for which the set

$$\{\lambda \in I : |g(\lambda)| \leq \rho\}$$

has Lebesgue measure at most $2\delta^{-1}\rho$, for any $\rho > 0$ and any g of the form

$$g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k)x^k,$$

where $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$ are sequences in $S_\gamma^+ = S_\gamma^-$. \square

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