

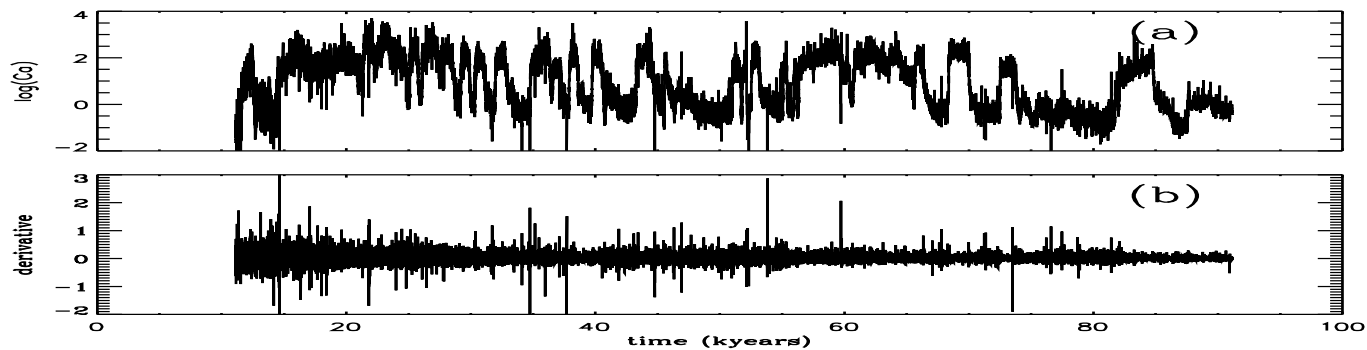
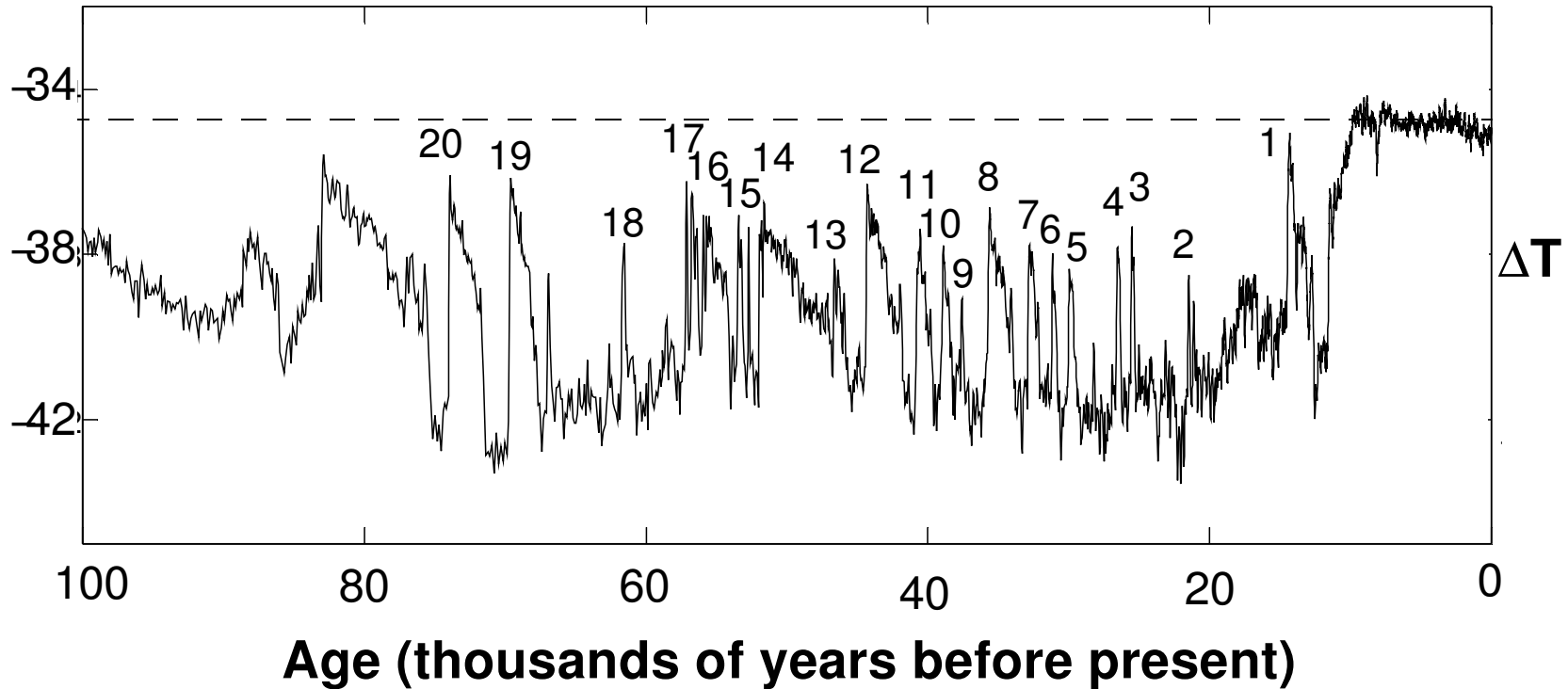
Transitions between meta-stable states of stochastic (partial) differential equations: model selection; the light tail limit of Lévy noise

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1. Dansgaard-Oeschger events: model selection



Ditlevsen (1999): which **noise model** fits best with **time series**?

2. p -Variation as test statistic

Ditlevsen's analysis: power spectrum of residua of time series

Problem: Stationarity?

Aim: better test statistics than peaks of power spectrum.

Model assumption: with some U interpret data as

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-)) ds + \varepsilon L(t) = Y^\varepsilon(t) + L^\varepsilon(t)$$

L Lévy process containing α -stable component with unknown α , Y^ε of bounded variation; test α

Idea: p -variation characteristic for fluctuation behavior of noise processes.

$$V_t^{p,n}(X) = \sum_{i=1}^{[nt]} \left| X\left(\frac{i}{n}\right) - X\left(\frac{i-1}{n}\right) \right|^p, \quad V_t^p = \lim_{n \rightarrow \infty} V_t^{p,n}$$

3. Stable Processes and their p -Variation

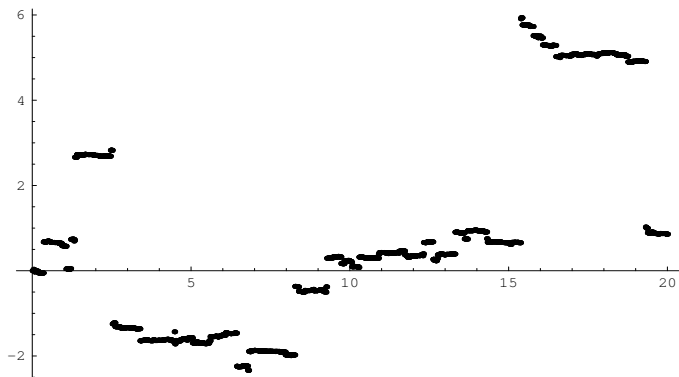
L Lévy process with characteristics (d, γ, ν) iff

$$E(\exp(iuL(t))) = \exp\left(t\left(-\frac{1}{2}du^2 + i\gamma u + \int_{\mathbf{R}} [e^{iuy} - 1 - iuy1_{\{|y|\leq 1\}}] \nu(dy)\right)\right), \quad u \in \mathbf{R}, t \geq 0,$$

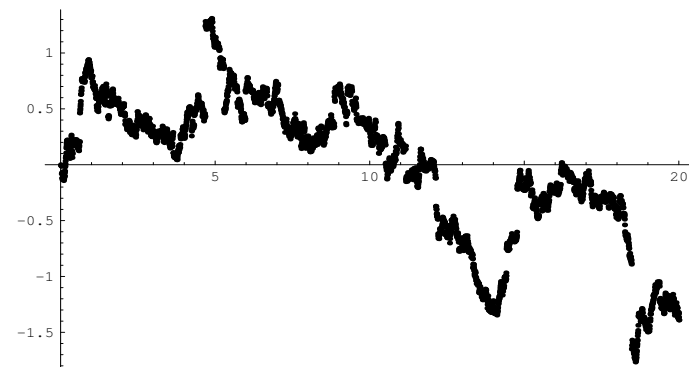
ν measure on Borel sets in \mathbf{R} with $\nu(\{0\}) = 0$, $\int_{\mathbf{R}} [|y|^2 \wedge 1] \nu(dy) < \infty$.

L α -stable symmetric Lévy process if

$$E(\exp(iuL(t))) = \exp(-c(\alpha)t|u|^\alpha), \quad \nu(dy) = \frac{1}{|y|^{\alpha+1}} dy, \quad u, y \in \mathbf{R}.$$



$\alpha = 0.75$



$\alpha = 1.75$

4. p -Variation and the Blumenthal-Gettoor Index

L α -stable process with jump measure ν ; then p -variation identified by

Blumenthal-Gettoor index

$$\beta_L = \inf\{s \geq 0 : \int_{\{|y| \leq 1\}} |y|^s \nu(dy) < \infty\}$$

$$\gamma_L = \inf\{p > 0 : V_1^p(L) < \infty\}$$

Thm 1

L symmetric α -stable. Then

$$\gamma_L = \beta_L = \alpha.$$

Problem: How to read $\gamma_L = \alpha$ off the sequence $(V_t^{p,n}(L))_{n \in \mathbb{N}}$?

Calls for results about the asymptotic behavior of the sequence.

5. The case $\alpha = 2$: Brownian motion

For $n \in \mathbf{N}$ $V_1^{p,n}(W)$ consists of n independent increments and

$$E(V_1^{p,n}(W)) = n^{1-\frac{p}{2}}E(|W(1)|^p)$$

Thm 2 (LLN type)

$$n^{-1+\frac{p}{2}}V_t^{p,n}(W) \rightarrow tE(|W(1)|^p) \quad \text{in probability,}$$

Y of bounded variation. Then also

$$n^{-1+\frac{p}{2}}V_t^{p,n}(W + Y) \rightarrow tE(|W(1)|^p) \quad \text{in probability.}$$

Thm 3 (CLT type)

$$\begin{aligned} (n^{\frac{1}{2}}[n^{-1+\frac{p}{2}}V_t^{p,n}(W) - tE(|W(1)|^p)])_{t \geq 0} &= (n^{-\frac{1}{2}+\frac{p}{2}}V_t^{p,n}(W) - n^{\frac{1}{2}}tE(|W(1)|^p))_{t \geq 0} \\ &\rightarrow ((\text{var}(|W(1)|^p))^{\frac{1}{2}}\tilde{W}(t))_{t \geq 0} \end{aligned}$$

weakly with respect to the Skorokhod metric, and an independent Brownian motion \tilde{W} .

6. The case $\alpha < 2$

(Lit: Corcuera, Nualart, Wörner '07; case $p < \alpha$ for LLN type, $p < \frac{\alpha}{2}$ for CLT type)

Problem: $p < \frac{\alpha}{2} < 1$ not satisfactory for **paleo-climatic data**! Beyond $\frac{\alpha}{2}$ no CLT type result available, no asymptotic normality, but **asymptotically of different type**.

Thm 4 (LT type)

L **α -stable** with $\alpha \in]0, 2[$. Then

$$(V_t^{p,n}(L) - B_t^n(\alpha, p))_{t \geq 0} \rightarrow \tilde{L}$$

weakly with respect to the Skorokhod metric, and an **independent $\frac{\alpha}{p}$ -stable process \tilde{L}** . Here

$$B_t^n(\alpha, p) = \begin{cases} n^{1-\frac{p}{\alpha}} t E(|L(1)|^p), & \frac{\alpha}{2} < p < \alpha, \\ nt^2 E(\sin((nt)^{-1}|L(1)|^p)), & p = \alpha, \\ 0, & \alpha < p. \end{cases}$$

Same result with $L + Y$ instead of L if Y is of finite p -variation and $\frac{\alpha}{2} < p < 1$ or $p > \alpha$.

7. Methods of Proof

- show that $(|L(n) - L(n-1)|^p)_{n \in \mathbf{N}}$ is in domain of attraction of $\frac{\alpha}{p}$ -stable law; use tails and characteristic functions

- use Aldous' criterion for tightness of sequence $(X^n)_{n \in \mathbf{N}}$ in Skorokhod metric:

$$(i) \lim_{K \rightarrow \infty} \sup_{n \in \mathbf{N}} P(\sup_{t \geq N} |X^n(t)| > K) = 0 \quad \text{for all } N \geq 0,$$

$$(ii) \lim_{\theta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{S \leq T \leq S + \theta} P(|X^n(T) - X^n(S)| \geq \varepsilon) = 0, \quad \text{for all } N \geq 0, \varepsilon > 0$$

S, T stopping times

- adding processes of smaller variation: new notion of Lipschitz continuity on large sets; comparison of small and large jumps

8. Gaussian versus α -stable revisited

W Wiener process

L symmetric α -stable Lévy process

Tail behavior

$$P(|W(1)| \geq x) \sim \exp(-cx^2)$$

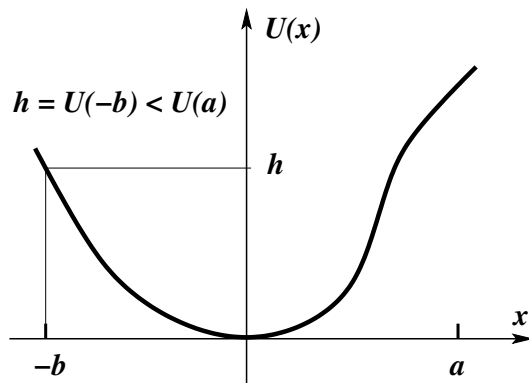
$$P(|L(1)| \geq x) \sim c \frac{1}{x^\alpha}, x \rightarrow \infty$$

$$\hat{X}^\varepsilon(t) = x - \int_0^t U'(\hat{X}^\varepsilon(s)) ds + \varepsilon W(t)$$

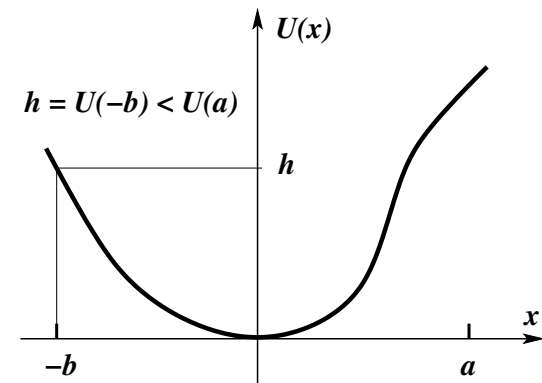
$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-)) ds + \varepsilon L(t)$$

$$\hat{\sigma} = \inf\{t \geq 0 : \hat{X}^\varepsilon(t) \notin [-b, a]\}$$

$$\sigma = \inf\{t \geq 0 : X^\varepsilon(t) \notin [-b, a]\}$$



$$\mathbf{E}_x \hat{\sigma} \approx \frac{\varepsilon \sqrt{\pi}}{|U'(-b)| \sqrt{U''(0)}} \exp\left(\frac{2h}{\varepsilon^2}\right)$$



$$\mathbf{E}_x \sigma \approx \frac{1}{\varepsilon^\alpha} \left(\int_{\mathbb{R} \setminus [-b, a]} \frac{dy}{|y|^{1+\alpha}} \right)^{-1}$$

9. Gaussian versus α -stable revisited

Conjecture: make **tails** of Lévy process **exponentially light** to recover Gaussian exit behavior.

Tail behavior

L Lévy process with jump measure having tails

$$P(|L(1)| \geq x) \sim \exp(-cx^\alpha), \quad x \rightarrow \infty$$

sub-exponential tails: $\alpha < 1$ super-exponential tails: $\alpha > 1$

Consider

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-)) ds + \varepsilon L(t)$$

$$\sigma(\varepsilon) = \inf\{t \geq 0 : X^\varepsilon(t) \notin [-1, 1]\}$$

Conjecture:

$$E_x(\sigma(\varepsilon)) \sim_{\varepsilon \rightarrow 0} \exp\left(\frac{c}{\varepsilon^2}\right) \quad \text{as } \alpha \uparrow 2.$$

10. The phase transition at $\alpha = 1$

Thm 5 [sub-exponential tails] For $\delta > 0$ there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0, t \geq 0$:

$$(1 - \delta) \exp(-C_\varepsilon^{1-\delta} t) \leq \sup_{|x| \leq 1} \mathbf{P}_x(\sigma(\varepsilon) > t) \leq \exp(-\frac{1}{2} C_\varepsilon t),$$

with $C_\varepsilon := 2\nu([\frac{1}{\varepsilon}, \infty))$. Hence for $|x| < 1$

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha \ln \mathbf{E}_x \sigma(\varepsilon) = 1.$$

Thm 6 [super-exponential tails] q_ε ε -quantile of jump measure ν . Then for $\delta > 0$ there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0, t \geq 0$:

$$(1 - \delta) \exp(-D_\varepsilon^{1-\delta} t) \leq \sup_{|x| \leq 1} \mathbf{P}_x(\sigma(\varepsilon) > t) \leq (1 + \delta) \exp(-D_\varepsilon^{1+\delta} t),$$

where $D_\varepsilon = \exp(-d_\alpha \frac{|\ln \varepsilon|}{\varepsilon q_\varepsilon})$ and $d_\alpha = \alpha(\alpha - 1)^{\frac{1}{\alpha}-1}$. Hence for $|x| < 1$

$$d_\alpha^{-1} \lim_{\varepsilon \rightarrow 0} \varepsilon |\ln \varepsilon|^{\frac{1}{\alpha}-1} \ln \mathbf{E}_x \sigma(\varepsilon) = 1.$$

11. α -stable towards Gaussian. The phase transition at $\alpha = 1$

Comparison of regimes for mean exit time

Power tails jump tails $\nu([x, \infty)) = x^{-r}$, $x \geq 1$ for some $r > 0$. Then

$$2 \lim_{\varepsilon \rightarrow 0} \varepsilon^r \mathbf{E}_x \sigma(\varepsilon) = 1.$$

Sub-exponential tails jump tails $\nu([u, \infty)) = \exp(-u^\alpha)$, $u \geq 1$, $\alpha < 1$. Then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha \ln \mathbf{E}_x \sigma(\varepsilon) = 1.$$

Super-exponential tails jump tails $\nu([u, \infty)) = \exp(-u^\alpha)$, $u \geq 1$, $\alpha > 1$. Then

$$d_\alpha^{-1} \lim_{\varepsilon \rightarrow 0} \varepsilon |\ln \varepsilon|^{\frac{1}{\alpha}-1} \ln \mathbf{E}_x \sigma(\varepsilon) = 1.$$

Gaussian diffusion no jumps, L one-dimensional Brownian motion. Then

$$\frac{1}{2}(U(-1) \wedge U(1))^{-1} \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \ln \mathbf{E}_x \sigma(\varepsilon) = 1.$$

12. Heuristics of exits: climbing versus jumping

The Brownian case

LD theory: diffusion has to climb potential in order to exit at lowest saddle point

The power tail case

for $\varepsilon > 0$ split $L = \eta^\varepsilon + \xi^\varepsilon$, compound Poisson pure jump part η^ε with jumps of height larger than $\frac{1}{\sqrt{\varepsilon}}$; small jump and Gaussian part ξ^ε with jumps not exceeding $\frac{1}{\sqrt{\varepsilon}}$; exit asymptotically due to one big jump, as shown in first talk

The case of exponential tails

for $\varepsilon > 0$ split $L = \eta^\varepsilon + \xi^\varepsilon$, compound Poisson pure jump part η^ε with jumps of height larger than g_ε ; small jump and Gaussian part ξ^ε with jumps not exceeding this bound;

choose g_ε individually according to sub- and super-exponential tails

show that exit before time T while not returning to an interval around stable fixed point 0 of radius $\delta > 0$ requires that either increments of ξ^ε exceed certain bounds (for which probability is small enough), or sum of large jumps before time T exceeds bound $1 - \delta$

in any case large jumps responsible for exits

13. Heuristics of exits: climbing versus jumping

N_T random number of large jumps before time T

W_i jump $n^o i$, $i \in \mathbf{N}$.

N_T Poisson with expectation $\beta_\varepsilon T$, where $\beta_\varepsilon = \nu([-g_\varepsilon, g_\varepsilon]^c) = 2 \exp(-x^\alpha)$

For n fixed, probability that sum of large jumps exceeds bound $1 - \delta$ estimated by

$$P(N_t > n) + \sum_{k=1}^n P(N_T = k) P\left(\sum_{i=1}^k |\varepsilon W_i| > 1 - \delta\right)$$

Idea for estimation:

for $n \in \mathbf{N}$

$$P(N_T > n) \leq (1 + \delta) \exp(-n \ln n) \quad (\text{Stirling's formula})$$

choose $n = n_\varepsilon$ suitably

14. Heuristics of exits: climbing versus jumping

essential term to estimate for $1 \leq k \leq n_\varepsilon$

$$P\left(\sum_{i=1}^k |\varepsilon W_i| > 1 - \delta\right)$$

law of i.i.d. random variables $(|W_i|)_{i \in \mathbb{N}}$: $\beta_\varepsilon^{-1} 2\nu|_{[g_\varepsilon, \infty[}$

hence

$$\begin{aligned} P\left(\sum_{i=1}^k |\varepsilon W_i| > 1 - \delta\right) &\leq \beta_\varepsilon^{-k} \exp\left(-\inf\left\{\sum_{i=1}^k x_i^\alpha : \sum_{i=1}^k x_i \geq \frac{1 - \delta}{\varepsilon}, x_i \in [g_\varepsilon, \infty[\right\}\right) \\ &= \beta_\varepsilon^{-k} \exp\left(-\inf\left\{\sum_{i=1}^k x_i^\alpha : \sum_{i=1}^k x_i = \frac{1 - \delta}{\varepsilon}, x_i \in [g_\varepsilon, \infty[\right\}\right) \end{aligned}$$

minimization problem in the exponent of this estimate causes phase transition

15. Heuristics of exits: climbing versus jumping

By suitable choice of g_ε : lower boundary for x_i in inf can be taken 0.

sub-exponential tails

$$\inf\left\{\sum_{i=1}^k x_i^\alpha : \sum_{i=1}^k x_i = 1, x_i \geq 0\right\} = 1$$

The **minimum** is taken on the **boundary of the simplex**, and $x_i = \frac{1}{n}, 1 \leq i \leq n$, corresponds to **maximum** of the function

$$(x_1, \dots, x_n) \mapsto \sum_{i=1}^n x_i^\alpha$$

16. Heuristics of exits: climbing versus jumping

Super-exponential tails

$$\inf \left\{ \sum_{i=1}^k x_i^\alpha : \sum_{i=1}^k x_i = 1, x_i \geq 0 \right\} = n \left(\frac{1}{n} \right)^\alpha$$

The **minimum** is taken for $x_i = \frac{1}{n}, 1 \leq i \leq n$, the **unique local minimum** of the function

$$(x_1, \dots, x_n) \mapsto \sum_{i=1}^n x_i^\alpha$$

Bifurcation in the asymptotic behavior:

phase transition due to switch from **concavity** to **convexity** at $\alpha = 1$ of

$$x \mapsto x^\alpha, \quad x \geq 0,$$

big jumps of the Lévy process **govern asymptotic behavior**

$\alpha < 1$: **biggest jump** responsible for exit

$\alpha > 1$: cumulative action of **several large jumps** responsible for exit