Fractal Geometry and Dynamics
Institut Mittag-Leffler

Exponential sums and Weighted ergodic Theorems

Ai-Hua FAN

University of Picardie, France

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Outline

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2. Four types of weight
3. Oscillating weights
4. Topological Polynomial WWT

Problems

A motivation

Multifractal analysis of weighted Birkhoff averages :

$$\dim_H\{x \in X : N^{-1}\sum_{n=0}^{N-1} w_n f(T^n x) \to \alpha\} = ?$$

- If T is shift on $\{0,1\}^\infty$, $f(x) = x_0$ and w_n =Möbius, then the dimension is equal to

$$1 - \frac{6}{\pi^2} + \frac{6}{\pi^2} H\left(\frac{1}{2} - \frac{\pi^2}{12}\right), \quad |\alpha| \le \frac{6}{\pi^2}$$

- But no result for $f(x) = x_0 x_1$!
- General method for $S_N(x) := \sum_{n=1}^N f_n(T^n x)$ (Fan 1997, JSP) :

$$\frac{e^{\beta S_N(x)}}{\int e^{\beta S_N(x)} dx} dx \longrightarrow \mu_\beta. \quad (\text{chaos} \;=\; \text{Gibbs measure})$$

- Hilbert ergodic transform $\sum_{n=1}^\infty \frac{f(T^n x)}{n}$ (Fan-Schmeling)
- Ergodic series $\sum a_n f(T^n x)$ (Fan, Cuny-Fan).

Weighted ergodic averages

- $T : X \to X$ a dynamics ; $w := (w_n) \subset \mathbb{C}$ a sequence of weights
- Object of study : weighted ergodic means

$$A_N^{(w)} f(x) := \frac{\sum_{n=0}^{N-1} w_n f(T^n x)}{\sum_{n=0}^{N-1} |w_n|}.$$

[we replace $\sum_0^{N-1} |w_n|$ by N if $\sum_0^{N-1} |w_n| \sim \sigma N$]

- TDS : X metric and compact ; T continuous.
- MPDS : $(X, \mathcal{B}, \nu)$ measure space, T measurable, $\nu = \nu \circ T^{-1}$.

Problem 1 : Let $(X, \mathcal{B}, \nu, T)$ be a MPDS and $f \in L^1(\nu)$.
Does $A_N^{(w)} f(x)$ converge for almost every $x \in X$?

Problem 2 : Assume (X, T) be a TDS and $f \in C(X)$.
Does $A_N^{(w)} f(x)$ converges for every $x \in X$?

Ergodic weights and Good weights

Definition 1 : (w_n) is L^2-ergodic if for any MPDS $(X, \mathcal{B}, \mu, T)$ and any $f \in L^2$,

$$L^2 - \lim A_N^{(w)} f = \mathbb{E}(f|\mathcal{J}).$$

(w_n) is L^2-good if the above limit exists (not necessarily equals to $\mathbb{E}(f|\mathcal{J})$).

Definition 2 : (w_n) is L^p-good for a.e. convergence $(1 \leq p \leq \infty)$ if for any MPDS $(X, \mathcal{B}, \mu, T)$ and any $f \in L^p$,

$$\mu - a.e. \quad \lim A_N^{(w)} f(x) \quad \text{exists.}$$

Three classical results

Birkhoff Ergodic Theorem (1931), $w_n \equiv 1$

Let $(X, \mathcal{B}, \nu, T)$ be a MPDS and $f \in L^1(\mu)$. Then ν-a.e.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) \quad \text{exists} \ .$$

Wiener-Wintner Ergodic Theorem (1941), $w_n = e^{2\pi i n \alpha}$

Let $(X, \mathcal{B}, \nu, T)$ be a MPDS and $f \in L^1(\mu)$. Then ν-a.e.

$$\forall \alpha \in \mathbb{R}, \quad \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i j \alpha} f(T^j x) \quad \text{exists} \ .$$

Krilov-Bogoliubov Theorem (1937), $w_n \equiv 1$

Let (X, T) be a uniquely ergodic TDS and $f \in C(X)$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \int f d\nu \quad \text{uniformly on x.}$$

Exponential sums

$$\widehat{A}_N^{(w)}(t) := \frac{\sum_{n=0}^{N-1} w_n e^{2\pi int}}{\sum_{n=0}^{N-1} |w_n|}.$$

Fact 1 :

$$\left\| \sum_{n=0}^{N-1} a_n f \circ T^n \right\|_{L^2(\mu)} = \left\| \sum_{n=0}^{N-1} a_n e^{2\pi int} \right\|_{L^2(\sigma_f)}.$$

Fact 2 : (w_n) is L^2-ergodic iff

$$\forall t \in (0,1), \quad \lim \widehat{A}_N^{(w)}(t) = 0.$$

(w_n) is L^2-good iff the above limit exists.

1. $w_n = 1_{\{n^2\}}(n)$: L^2-good but not L^2-ergodic (easy), L^p-good for a.e. convergence for all $p > 1$ (Bourgain 1988).
2. almost all return time sequences $1_A(S^n\omega)$ are L^1-good for a.e. convergence (Bourgain-Furstenberg-Katznelson-Ornstein 1989).

An exponential sum

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{2\pi i n^2 t} = 0 \text{ if } t \notin \mathbb{Q},$$

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{2\pi i n^2 t} = q^{-1} \sum_{r=0}^{q-1} e^{2\pi i r^2 p/q} \text{ if } t = p/q.$$

Four types of weight

Davenport weights

The Davenport exponent of (w_n), denoted $H((w_n))$, is the best $h > 0$ such that

$$\sup_{t\in[0,1)} \left| \sum_{n=0}^{N-1} w_n e^{2\pi i n t} \right| = O_h(N/\log^h N).$$

Fan (2016)

If $(w_n) \in \ell^\infty$ with $H((w_n)) > \frac{1}{2}$, then it is L^1-good for a.e. convergence. Actually, for any MPDS $(X, \mathcal{B}, \nu, T)$ and and $f \in L^1(\nu)$, we have ν-a.e.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^n x) = 0.$$

1. Sarnak : Möbius weight ; El Abdalaoui et al : $H > 1$.
2. Davenport-Erdös-LeVeque (1965) : $\lim_N \frac{1}{N} \sum_{n=1}^{N} \xi_n = 0$ a.s. if

$$\|\xi_n\|_\infty = O(1), \quad \sum \frac{\|\xi_1 + \cdots + \xi_n\|_2^2}{n^3} < \infty.$$

Gelfond weights

The Gelfond exponent of (w_n), denoted $D((w_n))$, is the best $d > 0$ such that

$$\sup_{n_0 \geq 0} \sup_{t \in [0,1)} \left| \sum_{n=n_0}^{n_0+N-1} w_n e^{2\pi i n t} \right| = O_d(N^d).$$

Fan (2016)

Suppose $(w_n) \in \ell^\infty$ and $D := D((w_n)) < 1$. Let $(X, \mathcal{B}, \nu, T)$ be a MPDS and $f \in L^2(\nu)$. Then ν-a.e.

$$\sum_{n=0}^{N-1} w_n f(T^n x) = o(N^D (\log N)^2 (\log \log N)^{1+\epsilon}).$$

1. Gelfond (1968) : $D((t_n)) = \frac{\log 3}{\log 4}$ for Thue-Morse sequence (t_n).
2. Fan : The Gelfond exponent < 1 for generalized Thue-Morse sequences
$$t_n^{(c)} := e^{2\pi i c s_2(n)}, \quad c \in (0,1).$$

Problem : Find the exact Gelfond exponent (Wei Xiao SHEN : $c = 1/4$).

Bernoulli weights

Let $(p_n) \subset [0,1]$ such that $\sum p_n = +\infty$. Let (ξ_n) be a sequence of independent random variables such that

$$P(\xi_n = 1) = p_n = 1 - P(\xi_n = 0).$$

The random infinite subset of integers

$$\Lambda(\omega) = \{n \in \mathbb{N} : \xi_n = 1\}$$

was first considered in harmonic analysis by Katznelson and Malliavin (1966).

1. Kahane and Katznelson (2008) : if $np_n = O(1)$, a.s $\Lambda(\omega)$ is not L^2-ergodic, not dense in the Bohr group, and is a Sidon set.
2. Kahane and Katznelson (2008) : if $np_n \to \infty$ and if np_n is slowly varying, a.s $\Lambda(\omega)$ is L^2-ergodic (equivalently uniformly distributed in the Bohr group).

Bernoulli weights (continued)

1. Fan and Schneider (2010) : Suppose $\frac{\sum_{n=1}^{N} p_n}{\log N} \to \infty$. Then

$$a.s.\ \Lambda(\omega) \text{ is } L^2\text{−ergodic} \iff \forall t \in (0,1),\ \lim_N \frac{\sum_{n=1}^{N} p_n e^{2\pi i n t}}{\sum_{n=1}^{N} p_n} = 0.$$

2. That is the case : $p_n = (\log\log\cdots\log n)^\delta / n$ with $\delta > 0$.
3. Bourgain (1988) : a.s $\Lambda(\omega)$ is L^p-good for a.e. convergence in the case

$$p_n = \frac{(\log\log n)^B}{n} \quad \text{with} \quad B > \frac{1}{p-1}.$$

4. Kahane-Katznelson (2008) : a.s. $\Lambda(\omega)$ is not L^2-ergodic if

$$\sup_N \left| \sum_{n=1}^{N} p_n e^{2\pi i n t} \right| \le A \log \frac{1}{|\sin \pi t|} + B.$$

Bernoulli weights : KK-multiplicative chaos

Bernouilli chaos :

$$\prod_{n=1}^{N} \frac{f(nt)^{\xi_n}}{1+p_n(f(nt)-1)} dt \longrightarrow \mu_\omega$$

Poisson chaos :

$$\prod_{n=1}^{N} \frac{f(nt)^{\eta_n}}{\exp(p_n(f(nt)-1))} dt \longrightarrow \nu_\omega$$

where f is the triangular function $\epsilon^{-2}(\epsilon - |t|)_+$ with mean value 1.

Conjecture. Assume $p_n \downarrow$. a.s $\Lambda(\omega)$ is L^2-ergodic iff $\log N = o(\sum_{n=1}^{N} p_n)$.

Counter-example : $p_{2n} = 1$, $p_{2n+1} = 0$.

DPP weights

Let $f : \mathbb{R}/\mathbb{Z} \to [0,1]$ such that $\sigma := \int f(x)dx \in (0,1)$. There exists a point process $\mathcal{X}$ on $\mathbb{Z}$ such that for any integers $n_1 < n_2 < \cdots < n_d$ we have

$$P(n_1, \cdots, n_d \in \mathcal{X}) = \det(\widehat{f}(n_i - n_j))_{1 \le i,j \le d}.$$

Define

$$\xi_n = 1_{\{n \in \mathcal{X}\}}.$$

Fan-Fan-Qiu (2017)

1. If $P \in \mathbb{Z}[X]$ with $P(\mathbb{N}) \subset \mathbb{N}$, a.s. $P(\mathcal{X} \cap \mathbb{N})$ is L^2-ergodic and L^p-good for a.e. convergence ($p > 1$).
2. The Gelfond exponent of $P(\mathcal{X} \cap \mathbb{N})$ equals a.s. to $\frac{1}{2}$.
3. a.s. $\mathcal{X}$ is not syndetic (has non bounded gaps).
4. a.s. $\mathcal{X}$ has density σ, but Banach density 1.
5. a.s. $\mathcal{X} + \mathcal{X} = \mathbb{Z}$.

Oscillating weights

Orthogonality

Def. Let $(a_n), (b_n) \subset \mathbb{C}$. We say $(a_n) \perp (b_n)$ if

$$\lim_N \frac{1}{N} \sum_{n=1}^{N} a_n b_n = 0.$$

Def. Let $(w_n) \subset \mathbb{C}$ and (X, T) a TDS. We say $(w_n) \perp (X, T)$ if

$$\forall f \in C(X), \forall x \in X, \quad (w_n) \perp (f(T^n x)).$$

Def. The Möbius function $\mu : \mathbb{N} \to \mathbb{N}$ is a multiplicative arithmetic function defined by

$$\mu(1) = 1,\ \mu(p) = -1 \text{ and } \mu(p^k) = 0 \text{ if } k \geq 2$$

Sarnak's conjecture. $h_{\text{top}}(X, T) = 0 \Longrightarrow \mu \perp (X, T)$.

Problem. Which weights can replace Möbius function ?

$$?(w_n) \perp (X, T); \quad (w_n) \perp ?(X, T)$$

Oscillating sequences

Def. A sequence $w = (w_n)$ is said to be d-oscillating ($d \geq 1$) if for any $P \in \mathbb{R}_d[t]$ we have $w \perp (e^{2\pi i P(n)})$, i.e.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} w_n e^{2\pi i P(n)} = 0.$$

We say w is ∞-oscillating if it is d-oscillating for all $d \geq 1$.

- 1-oscillating VS L^2-ergodic, Davenport, Gelfond.
- (Fan-Jiang 2015) : 1-oscillation/Sarnak's conjecture.
- (Lesigne 1993) : Almost all realizations $h(S^n x)$ of a totally ergodic dynamics are ∞-oscillating if $h \in L^1$ and $\int h = 0$.
- (R. X. Shi 2017) : for almost all $\beta > 1$, $e^{2\pi i \beta^n}$ shares the Chowla property and is then ∞-osc, and orthogonal to all TDS of zero entropy.

Multiple ergodic averages

Fan (2017)

Suppose (w_n) is ∞-oscillating and $Tx = Ax + b$ is an afffine map of zero entropy on a compact abelian group X. Let $f_1, \cdots, f_\ell \in C(X)$ and $q_1, \cdots, q_\ell \in \mathbb{Q}[t]$ taking values in $\mathbb{N}$. We have

$$\forall x \in X, \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} w_n f_1(T^{q_1(n)}x) \cdots f_\ell(T^{q_\ell(n)}x) = 0.$$

NB. Liu-Sarnak (2015) : $w_n = \mu(n)$, $\ell = 1$.

Topological Polynomial WWT

Robinson theorem

Robinson Theorem (1994)

Let (X, T) be a uniquely ergodic TDS with invariant measure ν. If $\lambda := e^{-2\pi i\alpha} \notin E_0(X) \setminus G_0(X)$ and $f \in C(T)$, then

$$\forall x, \quad \lim_N \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i\alpha} f(T^n x) \quad \text{exists.}$$

The limit is actually uniform on x to the projection onto the eigenspace of λ.

1. (Robinson) There exist (X, T), $\lambda \in E_0(X) \setminus G_0(X)$, $f \in C(X)$ such that the above limit doesn't exists for some $x \in X$.
2. If $E_0(X) = G_0(X)$, the above limit exists. If, moreover $\int w d\nu = 0$, then $(w(T^n x))$ is 1-oscillating for all $x \in X$.

Problem Find f such that $f(T^n x)$ is d-oscillating for EVERY x.

Quasi discrete spectrum of Parry-Hahn (1965)

Let (X, T) be a transitive TDS. Consider the group

$$G(X) = \{f \in C(X) : \forall x \in X, |f(x)| = 1\}.$$

Let $G_0(X)$ be the subgroup of all eigenvalues of $T : C(X) \to C(X)$ and inductively

$$G_k(X) = \{f \in C(X) : Tf \cdot \overline{f} \in G_{k-1}(X)\}, \quad \forall k \geq 1.$$

We say (X, T) has quasi discrete spectrum if $G_\infty(X) := \bigcup G_k(X)$ spans a dense space of $C(X)$.

Elements of $G_k(X)$ are called (continuous) k-th generalized eigenfunctions.
(Hoarse-Parry) Minimal affine maps on connected compact abelian groups have QDS.

Quasi discrete spectrum of Abramov (1962)

Let $(X, \mathcal{B}, \nu, T)$ be an ergodic MPDS. Consider the group

$$E(X) = \{f \in L^2(\nu) : \text{almost all } x \in X, |f(x)| = 1\}.$$

Let $E_0(X)$ be the subgroup of all eigenvalues of $T : L^2(\nu) \to L^2(\nu)$ and inductively

$$E_k(X) = \{f \in L^2(\nu) : Tf \cdot \overline{f} \in E_{k-1}(X)\}, \quad \forall k \geq 1.$$

We say $(X, \mathcal{B}, \nu, T)$ has quasi discrete spectrum if $E_\infty(X) := \bigcup E_k(X)$ spans a dense space of $L^2(\nu)$.

Elements of $E_k(X)$ are called (measurable) generalized eigenfunctions.

- (Halmos and von Neumann 1940's) introduction of Generalized Eigenfunctions.
- $(X, \mathcal{B}, \nu, T)$ has discrete spectrum if $E_1(X)$ spans a dense space.

Topological polynomial WWT

Theorem (Fan, 2016)

Suppose

- (Total unique ergodicity) (X, T^j) is unique ergodic for all $j \geq 1$;
- (Coincidence of spectra) $E_\ell(T) = G_\ell(T)$ for all $\ell \geq 1$

Let $k \geq 1$. Then for any $f \in C(X)$, we have

$$f \in G_k(T)^\perp \Longleftrightarrow \forall x \in X, \quad (f(T^n x)) \in OSC_k.$$

∞-oscillating nilsequences

Theorem (Fan, 2016)

Let G be a connected nilpotent Lie group, Γ a discrete cocompact subgroup of G and $g \in G$. Let T_g be the translation on $X := G/\Gamma$ defined by $x\Gamma \mapsto gx\Gamma$. Suppose (X, T_g) is uniquely ergodic. Then for any $F \in C(X)$ such that $F \perp G_\infty(T)$ and any $x \in X$, the sequence $(F(g^n x\Gamma))$ is ∞-oscillating.

Example : Heisenberg group

$$N = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ & 1 & \mathbb{R} \\ & & 1 \end{pmatrix}, \Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ & 1 & \mathbb{Z} \\ & & 1 \end{pmatrix}, g = \begin{pmatrix} 1 & \alpha & \delta \\ & 1 & \beta \\ & & 1 \end{pmatrix}, g^n = \begin{pmatrix} 1 & n\alpha & * \\ & 1 & n\beta \\ & & 1 \end{pmatrix}$$

- T_g is ergodic iff T_g is uniquely ergodic iff $1, \alpha, \beta$ are $\mathbb{Q}$-independent. So, T_g is totally uniquely ergodic. $e^{2\pi i \alpha n[\beta n]}$ is ∞-oscillating.
- $(N/\Gamma, T_g)$ doesn't have quasi discrete spectrum (Parry-Hahn).

A proof

Suppose $\sum p_n = \infty$. Let $\mathcal{X} := (\xi_n)$ be (p_n)-Bernoulli.

Let $(X, \mathcal{B}, \mu, T)$ be a MPDS and $f \in L^2(\mu)$. Then a.s.

$$\lim_{N\to\infty} \frac{1}{|\mathcal{X} \cap [1,N]|} \sum_{n\in\mathcal{X}\cap[1,N]} f(T^n x) = \mathbb{E}(f|\mathcal{J}) \quad \mu{-}a.e. \text{ and in } L^2.$$

Proof. Let $s_n = p_1 + \cdots + p_n$ and fix x, we consider the martingale

$$X_n^x = \sum_{k=1}^{n} \frac{(\xi_k - p_k) f(T^k x)}{s_k}.$$

$$\int \sup_n \mathbb{E}|X_n^x|^2 d\mu(x) = \int \sum_{k=1}^{\infty} \frac{|f(T^k x)|^2 p_k (1-p_k)}{s_k^2} d\mu(x) < \infty.$$

So, for μ-a.e. x, X_n^x is L^2-bounded then convergent a.s. Therefore $\sum \frac{(\xi_n - p_n) f(T^n x)}{s_n}$ converges a.e. a.s. We conclude by Fubini, Kronecker's lemma and Birkhoff's theorem.

But it is not true : a.s. $\forall$ MPDS $\forall f$ a.e.

Open problem : Existence of $p_n \downarrow 0$ s.t. L^1-goodness for a.e.convergence.